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ScienceDirect
Indagationes Mathematicae 21 (2011) 30–39
indagationes mathematicae
www.elsevier.com/locate/indag

Irreducibility of A-hypergeometric systems

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Received 17 September 2010; received in revised form 6 December 2010; accepted 17 December 2010

Communicated by Prof. T.H. Koornwinder

Abstract

We give an elementary proof of the Gel'fand–Kapranov–Zelevinsky theorem that non-resonant A-hypergeometric systems are irreducible. We also provide a proof of a converse statement.

1. Introduction

Let $A \subset \mathbb{Z}^r$ (with $r \geq 1$) be a finite set such that:

1. The $\mathbb{Z}$-span of A is $\mathbb{Z}^r$.
2. There exists a linear form h such that $h(\mathbf{a}) = 1$ for all $\mathbf{a} \in A$.

Let $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{C}^r$. At the end of the 1980's Gel'fand, Kapranov and Zelevinsky [4–6] developed a theory of hypergeometric functions and equations which uses A and α as starting data. It turns out that the resulting equations contain the classical cases of Appell, Horn, Lauricella and Aomoto hypergeometric functions.

Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$ (with $N \geq r$). Writing the vectors $\mathbf{a}_i$ in column form we get the so-called A-matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rN} \end{pmatrix}.$$

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0019-3577/$ - see front matter
doi:10.1016/j.indag.2010.12.002

For $i = 1, 2, \ldots, r$ consider the first-order differential operators

$$Z_i = a_{i1}v_1\partial_1 + a_{i2}v_2\partial_2 + \cdots + a_{iN}v_N\partial_N$$

where $\partial_j = \frac{\partial}{\partial v_j}$ for all j.

Let

$$L = \{(l_1, \ldots, l_N) \in \mathbb{Z}^N \mid l_1\mathbf{a}_1 + l_2\mathbf{a}_2 + \cdots + l_N\mathbf{a}_N = \mathbf{0}\}$$

be the lattice of integer relations between the elements of A. For every $\mathbf{l} \in L$ we define the so-called box operator

$$\Box_{\mathbf{l}} = \prod_{l_i>0} \partial_i^{l_i} - \prod_{l_i<0} \partial_i^{-l_i}.$$

The system of differential equations

$$\begin{aligned} &(Z_i - \alpha_i)\Phi = 0 \quad (i = 1, \ldots, r) \\ &\Box_{\mathbf{l}}\Phi = 0 \quad \mathbf{l} \in L \end{aligned}$$

is known as the system of A-*hypergeometric differential equations* and we denote it by $H_A(\alpha)$. We would like to remark that independently, and at around the same time, B. Dwork arrived at a similar setup for generalized hypergeometric functions. The system of A-hypergeometric equations is implicit in his book [3].

Let $K = \mathbb{C}(v_1, \ldots, v_N)$ and let $\mathcal{H}_A(\alpha)$ be the left ideal in $K[\partial_1, \ldots, \partial_N]$ generated by the operators from $H_A(\alpha)$. The quotient $K[\partial_1, \ldots, \partial_N]/\mathcal{H}_A(\alpha)$ is a K-module. Its K-rank is called the rank of the system $H_A(\alpha)$. Furthermore, the system is called *non-resonant* if the set $\alpha + \mathbb{Z}^r$ has empty intersection with the boundary of $C(A)$, the cone given by

$$C(A) = \{\lambda_1\mathbf{a}_1 + \cdots + \lambda_N\mathbf{a}_N | \lambda_i \in \mathbb{R}_{\geq 0}\}.$$

The system is called *resonant* if the intersection is non-empty.

In [6] (corrected in [8]) and [1, Corollary 5.20] the following theorem is shown.

Theorem 1.1 (*GKZ, Adolphson*). *Suppose at least one of the following conditions holds:*

1. *The toric ideal I_A in $\mathbb{C}[\partial_1, \ldots, \partial_N]$ generated by the box operators has the Cohen–Macaulay property.*
2. *The system $H_A(\alpha)$ is non-resonant.*

Then the rank of $H_A(\alpha)$ is finite and equals the volume of the convex hull $Q(A)$ of the points of A. The volume is normalized such that a minimal $(r-1)$-simplex with integer vertices in $h(\mathbf{x}) = 1$ has volume 1.

Let p be a generic point in $(\mathbb{C}^*)^N$ (the space with coordinates $v_1, \ldots, v_N$). Then it is known that the dimension of the $\mathbb{C}$-vector space of local power series solutions around p of $H_A(\alpha)$ equals the rank of $H_A(\alpha)$.

The K-module $K[\partial_1, \ldots, \partial_N]/\mathcal{H}_A(\alpha)$ has a natural left action by the operators ∂_i, so it is a D-module. We shall say that the system $H_A(\alpha)$ is *irreducible* if this D-module has no submodules beside 0 and the module itself. We call it *reducible* otherwise. Gel'fand, Kapranov and Zelevinsky proved in [7, Thm 2.11] the following beautiful theorem.

Theorem 1.2 (*GKZ, 1990*). *Suppose that the system $H_A(\alpha)$ is non-resonant. Then $H_A(\alpha)$ is irreducible.*

The proof uses the theory of perverse sheaves and is hard to follow for someone without this background. It is the purpose of the present paper to give a more elementary proof of this theorem. This is done in Section 5.

We say that the convex hull $Q(A)$ of A is a *pyramid* if there is a linear subspace of $\mathbb{R}^r$ of dimension $r-1$ which contains all points of A except one. Note that the case $r=1$ (and hence $N=1$) is considered a pyramid by this definition. We now formulate a statement converse to Theorem 1.2,

Theorem 1.3. *Suppose that the convex hull $Q(A)$ is not a pyramid. If the system $H_A(\alpha)$ is resonant, then it is reducible.*

As far as we could see the latter theorem is not stated as such in the papers of Gel'fand, Kapranov and Zelevinsky or any other papers. In an original version of the present paper, Theorem 1.3 contained the condition that the toric ideal I_A generated by the box operators should have the Cohen–Macaulay (CM) property. In a very recent manuscript, Schulze and Walther [10], after the appearance of a preprint of this text, have managed to prove Theorems 1.2 and 1.3 without the CM condition and where the homogeneity condition (2) on A is dropped. The results in that paper rely heavily on Walther's paper [13], which in its turn uses Koszul complexes and homological algebra around A-hypergeometric systems. In the present paper I decided to adopt the version of Theorem 1.3 without the CM condition, but with the homogeneity (2) still present. As for the proof, in the text I shall refer to the necessary places in [13], but give a self-contained proof for Theorem 1.3 under the CM condition. Since this proof is much easier than the proof of the unconditional theorem, I hope it has some value in itself.

To show that the pyramidal condition is really necessary, suppose for example that $Q(A)$ is a pyramid with top $\mathbf{a}_1$ and bottom the convex hull of $\tilde{A}=\{\mathbf{a}_2,\ldots,\mathbf{a}_N\}$. Suppose that $\alpha\in Q(\tilde{A})$, so our system is resonant. Introduce new coordinates in $\mathbb{R}^r$ such that all points of $\tilde{A}$ are in the space $x_1=0$. The new A-matrix now has the form

$$A=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rN} \end{pmatrix}$$

and $\alpha=(0,\alpha_2,\ldots,\alpha_r)$. Then one easily sees that the box operators do not contain ∂_1 and the first homogeneity equation reads $v_1\partial_1 F=0$, i.e. all solutions are independent of v_1. Hence all solutions are of the form $F(v_2,\ldots,v_N)$ and they satisfy the hypergeometric system $H_{\tilde{A}}(\tilde{\alpha})$ where $\tilde{\alpha}=(\alpha_2,\ldots,\alpha_r)$. If $\tilde{\alpha}$ does not lie in a face of $C(\tilde{A})$ modulo $\mathbb{Z}^{r-1}$, the system $H_{\tilde{A}}(\tilde{\alpha})$ is irreducible by Theorem 1.2 and so is $H_A(\alpha)$. This explains the pyramidal condition in Theorem 1.3.

2. Contiguity

Consider the system $H_A(\alpha)$,

$$\Box_{\mathbf{l}}\Phi=0,\quad \mathbf{l}\in L,\qquad Z_j\Phi=\alpha_j\Phi,\quad j=1,\ldots,r.$$

Apply the operator ∂_i from the left. We obtain

$$\Box_{\mathbf{l}}\partial_i\Phi=0,\quad \mathbf{l}\in L,\qquad Z_j\partial_i\Phi=(\alpha_j-a_{ji})\partial_i\Phi,\quad j=1,\ldots,r.$$

In other words, $F\mapsto\partial_i F$ maps the solution space of $H_A(\alpha)$ to the solution space of $H_A(\alpha-\mathbf{a}_i)$.

We can phrase this alternatively in terms of D-modules. Denote by $\mathcal{H}_A(\alpha)$ the left ideal in $K[\partial]$ generated by the hypergeometric operators $\Box_{\mathbf{l}}$ and Z_j. Then the map $P \mapsto P\partial_i$ gives a D-module homomorphism $K[\partial]/\mathcal{H}_A(\alpha - \mathbf{a}_i) \to K[\partial]/\mathcal{H}_A(\alpha)$. We are interested in the cases when this is a D-module isomorphism or, equivalently, whether $F \mapsto \partial_i F$ gives an isomorphism of solution spaces.

The following theorem was first proven by Dwork in his book [3, Thm 6.9.1]. Another proof was given in [2, Lemma 7.10]. We present an adaptation of Dwork's ideas into the notation of the present paper.

Theorem 2.1 (*Dwork*)**.** *Suppose $H_A(\alpha)$ is non-resonant. Then the map $F \mapsto \partial_i F$ yields an isomorphism between the solution spaces of $H_A(\alpha)$ and $H_A(\alpha - \mathbf{a}_i)$.*

For the proof we need an extra lemma and some notation. Suppose the positive cone $C(A)$ is given by a finite set $\mathcal{F}$ of linear inequalities $l(\mathbf{x}) \geq 0,\ l \in \mathcal{F}$. Assume moreover that the linear forms l are integral valued on $\mathbb{Z}^r$ and normalize them so that the greatest common divisor of all values is 1.

Consider the integral points in $C(A)$. It is not necessarily true that every point in $C(A) \cap \mathbb{Z}^r$ is a linear combination of the $\mathbf{a}_i$ with non-negative integer coefficients. However, we do have the following lemma.

Lemma 2.2. *There exists a point $\mathbf{p} \in C(A) \cap \mathbb{Z}^r$ such that $(\mathbf{p} + C(A)) \cap \mathbb{Z}^r \subset \mathbb{Z}_{\geq 0}A$ where $\mathbb{Z}_{\geq 0}A$ is the span of A with non-negative integer coefficients.*

This is a well-known lemma and we include a proof of it in order to make the paper self-contained.

Proof. It is clear that there exists a positive integer δ such that for any point $(\lambda_1, \ldots, \lambda_N) \in L \otimes \mathbb{R}$ there exists $(m_1, \ldots, m_n) \in L$ such that $|m_i - \lambda_i| \leq \delta$. Let us take $\mathbf{p} = \delta(\mathbf{a}_1 + \cdots + \mathbf{a}_N)$.

Suppose we are given a point $\mathbf{n} \in (\mathbf{p} + C(A)) \cap \mathbb{Z}^r$. Then there exist $\lambda_i \in \mathbb{R}_{\geq \delta}$ and integers $n_1 \ldots, n_N$ such that $\mathbf{n} = \lambda_1 \mathbf{a}_1 + \cdots + \lambda_N \mathbf{a}_N = n_1 \mathbf{a}_1 + \cdots + n_N \mathbf{a}_N$. The point $(\lambda_1 - n_1, \ldots, \lambda_N - n_N)$ lies in $L \otimes \mathbb{R}$. Hence there exists $(m_1, \ldots, m_N) \in L$ such that $|\lambda_i - n_i - m_i| \leq \delta$ for $i = 1, \ldots, N$. Since $\lambda_i \geq \delta$ for every i we find that $n_i + m_i \geq 0$. Hence $\mathbf{n} = n_1 \mathbf{a}_1 + \cdots + n_N \mathbf{a}_N = (n_1 + m_1)\mathbf{a}_1 + \cdots + (n_N + m_N)\mathbf{a}_N$, and hence $\mathbf{n} \in \mathbb{Z}_{\geq 0}A$. $\Box$

Proof of Theorem 2.1. We will construct an operator $P \in K[\partial]$ such that $P\partial_i \equiv 1 (\mathrm{mod}\ \mathcal{H}_A(\alpha))$. In particular, $F \mapsto P(F)$ would be the inverse of ∂_i, which establishes the isomorphism.

For any $l \in \mathcal{F}$ and any differential operator $\partial^{\mathbf{u}} = \partial_1^{u_1} \cdots \partial_N^{u_N}$ we define the valuation $val_l(\partial^{\mathbf{u}}) = \sum_{j=1}^N u_j l(\mathbf{a}_j)$. More generally, for any differential operator $P \in K[\partial]$ we define $val_l(P)$ to be the minimal valuation of all terms in P.

Let $\mathbf{p}$ be as in Lemma 2.2. Suppose $val_l(\partial^{\mathbf{w}}) \geq val_l(\partial^{\mathbf{u}}) + l(\mathbf{p})$ for every $l \in \mathcal{F}$. Hence $\sum_{j=1}^N l((w_j - u_j)\mathbf{a}_j) \geq l(\mathbf{p})$ for all $l \in \mathcal{F}$. So, according to Lemma 2.2, the sum $\sum_{j=1}^N (w_j - u_j)\mathbf{a}_j$ is a lattice point in $\mathbb{Z}_{\geq 0}A$. Hence there exist non-negative integers w'_j such that $\sum_{j=1}^N w'_j \mathbf{a}_j = \sum_{j=1}^N (w_j - u_j)\mathbf{a}_j$. Hence $\partial^{\mathbf{w}}$ is equivalent to $\partial^{\mathbf{w}'}\partial^{\mathbf{u}}$ modulo the box operator $\Box_{\mathbf{w}-\mathbf{w}'-\mathbf{u}}$.

Let $l \in \mathcal{F}$ be given. We show that modulo the ideal $\mathcal{H}_A(\alpha)$, the operator $\partial^{\mathbf{u}}$ is equivalent to an operator P such that $val_l(P) > val_l(\partial^{\mathbf{u}})$ and $val_{l'}(P) \geq v_{l'}(\partial^{\mathbf{u}})$ for all $l' \in \mathcal{F},\ l' \neq l$. Let

$Z_l = -l(\alpha) + \sum_{j=1}^{N} l(\mathbf{a}_j)v_j\partial_j$. Notice that $Z_l \in \mathcal{H}_A(\alpha)$ and $\partial^{\mathbf{u}} Z_l = Z_l\partial^{\mathbf{u}} + l(\mathbf{u})\partial^{\mathbf{u}}$. Hence,

$$\sum_{j=1}^{N} l(\mathbf{a}_j)v_j\partial_j\partial^{\mathbf{u}} \equiv l(\alpha - \mathbf{u})\partial^{\mathbf{u}} \pmod{\mathcal{H}_A(\alpha)}.$$

For each term on the left we have $l(\mathbf{a}_j) \neq 0 \Rightarrow val_l(\partial_j\partial^{\mathbf{u}}) > val_l(\partial^{\mathbf{u}})$. Since, by non-resonance, $l(\alpha - \mathbf{u}) \neq 0$, our assertion is proven. Choose $k_l \in \mathbb{Z}_{\geq 0}$ for every $l \in \mathcal{F}$. By repeated application of our principle we see that any monomial $\partial^{\mathbf{u}}$ is equivalent modulo $\mathcal{H}_A(\alpha)$ to an operator Q with $val_l(Q) \geq k_l + val_l(\partial^{\mathbf{u}})$ for all $l \in \mathcal{F}$.

In particular, there exists an operator Q, equivalent to 1 and $val_l(Q) \geq val_l(\partial_i) + l(\mathbf{p})$ for every $l \in \mathcal{F}$. Hence Q is equivalent to an operator $P\partial_i$. Summarizing, $1 \equiv P\partial_i (\mathrm{mod}\ \mathcal{H}_A(\alpha))$. So $F \mapsto \partial_i F$ is injective on the solution space of $H_A(\alpha)$. $\square$

There is another instance when $F \mapsto \partial_i F$ is an isomorphism of solution spaces.

Theorem 2.3. *Suppose that $Q(A)$ is not a pyramid and that $H_A(\alpha)$ is an irreducible system. Suppose also that the toric ideal I_A generated by the box operators has the Cohen–Macaulay property. Then $F \mapsto \partial_i F$ gives an isomorphism of solution spaces of $H_A(\alpha)$ and $H_A(\alpha - \mathbf{a}_i)$.*

It is very likely that this theorem also holds without the Cohen–Macaulay condition and that the ingredients for its proof are contained in [13]. However, we were not able to reconstruct it. We restrict our proof to the case with Cohen–Macaulay condition, which is self-contained.

Proof. Since $H_A(\alpha)$ is irreducible, the kernel of $F \mapsto \partial_i F$ is either trivial or the entire solution space. In the first case we are done, the map is injective and the solution spaces have the same dimension (because I_A has the Cohen–Macaulay property).

Now suppose we are in the second case, when $\partial_i F \equiv 0$ for every solution F of $H_A(\alpha)$. This is equivalent to the statement $\partial_i \in \mathcal{H}_A(\alpha)$ or equivalently, $v_i\partial_i \in \mathcal{H}_A(\alpha)$. Let us write

$$v_i\partial_i = \sum_{\lambda} A_\lambda \Box_\lambda + \sum_{j=1}^{r} B_j(Z_j - \alpha_j).$$

The summation over the $\lambda \in L$ is supposed to be a finite summation. Let us assume that we have chosen the A_λ and B_i such that the maximum of the orders of the B_i is minimal. Call this minimum m. We assert that $m = 0$. Suppose $m > 0$.

We now work over the polynomial ring $R = \mathbb{C}(\mathbf{v})[X_1, \ldots, X_N]$. For any differential operator P we write $P(\mathbf{X})$ for the polynomial that we get after we replace ∂_j by X_j for all j in P. Write I_A for the ideal in R generated by $\Box_{\mathbf{l}}(\mathbf{X})$. Since the quotient ring R/I_A is a Cohen–Macaulay ring, the linear forms $Z_i(\mathbf{X})$ form a regular sequence. In particular this means that if $P_1Z_1(\mathbf{X}) + \cdots + P_rZ_r(\mathbf{X}) = 0$ in R/I_A, then there exist polynomials η_{ij} with $\eta_{ij} = -\eta_{ji}$ such that $P_i = \sum_{j=1}^{r} \eta_{ij}Z_j(\mathbf{X})$ for $i = 1, \ldots, r$.

Let us return to the A_λ and B_j above. Note that $(A_\lambda\Box_\lambda)(\mathbf{X}) = A_\lambda(\mathbf{X})\Box_\lambda(\mathbf{X})$ since the box operators have constant coefficients. Denote the order m part of each B_j by $B_j^{(m)}$. Then the $m + 1$st degree part of $\sum_j (B_j(Z_j - \alpha_i))(\mathbf{X})$ reads $\sum_j B_j^{(m)}(\mathbf{X})Z_j(\mathbf{X})$. Since $m + 1 > 1$ this degree $m + 1$ part is zero in R/I_A. Hence there exist polynomials η_{jk} with $\eta_{jk} = -\eta_{kj}$ such that $B_j^{(m)}(\mathbf{X}) = \sum_{k=1}^{r} \eta_{jk}Z_k(\mathbf{X})$ in R/I_A. Denote by E_{jk} the differential operator which we get after we replace the variables X_b in η_{jk} by their counterparts ∂_b. Define $\tilde{B}_j = B_j - \sum_{k=1}^{r} E_{jk}(Z_k - \alpha_k)$

and note that $\tilde{B}_j$ has order $<m$. Moreover,

$$\sum_{j=1}^{r} B_j(Z_j-\alpha_j)=\sum_{j=1}^{r}\tilde{B}_j(Z_j-\alpha_j)+\sum_{j,k=1}^{r}E_{jk}(Z_j-\alpha_j)(Z_k-\alpha_k).$$

The last sum, by virtue of the antisymmetry of the E_{jk} and the fact that $Z_j-\alpha_j$ and $Z_k-\alpha_k$ commute for all j,k, is equal to zero in R/I_A. Hence

$$v_i\partial_i\equiv\sum_{j=1}^{r}\tilde{B}_j(Z_j-\alpha_j)\pmod{I_A}$$

where the $\tilde{B}_i$ have order $<m$. This contradicts the minimality of m. Therefore we conclude that $m=0$. In other words there exist $b_i\in\mathbb{C}(\mathbf{v})$ such that $v_i\partial_i\equiv\sum_{j=1}^{r}b_j(Z_j-\alpha_j)\pmod{I_A}$. Since the box operators all have order ≥ 2, this relation holds exactly.

We now show that the b_j are constant. To see this, write the operators Z_j explicitly and compare the parts of order 1. Writing $Z_j=a_{j1}v_1\partial_1+\cdots+a_{jN}v_N\partial_n$ (as in the introduction) we get

$$\begin{aligned}v_1\partial_i&=\sum_{j=1}^{r}b_j(a_{j1}v_1\partial_1+\cdots+a_{jN}v_N\partial_n)\\&=\sum_{k=1}^{N}(b_1a_{1k}+\cdots+b_ra_{rk})v_k\partial_k.\end{aligned}$$

Comparison of coefficients of the ∂_k gives us the equations

$$\sum_{j=1}^{r}b_ja_{jk}=\delta_{ik}\quad k=1,\ldots,N$$

in $b_1,\ldots,b_r$ where δ_{ik} is the Kronecker delta. We know that there exists a solution; the rank of the coefficient matrix is r and hence there is a unique solution. Since all coefficients are constant the b_j must be constant. In other words there exists a linear form m on $\mathbb{R}^r$ such that $m(\mathbf{a}_j)=0$ for all $j\neq i$ and $m(\mathbf{a}_i)=1$. But this implies that $Q(A)$ is a pyramid with $\mathbf{a}_i$ as a top. □

3. Resonant systems

In this section we prove Theorem 1.3. In the final stage of the proof we will need the following straightforward lemma.

Lemma 3.1. *Let F be a face of $Q(A)$ of codimension ≥ 1. If A is not a pyramid then the volume of F is strictly less than the volume of $Q(A)$.*

Proof of Theorem 1.3. Suppose that $H_A(\alpha)$ is resonant and irreducible. Then, by Theorem 3.7 of [13], $H_A(\beta)$ is irreducible for any $\beta\in\mathbb{R}^r$ with $\beta\equiv\alpha\pmod{\mathbb{Z}^r}$. In the case when the toric ideal I_A is Cohen–Macaulay this also follows from Theorem 2.3 which says that for any i the map $F\mapsto\partial_iF$ is an isomorphism of solution spaces of $H_A(\alpha)$ and $H_A(\alpha-\mathbf{a}_i)$ and the fact that the $\mathbf{a}_i$ span $\mathbb{Z}^r$.

Since the system is resonant there exists such a β in a face F of $Q(A)$. Suppose $A\cap F=\{\mathbf{a}_1,\ldots,\mathbf{a}_t\}$. We assert that there exist non-trivial solutions of the form $f=f(v_1,\ldots,v_t)$. Suppose that $s=\text{rank}(\mathbf{a}_1,\ldots,\mathbf{a}_t)$ (after reordering indices if necessary). By an $SL(r,\mathbb{Z})$ change of coordinates we can ensure that F is given by $x_{s+1}=\cdots=x_r=0$. Then the coordinate a_{ij} of $\mathbf{a}_j$ is zero for $i=s+1,\ldots,r$ and $j=1,\ldots,t$. Also, $\beta_{s+1}=\cdots=\beta_r=0$. A solution of the form $f = f(v_1, \ldots, v_t)$ satisfies the homogeneity equations

$$\left(-\beta_i + \sum_{j=1}^{t} a_{ij} v_j \partial_j\right) f = 0, \quad i = 1, \ldots, s.$$

Notice that the homogeneity equations with $i = s+1, \ldots, r$ are trivially satisfied by f.

Consider the box operator $\Box_\lambda$ with $\lambda \in L$. Write $\lambda = (\lambda_1, \ldots, \lambda_N)$. The positive support is the set of indices i where $\lambda_i > 0$; the negative support is the set of indices i where $\lambda_i < 0$.

Suppose the positive support is contained in $1, 2, \ldots, t$. Then $\sum_{\lambda_i > 0} \lambda_i \mathbf{a}_i$ is in $\mathcal{F}$. Hence $-\sum_{\lambda_i < 0} \lambda_i \mathbf{a}_i$ is also in F. Since F is a face, all non-zero terms of the latter have index $\le t$. So the negative support is also in $1, 2, \ldots, t$. Hence

$$\text{negative support} \subset \{1, \ldots, t\} \iff \text{positive support} \subset \{1, \ldots, t\}.$$

If the positive and negative supports of λ contain indices $>t$ then $f(v_1, \ldots, v_t)$ satisfies $\Box_\lambda f = 0$ trivially.

Define a new set $\tilde{A} = \{\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_t\} \subset \mathbb{Z}^s$ where $\tilde{\mathbf{a}}_j$ is the projection of $\mathbf{a}_j$ on its first s coordinates. Define a new parameter $\tilde{\beta}$ similarly. The solutions of the form $f(v_1, \ldots, v_t)$ of the original GKZ system satisfy the new GKZ system corresponding to $H_{\tilde{A}}(\tilde{\beta})$. We make the additional assumption that we choose β and F in such a way that t is minimal. It can be 0 when $\alpha \in \mathbb{Z}^r$ and $\beta = 0$. When $t > 0$ the new system $H_{\tilde{A}}(\tilde{\beta})$ is non-resonant and the rank equals the volume of F. However, by Lemma 3.1 this volume is strictly less than $Q(A)$ and therefore strictly less than the rank of $H_A(\beta)$. Hence the additional equations $\partial_i F = 0$ for $i > t$ define a proper subspace of the solution space of $H_A(\beta)$. So the system $H(\beta)$ is reducible and we have a contradiction.

In the extreme case when $t = 0$ we have $\alpha \in \mathbb{Z}^r$ and we can take $\beta = \mathbf{0}$. The system then has the constant solution 1. Since, by the pyramidal condition, the rank of $H_A(\beta)$ must be at least 2, we again get a contradiction with irreducibility of $H_A(\beta)$. $\Box$

4. Series solutions

Just as in the classical literature we like to be able to display explicit series solutions for the A-hypergeometric system. In GKZ theory one chooses $\gamma = (\gamma_1, \ldots, \gamma_N)$ such that $\alpha = \gamma_1 \mathbf{a}_1 + \cdots + \gamma_N \mathbf{a}_N$ and one takes as the starting point the formal Laurent series

$$\Phi_{L,\gamma}(v_1, \ldots, v_N) = \sum_{\mathbf{l} \in L} \frac{\mathbf{v}^{\mathbf{l}+\gamma}}{\Gamma(\mathbf{l} + \gamma + \mathbf{1})}$$

where we use the shorthand notation

$$\frac{\mathbf{v}^{\mathbf{l}+\gamma}}{\Gamma(\mathbf{l} + \gamma + \mathbf{1})} = \frac{v_1^{l_1+\gamma_1} \cdots v_N^{l_N+\gamma_N}}{\Gamma(l_1 + \gamma_1 + 1) \cdots \Gamma(l_N + \gamma_N + 1)}.$$

Note that there is a freedom of choice in γ through shifts over $L \otimes \mathbb{R}$. A priori this series is formal, i.e. there is no convergence. However by making proper choices for γ we do end up with series that have an open domain of convergence in $\mathbb{C}^N$.

Choose a subset $\mathcal{I} \subset \{1, 2, \ldots, N\}$ with $|\mathcal{I}| = N - r$ such that $\mathbf{a}_i$ with $i \notin \mathcal{I}$ are linearly independent. In [5, Prop 1] we find the following proposition (albeit in a different formulation).

Proposition 4.1. *Define $\pi_{\mathcal{I}} : L \to \mathbb{Z}^{N-r}$ by $\mathbf{l} \mapsto (l_i)_{i \in \mathcal{I}}$. Then $\pi_{\mathcal{I}}$ is injective and its image is a sublattice of $\mathbb{Z}^{N-r}$ of index $|\det(\mathbf{a}_i)_{i \notin \mathcal{I}}|$.*

We define $\Delta_{\mathcal{I}} = |\det(\mathbf{a}_i)_{i \notin \mathcal{I}}|$. Choose γ such that $\gamma_i \in \mathbb{Z}$ for $i \in \mathcal{I}$. The formal solution series

$$\Phi = \sum_{\mathbf{l} \in L} \prod_{i \in \mathcal{I}} \frac{v_i^{l_i+\gamma_i}}{\Gamma(l_i+\gamma_i+1)} \prod_{i \notin \mathcal{I}} \frac{v_i^{l_i+\gamma_i}}{\Gamma(l_i+\gamma_i+1)}$$

is now a power series because the summation runs over the polytope $l_i+\gamma_i \geq 0$ for $i \in \mathcal{I}$ and the other l_j are dependent on l_i, $i \in \mathcal{I}$. Terms where $l_i+\gamma_i < 0$ do not occur because $1/\Gamma(l_i+\gamma_i+1)$ is zero when $l_i+\gamma_i$ is a negative integer. By slight abuse of language will call the corresponding simplicial cone $l_i \geq 0$ for $i \in \mathcal{I}$ the *sector of summation* with index $\mathcal{I}$.

Denote the resulting series expansion by $\Phi_{\mathcal{I},\gamma}$. The following statement, which is a direct consequence of estimates using Stirling's formula for Γ, says that there is a non-trivial region of convergence.

Proposition 4.2. *Let $(\rho_1, \ldots, \rho_N) \in \mathbb{R}^N$ be such that $\rho_1 l_1 + \cdots + \rho_N l_N > 0$ for all $\mathbf{l} \in L$ with $\forall i \in \mathcal{I} : l_i \geq 0$. Then $\Phi_{\mathcal{I},\gamma}$ converges for all $\mathbf{v} \in \mathbb{C}^N$ with $|v_i| = t^{\rho_i}$ for sufficiently small $t \in \mathbb{R}_{>0}$.*

A proof can be found for example in [12]. An N-tuple ρ such that $\rho_1 l_1 + \cdots + \rho_N l_N > 0$ for all $\mathbf{l} \in L$ with $\forall i \in \mathcal{I} : l_i \geq 0$ will be called a *convergence direction*.

The following statement is a direct corollary of Proposition 4.1.

Corollary 4.3. *With the notation as above, the number of distinct choices modulo L for γ such that $\forall i \in \mathcal{I} : \gamma_i \in \mathbb{Z}$ is $\Delta_{\mathcal{I}}$.*

There is one important assumption that we need in order to make this approach work, namely the guarantee that not too many of the arguments $l_i + \gamma_i$ are a negative integer. Otherwise we might even end up with a power series which is identically zero. The best way to do is to impose the condition $\gamma_i \notin \mathbb{Z}$ for $i \notin \mathcal{I}$. Geometrically, since $\alpha = \sum_{i=1}^N \gamma_i \mathbf{a}_i \equiv \sum_{i \notin \mathcal{I}} \gamma_i \mathbf{a}_i \pmod{\mathbb{Z}^r}$, this condition comes down to the requirement that $\alpha + \mathbb{Z}^r$ does not contain points in a face of the simplicial cone spanned by $\mathbf{a}_i$ with $i \notin \mathcal{I}$. Unfortunately this is stronger than the requirement of non-resonance of $H_A(\alpha)$, as faces of the individual simplicial cones, not necessarily on the boundary of $C(A)$, are involved. However, the condition of non-resonance does turn out to be useful.

Proposition 4.4. *Let $\mathcal{I}$ be as above and suppose that the system $H_A(\alpha)$ is non-resonant. Then there exists an open cone C in $L \otimes \mathbb{R}$ such the series $\Phi_{\mathcal{I},\gamma}$ has non-zero terms for all $\mathbf{l} \in C$.*

Proof. We will use the following observation. The ith coordinate l_i of $\mathbf{l} \in L$ can be considered as a linear form on L. We shall do so in this proof. Suppose we have a relation $\sum_{i=1}^N \lambda_i l_i = 0$ with $\lambda_i \in \mathbb{R}$. Then there exists a linear form m on $\mathbb{R}^r$ such that $m(\mathbf{a}_i) = \lambda_i$ for $i = 1, \ldots, N$.

Denote the set of indices i for which $\gamma_i \notin \mathbb{Z}$ by R. When $|R| = r$ all terms of $\Phi_{\mathcal{I},\gamma}$ are non-zero and our statement is proven. Suppose $|R| < r$. Then there exist linear relations between the forms l_i with $\lambda_i = 0$ when $i \in R$. Consider the convex hull D of the forms l_i for $i \notin R$. Suppose that this hull contains the trivial form $\mathbf{0}$. In other words, there exists a relation with coefficients $\lambda_i \in \mathbb{R}_{\geq 0}$, not all zero, with $\lambda_i = 0$ for all $i \in R$. Hence, by our observation, there exists a non-trivial form m on $\mathbb{R}^r$ such that $m(\mathbf{a}_i) = \lambda_i$ for all i. Hence we have found a non-trivial form with $m(\mathbf{a}_i) \geq 0$ for all i and $m(\mathbf{a}_i) = 0$ for $i \in R$. Therefore the $\mathbb{R}_{\geq 0}$-span of $\mathbf{a}_i$, $i \in R$, is contained in a face F of $C(A)$. Furthermore, $\alpha = \sum_{i=1}^N \gamma_i \mathbf{a}_i \equiv \sum_{i \in R} \gamma_i \mathbf{a}_i \pmod{\mathbb{Z}^r}$. Hence modulo $\mathbb{Z}^r$

the vector α lies in the face F. This contradicts our non-resonance assumption and therefore the convex hull D does not contain $\mathbf{0}$. Consequently, the set of inequalities $l_i \geq 0,\ i \notin R$, has a polyhedral cone with non-empty interior as the solution space in $\mathbb{R}^{N-r}$. The terms in $\Phi_{\mathcal{I},\gamma}$ with indices inside this cone are non-zero. $\square$

The following theorem was one of the discoveries made by Gel'fand, Kapranov and Zelevinsky.

Theorem 4.5. *Let ρ be a convergence direction. Then there exists a regular triangulation T of A such that the summation sectors for which ρ is a convergence direction are given by J^c where J runs through the $(r-1)$-simplices in T.*

In order to proceed it is now important that different choices of summation sectors give independent series solutions. For this we require the following condition.

Definition 4.6. For any subset $J \subset \{1, 2, \ldots, N\}$ define $A_J = \{\mathbf{a}_j | j \in J\}$ and let $Q(A_J)$ be the convex hull of the points in A_J.

Let T be a regular triangulation of A. The parameter α will be called T-non-resonant if $\alpha + \mathbb{Z}^r$ does not contain a point on the boundary of any cone over an $(r-1)$-simplex $Q(A_J)$ with $J \in T$. We call the system T-resonant otherwise.

Notice that the T-non-resonance condition implies the non-resonance condition. Let us assume that α is T-non-resonant. For any $\mathcal{I} = J^c$ with $J \in T$ and one of the $\mathrm{Vol}(Q(A_J))$ choices of γ we get the series $\Phi_{\mathcal{I},\gamma}$.

Theorem 4.7. *Under the T-non-resonance condition the power series solutions just constructed form a basis of solutions of $H_A(\alpha)$.*

Proof. To show that the solutions are independent it suffices to show that for any two distinct summation sectors $\mathcal{I}$ and $\mathcal{I}'$ the values of $\gamma_1, \ldots, \gamma_N$, as chosen in $\Phi_{\mathcal{I}}$ and $\Phi_{\mathcal{I}'}$, are distinct modulo the lattice L. Suppose they are not distinct modulo L. Then there exists an index $i \in \mathcal{I}'$, but $i \notin \mathcal{I}$, such that $\gamma_i \in \mathbb{Z}$. But this is contradicted by our T-non-resonance assumption.

For every $J \in T$ we get $\mathrm{Vol}(Q(A_J))$ solutions by the different choices of γ. Summing over $J \in T$ shows that we obtain $\sum_{J \in T} \mathrm{Vol}(Q(A_J)) = \mathrm{Vol}(Q(A))$ independent solutions. $\square$

Given a regular triangulation we can consider the union of all summation domains in L. More precisely, define $\mathrm{supp}(T)$ to be the convex closure of $\cup_{J \in T}\{\mathbf{l} \in L | l_i \geq 0 \text{ for all } i \in J^c\}$. Then $\mathrm{supp}(T)$ will be the common support of all series $\Phi_{\mathcal{I}}$ with $I^c \in T$. More precisely, denote the set of power series in $\mathbf{v}$ with support in $\mathrm{supp}(T)$ by $\mathbb{C}[[\mathbf{v}]]_T$. Note that this set forms a ring by the obvious multiplication. The coefficient ring $\mathbb{C}$ can be extended to the ring of finite linear combinations of powers $\mathbf{v}^\gamma$ to get the ring denoted by $\mathbb{C}[\mathbf{v}^\gamma][[\mathbf{v}]]_T$. Note that the series constructed above all belong to this ring. In order to incorporate solutions in T-resonant cases we need to introduce logarithms of the v_i and extend our ring to the so-called Nilsson ring $\mathbb{C}[\log(\mathbf{v}), \mathbf{v}^\gamma][[\mathbf{v}]]_T$ (see [11]). We quote the following theorem.

Theorem 4.8 (*Saito–Sturmfels–Takayama*)**.** *Suppose $H_A(\alpha)$ is non-resonant. For any regular triangulation of $Q(A)$ there exists a space of solutions to $H_A(\alpha)$ in the ring $\mathbb{C}[\log(\mathbf{v}), \mathbf{v}^\gamma][[\mathbf{v}]]_T$ of $\mathbb{C}$-dimension* $\mathrm{Vol}(A)$.

By a theorem of Adolphson [1, Corollary 5.20] the rank of $H_A(\alpha)$ equals $\mathrm{Vol}(A)$ when the system is non-resonant. Hence we get the following.

Corollary 4.9. *When $H_A(\alpha)$ is non-resonant the system of solutions in Theorem 4.8 provides a basis of solutions to $H_A(\alpha)$ in $\mathbb{C}[\log(\mathbf{v}), \mathbf{v}^\gamma][[\mathbf{v}]]_T$.*

5. Non-resonant systems

In this section we prove Theorem 1.2. Suppose we have a non-resonant system and an operator $P \in K[\partial]$ which annihilates a non-trivial solution f in the solution space of $H_A(\alpha)$.

First we show the existence of such an f which has the form of a power series of the type Φ_γ, as in the previous section. Fix a convergence direction $\rho_1, \ldots, \rho_N$ and let T be the corresponding regular triangulation of $Q(A)$.

Corollary 4.9 provides a basis of solutions in $\mathbb{C}[\log(\mathbf{v}), \mathbf{v}^\gamma][[\mathbf{v}]]_T$. Consider these solutions as analytic functions on an open neighbourhood of the set V given by $|v_1| = t^{\rho_1}, \ldots, |v_N| = t^{\rho_N}$ for t sufficiently small. The fundamental group $\pi_1(V)$ is generated by $v_j = t^{\rho_j} e^{2\pi i x}$, $x \in [0, 1]$ for any j and v_i fixed for all $i \neq j$. The corresponding monodromy group is an abelian group and so is its restriction to the common solution space of $H_A(\alpha)$ and $P(f) = 0$. Since the monodromy group is abelian, there exists a one-dimensional invariant subspace. The character, with which $\pi_1(V)$ acts on this space, uniquely determines a solution of the form Φ_γ.

In the terminology of [9, Thm 2.7] the solution Φ_γ is a fully supported solution by virtue of Proposition 4.4. Theorem 2.7 of [9] implies that the operator P lies in $\mathcal{H}_A(\alpha)$. Hence we conclude that $H_A(\alpha)$ is irreducible. □

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