Hele-Shaw Flows with Anisotropic Surface Tension

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Abstract—In this note, we study Hele-Shaw flows in the presence of anisotropic surface tension when the fluid domain is bounded. The flows are driven by a sink, by a multipole, or solely by anisotropic surface tension. For a sink flow, we show that if the center of mass of the initial domain is not located at a certain point which is determined by the anisotropic surface tension and intensity of the sink, then either the solution will break down before all the fluid is sucked out or the fluid domain will eventually become unbounded in diameter. For a multipole driven flow, we prove that if the anisotropic surface tension, the order, and intensity of the multipole do not satisfy a certain equality, either the flow will develop finite-time singularities or the fluid domain will become unbounded in diameter as time goes to infinity. For a flow driven purely by anisotropic surface tension, we show that the center of mass of the fluid domain moves in a constant velocity, which is determined explicitly.

Keywords—Hele-Shaw, Anisotropy, Singularity, Multipole.

1. INTRODUCTION

Flow in a porous medium is a challenging scientific problem of great technological importance. For the case of two immiscible fluids, it is well known that the interface will be unstable when a less viscous fluid drives a more viscous fluid. This instability is responsible for water flooding of oil wells, and is of great interest to oil reservoir engineering and many other applications.

When surface tension is neglected, all two-dimensional sink flows except the circular solution break down before all the fluid is sucked out from the sink and all multipole driven flows develop finite time singularities [1,2]. In fact, cusp formations are observed in generic Hele-Shaw solutions when surface tension is absent [3].

Surface tension is naturally included to mimic the realistic physical situations. When surface tension is isotropic, i.e., it is a nonzero constant, only a few rigorous analytical results have been obtained [1,4–10] while tremendous numerical and asymptotic progress has been made [11,12].

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In particular, analytical, numerical, and experimental evidence shows that generic sink flows and multipole driven flows develop singularities via interface reaching the sink or multipole [7,9,13-15]. When surface tension is not a constant, there are some experimental, numerical, and asymptotic studies of Hele-Shaw flows [16-19]. However, no rigorous analytical result has been obtained up to this point.

The purpose of this research is to study the behavior of Hele-Shaw flows in the presence of anisotropic surface tension. Anisotropy can be imposed on Hele Shaw flows by engraving a grid on one of the grass plates [18-21]. The Hele-Shaw model can also be considered as the "quasistatic" limit of a solidification model, where the surface tension term of (2) has an interpretation as the Gibbs-Thomson surface energy. In solidification, anisotropy comes from the existence of a crystal lattice structure [16,22,23].

In this note, we show that, for a sink flow, if the center of mass of the initial domain is not located at a certain point which is determined by the anisotropic surface tension and intensity of the sink, either the flow will break down before all the fluid is sucked out or the fluid domain will eventually become unbounded in diameter. For a multipole driven flow, we prove that if the anisotropic surface tension, the order, and intensity of the multipole do not satisfy a certain equality, either the flow will develop finite time singularities or the fluid domain will become unbounded as time goes to infinity. We do not determine the mechanism of the singularities. There are several possibilities. It could be caused by cusp formation at the portion of the interface where anisotropic surface tension vanishes, by the interface reaching the sink, or perhaps by some other means.

Finally, we study flows driven solely by anisotropic surface tension. We show that the center of mass of the fluid domain travels in a constant velocity. For some anisotropic surface tension, the constant velocity is not zero. We discuss the possibility of finite time singularities and possibility of solution existing for all time.

2. EQUATIONS OF MOTION

The mathematics of two-dimensional Hele-Shaw flows driven by a sink or multipole is as follows [3,4,9,24]. First, one solves a Dirichlet problem for the Laplace equation in a domain with a given singularity at a fixed interior point, say, the origin

\[ \phi_{xx} + \phi_{yy} = 0, \quad \text{in } \Omega(t) \setminus (0,0), \]  
\[ \phi = \kappa, \quad \text{on } \partial \Omega(t), \]  
\[ \phi \approx Q \log \sqrt{x^2 + y^2} + O(1), \quad \text{as } (x,y) \to (0,0), \]  
\[ \text{or } \phi \approx -M \Re \left\{ \frac{1}{(x+iy)^p} \right\} + O(1), \quad \text{as } (x,y) \to (0,0), \]

where \( \Omega(t) \) is a simply connected domain with a smooth boundary \( \partial \Omega(t) \), \( \kappa \) is the curvature with the requirement that it is negative when \( \partial \Omega(t) \) is a circle, and \( \tau \) is the surface tension coefficient. For anisotropic Hele-Shaw flows, \( \kappa \) is directionally dependent. We may assume that \( \kappa \) is a nonnegative function of normal angle \( \theta \) of the boundary \( \partial \Omega(t) \); i.e., \( \kappa = \tau(\theta) \). This normal angle \( \theta \) is taken as the angle between the outward normal vector \( \vec{n} \) and the x-axis. The solution uniquely exists, and depends smoothly on time if \( \partial \Omega(t) \) does so. One then uses the solution \( \phi(x,y,t) \) to determine the motion of the boundary \( \partial \Omega(t) \) by

\[ V_n = \frac{\partial \phi}{\partial n}, \quad \text{on } \partial \Omega(t), \]

where \( \vec{n} \) is the outward normal vector to \( \partial \Omega(t) \), and \( V_n \) is the normal component of the velocity of \( \partial \Omega(t) \).
Physically, the Hele-Shaw problem involves two fluids: one with high viscosity and the other with low viscosity to be regarded as inviscid, each trapped between two glass plates separated by a tiny distance. The viscous fluid occupies the domain $\Omega(t)$, and the inviscid one its complement. The function $\phi$ is the viscous fluid velocity potential which can be identified with the fluid-pressure field. Equation (1) follows from the fact that $\phi$ is the velocity potential and that the viscous fluid is incompressible. The pressure in the nonviscous fluid is taken to be constant, say, zero, and relation (2) thus represents the jump of pressure across the interface $\partial \Omega(t)$. $Q$ is a negative constant and $M$ a complex constant. Equation (3) (or equation (4)) indicates that the origin is a sink (or a multipole of order $p$). Finally, equation (5) expresses that $\partial \Omega(t)$ is a material curve.

Hele-Shaw flows with constant surface tension have been studied extensively. One of the ideas is to capture the motion of the interface by examining the evolution of the complex moments of the fluid domain $\Omega(t)$ \cite{1,2,15,24}

$$\int \int_{\Omega(t)} z^m \, dx \, dy, \quad z = x + iy,$$

for $m = 0, 1, 2, \ldots$. In this paper, we employ this approach to study anisotropic Hele-Shaw flows.

We first calculate the time derivative of the moments in the standard way. More precisely,

$$\frac{d}{dt} \left[ \int \int_{\Omega(t)} z^m \, dx \, dy \right] = \int \int_{\partial \Omega(t)} z^m V_n \, ds = \int \int_{\partial \Omega(t)} z^m \frac{\partial \phi}{\partial n} \, ds,$$

for nonnegative integer $m$ where we have used equation (5) in the last equality. Choosing a small circular disk $D(\epsilon)$ with radius $\epsilon$ and center at the origin, and applying Green's theorem to the region $\Omega(t) \setminus D$, we get

$$\int_{\partial \Omega(t) \cup \partial D} z^m \frac{\partial \phi}{\partial n} \, ds = \int_{\partial \Omega(t) \cup \partial D} \phi \frac{\partial z^m}{\partial n} \, ds = \int_{\partial D} \phi \frac{\partial z^m}{\partial n} \, ds + \int_{\partial \Omega(t)} \tau \kappa \frac{\partial z^m}{\partial n} \, ds,$$

where we have used equation (2) in the last equality.

If the flow is driven by a sink, we use equation (3) and let $\epsilon \to 0$ in equation (7) to obtain

$$\int_{\Omega(t)} z^m \frac{\partial \phi}{\partial n} \, ds = \int_{\Omega(t)} \tau \kappa \frac{\partial z^m}{\partial n} \, ds + 2Q\pi \delta_{m0}, \quad m = 0, 1, 2, \ldots.$$

Here $\delta_{m0} = 1$ for $m = 0$ and vanishes otherwise. This together with equation (6) gives

$$\frac{d}{dt} \left[ \int \int_{\Omega(t)} z^m \, dx \, dy \right] = \int_{\Omega(t)} \tau \kappa \frac{\partial z^m}{\partial n} \, ds + 2Q\pi \delta_{m0}, \quad m = 0, 1, 2, \ldots \quad (8)$$

Similarly, if the flow is driven by a multipole, we then use (4) in (7) and we finally obtain

$$\frac{d}{dt} \left[ \int \int_{\Omega(t)} z^m \, dx \, dy \right] = \int_{\partial \Omega(t)} \tau \kappa \frac{\partial z^m}{\partial n} \, ds + 2PM \pi \delta_{mp}, \quad m = 0, 1, 2, \ldots, \quad (9)$$

where $\delta_{mp} = 1$ for $m = p$ and vanishes otherwise.

Conversely, a simple argument similar to that of \cite{2} shows that equations (8) or (9) contain all the information about a Hele-Shaw flow.

**Theorem 1.** Suppose that $\{\Omega(t)\}$ is a smooth family of simply connected domains. $\{\Omega(t)\}$ is a solution for the Hele-Shaw flow driven by a sink (or by a multipole) if and only if its complex moments satisfy equation (8) (or equation (9)).

The infinitely many equations (8) or (9) are in general difficult to handle due to surface tension terms. However, the first two equations are easy even when surface tension is present. In the next section, we will see what information we can get from these two equations. Our method is a modification of those used in \cite{7,9} for the isotropic case.
3. SINGULARITY FORMATION

We first study sink flows. Equation (8) when \( m = 0 \) is

\[
\frac{d}{dt} \text{Area}(\Omega(t)) = 2\pi Q. \tag{10}
\]

When \( m = 1 \), it becomes

\[
\frac{d}{dt} \left[ \int \int_{\Omega(t)} z \, dx \, dy \right] = \int_{\partial \Omega(t)} \tau(\theta) \kappa [y'(s) - iz'(s)] \, ds \\
= -\int_{\partial \Omega(t)} \tau(\theta) \theta e^{i\theta} \, ds \\
= -\int_0^{2\pi} \tau(\theta) e^{i\theta} \, d\theta, \tag{11}
\]

where we have used curvature \( \kappa = -\theta_s \) and the unit outward normal vector \( y'(s) - iz'(s) = e^{i\theta} \), here \( \theta \) is the normal angle and \( s \) is the arclength. Notice that the right-hand side is a time-independent constant. It is because of this that we are able to integrate (11) explicitly.

We will next see how much information we can get from these two equations. The first equation says that the total mass of the fluid blob decreases at a fixed rate. The maximum possible life span for a solution \( \{\Omega(t)\} \) is then given by

\[
T^* = \frac{\text{Area}(\Omega(0))}{2\pi|Q|.} \tag{12}
\]

Suppose a solution \( \{\Omega(t)\} \) has the maximum life span \( T^* \). We integrate equation (11) to obtain

\[
\int \int_{\Omega(t)} z \, dx \, dy = \int \int_{\Omega(0)} z \, dx \, dy + t \int_0^{2\pi} \tau(\theta) e^{i\theta} \, d\theta,
\]

where we have used the time-independence of the right-hand side of (11).

If \( \Omega(t) \) remains uniformly bounded for all possible time, the integral on the left-hand side will tend to zero as \( t \to T^* \). The linear moment at the initial time must then be

\[
\int \int_{\Omega(0)} z \, dx \, dy = T^* \int_0^{2\pi} \tau(\theta) e^{i\theta} \, d\theta.
\]

Using formula (12), the initial center of mass must then be located at the point

\[
\frac{1}{2\pi|Q|} \int_0^{2\pi} \tau(\theta) e^{i\theta} \, d\theta. \tag{13}
\]

**Theorem 2.** If \( \{\Omega(t)\}_{t \geq 0} \) is a solution for a Hele-Shaw flow driven by a sink of intensity \( Q \) with anisotropic surface tension \( \tau(\theta) \) and if it has the maximum life span \( T^* \), either the center of mass of the initial fluid domain is located at the point (13) or the domain \( \Omega(t) \) eventually becomes unbounded in diameter. In other words, if the center of mass the initial fluid domain is not located at the point (13), then either the Hele-Shaw solution will break down before all the fluid is sucked out or the fluid domain will eventually extend to infinity.

**Remark 1.** For an isotropic Hele-Shaw flow, the point (13) is at the origin at which the sink is located. Therefore, if the initial center of mass is not at the sink, then a sink flow with positive constant surface tension will either develop singularities before all the fluid is gone or have an unbounded fluid domain. This result was first obtained in [9].
REMARK 2. The point (13) is also at the origin even for some anisotropic flows whose surface tension is \( \tau(\theta) = \tau_0(1 - \epsilon \cos m\theta) \), here the integer \( m \neq 1 \). This type of anisotropic surface tension has been studied numerically [10].

We now turn our attention to multipole driven flows. The first equation of (9) says that the mass of the fluid blob is time-independent. The second one becomes

\[
\frac{d}{dt} \left[ \int \int_{\Omega(t)} z \, dx \, dy \right] = - \int_{\partial \Omega(t)} \tau(\theta)e^{i\theta} \, d\theta + 2\pi M \delta_{1p}. \tag{14}
\]

We see from this equation that if the right-hand side of (14) does not vanish, then the center of mass is linear in \( t \), and thus, moves toward infinity as \( t \) becomes large. Since the multipole is always at the origin, either the solution develops finite time singularities or the fluid domain becomes unbounded in diameter as \( t \to +\infty \).

THEOREM 3. For a Hele-Shaw flow driven by a multipole of order \( p \) and intensity \( M \) with anisotropic surface tension \( \tau(\theta) \), if

\[
\int_{\partial \Omega(t)} \tau(\theta)e^{i\theta} \, d\theta = 2\pi M \delta_{1p},
\]

then either Hele-Shaw solution will develop finite singularities or have unbounded fluid domain as \( t \) goes to infinity.

REMARK. When \( p = 1 \) and \( \tau \) is isotropic, inequality (15) is always true. Therefore, a dipole driven flow with positive constant surface tension will either develop finite time singularities or have an unbounded fluid domain as \( t \to +\infty \) [7].

Can a flow driven by a sink or multipole with anisotropic surface tension have an unbounded fluid domain evolving from a bounded initial fluid domain? It is unlikely. When surface tension is zero, such a flow always has a bounded fluid domain. When surface tension is a nonzero constant, numerical simulations indicate that the fluid domain is also uniformly bounded [7,13-15]. However, we would like to mention that a rigorous proof of uniform boundedness of the fluid domain is not yet available for anisotropic Hele-Shaw flows.

Finally, we will discuss how a sink flow develops singularities before all the fluid is sucked out and also how a multipole driven flow forms finite time singularities. We can think of two possibilities. Singularities can be formed at the portion of the interface where anisotropic surface tension vanishes. In this case, cusp formation is likely. The second possibility is that the flow may develop singularities via interface reaching the sink or multipole, as in the isotropic cases [7,13-15].

4. FLOWS DRIVEN PURELY BY ANISOTROPIC SURFACE TENSION

When the sink and multipole are absent, i.e., \( Q \) of (3) is zero or equivalently \( M \) of (4) is zero, the flow is driven solely by anisotropic surface tension. The motion of the interface is partially determined by equations (10) and (11) with \( Q = 0 \). It immediately follows from these two equations that the center of mass moves in a constant velocity which is given by

\[
\frac{\int_0^{2\pi} \tau(\theta)e^{i\theta} \, d\theta}{\text{Area} \left( \Omega(0) \right)}.
\]
THEOREM 4. For a Hele-Shaw flow driven solely by anisotropic surface tension $\tau(\theta)$, the center of mass of the fluid domain $\mathbf{c}(t)$ moves in a constant velocity. This velocity is given by (16).

Suppose we choose the surface tension so that the integral in (16) is not zero. The center of mass thus travels in a nonzero constant velocity. Finite time singularities are possible: cusp formation at the portion of the interface where $\tau(\theta) = 0$ or topological change of fluid blob. The latter phenomenon has been observed in Hele-Shaw flows driven purely by constant surface tension [25]. It is also possible that the flow exists for all time. In this case, the center should go to infinity as $t \rightarrow +\infty$, but it is an open problem as to whether the whole fluid blob or only a portion of it moves to infinity with its center of mass.

5. CONCLUSIONS

We have derived equations of motion for anisotropic Hele-Shaw flows driven by a sink or multipole. By analyzing two of the equations, we have found a breakdown of the flow when a certain condition is not satisfied.

We have also studied flows driven purely by anisotropic surface tension. For some choices of anisotropic surface tension, it is possible that the whole fluid blob or a portion of it must eventually move to infinity.

The results obtained in this paper concern simply-connected fluid domains. They can easily be generalized to multiply-connected domains in the same way as what was done in [26] for isotropic Hele-Shaw flows.

REFERENCES


