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Lifting KK-elements, asymptotic unitary equivalence and classification of simple C*-algebras

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Abstract

Let A and C be two unital simple C*-algebras with tracial rank zero. Suppose that C is amenable and satisfies the Universal Coefficient Theorem. Denote by $KK_e(C,A)^{++}$ the set of those κ in $KK(C,A)$ for which $\kappa(K_0(C)_+ \setminus \{0\}) \subset K_0(A)_+ \setminus \{0\}$ and $\kappa([1_C]) = [1_A]$. Suppose that $\kappa \in KK_e(C,A)^{++}$. We show that there is a unital monomorphism $\phi : C \to A$ such that $[\phi] = \kappa$. Suppose that C is a unital AH-algebra and $\lambda : \mathrm{T}(A) \to \mathrm{T}_{\mathfrak{f}}(C)$ is a continuous affine map for which $\tau(\kappa([p])) = \lambda(\tau)(p)$ for all projections p in all matrix algebras of C and any $\tau \in \mathrm{T}(A)$, where $\mathrm{T}(A)$ is the simplex of tracial states of A and $\mathrm{T}_{\mathfrak{f}}(C)$ is the convex set of faithful tracial states of C. We prove that there is a unital monomorphism $\phi : C \to A$ such that ϕ induces both κ and λ.

Suppose that $h : C \to A$ is a unital monomorphism and $\gamma \in \mathrm{Hom}(K_1(C), \mathrm{Aff}(A))$. We show that there exists a unital monomorphism $\phi : C \to A$ such that $[\phi] = [h]$ in $KK(C,A)$, $\tau \circ \phi = \tau \circ h$ for all tracial states τ and the associated rotation map can be given by γ. Denote by $KKT(C,A)^{++}$ the set of compatible pairs (κ, λ), where $\kappa \in KL_e(C,A)^{++}$ and λ is a continuous affine map from $\mathrm{T}(A)$ to $\mathrm{T}_{\mathfrak{f}}(C)$. Together with a result on asymptotic unitary equivalence in [H. Lin, Asymptotic unitary equivalence and asymptotically inner automorphisms, arXiv:math/0703610, 2007], this provides a bijection from the asymptotic unitary equivalence classes of unital monomorphisms from C to A to $(KKT(C,A)^{++}, \mathrm{Hom}(K_1(C), \mathrm{Aff}(\mathrm{T}(A)))/\mathcal{R}_0)$, where $\mathcal{R}_0$ is a subgroup related to vanishing rotation maps.

As an application, combining these results with a result of W. Winter [W. Winter, Localizing the Elliott conjecture at strongly self-absorbing C*-algebras, arXiv:0708.0283v3, 2007], we show that two unital amenable simple $\mathcal{Z}$-stable C*-algebras are isomorphic if they have the same Elliott invariant and the tensor

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products of these C*-algebras with any UHF-algebra have tracial rank zero. In particular, if A and B are two unital separable simple $\mathcal{Z}$-stable C*-algebras with unique tracial states which are inductive limits of C*-algebras of type I, then they are isomorphic if and only if they have isomorphic Elliott invariants.

Keywords: Simple C*-algebras; Classification; K-theory

1. Introduction

Let A and B be two unital separable amenable simple C*-algebras satisfying the Universal Coefficient Theorem (UCT). It has been shown (see [14] and [19], and also [3] and [4]) that, if in addition, A and B have tracial rank one or zero, then A and B are isomorphic if their Elliott invariants are isomorphic. There are interesting simple amenable C*-algebras with stable rank one which do not have tracial rank zero or one and some classification theorems have been established too (see for example [9,8,22,5]). One of the interesting classes of simple amenable C*-algebras which satisfy the UCT are the simple ASH-algebras (approximate subhomogeneous C*-algebras), or even more generally, the simple C*-algebras which are inductive limits of type I C*-algebras. In the case of unital $\mathcal{Z}$-stable simple ASH-algebras, if in addition, their projections separate traces, a classification theorem can be given (see [26] and [18]). More precisely, let A and B be two unital simple $\mathcal{Z}$-stable ASH-algebras whose projections separate the traces. Then $A \cong B$ if and only if $(K_0(A), K_0(A)_+, [1_A], K_1(A))$ is isomorphic to $(K_0(B), K_0(B)_+, [1_B], K_1(B))$ provided that the groups $K_i(A)$ are finitely generated (or $K_i(A)$ contain their torsion part as a direct summand). It should be noted that there are known examples of ASH-algebras whose projections separate the traces but have real rank other than zero. In particular, A and B may not have any non-trivial projections as long as each one has a unique tracial state (for instance, the Jiang-Su algebra $\mathcal{Z}$). It is clearly important to remove the above mentioned restriction on the K-groups. The original purpose of this paper was just to remove these restrictions.

One of the technical tools used in the proof of [26] and [18] is a theorem which determines when two unital monomorphisms from a unital AH-algebra C to a unital simple C*-algebra A are asymptotically unitarily equivalent. In the case that A has tracial rank zero, it was shown in [17] that two such monomorphisms are asymptotically unitarily equivalent if they induce the same element in $KK(C, A)$ and the same affine map on the tracial state spaces, and the rotation map associated with this pair of monomorphisms vanishes. This result serves as the uniqueness theorem for the classification theorem ([26] and [18]) above. It is equally important to determine the range of the asymptotic unitary equivalence classes of such monomorphisms which will serve the existence theorem for the above mentioned classification theorem. When the above mentioned restriction on K-groups holds, the existence theorem can be easily obtained so the classification result can be established. The general case will follow once the range of the asymptotic unitary equivalence classes of monomorphisms can be determined, or the existence theorem can be established (see Section 5).

The first question we need to answer is the following: Let A and C be two unital simple amenable C*-algebras with tracial rank zero and let C satisfy the UCT. Suppose that $\kappa \in KK_e(C, A)^{++}$ (i.e., $\kappa \in KK(C, A)$ and $\kappa(K_0(C)_+ \setminus \{0\}) \subseteq K_0(A)_+ \setminus \{0\}$, $\kappa([1_C]) = [1_A]$). Is there a unital monomorphism $\phi : C \to A$ such that $[\phi] = \kappa$? It is known (see [11]) that if the groups $K_i(C)$ are finitely generated then the answer is affirmative. It was proved in [12] that there exists a unital monomorphism $\phi : C \to A$ such that $[\phi] - \kappa = 0$ in $KL(C, A)$. The problem remained open whether one can choose ϕ so that $[\phi] = \kappa$ in $KK(C, A)$. It also has been known that there are several significant consequences if such a ϕ can be found.

The second question is the following: Let $\phi : C \to A$ be a unital monomorphism and $\gamma \in \mathrm{Hom}(K_1(C), \mathrm{Aff}(\mathrm{T}(A)))$. Can we find a unital monomorphism $\psi : C \to B$ with $[\psi] = [\phi]$ in $KK(C, A)$ such that the associated rotation map is γ?

In this paper, we will give affirmative answers to both questions, in the case that C is an AH-algebra. Among other consequences, we give the following: Let A and B be two unital simple C*-algebras with unique tracial states which are inductive limits of type I C*-algebras. Suppose that

$$\big(K_0(A), K_0(A)_+, [1_A], K_1(A)\big) \cong \big(K_0(B), K_0(B)_+, [1_B], K_1(B)\big).$$

Then A and B are $\mathcal{Z}$-stably isomorphic (see 5.6 below).

For the first question, as mentioned above, an affirmative answer was known for C with finitely generated K-groups. Passing to inductive limits, the first author showed in [12] that any strictly positive element in $KL(C, A)$ can be lifted for any unital AH-algebra C. However, since the KK-functor is not continuous with respect to inductive limits, it remained somewhat mysterious how to move from $KL(C, A)$ to $KK(C, A)$ until a clue was given by a paper of Kishimoto and Kumjian [10], where they studied simple $A\mathbb{T}$-algebras with real rank zero. The important advantage of simple $A\mathbb{T}$-algebras is that their K-groups are torsion free. In this case a hidden Bott-like map was revealed. Kishimoto and Kumjian navigated this hurdle using the so-called Basic Homotopy Lemma for simple C*-algebras of real rank zero. We will adapt the ideas of Kishimoto and Kumjian to the case that the domain algebras are no longer assumed to be the same as the targets. More generally, we will not assume that C has real rank zero, nor will we assume it is simple. Furthermore, we will allow C to have torsion in its K-groups. Therefore, for the general case, the Bott-like maps involve the K-groups with coefficients and demand a much more general Basic Homotopy Lemma. For this we apply the recently established results of [16] and [17]. It is interesting to note that the second problem is closely related to the first one and its proof is also closely related.

We will consider the case that C is a general unital AH-algebra (which may have arbitrary stable rank and other properties even in the case it is simple (see, for example, [24,25,6])). Denote by $\mathrm{T}_{\mathrm{f}}(C)$ the convex set of all faithful tracial states of C. Let $\kappa \in KK_e(C, A)^{++}$ and let $\lambda : \mathrm{T}(A) \to \mathrm{T}_{\mathrm{f}}(C)$ be a continuous affine map. We shall say that λ is compatible with κ if $\lambda(\tau)(p) = \tau(\kappa([p]))$ for any projection p in a matrix algebra over C and for any $\tau \in \mathrm{T}(A)$. We shall show that for any such compatible pair (κ, λ), there exists a unital monomorphism $\phi : C \to A$ such that $[\phi] = \kappa$ and $\phi_{\mathrm{T}} = \lambda$, where $\phi_{\mathrm{T}} : \mathrm{T}(A) \to \mathrm{T}_{\mathrm{f}}(C)$ is the continuous affine map induced by ϕ. It is worth pointing out that the information λ is essential since there are examples of compact metric spaces X, unital simple AF-algebras and $\kappa \in K_e(C, A)^{++}$ where $C = \mathrm{C}(X)$ for which there is no unital monomorphism h such that $[h] = \kappa$ (see [20]). Furthermore, given a pair $([\phi], \phi_T)$ and $\gamma \in \mathrm{Hom}(K_1(C), \mathrm{Aff}(\mathrm{T}(A)))$, we shall also show that there is a unital monomorphism $\psi : C \to A$ such that $([\psi], \psi_{\mathrm{T}}) = ([\phi], \phi_{\mathrm{T}})$ and the rotation map from $K_1(C)$ to $\mathrm{Aff}(\mathrm{T}(A))$ associated with ϕ and ψ is exactly γ.

The paper is organized as follows: Some preliminaries and notation are given in Section 2. In Section 3, it is shown that any element $\kappa \in KK_e(C, A)^{++}$ which is compatible with a continuous affine map $\lambda : \mathrm{T}(A) \to \mathrm{T}_{\mathrm{f}}(C)$ can be represented, together with the map λ, by a monomorphism α if C is a unital AH-algebra and A is a unital simple C*-algebra with tracial rank zero. It is shown that if $A = C$, then α can be chosen to be an automorphism. Then, in Section 4, it is proved that, for any monomorphism $\iota : C \to A$, one can realize any homomorphism ψ from $K_1(C)$ to $\mathrm{Aff}(\mathrm{T}(A))$ as a rotation map without changing the KK-class of ι, that is, there is a monomorphism $\alpha : C \to A$ such that $[\iota] = [\alpha]$ in $KK(C, A)$ and $\tilde{\eta}_{\iota,\alpha} = \psi$. Moreover, we also give a description of the asymptotic unitary equivalence classes of the maps inducing the same KK-element. In Section 5, we give an application of the results of the previous sections to the classification program. Combined with the work [26] of W. Winter and that of [17], it is shown that certain $\mathcal{Z}$-stable C*-algebras can be classified by their K-theory information.

2. Preliminaries and notation

2.1. Let A be a unital stably finite C*-algebra. Denote by $\mathrm{T}(A)$ the simplex of tracial states of A and denote by $\mathrm{Aff}(\mathrm{T}(A))$ the space of all real affine continuous functions on $\mathrm{T}(A)$. Suppose that $\tau \in \mathrm{T}(A)$ is a tracial state. We will also use τ for the trace $\tau \otimes \mathrm{Tr}$ on $\mathrm{M}_k(A) = A \otimes \mathrm{M}_k(\mathbb{C})$ (for every integer $k \geqslant 1$), where Tr is the standard trace on $\mathrm{M}_k(\mathbb{C})$. A trace τ is faithful if $\tau(a) > 0$ for any $a \in A_+ \setminus \{0\}$. Denote by $\mathrm{T}_{\mathfrak{f}}(A)$ the convex subset of $\mathrm{T}(A)$ consisting of all faithful tracial states.

Denote by $\mathrm{M}_\infty(A)$ the set $\bigcup_{k=1}^{\infty} \mathrm{M}_k(A)$, where $\mathrm{M}_k(A)$ is regarded as a C*-subalgebra of $\mathrm{M}_{k+1}(A)$ by the embedding

$$\mathrm{M}_k(A) \ni a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_{k+1}(A).$$

Define the positive homomorphism $\rho_A : K_0(A) \to \mathrm{Aff}(\mathrm{T}(A))$ by $\rho_A([p])(\tau) = \tau(p)$ for any projection p in $\mathrm{M}_k(A)$. Denote by $\mathrm{S}(A)$ the suspension of A, by $\mathrm{U}(A)$ the unitary group of A, and by $\mathrm{U}_0(A)$ the connected component of $\mathrm{U}(A)$ containing the identity.

Suppose that C is another unital C*-algebra and $\phi : C \to A$ is a unital *-homomorphism. Denote by $\phi_{\mathrm{T}} : \mathrm{T}(A) \to \mathrm{T}(C)$ the continuous affine map induced by ϕ, i.e., $\phi_{\mathrm{T}}(\tau)(c) = \tau \circ \phi(c)$ for all $c \in C$ and $\tau \in \mathrm{T}(A)$.

Definition 2.2. Let A be a unital C*-algebra and let $B \subseteq A$ be a unital C*-subalgebra. For any $u \in \mathrm{U}(A)$, the *-homomorphism $\mathrm{Ad}(u)$ is defined by

$$\mathrm{Ad}(u) : B \ni b \mapsto u^* b u \in A.$$

Denote by $\overline{\mathrm{Inn}}(B, A)$ the closure of $\{\mathrm{Ad}(u);\ u \in \mathrm{U}(A)\}$ in $\mathrm{Hom}(B, A)$ with the pointwise convergence topology. Note that $\overline{\mathrm{Inn}}(A) \subseteq \overline{\mathrm{Inn}}(B, A)$.

2.3. An extension of abelian groups

$$0 \longrightarrow G \longrightarrow E \xrightarrow{\pi} H \longrightarrow 0$$

is pure if it is locally trivial, i.e., for any finitely generated subgroup $H' \subseteq H$, there is a homomorphism $\theta' : H' \to E$ such that $\pi \circ \theta' = \mathrm{id}_{H'}$. Denote by $\mathrm{Pext}(H, G)$ the group of pure extensions of H by G.

Definition 2.4. Let A and B be two unital C*-algebras, and let ψ and ϕ be two unital *-monomorphisms from B to A. Then the mapping torus $M_{\psi,\phi}$ is the C*-algebra defined by

$$M_{\psi,\phi} := \big\{ f \in \mathrm{C}\big([0,1], A\big);\ f(0) = \psi(b) \text{ and } f(1) = \phi(b) \text{ for some } b \in B \big\}.$$

If $B \subseteq A$ is a unital C*-subalgebra with ι the inclusion map, then, for any unital *-monomorphism α from B to A, we shall just denote $M_{\iota,\alpha}$ by M_α.

For any $\psi, \phi \in \mathrm{Hom}(B, A)$, denoting by π_0 the evaluation of $M_{\psi,\phi}$ at 0, we have the short exact sequence

$$0 \longrightarrow \mathrm{S}(A) \stackrel{\iota}{\longrightarrow} M_{\psi,\phi} \stackrel{\pi_0}{\longrightarrow} B \longrightarrow 0,$$

and hence the six-term exact sequence

$$\begin{array}{ccccc} K_0(\mathrm{S}(A)) & \xrightarrow{\iota_0} & K_0(M_{\psi,\phi}) & \xrightarrow{[\pi_0]_0} & K_0(B) \\ {\scriptstyle [\psi]_1-[\phi]_1}\uparrow & & & & \downarrow{\scriptstyle [\psi]_0-[\phi]_0} \\ K_1(B) & \xleftarrow[{[\pi_0]_1}]{} & K_1(M_{\psi,\phi}) & \xleftarrow[\iota_1]{} & K_1(\mathrm{S}(A)). \end{array}$$

If $[\psi]_* = [\phi]_*$ (in particular, if $B \subseteq A$ and $\alpha \in \overline{\mathrm{Inn}}(B, A)$), then the six-term exact sequence above breaks down to the following two extensions:

$$\eta_0(M_{\psi,\phi}): \quad 0 \longrightarrow K_1(A) \longrightarrow K_0(M_{\psi,\phi}) \longrightarrow K_0(B) \longrightarrow 0,$$

and

$$\eta_1(M_{\psi,\phi}): \quad 0 \longrightarrow K_0(A) \longrightarrow K_1(M_{\psi,\phi}) \longrightarrow K_1(B) \longrightarrow 0.$$

Moreover, if $[\psi] = [\phi]$ in $KL(B, A)$ and B satisfies the UCT (see Definition 2.9 below), then the extensions $\eta_0(M_{\psi,\phi})$ and $\eta_1(M_{\psi,\phi})$ are pure.

2.5. Suppose that, in addition,

$$\tau \circ \phi = \tau \circ \psi \quad \text{for all } \tau \in \mathrm{T}(A). \tag{2.1}$$

For any piecewise smooth path of unitaries $u(t) \in M_{\psi,\phi}$, the integral

$$R_{\phi,\psi}\big(u(t)\big)(\tau) = \frac{1}{2\pi i} \int_0^1 \tau\big(\dot{u}(t)u^*(t)\big)\, dt$$

defines an affine function on $\mathrm{T}(A)$, and it depends only on the homotopy class of $u(t)$. Therefore, it induces a map, denoted by $R_{\psi,\phi}$, from $K_1(M_\alpha)$ to $\mathrm{Aff}(\mathrm{T}(A))$, and we call it the rotation map associated to the pair ψ, ϕ. The map $R_{\psi,\phi}$ is in fact a homomorphism. Early study of the rotation maps can be found in [7] and [1] among others.

2.6. If p and q are two mutually orthogonal projections in $\mathrm{M}_l(A)$ for some integer $l \geqslant 1$, define a unitary $u \in \mathrm{U}(\widetilde{\mathrm{M}_l(\mathrm{S}(A))})$ by

$$u(t) = \big(e^{2\pi i t} p + (1-p)\big)\big(e^{-2\pi i t} q + (1-q)\big) \quad \text{for } t \in [0,1].$$

One computes that

$$\int_0^1 \tau\left(\frac{du(t)}{dt} u(t)^*\right) dt = \tau(p) - \tau(q) \quad \text{for all } t \in [0,1].$$

Note that if $v(t) \in \widetilde{\mathrm{M}_l(\mathrm{S}A)}$ is another piecewise smooth unitary which is homotopic to $u(t)$, then, as mentioned above,

$$\int_0^1 \tau\left(\frac{dv(t)}{dt}\right) dt = \tau(p) - \tau(q).$$

It follows that, for any two projections p and q in $\mathrm{M}_l(A)$,

$$R_{\phi,\psi}\big(\imath_1\big([p]-[q]\big)\big)(\tau) = \tau(p) - \tau(q) \quad \text{for all } \tau \in \mathrm{T}(A).$$

In other words,

$$R_{\phi,\psi}\big(\imath_1\big([p]-[q]\big)\big) = \rho_A\big([p]-[q]\big).$$

Thus one has, exactly as in 2.2 of [10], the following:

Lemma 2.7. *When* (2.1) *holds, the following diagram commutes:*

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\quad \imath_1 \quad} & K_1(M_{\phi,\psi}) \\ & {}_{\rho_A}\searrow \quad \swarrow_{R_{\phi,\psi}} & \\ & \mathrm{Aff}(\mathrm{T}(A)) & \end{array}$$

Definition 2.8. Let A and C be two unital C^*-algebras and let $\phi, \psi : C \to A$ be two unital homomorphisms. One says that ϕ and ψ are asymptotically unitarily equivalent if they are one-parameter approximately unitarily equivalent, i.e., if there exists a continuous path of unitaries $\{u(t)\colon t \in [0,\infty)\}$ such that

$$\lim_{t\to\infty} \mathrm{Ad}\big(u(t)\big) \circ \phi(c) = \psi(c) \quad \text{for all } c \in C.$$

Definition 2.9. Let A be a unital C*-algebra and let C be a separable C*-algebra which satisfies the Universal Coefficient Theorem. By [2] of Dădărlat and Loring,

$$KL(C,A) = \mathrm{Hom}_\Lambda\big(\underline{K}(C), \underline{K}(A)\big), \tag{2.2}$$

where, for any C*-algebra B,

$$\underline{K}(B) = \big(K_0(B) \oplus K_1(B)\big) \oplus \left(\bigoplus_{n=2}^{\infty}\big(K_0(B,\mathbb{Z}/n\mathbb{Z}) \oplus K_1(B,\mathbb{Z}/n\mathbb{Z})\big)\right).$$

We will identify the two objects in (2.2). Note that one may view $KL(C,A)$ as a quotient of $KK(C,A)$.

Denote by $KL(C,A)^{++}$ the set of those $\bar{\kappa} \in \mathrm{Hom}_\Lambda(\underline{K}(C), \underline{K}(A))$ such that

$$\bar{\kappa}\big(K_0(C)_+ \setminus \{0\}\big) \subseteq K_0^+(A) \setminus \{0\}.$$

Denote by $KL_e(C,A)^{++}$ the set of those elements $\bar{\kappa} \in KL(C,A)^{++}$ such that $\bar{\kappa}([1_C]) = [1_A]$.

Suppose that both A and C are unital and $\mathrm{T}(C) \neq \emptyset$ and $\mathrm{T}(A) \neq \emptyset$. Let $\lambda : \mathrm{T}(A) \to \mathrm{T}(C)$ be a continuous affine map. We say λ is compatible with $\bar{\kappa}$ if for any projection $p \in \mathrm{M}_\infty(C)$, one has that $\lambda(\tau)(p) = \tau(\bar{\kappa}([p])$ for all $\tau \in \mathrm{T}(A)$.

Denote by $KLT(C,A)^{++}$ the set of those pairs $(\bar{\kappa}, \lambda)$, where $\bar{\kappa} \in KL_e(C,A)^{++}$ and $\lambda : \mathrm{T}(A) \to \mathrm{T}_{\mathfrak{f}}(C)$ is a continuous affine map which is compatible with $\bar{\kappa}$.

Definition 2.10. Denote by $KK(C,A)^{++}$ the set of those elements $\kappa \in KK(C,A)$ for which the image $\bar{\kappa}$ is in $KL(C,A)^{++}$. Denote by $KK_e(C,A)^{++}$ the set of those $\kappa \in KK(C,A)^{++}$ for which $\bar{\kappa} \in KL_e(C,A)^{++}$.

Denote by $KKT(C,A)^{++}$ the set of pairs (κ, λ) such that $(\bar{\kappa}, \lambda) \in KLT(C,A)^{++}$.

2.11. Let A and B be two unital C*-algebras. Let $h : A \to B$ be a homomorphism and $v \in \mathrm{U}(B)$ be such that

$$\big[h(g), v\big] = 0 \quad \text{for any } g \in A.$$

We then have a homomorphism $\bar{h} : A \otimes \mathrm{C}(\mathbb{T}) \to B$ defined by $f \otimes g \mapsto h(f)g(v)$ for any $f \in A$ and $g \in \mathrm{C}(\mathbb{T})$. The tensor product induces two injective homomorphisms:

$$\beta^{(0)} : K_0(A) \to K_1\big(A \otimes \mathrm{C}(\mathbb{T})\big),$$

and

$$\beta^{(1)} : K_1(A) \to K_0\big(A \otimes \mathrm{C}(\mathbb{T})\big).$$

The second one is the usual Bott map. Note that, in this way, one writes

$$K_i\big(A \otimes \mathrm{C}(\mathbb{T})\big) = K_i(A) \oplus \beta^{(i-1)}\big(K_{i-1}(A)\big).$$

Let us use $\widehat{\beta^{(i)}} : K_i(A \otimes \mathrm{C}(\mathbb{T})) \to \beta^{(i-1)}(K_i(A))$ to denote the projection.

For each integer $k \geqslant 2$, one also has the following injective homomorphisms:

$$\beta_k^{(i)} : K_i(A, \mathbb{Z}/k\mathbb{Z}) \to K_{i-1}\big(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}\big), \quad i = 0, 1.$$

Thus, we write

$$K_i\big(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}\big) = K_i(A, \mathbb{Z}/k\mathbb{Z}) \oplus \beta^{(i-1)}\big(K_{i-1}(A), \mathbb{Z}/k\mathbb{Z}\big).$$

Denote by $\widehat{\beta_k^{(i)}} : K_i(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}) \to \beta^{(i-1)}(K_{i-1}(A), \mathbb{Z}/k\mathbb{Z})$ the map analogous to that of $\widehat{\beta^{(i)}}$. If $x \in \underline{K}(A)$, we use $\boldsymbol{\beta}(x)$ for $\beta^{(i)}(x)$ if $x \in K_i(A)$ and for $\beta_k^{(i)}(x)$ if $x \in K_i(A, \mathbb{Z}/k\mathbb{Z})$. Thus we have a map $\boldsymbol{\beta} : \underline{K}(A) \to \underline{K}(A \otimes C(\mathbb{T}))$ as well as $\widehat{\boldsymbol{\beta}} : \underline{K}(A \otimes C(\mathbb{T})) \to \boldsymbol{\beta}(\underline{K}(A))$. Therefore, we may write $\underline{K}(A \otimes C(\mathbb{T})) = \underline{K}(A) \oplus \boldsymbol{\beta}(\underline{K}(A))$.

On the other hand, $\bar{h}$ induces homomorphisms

$$\bar{h}_{*i,k} : K_i\big(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}\big) \to K_i(B, \mathbb{Z}/k\mathbb{Z}),$$

$k = 0, 2, \ldots,$ and $i = 0, 1$.

We use $\mathrm{Bott}(h, v)$ for all homomorphisms $\bar{h}_{*i,k} \circ \beta_k^{(i)}$, and we use $\mathrm{bott}_1(h, v)$ for the homomorphism $\bar{h}_{1,0} \circ \beta^{(1)} : K_1(A) \to K_0(B)$, and $\mathrm{bott}_0(h, v)$ for the homomorphism $\bar{h}_{0,0} \circ \beta^{(0)} : K_0(A) \to K_1(B)$. We also use $\mathrm{bott}(u, v)$ for the Bott element when $[u, v] = 0$.

2.12. Given a finite subset $\mathcal{P} \subset \underline{K}(A)$, there exists a finite subset $\mathcal{F} \subset A$ and $\delta_0 > 0$ such that

$$\mathrm{Bott}(h, v)|_{\mathcal{P}}$$

is well defined if

$$\big\|\big[h(a), v\big]\big\| < \delta_0$$

for all $a \in \mathcal{F}$. See 2.11 of [17] and 2.10 of [16] for more details.

Definition 2.13. A unital simple C*-algebra A has tracial rank zero, denoted by $\mathrm{TR}(A) = 0$, if for any finite subset $\mathcal{F} \subset A$, any $\varepsilon > 0$, and nonzero $a \in A^+$, there exist a nonzero projection $p \in A$ and a finite dimensional C*-subalgebra F with $1_F = p$, such that

1. $\|[x, p]\| \leqslant \varepsilon$ for any $x \in \mathcal{F}$,
2. for any $x \in \mathcal{F}$, there is $x' \in F$ such that $\|pxp - x'\| \leqslant \varepsilon$, and
3. $1 - p$ is Murray–von Neumann equivalent to a projection in $\overline{aAa}$.

2.14. Finally, we will write $a \approx_\varepsilon b$ if $\|a - b\| < \varepsilon$.

3. *KK*-lifting

Let A be a unital C*-algebra. Fix $0 < \delta^{\mathrm{p}} \leqslant \min\{\delta_1, \delta_2\}$ where δ_1 and δ_2 are the constants δ of Lemma 9.6 and Lemma 9.7 of [17] respectively. Note that δ^{p} is universal and $\delta^{\mathrm{p}} < 1/4$ (therefore, the Bott element $\mathrm{bott}(u, v)$ is well defined for any unitaries u and v with $\|[u, v]\| < \delta^{\mathrm{p}}$).

For some integer $l \geqslant 1$, let z be a unitary in $\mathrm{M}_l(A)$, and let $U(t)$ be a path of unitaries $U(t) \in \mathrm{C}([0,1], \mathrm{U}(\mathrm{M}_l(A)))$ with $U(0) = 1$ and $\|[U(1), z]\| \leqslant \delta^{\mathrm{p}}$. We have that

$$U(1)zU^*(1)z^* = \exp(i\omega)$$

where $\omega = (1/i)(\log(U(1)zU^*(1)z^*)) \in A_{s.a}$. Define the element $\mathtt{p}(U, z)$ as follows:

$$\mathtt{p}(U,z)(t) = \begin{cases} U((8/7)t)zU^*((8/7)t)z^* & \text{for all } t \in [0, 7/8]; \\ \exp(i8(1-t)\omega) & \text{for all } t \in [7/8, 1]. \end{cases}$$

Then, $\mathtt{p}(U,z)(t)$ defines a loop of unitaries in A, and this gives a well-defined element in $K_1(\mathrm{S}(A)) \cong K_0(A)$.

Remark 3.1. Given a path of unitaries $U(t) \in \mathrm{C}([0,1], \mathrm{U}(\mathrm{M}_l(A)))$ for some integer $l \geqslant 1$, write

$$S(t) := \begin{pmatrix} U(t) & \\ & U^*(t) \end{pmatrix} \quad \text{and} \quad Z := \begin{pmatrix} z & \\ & z \end{pmatrix}.$$

If $U(0) = 1$ and $\|[U(1), z]\| \leqslant \delta^{\mathrm{p}}$, then $S(0) = 1$ and $\|[S(1), Z]\| \leqslant \delta^{\mathrm{p}}$, and we can consider the K-element $[\mathtt{p}(S, Z)]$ and we have

$$\big[\mathtt{p}(S,Z)\big] = \big[\mathtt{p}(U,z)\big] + \big[\mathtt{p}(U^*,z)\big].$$

Write

$$R(t,s) := \begin{pmatrix} U(t) & \\ & 1 \end{pmatrix} R(s) \begin{pmatrix} 1 & \\ & U^*(t) \end{pmatrix} R^*(s)$$

where

$$R(s) = \begin{pmatrix} \cos(\frac{\pi s}{2}) & \sin(\frac{\pi s}{2}) \\ -\sin(\frac{\pi s}{2}) & \cos(\frac{\pi s}{2}) \end{pmatrix}.$$

Then one has that $R(t,0) = S(t)$, $R(t,1) = 1$, and for any $s \in [0,1]$, $R(0,s) = 1$ and $\|[R(1,s), Z]\| \leqslant 2\delta^{\mathrm{p}}$. Therefore, the path $\mathtt{p}(S,Z)(t)$ is homotopic to the identity in $\mathrm{M}_2(\widetilde{\mathrm{S}(A)})$ by a small perturbation of

$$W(t,s) := R(t,s)ZR^*(t,s)Z^*,$$

and hence

$$\big[\mathtt{p}(U,z)\big] + \big[\mathtt{p}(U^*,z)\big] = \big[\mathtt{p}(S,Z)\big] = 0 \quad \text{in } K_1\big(\mathrm{S}(A)\big).$$

Lemma 3.2. *Let $U(t)$ and $V(t)$ be two continuous and piecewise smooth paths of unitaries. Let z_1 and z_2 be unitaries in A with $\|[U(1), z_1]\| < \delta^{\mathrm{p}}$ and $\|[V(1), z_2]\| < \delta^{\mathrm{p}}$. If $[\mathtt{p}(U, z_1)] = [\mathtt{p}(V, z_2)]$, then one has that* $\mathrm{bott}(U(1), z_1) = \mathrm{bott}(V(1), z_2)$.

Proof. Consider the path of unitaries

$$S(t) = \mathrm{diag}\big(U(t), V^*(t)\big)$$

and the unitary

$$Z := \mathrm{diag}(z_1, z_2)$$

in $\mathrm{M}_2(A)$. By Remark 3.1, one has

$$\big[\mathtt{p}(V^*, z_2)\big] + \big[\mathtt{p}(V, z_2)\big] = 0,$$

and hence

$$\big[\mathtt{p}(S^*, Z)\big] = \big[\mathtt{p}(U^*, z_1)\big] + \big[\mathtt{p}(V, z_2)\big] = 0.$$

Therefore, $\mathtt{p}(S^*, Z)$ is homotopic to the identity in a matrix algebra of $\widetilde{(\mathrm{S}(A))}$. Hence, passing to a matrix algebra if necessary, there is a continuous path of unitaries $W(t, s)$ such that $W(t, 0) = 1 = W(t, 1)$ for any $t \in [0, 1]$ and $W(1, t) = \mathtt{p}(S^*, Z)(t)$ for any $t \in [0, 1]$, and $W(t, 1) = 1$.

Note that $\mathtt{p}(S^*, Z)(t) = S^*(\frac{8}{7}t)ZS(\frac{8}{7}t)Z^*$ for $t \in [0, 7/8]$ and

$$\big\|\mathtt{p}(S^*, Z)(7/8) - 1\big\| < \delta^{\mathtt{p}}.$$

Thus, by Lemma 9.6 of [17],

$$0 = \mathrm{bott}\big(S(1), Z\big) = \mathrm{bott}\big(U(1), z_1\big) - \mathrm{bott}\big(V(1), z_2\big),$$

as desired. □

Lemma 3.3. *Let $U(t)$ and $V(t)$ be two continuous and piecewise smooth paths of unitaries. Let z_1 and z_2 be unitaries in A with $\|[U(t), z_1]\| < \delta^{\mathtt{p}}$ and $\|[V(t), z_2]\| < \delta^{\mathtt{p}}$. If* $\mathrm{bott}(U(1), z_1) = \mathrm{bott}(V(1), z_2)$*, then one has that* $[\mathtt{p}(U, z_1)] = [\mathtt{p}(V, z_2)]$.

Proof. Consider the path of unitaries

$$W(t) := \mathrm{diag}\big(U(t), V^*(t)\big)$$

and the unitary

$$Z := \mathrm{diag}(z_1, z_2)$$

in $\mathrm{M}_2(A)$. Then, one has that $\mathrm{bott}(W(1), Z) = \mathrm{bott}(U(1), z_1) - \mathrm{bott}(V(1), z_2) = 0$. By Lemma 9.7 of [17], one has

$$\big[\mathtt{p}(W^*, Z)\big] = 0,$$

and hence

$$\big[\mathtt{p}(U^*, z_1)\big] + \big[\mathtt{p}(V, z_2)\big] = 0. \tag{3.1}$$

On the other hand, by Remark 3.1,

$$\big[\mathrm{p}(V^*, z_2)\big] + \big[\mathrm{p}(V, z_2)\big] = 0.$$

Together with (3.1), one has

$$\big[\mathrm{p}(U, z_1)\big] = \big[\mathrm{p}(V, z_2)\big],$$

as desired. □

3.4. Denote by $C(A)$ the subset of $K_1(\mathrm{S}(A))$ consisting of $[\mathrm{p}(U, z)(t)]$ for a path of unitaries $U(t) \in \mathrm{C}([0,1], \mathrm{M}_l(A))$ and a unitary $z \in \mathrm{M}_l(A)$ for some $l \geqslant 1$ with $\|[U(1), z]\| < \delta^{\mathrm{p}}$. It is easy to verify that $C(A)$ is a subgroup. It follows from Lemma 3.2 and Lemma 3.3 that there is an injective map $\Lambda : C(A) \to K_0(A)$ defined by

$$\Lambda : \big[\mathrm{p}(U, z)\big] \mapsto \mathrm{bott}\big(U(1), z\big).$$

Moreover, the map Λ is a homomorphism.

Lemma 3.5. *Consider a path of unitaries $U(t) \in \mathrm{C}([0,1], \mathrm{M}_\infty(A))$ with $U(0) = 1$ and $\|[z, U(1)]\| < \delta^{\mathrm{p}}$. Then we have*

$$\tau\big(\big[\mathrm{p}(U, z)\big]\big) = \tau\big(\mathrm{bott}_1\big(U(1), z\big)\big)$$

for any $\tau \in \mathrm{T}(A)$ if one considers $[\mathrm{p}(U, z)]$ as an element in $K_0(A) \cong K_1(\mathrm{S}(A))$. In other words, $\tau(\Lambda(h)) = \tau(h)$ for any $h \in C(A)$.

Proof. Denote by $\omega = U(1)zU^*(1)z^*$, and note that $[\mathrm{p}(U, z)] \in K_1(\mathrm{S}(A)) = K_0(A)$. Denote by

$$r(t) = \exp\big(\log(\omega)(1-t)\big).$$

We then have that (see 2.6), for any $\tau \in \mathrm{T}(A)$,

$$\begin{aligned}
\tau\big(\mathrm{p}(U, z)\big) &= \frac{1}{2\pi i}\int_0^{7/8} \tau\Big(\big(U\big((8/7)t\big)zU^*\big((8/7)t\big)z^*\big)'\big(zU\big((8/7)t\big)z^*U^*\big((8/7)t\big)\big)\Big)\,dt \\
&\quad + \frac{1}{2\pi i}\int_{7/8}^{1} \tau\big(\dot r\big(8(t-7/8)\big)r^*\big(8(t-7/8)\big)\big)\,dt \\
&= \frac{1}{2\pi i}\int_0^{7/8} \tau\Big(\big(U\big((8/7)t\big)zU^*\big((8/7)t\big)\big)'\big(U\big((8/7)t\big)z^*U^*\big((8/7)t\big)\big)\Big)\,dt \\
&\quad + \frac{1}{2\pi i}\int_{7/8}^{1} \tau\big(\dot r\big(8(t-7/8)\big)r^*\big(8(t-7/8)\big)\big)\,dt
\end{aligned}$$

$$= \frac{1}{2\pi i} \int_{7/8}^{1} \tau\bigl(\dot{r}\bigl(8(t-7/8)\bigr)r^*\bigl(8(t-7/8)\bigr)\bigr)\,dt \quad \bigl(\text{by Lemma 4.2 of [17]}\bigr)$$

$$= -\frac{1}{2\pi i}\tau\bigl(\log(w)\bigr)$$

$$= \tau\bigl(\mathrm{bott}\bigl(U(1), z\bigr)\bigr) \quad \bigl(\text{by Theorem 3.6 of [17]}\bigr),$$

as desired. □

Definition 3.6. Let A be a unital C*-algebra with $\mathrm{T}(A) \neq \emptyset$. Let us say that A has Property (B1) if the following holds: For any unitary $z \in \mathrm{U}(\mathrm{M}_k(A))$ (for some integer $k \geqslant 1$) with $\mathrm{sp}(z) = \mathbb{T}$, there is a non-decreasing function $1/4 > \delta_z(t) > 0$ on $[0, 1]$ with $\delta_z(0) = 0$ such that for any $x \in K_0(A)$ with $|\tau(x)| \leqslant \delta_z(\varepsilon)$ for all $\tau \in \mathrm{T}(A)$, there exists a unitary $u \in \mathrm{M}_k(A)$ such that

$$\bigl\|[u, z]\bigr\| < \min\left\{\varepsilon, \frac{1}{4}\right\} \quad \text{and} \quad \mathrm{bott}_1(u, z) = x. \tag{3.2}$$

Let C be a unital separable C*-algebra. Let $1/4 > \Delta_c(t, \mathcal{F}, \mathcal{P}_0, \mathcal{P}_1, h) > 0$ be a function defined on $t \in [0, 1]$, the family of all finite subsets $\mathcal{F} \subset C$, and the family of all finite subsets $\mathcal{P}_0 \subset K_0(C)$, and family of all finite subsets $\mathcal{P}_1 \subset K_1(C)$, and the set of all unital monomorphisms $h : C \to A$. Let us say that A has Property (B2) associated with C and Δ_c if the following holds: For any unital monomorphism $h : C \to A$, any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset C$, any finite subset $\mathcal{P}_0 \subset K_0(C)$, and any finite subset $\mathcal{P}_1 \subset K_1(C)$, there are finitely generated subgroups $G_0 \subset K_0(C)$ and $G_1 \subset K_1(C)$ with finite sets of generators $\mathcal{G}_0$ and $\mathcal{G}_1$ respectively, such that $\mathcal{P}_0 \subset G_0$ and $\mathcal{P}_1 \subset G_1$, satisfying the following: for any homomorphisms $b_0 : G_0 \to K_1(A)$ and $b_1 : G_1 \to K_0(A)$ such that

$$\bigl|\tau \circ b_1(g)\bigr| < \Delta_c(\varepsilon, \mathcal{F}, \mathcal{P}_0, \mathcal{P}_1, h) \tag{3.3}$$

for any $g \in \mathcal{G}_1$ and any $\tau \in \mathrm{T}(A)$, there exists a unitary $u \in \mathrm{U}(A)$ such that

$$\mathrm{bott}_0(h, u)|_{\mathcal{P}_0} = b_0|_{\mathcal{P}_0}, \qquad \mathrm{bott}_1(h, u)|_{\mathcal{P}_1} = b_1|_{\mathcal{P}_1} \quad \text{and} \tag{3.4}$$

$$\bigl\|[h(c), u]\bigr\| < \varepsilon \quad \text{for all } c \in \mathcal{F}. \tag{3.5}$$

Remark 3.7. Note that in the definition of Property (B2), b_0 and b_1 are defined on G_0 and G_1, respectively, not on the subgroups generated by $\mathcal{P}_0$ and $\mathcal{P}_1$. One should also note that if A has Property (B2) associated with any C*-subalgebra $\mathrm{C}(\mathbb{T})$, then A also has Property (B1).

Remark 3.8. Let A be a C*-algebra with Property (B2) associated with C and $\Delta_c(t, \mathcal{F}, \mathcal{P}_0, \mathcal{P}_1, h)$. Then the function Δ'_c and the subgroups G_0 and G_1 can be chosen so that they only depend on the unitary conjugacy class of the unital embedding h.

Indeed, pick one representative h_λ for each conjugacy class of the unital embedding of C to A, and define the function

$$\Delta'_c(t, \mathcal{F}, \mathcal{P}_0, \mathcal{P}_1, h) = \Delta_c(t, \mathcal{F}, \mathcal{P}_0, \mathcal{P}_1, h_\lambda)$$

where h_λ is in the unitary conjugacy class of h. It is clear that Δ'_c only depends on the conjugacy class of h. Let us show that A has Property (B2) associated with C and Δ'_c and G_0 and G_1 can be chosen so that they only depend on h_λ.

Fix a unital monomorphism $h : C \to A$, any $\varepsilon > 0$, a finite subset $\mathcal{F} \subset C$, a finite subset $\mathcal{P}_0 \subset K_0(C)$, and a finite subset $\mathcal{P}_1 \subset K_1(C)$. There is a unitary $w \in A$ such that $\mathrm{Ad}(w) \circ h = h_\lambda$ for some λ.

Since A has Property (B2) associated with C and Δ_c, there are finitely generated subgroups $G_0 \subset K_0(C)$ and $G_1 \subset K_1(C)$ with finite sets of generators $\mathcal{G}_0$ and $\mathcal{G}_1$ respectively, such that $\mathcal{P}_0 \subset G_0$ and $\mathcal{P}_1 \subset G_1$ satisfying the following: for any homomorphisms $b_0 : G_0 \to K_1(A)$ and $b_1 : G_1 \to K_0(A)$ such that

$$\left|\tau \circ b_1(g)\right| < \Delta_c(\varepsilon, \mathcal{F}, \mathcal{P}_0, \mathcal{P}_1, h_\lambda) \tag{3.6}$$

for any $g \in \mathcal{G}_1$ and any $\tau \in \mathrm{T}(A)$, there exists a unitary $u \in \mathrm{U}(A)$ such that

$$\mathrm{bott}_0(h_\lambda, u)|_{\mathcal{P}_0} = b_0|_{\mathcal{P}_0}, \qquad \mathrm{bott}_1(h_\lambda, u)|_{\mathcal{P}_1} = b_1|_{\mathcal{P}_1} \quad \text{and} \tag{3.7}$$

$$\left\|[h_\lambda(c), u]\right\| < \varepsilon \quad \text{for all } c \in \mathcal{F}. \tag{3.8}$$

Then,

$$\mathrm{bott}_0(h, wuw^*) = \mathrm{bott}_0(h_\lambda, u)|_{\mathcal{P}_0} = b_0|_{\mathcal{P}_0},$$

$$\mathrm{bott}_1(h, wuw^*) = \mathrm{bott}_1(h_\lambda, u)|_{\mathcal{P}_1} = b_1|_{\mathcal{P}_1} \tag{3.9}$$

$$\text{and} \quad \left\|\left[h(c), wuw^*\right]\right\| < \varepsilon \quad \text{for all } c \in \mathcal{F}. \tag{3.10}$$

In other words, A has Property (B2) associated with Δ'_c. Therefore, we can always assume that the function Δ_c and the subgroups G_0 and G_1 only depend on the conjugacy classes of embeddings of C into A.

Lemma 3.9. *Let A be a unital* C*-*algebra which contains a positive element b with* $\mathrm{sp}(b) = [0, 1]$, *and assume that A has Property* (B1). *There exists $\delta > 0$ such that for any $a \in K_0(A)$, if $|\tau(a)| \leqslant \delta$ for any $\tau \in \mathrm{T}(A)$, then one has*

$$a = \left[\mathrm{p}(U, z)\right] \in K_1\left(\mathrm{S}(A)\right)$$

for some unitary $z \in \mathrm{M}_k(A)$ (for some $k \geqslant 1$) and some $U(t) \in \mathrm{C}([0, 1], \mathrm{U}(\mathrm{M}_k(A)))$ with $U(0) = 1$ and $\|[U(1), z]\| \leqslant \delta^{\mathrm{p}}$. In other words, $a \in C(A)$.

Proof. Since A has Property (B1), for any $1/4 > \varepsilon > 0$ and any unitary z in a matrix of A with $\mathrm{sp}(z) = \mathbb{T}$, there is a non-decreasing positive function $\delta_z(t)$ such that if $a \in K_0(A)$ with $|\tau(a)| \leqslant \delta_z(\varepsilon)$ for any trace τ, then there is a unitary u in a matrix algebra over A such that $\|[u, z]\| < \varepsilon$ and $\mathrm{bott}_1(u, z) = a$.

For any ε, there is $\delta_e(\varepsilon)$ such that if $|x - y| \leqslant \delta_e(\varepsilon)$ with $x, y \in \{c \in \mathbb{C};\ |c - 1| \leqslant \frac{1}{2}\}$, one has that $|\log(x) - \log(y)| \leqslant \varepsilon$. We also regard $\delta_e(\varepsilon)$ as a positive function of ε with $\delta_e(0) = 0$ and $\delta_e(\varepsilon) > 0$ if $\varepsilon > 0$.

For any natural number k and any $\varepsilon > 0$, define

$$\Delta_z(\varepsilon, k) := \begin{cases} \min\{\delta_z(\frac{1}{2}\Delta_z(\varepsilon, k-1)), \frac{1}{2}\delta_e(\frac{1}{2}\Delta_z(\varepsilon, k-1)), \delta^{\mathrm{p}}, \varepsilon\} & \text{if } k \geqslant 2, \\ \min\{\delta_z(\varepsilon), \delta^{\mathrm{p}}\} & \text{if } k = 1. \end{cases}$$

It is a positive function of ε and k with $\Delta_z(\varepsilon, k) > 0$ if $\varepsilon > 0$.

Fix $0 < \varepsilon < 1/12$. Then, there exists $m \in \mathbb{N}$ such that there is a partition $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1$ such that for any $W(t) = (e^{2\pi it}p + (1-p))(e^{-2\pi it}q + (1-q)) \in \mathrm{U}_\infty(\widetilde{\mathrm{S}(A)})$ where p and q are any projections, one has that $\|W(t_{i-1}) - W(t_i)\| \leqslant \varepsilon$ for each $1 \leqslant i \leqslant m$. Fix such a partition.

Let $z_0 = \exp(i2\pi b)$. Since $\mathrm{sp}(b) = [0, 1]$, z_0 is a unitary in A with $\mathrm{sp}(z_0) = \mathbb{T}$. Set

$$\delta = \frac{1}{2}\min\bigl\{\Delta_{z_0}(\varepsilon, 2m), \Delta_{z_0}(\varepsilon, 2m-1), \ldots, \Delta_{z_0}(\varepsilon, 1)\bigr\}.$$

Let a be an element in $K_0(A)$ with $|\tau(a)| < \delta$ for any $\tau \in \mathrm{T}(A)$. Without loss of generality, we may assume that $a = [p] - [q]$ for some projections $p, q \in \mathrm{M}_k(A)$ for some integer $k \geqslant 1$. Note that

$$\bigl|\tau(p) - \tau(q)\bigr| < \delta.$$

Put $W_0(t) = (e^{2\pi it}p + (1-p))(e^{-2\pi it}q + (1-q))$. To simplify notation, by replacing A by $\mathrm{M}_k(A)$, without loss of generality, we may assume that $p, q \in A$ and $W_0(t) \in A$ for each $t \in [0, 1]$.

It follows from Lemma 8 of [21] that, for some large $n \geqslant 1$, there is a unitary $V_t \in M_{n+1}(A)$, for each $t \in [0, 1]$,

$$\bigl\|V_t^* z V_t - W(t)z\bigr\| < \delta/2, \tag{3.11}$$

where

$$W(t) = \mathrm{diag}\bigl(W_0(t), \overbrace{1, 1, \ldots, 1}^{n}\bigr),$$

and

$$z = \mathrm{diag}\bigl(z_0, \omega, \omega^2, \ldots, \omega^n\bigr).$$

and where $\omega = e^{2\pi i/n+1}$.

We assert that one can find unitaries $V_0, V_1, \ldots, V_m$ such that

$$\bigl\|V_i z V_i^* - W(t_i)z\bigr\| < \Delta_{z_0}(\varepsilon, 2m+1-2i) \quad \text{for any } 0 \leqslant i \leqslant m,$$

and

$$\mathrm{bott}_1\bigl(V_{i+1}^* V_i, z\bigr) = 0 \quad \text{for any } 0 \leqslant i \leqslant m-1.$$

Assume that the unitaries $V_0, V_1, \ldots, V_i$ satisfy the condition above. As indicated above, one can find a unitary V_{i+1} such that

$$\left\| V_{i+1} z V_{i+1}^* - W(t_{i+1}) z \right\| < \min\left\{ \frac{1}{2} \delta_e\left(\frac{1}{2} \Delta_{z_0}(\varepsilon, 2m-2i) \right), \frac{1}{2} \Delta_{z_0}(\varepsilon, 2m-2i-1) \right\},$$

and set

$$b_i := \mathrm{bott}_1\big(V_{i+1}^* V_i, z\big).$$

By Theorem 3.6 of [17], one has that for any $\tau \in \mathrm{T}(A)$

$$\begin{aligned} \left|\tau(b_i)\right| &= \frac{1}{2\pi} \left|\tau\big(\log\big(z^* V_{i+1}^* V_i z V_i^* V_{i+1}\big)\big)\right| \\ &= \frac{1}{2\pi} \left|\tau\big(\log\big(V_{i+1} z^* V_{i+1}^* V_i z V_i^*\big)\big)\right|. \end{aligned}$$

Note that

$$\begin{aligned} \left\| V_{i+1} z^* V_{i+1}^* V_i z V_i^* - z^* W^*(t_{i+1}) W(t_i) z \right\| &< \frac{1}{2} \delta_e\left(\frac{1}{2} \Delta_{z_0}(\varepsilon, 2m-2i) \right) + \Delta_{z_0}(\varepsilon, 2m-2i+1) \\ &< \delta_e\left(\frac{1}{2} \Delta_{z_0}(\varepsilon, 2m-2i) \right). \end{aligned}$$

Therefore, we have

$$\left\| \log\big(V_{i+1} z^* V_{i+1}^* V_i z V_i^*\big) - \log\big(z^* W^*(t_{i+1}) W(t_i) z\big) \right\| < \frac{1}{2} \Delta_{z_0}(\varepsilon, 2m-2i),$$

and hence

$$\begin{aligned} \left|\tau(b_i)\right| &< \frac{1}{2\pi} \left|\tau\big(\log\big(W^*(t_{i+1}) W(t_i)\big)\big)\right| + \frac{1}{2} \Delta_{z_0}(\varepsilon, 2m-2i) \\ &= \frac{1}{2\pi} (t_{i+1} - t_i) \left|\big(\tau(p) - \tau(q)\big)\right| + \frac{1}{2} \Delta_{z_0}(\varepsilon, 2m-2i) \\ &< \Delta_{z_0}(\varepsilon, 2m-2i) < \delta_{z_0}\left(\frac{1}{2} \Delta_{z_0}(\varepsilon, 2m-2i-1) \right). \end{aligned}$$

Since A has Property (B1), there is a unitary $v'_{i+1} \in A$ such that

$$\left\| v'_{i+1} z_0 \big(v'_{i+1}\big)^* - z_0 \right\| < \frac{1}{2} \Delta_{z_0}(\varepsilon, 2m-2i-1) \quad \text{and} \quad \mathrm{bott}_1\big(v'_{i+1}, z_0\big) = b_i.$$

Set

$$v_{i+1} = \mathrm{diag}\big(v'_{i+1}, 1, \ldots, 1\big).$$

We then have

$$\left\|v_{i+1}zv_{i+1}^*-z\right\|=\left\|v_{i+1}'z_0\left(v_{i+1}'\right)^*-z_0\right\|<\frac{1}{2}\Delta_{z_0}(\varepsilon,2m-2i-1)$$

and

$$\mathrm{bott}_1(v_{i+1},z)=\mathrm{bott}_1\left(v_{i+1}',z_0\right)=b_i.$$

Therefore

$$\begin{aligned}\left\|(V_{i+1}v_{i+1})z\left(v_{i+1}^*V_{i+1}^*\right)-W(t_{i+1})z\right\|&<\frac{1}{2}\Delta_{z_0}(\varepsilon,2m-2i-1)+\left\|V_{i+1}zV_{i+1}^*-W(t_{i+1})z\right\|'\\&<\Delta_{z_0}(\varepsilon,2m-2i-1)\end{aligned}$$

and

$$\mathrm{bott}_1\left(\left(v_{i+1}^*V_{i+1}^*\right)V_i,z\right)=\mathrm{bott}_1\left(v_{i+1}^*,z\right)+\mathrm{bott}_1\left(V_{i+1}^*V_i,z\right)=0.$$

Replacing V_{i+1} by $V_{i+1}v_{i+1}$, we have proved the assertion.

Then, for each V_i and V_{i+1}, there is a path of unitary $V^{(i)}(t)$ such that

$$\left\|\left[V^{(i)}(t),z\right]\right\|<\varepsilon,\qquad V^{(i)}(0)=1,\qquad V^{(i)}(1)=V_{i+1}^*V_i.$$

Setting $U^{(i)}(t)=V_{i+1}V^{(i)}(1-t)$, we have that

$$U^{(i)}(0)=V_i,\qquad U^{(i)}(1)=V_{i+1},$$

and for each t,

$$\begin{aligned}\left\|U^{(i)}(t)z\left(U^{(i)}(t)\right)^*-W(t)z\right\|&=\left\|V_{i+1}V^{(i)}(1-t)z\left(V^{(i)}(1-t)\right)^*V_{i+1}^*-W(t)z\right\|\\&<\varepsilon+\left\|V_{i+1}zV_{i+1}^*-W(t)z\right\|\\&\leqslant 2\varepsilon\leqslant\frac{1}{4}.\end{aligned}$$

By connecting all $U^{(i)}(t)$, we get a path of unitary $U(t)$ such that for any t,

$$\left\|U(t)zU^*(t)-W(t)z\right\|<\frac{1}{4}.$$

In particular,

$$a=\left[W(t)\right]_1=\left[\mathrm{p}(U,z)\right].$$

Moreover,

$$\left\|\left[U(1),z\right]\right\|=\left\|[V_m,z]\right\|<\delta^{\mathrm{p}},$$

as desired. □

Corollary 3.10. *Let A be a unital separable simple* C^**-algebra with* $\mathrm{TR}(A)=0$. *Then there exists $\delta>0$ such that for any element $a\in K_0(A)$, if $|\tau(a)|\leqslant\delta$ for any $\tau\in\mathrm{T}(A)$, then one has*

$$a=\big[\mathrm{p}(U,z)\big]\in K_1\big(\mathrm{S}(A)\big)$$

for some unitary $z\in\mathrm{M}_k(A)$ and some $U(t)\in\mathrm{C}([0,1],\mathrm{U}(\mathrm{M}_k(A)))$ (for some $k\geqslant 1$) with $U(0)=1$ and $\|[U(1),z]\|<\delta^{\mathfrak{p}}$.

Proof. By Lemma 5.2 of [17], A has Property (B1). Then, the statement follows from Lemma 3.9. □

Corollary 3.11. *Let A be a unital separable simple* C^**-algebra with* $\mathrm{TR}(A)=0$. *Then $C(A)=K_0(A)$.*

Proof. Denote by δ the constant of Corollary 3.10. For any $a\in K_0(A)$, since $K_0(A)$ is tracially approximately divisible (see Definition 3.15), one has

$$a=a_1+a_2+\cdots+a_n$$

with $|\tau(a_i)|<\delta$ for each $1\leqslant i\leqslant n$. Therefore, $a_i\in C(A)$. Since $C(A)$ is a group, one has that $a\in C(A)$, as desired. □

Definition 3.12. For any unitary u in a C^*-algebra A, denote by $R(u,t)$ the unitary path in $\mathrm{M}_2(A)$ defined by

$$R(u,t):=\begin{pmatrix}u&0\\0&1\end{pmatrix}\begin{pmatrix}\cos(\frac{\pi t}{2})&\sin(\frac{\pi t}{2})\\-\sin(\frac{\pi t}{2})&\cos(\frac{\pi t}{2})\end{pmatrix}\begin{pmatrix}u^*&0\\0&1\end{pmatrix}\begin{pmatrix}\cos(\frac{\pi t}{2})&-\sin(\frac{\pi t}{2})\\\sin(\frac{\pi t}{2})&\cos(\frac{\pi t}{2})\end{pmatrix},\quad t\in[0,1].$$

Note that

$$R(u,0)=\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad\text{and}\quad R(u,1)=\begin{pmatrix}u&0\\0&u^*\end{pmatrix}.$$

Lemma 3.13. *Let A be a unital* C^**-algebra. Let $B\subseteq A$ be a unital separable* C^**-subalgebra. Write*

$$K_0(B)_+=\{k_1,k_2,\dots,k_n,\dots\}$$

and

$$K_1(B)=\{h_1,h_2,\dots,h_n,\dots\},$$

and denote by $K_n=\langle k_1,\dots,k_n\rangle$ the subgroup generated by $\{k_1,\dots,k_n\}$, and by $H_n=\langle h_1,\dots,h_n\rangle$ the subgroup generated by $\{h_1,\dots,h_n\}$. Let $\{\mathcal{F}_i\}$ be an increasing family of finite subsets whose union is dense in B. Assume that for each i, there exist a projection $p_i\in\mathrm{M}_{r_i}(\mathcal{F}_i)$ and a unitary $z_i\in\mathrm{M}_{r_i}(\mathcal{F}_i)$ with $[p_i]_0=k_i$ and $[z_i]_1=h_i$. Let $\{u_n\}$ be a sequence of unitaries such that

$$\big\|\big[u_{n+1}^{(r_n)},a\big]\big\|\leqslant\frac{\delta^{\mathfrak{p}}}{2^{n+1}}$$

for any $a \in \mathrm{M}_{r_n}(w_n^* \mathcal{F}_n w_n)$, *where* $w_n = u_1 \cdots u_n$ *and* $u_k^{(r_j)} = \mathrm{diag}(\overbrace{u_k, u_k, \ldots, u_k}^{r_j})$. *Then*

$$\alpha = \lim_{n\to\infty} \mathrm{Ad}(w_n)$$

defines a monomorphism from B *to* A, *and the extensions* $\eta_0(M_\alpha)$ *and* $\eta_1(M_\alpha)$ *are determined by the inductive limits*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_1(A) & \longrightarrow & K_1(A) \oplus K_{n+1} & \longrightarrow & K_{n+1} & \longrightarrow & 0 \\
 & & \| & & \| \;\; \nwarrow^{\gamma_n^0} \;\; \uparrow \iota_{n,n+1} & & \uparrow \iota_{n,n+1} & & \\
0 & \longrightarrow & K_1(A) & \longrightarrow & K_1(A) \oplus K_n & \longrightarrow & K_n & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A) \oplus H_{n+1} & \longrightarrow & H_{n+1} & \longrightarrow & 0 \\
 & & \| & & \| \;\; \nwarrow^{\gamma_n^1} \;\; \uparrow \iota_{n,n+1} & & \uparrow \iota_{n,n+1} & & \\
0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A) \oplus H_n & \longrightarrow & H_n & \longrightarrow & 0
\end{array}$$

respectively, where $\gamma_n^0 : K_n \to K_1(A)$ *is defined by* $k_i \mapsto [(w_n^* p_i w_n^*) u_{n+1} (w_n^* p_i w_n) + (1 - w_n^* p_i w_n)]$ *and* $\gamma_n^1 : H_n \to K_0(A)$ *is defined by* $h_i \mapsto [\mathrm{p}(R^*(u_{n+1}, t), w_n^* z_i w_n)]$.

Proof. It is clear that we may assume that $r_n \leqslant r_{n+1}$, $n = 1, 2, \ldots$.

Note that for any $a \in \mathcal{F}_n$,

$$\begin{aligned}
\big\| \mathrm{Ad}(w_{n+k})(a) - \mathrm{Ad}(w_n)(a) \big\| &\leqslant \sum_{i=1}^{k} \big\| \mathrm{Ad}(u_1 u_2 \cdots u_{n+i})(a) - \mathrm{Ad}(u_1 u_2 \cdots u_{n+i-1})(a) \big\| \\
&\leqslant \sum_{i=1}^{k} \frac{1}{2^{n+i}} \leqslant \frac{1}{2^n}.
\end{aligned}$$

Therefore $\lim \mathrm{Ad}(w_n)$ exists if $n \to \infty$. Denote by $\alpha_n = \mathrm{Ad}(w_n)$, and denote its limit by α. Note that α is a monomorphism. Moreover,

$$[\alpha] = [\iota] \quad \text{in } KL(B, A), \tag{3.12}$$

where $\iota : B \to A$ is the embedding. It follows that the six-term exact sequence in 2.4 splits.

To simplify notation, without loss of generality, in what follows, we may replace $\alpha \otimes \mathrm{id}_{\mathrm{M}_{r_i}}$ by α, $\alpha_n \otimes \mathrm{id}_{\mathrm{M}_{r_i}}$ by α_n, $u_n^{(r_n)}$ by u_n, and $w_n^{(r_n)}$ by w_n respectively (and write $p_i \in A$).

Consider $K_0(M_\alpha(A))$ first. Fix n, and note that

$$\big\| \alpha_n(p_i) - \alpha(p_i) \big\| \leqslant \frac{1}{4}, \quad \text{for any } 1 \leqslant i \leqslant n.$$

Therefore, for each i, there is a unitary v_i with $\|v_i - 1\| \leqslant \frac{1}{2}$ such that $\alpha(p_i) = v_i^* \alpha_n(p_i) v_i$. In particular, there is a path $r_i^{(n)}(t)$ of projections with $r_i^{(n)}(0) = \alpha_n(p_i)$, $r_i^{(n)}(1) = \alpha(p_i)$, and

$\|r_i^{(n)}(t)-\alpha(p_i)\| \leqslant \frac{1}{2}$. Then there is a homomorphism $\psi_n^{(0)} : K_1(A)\oplus K_n \to K_0(M_\alpha)$ defined by

$$\psi_n^{(0)} : \big(\big[Q(t)\big],[p_i]\big) \mapsto \big[Q(t)\big]+\big[P_i^{(n)}(t)\big],$$

where $P_i^{(n)}(t)$ is the path

$$P_i^{(n)}(t)=\begin{cases} R^*(w_n,2t)p_iR(w_n,2t), & 0\leqslant t\leqslant \frac{1}{2},\\ r_i^{(n)}(2t-1), & \frac{1}{2}\leqslant t\leqslant 1,\end{cases}$$

and $[Q(t)]\in K_0(\mathrm{S}(A))$. Since $[\pi]_0\circ\psi_n^{(0)}=\mathrm{id}_{K_n}$, the map $\psi_n^{(0)}$ is injective.

We then have

$$\psi_{n+1}^{(0)}\big(\big[Q(t)\big],[p_i]\big)-\psi_n^{(0)}\big(\big[Q(t)\big],[p_i]\big)=\big[P_i^{(n+1)}(t)\big]-\big[P_i^{(n)}(t)\big]\in K_0\big(\mathrm{S}(A)\big). \tag{3.13}$$

Moreover, it is easy to see that

$$\big[P_i^{(n+1)}(t)\big]-\big[P_i^{(n)}(t)\big]=\big[R^*(u_{n+1},t)w_n^*p_iw_nR(u_{n+1},t)\big]-\big[w_n^*p_iw_n\big]\in K_0\big(\mathrm{S}(A)\big).$$

Define a homomorphism $\psi_{n,n+1}^{(0)} : K_1(A)\oplus K_n\to K_1(A)\oplus K_{n+1}$ by

$$\psi_{n,n+1}^{(0)} : \big(Q(t),\big[p_i(t)\big]\big)\mapsto \big(\big[Q(t)\big]+\big(\big[P_i^{(n)}(t)\big]-\big[P_i^{(n+1)}(t)\big]\big),\big[p_i(t)\big]\big).$$

By the construction we have that $\psi_{n+1}^{(0)}\circ\psi_{n,n+1}^{(0)}=\psi_n^{(0)}$. Thus, we obtain a homomorphism

$$\psi^{(0)} : \lim_{\to}\big(K_0(M_{\alpha_n}),\psi_{n,n+1}\big)\to K_0\big(M_\alpha(A)\big).$$

Let us show that $\psi^{(0)}$ is surjective. For each projection $p(t)\in M_\alpha$, we can assume that $[p(0)]_0=k_i=[p_i]_0\in K_n$ for some $i\leqslant n$, and if we denote by v the partial isometry with $v^*p_iv=p(0)$, then $\|w_n^*vw_n-v\|\leqslant\frac{1}{2}$. Then there is a path of partial isometries $v(t)\in M_\alpha$ such that $v(0)=v$ and $v(1)=\alpha(v)$. Then

$$h:=\big[p(t)\big]-\big[v^*(t)P_i^{(n)}(t)v(t)\big]$$

is an element in $K_0(\mathrm{S}(A))$ and $[p(t)]_0=\psi^{(0)}(h,k_i)$. Therefore, $\psi^{(0)}$ is surjective. The injectivity of $\psi^{(0)}$ follows from the injectivity of each $\psi_n^{(0)}$. Thus, $\psi^{(0)}$ is an isomorphism.

Let us show that $\psi_{n,n+1}^{(0)}$ has the desired form.

Consider the invertible element $c=(w_n^*p_iw_n)u_{n+1}(w_n^*p_iw_n)+(1-(w_n^*p_iw_n))\in A$ (which is close to a unitary). Let us calculate the corresponding element in $\mathrm{S}(A)$. Consider the path

$$Z(t)=\begin{pmatrix} w_n^*p_iw_n & 0\\ 0 & w_n^*p_iw_n\end{pmatrix}R(u_{n+1},t)\begin{pmatrix} w_n^*p_iw_n & 0\\ 0 & w_n^*p_iw_n\end{pmatrix} + \begin{pmatrix} 1-w_n^*p_iw_n & 0\\ 0 & 1-w_n^*p_iw_n\end{pmatrix}.$$

Note that $Z(0) = \mathrm{diag}(1,1)$ and $Z(1) = \mathrm{diag}(c, c^*)$ and $Z(t)$ is invertible for any t with $\|Z^*(t)Z(t) - 1\| \leqslant \frac{1}{2^n}$. Set

$$e(t) = \big(Z^*(t)Z(t)\big)^{-\frac{1}{2}} Z^*(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Z(t)\big(Z^*(t)Z(t)\big)^{-\frac{1}{2}}.$$

Then the element in $K_0(\mathrm{S}(A))$ which corresponds to c is $[e] - \big[\big(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\big)\big]$. However, since

$$\left\| \left[\begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & w_n^* p_i w_n \end{pmatrix}, R(u_{n+1}, t) \right] \right\| \leqslant 2 \big\| [w_n^* p_i w_n, u_{n+1}] \big\| \leqslant \frac{1}{2^n},$$

a direct calculation shows

$$\begin{aligned} e(t) &\approx_{2/(2^n-1)} \begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & w_n^* p_i w_n \end{pmatrix} R^*(u_{n+1}, t) \begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & 0 \end{pmatrix} R(u_{n+1}, t) \\ &\qquad \times \begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & w_n^* p_i w_n \end{pmatrix} + \begin{pmatrix} 1 - w_n^* p_i w_n & 0 \\ 0 & 0 \end{pmatrix} \\ &\approx_{2/2^n} \quad R^*(u_{n+1}, t) \begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & w_n^* p_i w_n \end{pmatrix} \begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & 0 \end{pmatrix} \\ &\qquad \times \begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & w_n^* p_i w_n \end{pmatrix} R(u_{n+1}, t) + \begin{pmatrix} 1 - w_n^* p_i w_n & 0 \\ 0 & 0 \end{pmatrix} \\ &= \quad R^*(u_{n+1}, t) \begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & 0 \end{pmatrix} R(u_{n+1}, t) + \begin{pmatrix} 1 - w_n^* p_i w_n & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

and therefore, for $n > 2$,

$$\begin{aligned} [e] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] &= \left[R^*(u_{n+1}, t) \begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & 0 \end{pmatrix} R(u_{n+1}, t) \right] - \left[\begin{pmatrix} w_n^* p_i w_n & 0 \\ 0 & 0 \end{pmatrix} \right] \\ &= \big[R^*(u_{n+1}, t) w_n^* p_i w_n R(u_{n+1}, t) \big] - \big[w_n^* p_i w_n \big] \\ &= \big[P_i^{(n+1)}(t) \big] - \big[P_i^{(n)}(t) \big]. \end{aligned}$$

Therefore, the corresponding element of $[P_i^{(n+1)}(t)] - [P_i^{(n)}(t)]$ in $K_1(A)$ is

$$\big(w_n^* p_i w_n\big) u_{n+1} \big(w_n^* p_i w_n\big) + \big(1 - \big(w_n^* p_i w_n\big)\big),$$

and hence the map $\psi_{n,n+1}^{(0)}$ has the desired form.

Now, let us consider $K_1(M_\alpha)$. For each n, consider the unitaries $\{z_i;\ 1 \leqslant i \leqslant n\}$. Note that

$$\big\| \alpha(z_i) - \alpha_n(z_i) \big\| \leqslant \frac{1}{2^n}$$

for each i. Then, there are paths of unitaries $s_i^{(n)}$ such that $s_i^{(n)}(0) = \alpha_n(v_i)$, $s_i^{(n)}(1) = \alpha(v_i)$, and

$$\big\| s_i^{(n)}(t) - \alpha(v_i) \big\| \leqslant \frac{1}{2^{n-1}}$$

for each $t \in [0,1]$. Define a homomorphism $\psi_n^{(1)} : K_0(A) \oplus H_n \to K_1(M_\alpha(A))$ by

$$\psi_n^{(1)} : \big(\big[S(t)\big], z_i\big) \mapsto \big[S(t)\big] + \big[V_i^{(n)}(t)\big]$$

where $V_i^{(n)}$ is the path

$$V_i^{(n)}(t) = \begin{cases} R^*(w_n, 2t) z_i R(w_n, 2t), & 0 \leqslant t \leqslant \frac{1}{2}, \\ s_i^{(n)}(2t-1), & \frac{1}{2} \leqslant t \leqslant 1, \end{cases}$$

and $S(t)$ is a unitary in $\mathrm{U}_\infty(\widetilde{\mathrm{S}(A)})$.

We then have

$$\begin{aligned} \psi_{n+1}\big(\big[S(t)\big], [v_i]\big) - \psi_n\big(\big[S(t)\big], [v_i]\big) &= \big[V_i^{(n+1)}(t)\big] - \big[V_i^{(n)}(t)\big] \\ &= \big[\mathrm{p}\big(R^*(u_{n+1}, t), w_n^* v_i w_n\big)\big] \in K_1\big(\mathrm{S}(A)\big). \end{aligned}$$

Define homomorphism $\psi_{n,n+1}^{(1)} : K_0(A) \oplus H_n \to K_0(A) \oplus H_{n+1}$ by

$$\psi_{n,n+1}^{(0)} : \big(S(t), [v_i]\big) \mapsto \big(\big[S(t)\big] + \big(\big[V_i^{(n)}(t)\big] - \big[V_i^{(n+1)}(t)\big]\big), [v_i]\big).$$

By the construction we have $\psi_{n+1}^{(1)} \circ \psi_{n,n+1}^{(1)} = \psi_n^{(1)}$. Thus, there is a homomorphism

$$\psi^{(1)} : \varinjlim\big(K_0(M_{\alpha_n}), \psi_{n,n+1}\big) \to K_0(M_\alpha).$$

By the same argument as that for $K_0(M)_\alpha$, the map $\psi^{(1)}$ is an isomorphism, and moreover, it has the desired form. □

3.14. Let

$$0 \longrightarrow G \longrightarrow E \xrightarrow{\pi} H \longrightarrow 0$$

be a pure extension of abelian groups (cf. 2.3). Suppose that H is countable and write $H = \{h_1, h_2, \ldots\}$. Consider $H_n = \langle h_1, \ldots, h_n \rangle$, the subgroup generated by $\{h_1, \ldots, h_n\}$. Since E is a pure extension, there is a map $\theta_n : H_n \to E$ such that $\pi \circ \theta_n = \mathrm{id}_{H_n}$. Let us call this map θ_n a partial splitting map. Denote by ι_n the inclusion map from H_n to H_{n+1}, and consider the map $\gamma_n : H_n \to E$ defined by

$$\gamma_n = \theta_{n+1} \circ \iota_n - \theta_n.$$

It is clear that $\gamma_n(H_n) \subseteq G \subseteq E$. Therefore, let us regard γ_n as a map from H_n to G.

Definition 3.15. A partially ordered group G with an order unit is tracially approximately divisible if for any $a \in G$, any $\varepsilon > 0$, and any natural number n, there exists $b \in G$ such that $|\tau(a - nb)| \leqslant \varepsilon$ for any state τ of G.

Lemma 3.16. *In the setting of* 3.14*, if, in addition, G is a partially ordered group with an order unit which is tracially approximately divisible, then, for any $\varepsilon > 0$ and any given partial splitting map θ_n with $\pi \circ \theta_n = \mathrm{id}_{H_n}$, one can choose a partial splitting map θ_{n+1} such that $|\tau(\gamma_n(h_i))| < \varepsilon$ for any $1 \leqslant i \leqslant n$ and any state τ of G.*

Proof. Pick any partial splitting map $\theta'_{n+1} : H_{n+1} \to E_1$, and consider the map $\gamma'_n : H_n \to G$ defined by

$$\gamma'_n = \theta'_{n+1} \circ \iota_n - \theta_n.$$

Consider the $\mathbb{Q}$-linear map

$$\gamma'_n \otimes \mathrm{id} : H_n \otimes \mathbb{Q} \to G \otimes \mathbb{Q}.$$

Then it can be extended to $H_{n+1} \otimes \mathbb{Q}$. That is, there is an linear map $\psi : H_{n+1} \otimes \mathbb{Q} \to G \otimes \mathbb{Q}$ such that

$$\begin{array}{ccc} G \otimes \mathbb{Q} & \xleftarrow{\ \psi\ } & H_{n+1} \otimes \mathbb{Q} \\ & \nwarrow^{\gamma'_n \otimes \mathrm{id}} & \uparrow{\scriptstyle \iota_n \otimes \mathrm{id}} \\ & & H_n \otimes \mathbb{Q} \end{array}$$

commutes.

Since H_{n+1} is finitely generated, $H_{n+1} \cong \mathbb{Z}^{k_{n+1}} \oplus T_{n+1}$ for some finite abelian group T_{n+1}. Denote by $\{e_1, e_2, \ldots, e_{k_{n+1}}\}$ the standard generators for the torsion free part of H_{n+1}. Then, for each $1 \leqslant i \leqslant k_{n+1}$, we have

$$\psi(e_i) = \sum_{j=1}^{l_i} r_j^{(i)} g_j^{(i)}$$

for some $r_j^{(i)} \in \mathbb{Q}$ and $g_j^{(i)} \in G$.

Write $(\iota_n(h_s))_{\mathrm{free}} = \sum_i m_i^{(s)} e_i$, and denote by $m = \max\{m_i^{(s)}; 1 \leqslant s \leqslant n, 1 \leqslant i \leqslant k_{n+1}\}$. Since G is tracially approximately divisible, for each e_i, one can find $p_i \in G$ such that

$$\tau\big(\psi(e_i) - p_i\big) < \frac{\varepsilon}{k_{n+1} m}$$

for any state τ of G.

Define the map $\phi : H_{n+1} \to K_0(A) \subset E_1$ by sending e_i to p_i and T_{n+1} to $\{0\}$, and let us consider the map

$$\theta_{n+1} := \theta'_{n+1} - \phi.$$

Then, the map θ_{n+1} satisfies the lemma. Indeed, it is clear that $\pi \circ \theta_{n+1} = \mathrm{id}_{H_{n+1}}$. Moreover, for any h_s and any τ, we have

$$\begin{aligned}
\left|\tau\big(\theta_{n+1}\circ\iota_n(h_s)-\theta_n(h_s)\big)\right| &= \left|\tau\big(\theta'_{n+1}\circ\iota_n(h_s)-\phi\circ\iota_n(h_s)-\theta_n(h_s)\big)\right| \\
&= \left|\tau\big(\gamma'_n(h_s)-\phi\circ\iota_n(h_s)\big)\right| \\
&= \left|\tau\big(\psi\circ\iota_n(h_s)-\phi\circ\iota_n(h_s)\big)\right| \\
&= \left|\sum_i m_i^{(s)}\tau\big(\psi(e_i)-p_i\big)\right| \\
&< \varepsilon,
\end{aligned}$$

as desired. □

Theorem 3.17. *Let A be a unital simple* C*-*algebra and let $B\subseteq A$ be a unital separable* C*-*subalgebra. Suppose that A contains a positive element b with* $\mathrm{sp}(b)=[0,1]$*, that A has Property* (B1)*, and that Property* (B2) *associated with B and Δ_B and $K_0(A)$ is tracially approximately divisible. For any $E_0\in\mathrm{Pext}(K_1(B),K_0(A))$ and $E_1\in\mathrm{Pext}(K_0(B),K_1(A))$, there exists $\alpha\in\overline{\mathrm{Inn}}(B,A)$ such that $\eta_0(M_\alpha)=E_0$ and $\eta_1(M_\alpha)=E_1$.*

Proof. Write

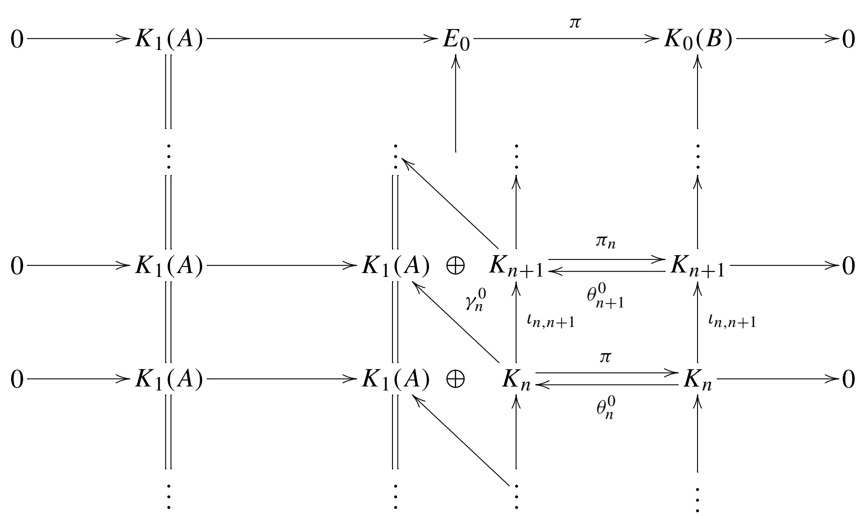
$$K_0(B)_+=\{k_1,k_2,\ldots,k_n,\ldots\}$$

and

$$K_1(B)=\{h_1,h_2,\ldots,h_n,\ldots\},$$

and consider the subgroups $K_n:=\langle k_1,\ldots,k_n\rangle$ and $H_n:=\langle h_1,\ldots,h_n\rangle$. Let $\{\mathcal{F}_i\}$ be an increasing family of finite subsets in the unit ball of B with the union dense in the unit ball of B. Assume that for each i, there is a projection $p_i\in\mathrm{M}_{r_i}(\mathcal{F}_i)$ and unitary $z_i\in\mathrm{M}_{r_i}(\mathcal{F}_i)$ with $[p_i]_0=k_i$ and $[z_i]_1=h_i$. We may assume that $r_i\leqslant r_{i+1}$, $i\in\mathbb{N}$.

We assert that there are unitaries $\{u_n\}$ and diagrams



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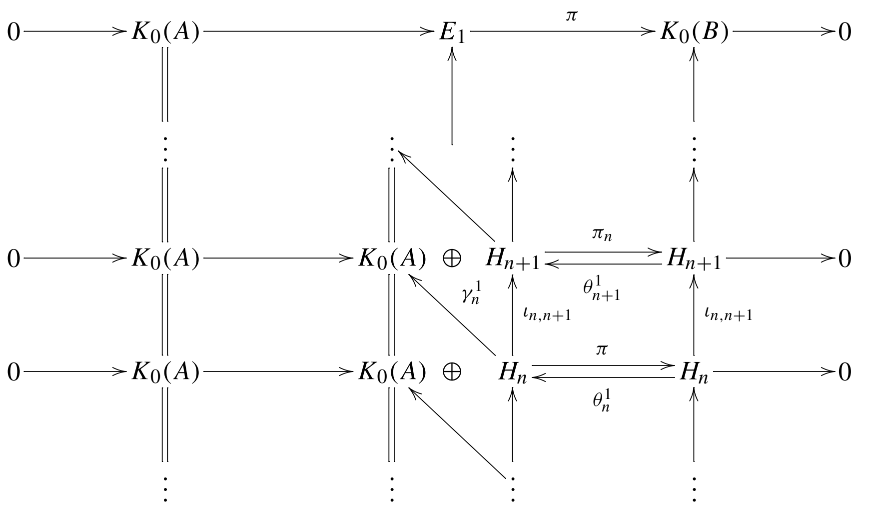


such that

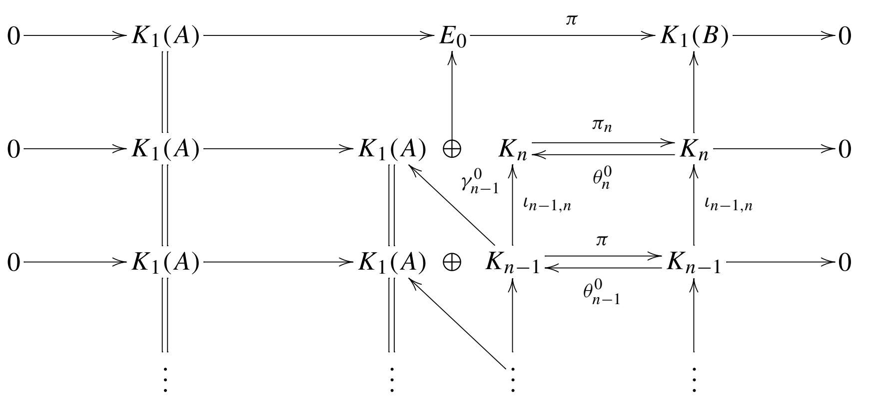
$$\big\|[u_{n+1}, a]\big\| \leqslant \frac{\delta^{\mathfrak{p}}}{r_n^2 \cdot 2^{n+1}}$$

for any $a \in \mathrm{M}_{r_n}(w_n^* \mathcal{F}_n w_n)$, where $w_n = u_1 \cdots u_n$ and $u_1 = 1$. The image of each γ_n^1 lies inside $C(A)$ so that $\Lambda \circ \gamma_n^1$ is well defined, and

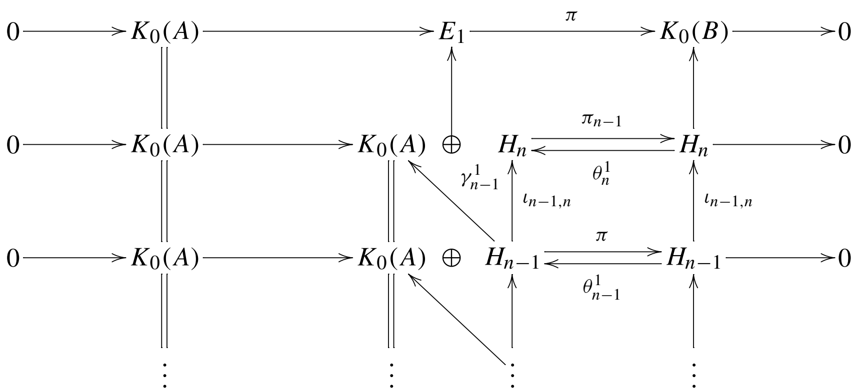
$$\mathrm{bott}_1\big(w_n^* z_i w_n, u_{n+1}\big) = \Lambda \circ \gamma_n^1(h_i) \quad \text{and} \quad \mathrm{bott}_0\big(w_n^* p_i w_n, u_{n+1}\big) = \gamma_n^0(k_i).$$

Moreover, each partial splitting map θ_n^i $(i = 0, 1)$ can be extended to a partial splitting map $\tilde{\theta}_n^i$ $(i = 0, 1)$ defined on the subgroup generated by $K_n \cup \mathcal{G}_0^n$ or $H_n \cup \mathcal{G}_1^n$, where $\mathcal{G}_i$ $(i = 0, 1)$ is the set of generators of G_i $(i = 0, 1)$ of Definition 3.6 with respect to $\frac{\delta^{\mathfrak{p}}}{r_{n+1}^2 \cdot 2^{n+1}}$, $\mathcal{F}_n$, $\mathcal{P}_0^{(n)}$, $\mathcal{P}_1^{(n)}$, and ι, where $\mathcal{P}_0 = \{[p_1], [p_2], \ldots, [p_n]\}$ and $\mathcal{P}_1 = \{[z_1], [z_2], \ldots, [z_n]\}$. Denote the subgroups generated by $K_n \cup \mathcal{G}_0$ and $H_n \cup \mathcal{G}_1$ by $\widetilde{K}_n$ and $\widetilde{H}_n$ respectively.

Assume that we have constructed unitaries $\{u_1 = 1, u_2, \ldots, u_n\}$ and diagrams



and



satisfying the above assertion.

Denote by

$$\delta_n = \Delta_B\left(\frac{\delta^{\mathrm{p}}}{r_{n+1}^2 \cdot 2^{n+1}}, \mathcal{F}_n, \mathcal{P}_0^{(n)}, \mathcal{P}_1^{(n)}, \mathrm{Ad}(w_n) \circ \imath\right),$$

where $\mathcal{P}_0 = \{[p_1], [p_2], \dots, [p_n]\}$ and $\mathcal{P}_1 = \{[z_1], [z_2], \dots, [z_n]\}$. Set

$$w_n^{(r_n)} = \mathrm{diag}(\overbrace{w_n, w_n, \dots, w_n}^{r_n}), \quad n = 1, 2, \dots.$$

We note that $[p_i] = [(w_n^{(r_n)})^* p_i w_n^{(r_n)}]$ and $[z_i] = [(w_n^{(r_n)})^* z_i w_n^{(r_n)}]$, $i = 1, 2, \dots$.

Since E_0 and E_1 are pure extensions, there are partial splitting maps $\tilde{\theta}_{n+1}^0 : \widetilde{K}_{n+1} \to E_0$ and $\tilde{\theta}_{n+1}^1 : \widetilde{H}_{n+1} \to E_1$. Since $K_0(A)$ is tracially approximately divisible, by Lemma 3.16, the partial splitting map $\tilde{\theta}_{n+1}^1$ can be chosen so that for any $g \in \mathcal{G}_i^n \cup \{g_1, \dots, g_n\}$,

$$\left|\tau\left(\gamma_n^1(g)\right)\right| \leqslant \min\{\delta, \delta_n\}, \quad \text{for all } \tau \in \mathrm{T}(A),$$

where δ is the constant of Lemma 3.9 (since A has Property (B1)). Note that the maps γ_n^0 and γ_n^1 are defined on $\widetilde{K}_n$ and $\widetilde{H}_n$ respectively, in particular, on G_0 and G_1 respectively.

By Lemma 3.9, one has

$$\gamma_n^1(h_i) = \left[U^*(t) z U(t) z^*\right]_1 \in K_1\left(\mathrm{S}(A)\right)$$

for a unitary $z \in A$ and a path $U(t) \in \mathrm{C}([0,1], \mathrm{U}_\infty(A))$ with $U_0 = 1$ and $\|[U(1), z]\| \leqslant \delta^{\mathrm{p}}$. Therefore, the map

$$\Lambda \circ \gamma_n^1\big|_{H_n} : H_n \to K_0(A)$$

is well defined, and by Lemma 3.5,

$$\left|\tau\left(\Lambda \circ \gamma_n^1(h_i)\right)\right| = \left|\tau\left(\gamma_n^1(h_i)\right)\right| \leqslant \delta_n$$

for any $\tau \in \mathrm{T}(A)$ and $1 \leqslant i \leqslant n$.

Put $b_0 = \gamma_n^0$ and $b_1 = \Lambda \circ \gamma_n^1$. By the assumption that A has Property (B2) associated with B and Δ_B, there is a unitary $u_{n+1} \in A$ such that

$$\big\|[u_{n+1}, a]\big\| \leqslant \frac{1}{r_n^2 \cdot 2^{n+1}}$$

for any $a \in \mathrm{M}_{r_n}(w_n^* \mathcal{F}_n w_n)$ and

$$\mathrm{bott}_1\big(w_n^* z_i w_n, u_{n+1}\big) = \Lambda \circ \gamma_n^1(h_i) \quad \text{and} \quad \mathrm{bott}_0\big(w_n^* p_i w_n, u_{n+1}\big) = \gamma_n^0(k_i).$$

Denote by θ_{n+1}^i $(i = 0, 1)$ the restriction of $\tilde{\theta}_{n+1}^i$ $(i = 0, 1)$ to K_{n+1} and H_{n+1} respectively. Repeating this procedure, we get a sequence of unitaries $\{u_n\}$ and diagrams satisfying the assertion.

By Lemma 3.13, the inner automorphisms $\{\mathrm{Ad}(u_1 \cdots u_n)\}$ converge to a monomorphism α, and the extensions $\eta_0(M_\alpha)$ and $\eta_1(M_\alpha)$ are determined by the inductive limits of

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_1(A) & \longrightarrow & K_1(A) \oplus K_{n+1} & \longrightarrow & K_{n+1} & \longrightarrow & 0 \\
 & & \| & & \| \ \ \nwarrow^{\tilde{\gamma}_n^0} \ \ \uparrow^{\iota_{n,n+1}} & & \uparrow^{\iota_{n,n+1}} & & \\
0 & \longrightarrow & K_1(A) & \longrightarrow & K_1(A) \oplus K_n & \longrightarrow & K_n & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A) \oplus H_{n+1} & \longrightarrow & H_{n+1} & \longrightarrow & 0 \\
 & & \| & & \| \ \ \nwarrow^{\tilde{\gamma}_n^1} \ \ \uparrow^{\iota_{n,n+1}} & & \uparrow^{\iota_{n,n+1}} & & \\
0 & \longrightarrow & K_0(A) & \longrightarrow & K_0(A) \oplus H_n & \longrightarrow & H_n & \longrightarrow & 0
\end{array}$$

respectively, where

$$\tilde{\gamma}_n^0(k_i) = \tilde{\gamma}_n^0\big([p_i]\big) = \big[\big(w_n^* p_i w_n\big) u_{n+1} \big(w_n^* p_i w_n\big) + \big(1 - w_n^* p_i w_n\big)\big]$$

and

$$\tilde{\gamma}_n^1(h_i) = \tilde{\gamma}_n^1\big([z_i]\big) = \big[\mathrm{p}\big(R^*(u_{n+1}, t), w_n^* z_i w_n\big)\big],$$

and therefore,

$$\tilde{\gamma}_n^0(k_i) = \mathrm{bott}_0\big(w_n^* p_i w_n, u_{n+1}\big) = \gamma_n^0(k_i), \quad 1 \leqslant i \leqslant n,$$

that is, $\tilde{\gamma}_n^0 = \gamma_n^0$. For each cross map $\tilde{\gamma}_n^1$, one has

$$\Lambda \circ \tilde{\gamma}_n^1(h_i) = \mathrm{bott}\big(w_n^* z_i w_n, R(u_{n+1}, 1)\big) = \mathrm{bott}\big(w_n^* z_i w_n, u_{n+1}\big) = \Lambda \circ \gamma_n^1(h_i), \quad 1 \leqslant i \leqslant n.$$

Since Λ is injective, we have that $\tilde{\gamma}_n^1 = \gamma_n^1$. Hence, one has that $\eta_0(M_\alpha) = E_0$ and $\eta_1(M_\alpha) = E_1$, as desired. □

Proposition 3.18. *Let C be a unital AH-algebra and let A be a unital separable simple* C^*-*algebra with* $\mathrm{TR}(A)=0$. *Suppose that there is a unital monomorphism $h: C \to A$. Then A has Property* (B2) *associated with C for some Δ_C as described in* 3.6.

Proof. Fix $\varepsilon > 0$, and finite subsets $\mathcal{F} \subset C$, $\mathcal{P}_0 \subset K_0(C)$ and $\mathcal{P}_1 \subset K_1(C)$. Write $C = \lim_{\to}(C_n, \psi_n)$ so that C_n and ψ_n satisfy the conditions in 7.2 of [17]. Let $\Delta_C(\varepsilon, \mathcal{F}, \mathcal{P}_0, \mathcal{P}_1, h)$ be δ as required by Lemma 7.5 of [17] for the above ε, $\mathcal{F}$, $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1$. Let $n \geqslant 1$ be an integer given in 7.5 of [17] so we may assume that

$$\mathcal{P} \subset \bigcup_{i=0,1} (\psi_{n,\infty})_{*i}\big(K_i(C_n)\big)$$

and let $k(n) \geqslant n$ be as in 7.5 of [17].

Put $G_i = (\psi_{k(n),\infty})_{*i}(K_i(C_{k(n)}))$, $i = 0, 1$. In particular, G_i is finitely generated and $\mathcal{P}_i \subset G_i$, $i = 0, 1$.

Let $b_i : G_i \to K_{i-1}(A)$ be given, $i = 0, 1$. Write

$$K_i\big(C_{k(n)} \otimes \mathrm{C}(\mathbb{T})\big) = K_i(C_{k(n)}) \oplus \boldsymbol{\beta}\big(K_{i-1}(C_{k(n)})\big), \quad i = 0, 1, \quad \text{and} \tag{3.14}$$

$$\underline{K}\big(C_{k(n)} \otimes \mathrm{C}(\mathbb{T})\big) = \underline{K}(C_{k(n)}) \oplus \boldsymbol{\beta}\big(\underline{K}(C_{k(n)})\big) \tag{3.15}$$

(see 2.10 of [16]).

Define $\kappa^{(0)} : \underline{K}(C_{k(n)}) \to \underline{K}(A)$ by $\kappa^{(0)} = [h \circ \psi_{k(n),\infty}]$. Define $\kappa_i^{(1)} : \boldsymbol{\beta}(K_i(C_{k(n)})) \to K_{i-1}(A)$ by

$$\kappa_i^{(1)} \circ \boldsymbol{\beta}(x) = b_{i-1}(x)$$

for $x \in K_{i-1}(C_{k(n)})$, $i = 0, 1$. Since C satisfies the UCT, there is $\kappa^{(1)} \in KK(\mathrm{S}(C_{k(n)}), A)$ such that $\Gamma(\kappa^{(1)}) = \kappa_i^{(1)}$. Define $\kappa : \underline{K}(C_{k(n)} \otimes \mathrm{C}(\mathbb{T})) \to \underline{K}(A)$ by $\kappa|_{\underline{K}(C_{k(n)})} = \kappa^{(0)}$ and $\kappa|_{\boldsymbol{\beta}(\underline{K}(C_{k(n)}))} = \kappa^{(1)}$.

The lemma then follows from Lemma 7.5 of [17]. □

Lemma 3.19. *Let B be a unital separable simple amenable* C^*-*algebra with* $\mathrm{TR}(B)=0$ *which satisfies the UCT, and let A be a unital simple* C^*-*algebra with real rank zero, stable rank one and weakly unperforated K_0-group. Suppose that $\bar{\kappa} \in KL(B, A)^{++}$ with $\bar{\kappa}([1_B]) \leqslant [1_A]$ in $K_0(A)$. Then there is a monomorphism $\alpha : B \to A$ such that*

$$[\alpha] = \bar{\kappa} \quad \text{in } KL(B, A). \tag{3.16}$$

Proof. It follows from the classification theorem [14] that B is a unital simple AH-algebra with slow dimension growth and with real rank zero. Then the lemma follows from Theorem 4.6 of [12] immediately. □

Theorem 3.20. *Let A and B be unital simple* C^*-*algebras with* $\mathrm{TR}(A)=0$ *and* $\mathrm{TR}(B)=0$. *Assume that B is separable amenable and satisfies the UCT. Then, for any $\kappa \in KK(B, A)^{++}$ with $\kappa([1_B]) \leqslant [1_A]$ in $K_0(A)$, there is a monomorphism $\alpha : B \to A$ such that $[\alpha] = \kappa$ in $KK(B, A)$.*

Proof. Denote by $\bar{\kappa}$ the image of κ in $KL(B,A)^+$. It follows from Lemma 3.19 that there exists $\alpha_1 \in \mathrm{Hom}(B,A)$ such that $[\alpha_1]_{KL} = \bar{\kappa}$. By considering the cut-down of A by $\alpha_1(1_B)$, we can regard B as a unital C*-subalgebra of A with embedding α_1. Since $\kappa - [\alpha_1]_{KK} \in \mathrm{Pext}(K_*(B), K_{*+1}(A))$, by Theorem 3.17, there is an approximately inner monomorphism α_2 of from B to A such that $[\alpha_2 \circ \alpha_1]_{KK} - [\alpha_1]_{KK} = \kappa - [\alpha_1]_{KK}$. Then, $\alpha := \alpha_2 \circ \alpha_1$ is the desired homomorphism. □

Let us recall the following theorem from [20]:

Theorem 3.21. *(See [20, Theorem 5.2].) Let C be a unital AH-algebra and let A be a unital simple* C*-*algebra with* $\mathrm{TR}(A) = 0$. *Suppose that there is a pair $\kappa \in KL_e(C,A)^{++}$ and a continuous affine map $\lambda : \mathrm{T}(A) \to \mathrm{T}_{\mathfrak{f}}(C)$ which is compatible with κ. Then there exists a unital monomorphism $h : C \to A$ such that*

$$[h] = \kappa \quad \textit{and} \quad h_{\mathrm{T}} = \lambda.$$

Theorem 3.22. *Let C be a unital AH-algebra and let A be a unital simple* C*-*algebras with* $\mathrm{TR}(A) = 0$. *Suppose that $(\kappa, \lambda) \in KKT(C,A)^{++}$. Then there is a monomorphism $\phi : C \to A$ such that $[\phi] = \kappa$ and $\phi_{\mathrm{T}} = \lambda$.*

Proof. It follows from Theorem 3.21 that there exists a unital monomorphism $\psi : C \to A$ such that

$$[\psi]_{KL} = \bar{\kappa} \quad \text{in } KL(C,A) \quad \text{and} \quad \phi_{\mathrm{T}} = \lambda.$$

Then $\kappa - [\psi] \in \mathrm{Pext}(K_*(C), K_*(A))$. As in the proof of 3.20, we obtain a unital monomorphism $\alpha : \psi(C) \to A$ for which

$$[\alpha \circ \psi] = [\kappa] \quad \text{in } KK(C,A)$$

and there exists a sequence of unitaries $\{u_n\} \subset \mathrm{U}(A)$ such that

$$\lim_{n\to\infty} \mathrm{Ad}\, u_n \circ \phi(c) = \alpha \circ \phi(c) \quad \text{for all } c \in C.$$

Put $\phi = \alpha \circ \psi$. Then

$$\lambda(\tau)(c) = \tau \circ \psi(c) = \lim_{n\to\infty} \tau\big(\mathrm{Ad}\, u_n \circ \psi(c)\big) \tag{3.17}$$

$$= \tau\big(\alpha \circ \psi(c)\big) = \tau \circ \phi(c) \quad \text{for all } c \in C. \tag{3.18}$$

It follows that

$$\phi_{\mathrm{T}} = \lambda,$$

as desired. □

Remark 3.23. It was shown in [20] that there are a compact metric space X, a unital simple AF-algebra A and $\bar{\kappa} \in KL_e(\mathrm{C}(X), A)^{++}$ for which there is no unital homomorphism $h : \mathrm{C}(X) \to A$ such that $[h] = \bar{\kappa}$. Thus, the information on λ is essential in general. In the case that C is also simple and has real rank zero, the map λ is completely determined by $\bar{\kappa}$ since $\rho_C(K_0(C))$ is dense in $\mathrm{Aff}(\mathrm{T}(C))$ and $\mathrm{T}(C) = \mathrm{T}_{\mathfrak{f}}(C)$. If the C*-algebra C is of real rank zero and exact, then, without assuming simplicity, $\mathrm{T}(C) = \mathrm{S}(K_0(C))$, and hence one can define the map $r : \mathrm{T}(A) \to \mathrm{T}(C)$ by factoring through $\mathrm{S}(K_0(A))$. It is obvious that r is compatible with $\bar{\kappa}$. Moreover, $r(\tau)$ is faithful on C for any $\tau \in \mathrm{T}(A)$. Indeed, if $r(\tau)(c) = 0$ for some nonzero positive element $c \in C$, then $r(\tau)(\overline{cCc}) = \{0\}$. In particular, $r(\tau)(p) = 0$ for some nonzero projection $p \in \overline{cCc}$, which contradicts the strict positivity of $\bar{\kappa}$.

Lemma 3.24. *Let X be a Banach space, and let $\{\alpha_n\}$ be a sequence of isometries. If each α_n is invertible, and $\lim_{n\to\infty}\alpha_n(x)$ and $\lim_{n\to\infty}\alpha_n^{-1}(x)$ exist for any $x \in X$, then $\alpha := \lim_{n\to\infty}\alpha_n$ is invertible.*

Proof. Denote by $\beta = \lim_{n\to\infty} \alpha_n^{-1}$. It is clear that α and β are isometries. Fix an element $x \in X$. For any $\varepsilon > 0$, there exists N such that

$$\left\|\beta(x) - \alpha_n^{-1}(x)\right\| \leqslant \varepsilon \quad \text{and} \quad \left\|\alpha \circ \beta(x) - \alpha_n \circ \beta(x)\right\| \leqslant \varepsilon$$

for any $n \geqslant N$. Then

$$\begin{aligned}
\left\|\alpha \circ \beta(x) - x\right\| &\leqslant \left\|\alpha \circ \alpha_n^{-1}(x) - x\right\| + \left\|\alpha \circ \beta(x) - \alpha \circ \alpha_n^{-1}(x)\right\| \\
&\leqslant \left\|\alpha \circ \alpha_n^{-1}(x) - \alpha_n \circ \alpha_n^{-1}(x)\right\| + \varepsilon \\
&\leqslant \left\|\alpha \circ \alpha_n^{-1}(x) - \alpha \circ \beta(x)\right\| + \left\|\alpha \circ \beta(x) - \alpha_n \circ \beta(x)\right\| \\
&\quad + \left\|\alpha_n \circ \beta(x) - \alpha_n \circ \alpha_n^{-1}(x)\right\| + \varepsilon \\
&\leqslant 4\varepsilon.
\end{aligned}$$

Since ε is arbitrary, one has that $\alpha \circ \beta(x) = x$, and hence $\alpha \circ \beta = \mathrm{id}$. The same argument as above shows that $\beta \circ \alpha = \mathrm{id}$. Therefore, α is invertible, as desired. □

Definition 3.25. Let A be a unital C*-algebra. Denote by $KK_e^{-1}(A, A)^{++}$ the set of those elements $\kappa \in KK_e(A, A)^{++}$ such that κ induces an ordered isomorphism between $K_0(A)$ and isomorphisms between $\underline{K}(A)$.

Corollary 3.26. *Let A be a unital separable amenable simple* C*-*algebra with* $\mathrm{TR}(A) = 0$ *satisfying the UCT. Then, for any $\kappa \in KK_e^{-1}(A, A)^{++}$, there exists an automorphism $\alpha \in \mathrm{Aut}(A)$ such that*

$$[\alpha] = \kappa.$$

Proof. By Theorem 3.20 and its remark, there is a monomorphism $\alpha = \alpha_2 \circ \alpha_1$ such that $[\alpha] = \kappa$ in $KK(A, A)$. Let us show that α can be chosen to be an automorphism. Let us first show that α_1 can be chosen to be an automorphism. We first choose a unital monomorphism $\alpha_1' : A \to A$ so that $[\alpha'] = \kappa$ by 3.20 and its remark.

By the UCT, there is $\kappa_1 \in KK_e^{-1}(A, A)^{++}$ such that $\kappa_1 \times \kappa = \kappa \times \kappa_1 = [\mathrm{id}_A]$. So there is a unital monomorphism $\beta : A \to A$ such that $[\beta] = \kappa_1$. Then $[\beta \circ \alpha'] = [\alpha' \circ \beta] = [\mathrm{id}_A]$. By the uniqueness theorem (2.3 of [13]), both $\beta \circ \alpha'$ and $\alpha' \circ \beta$ are approximately unitarily equivalent to the identity. Then, by a standard intertwining argument of Elliott, one obtains an isomorphism $\alpha_1 : A \to A$ which is approximately unitarily equivalent to α' (see for example Theorem 3.6 of [13]). In particular, $\overline{[\alpha_1]} = \bar{\kappa}$ in $KL(A, A)$. (This also follows from the proof of Theorem 3.7 of [13]: Using that fact that A is pre-classifiable in the sense of [13], there is an isomorphism $\alpha_1 : A \to A$ such that $[\alpha_1] = \bar{\kappa}$ in $KL(A, A)$. See Theorem 4.2 of [14].)

Consider the map α_2. Note that if $A = B$, then, in the proof of Theorem 3.17, the union of the finite subsets $\mathcal{F}_n$ is dense in A, and the inner automorphisms $\{\mathrm{Ad}(u_1 \cdots u_n)\}$ satisfy Lemma 3.24. Therefore, by Lemma 3.24, the monomorphism $\alpha_2 = \lim_{n\to\infty} \mathrm{Ad}(u_1 \cdots u_n)$ is an automorphism of A.

Therefore, the map $\alpha = \alpha_2 \circ \alpha_1$ is an automorphism of A. □

4. Rotation maps

Lemma 4.1. *Let H be a finitely generated abelian group, and let A be a* C*-*algebra with $\rho_A(K_0(A))$ dense in* Aff(T(A)). *Let $\psi \in \mathrm{Hom}(H, \mathrm{Aff}(\mathrm{T}(A)))$. Fix $\{g_1, \ldots, g_n\} \subseteq H$. Then, for any $\varepsilon > 0$, there is a homomorphism $h : H \to K_0(A)$ such that*

$$\bigl|\psi(g_i) - \rho_A\bigl(h(g_i)\bigr)\bigr| < \varepsilon$$

for any $1 \leqslant i \leqslant n$.

Proof. Since the map ψ factors through H/H_{Tor}, we may assume that $H = \bigoplus^k \mathbb{Z}$ for some k and prove the lemma for $\{g_i;\ i = 1, \ldots, n\}$ being replaced by the standard basis $\{e_i;\ i = 1, \ldots, k\}$. Since $\rho_A(K_0(A))$ is dense in Aff(T(A)), there exist $a_1, \ldots, a_n \in K_0(A)$ such that

$$\bigl|\psi(e_i) - \rho_A(a_i)\bigr| < \varepsilon, \quad i = 1, \ldots, k.$$

Then the map $h : H \to K_0(A)$ defined by

$$e_i \mapsto a_i$$

satisfies the lemma. □

Theorem 4.2. *Let A be a unital simple* C*-*algebra with $\rho_A(K_0(A))$ dense in* Aff(T(A)). *Assume that there is a positive element $b \in A$ with* sp(b) = [0, 1]. *Let $B \subseteq A$ be a unital* C*-*subalgebra and denote by ι the inclusion map. Suppose that A has Property* (B1) *and Property* (B2) *associated with B and Δ_B. For any $\psi \in \mathrm{Hom}(K_1(B), \mathrm{Aff}(\mathrm{T}(A)))$, there exists $\alpha \in \overline{\mathrm{Inn}}(B, A)$ such that there are maps $\theta_i : K_i(B) \to K_i(M_\alpha)$ with $\pi_0 \circ \theta_i = \mathrm{id}_{K_i(B)}$, $i = 0, 1$, and the rotation map $R_{\iota,\alpha} : K_1(M_\alpha) \to \mathrm{Aff}(\mathrm{T}(A))$ is given by*

$$R_{\iota,\alpha}(c) = \rho_A\bigl(c - \theta_1\bigl([\pi_0]_1(c)\bigr)\bigr) + \psi\bigl([\pi_0]_1(c)\bigr) \quad \text{for all } c \in K_1\bigl(M_\alpha(A)\bigr).$$

In other words,

$$[\alpha] = [\iota]$$

in $KK(B,A)$, and the rotation map $R_{\iota,\alpha}: K_1(M_\alpha) \to \mathrm{Aff}(\mathrm{T}(A))$ is given by

$$R_{\iota,\alpha}(a,b) = \rho_A(a) + \psi(b)$$

for some identification of $K_1(M_\alpha)$ with $K_0(A) \oplus K_1(B)$.

Proof. Write

$$K_1(B) = \{g_1, g_2, \ldots, g_n, \ldots\},$$

$$K_0(B)_+ = \{k_1, k_2, \ldots, k_n, \ldots\},$$

and denote by $H_n = \langle g_1, \ldots, g_n \rangle$ the subgroup generated by $\{g_1, \ldots, g_n\}$, and by $K_n = \langle k_1, \ldots, k_n \rangle$ the subgroup generated by $\{k_1, \ldots, k_n\}$. Let $\{\mathcal{F}_i\}$ be an increasing family of finite subsets with union dense in B. Assume that for each i, there is a unitary z_i and a projection p_i in $\mathrm{M}_{r_i}(\mathcal{F}_i)$ for some natural number r_i such that $[z_i]_1 = g_i$ and $[p_i] = k_i$. Without loss of generality, we may assume that $r_i \leqslant r_{i+1}$, $i \in \mathbb{N}$. In what follows, if $v \in A$, by $v^{(m)}$, we mean $v^{(m)} = \mathrm{diag}(\overbrace{v, v, \ldots, v}^{m})$.

We assert that there are maps $\{h_i : H_i \to K_0(A);\ i \in \mathbb{N}\}$ and unitaries $\{u_i;\ i \in \mathbb{N}\}$ such that for any n, writing $w_n = u_1 \cdots u_{n-1}$ (assume $u_{-1} = u_0 = 1$), one has

1. For any $x \in \{g_1, \ldots, g_i\} \cup \mathcal{G}_1^n$,

   $$\left|\rho_A \circ h_n(x) - \psi(x)\right| < \frac{\delta_n \delta}{2^n}$$

   where

   $$\delta_n = \Delta_B\left(\frac{\delta^{\mathrm{p}}}{r_n^2 \cdot 2^{n+1}}, \mathcal{F}_n, \mathcal{P}_n^{(0)}, \mathcal{P}_n^{(1)}, \mathrm{Ad}(w_{n-1}) \circ \iota\right)$$

   for $\mathcal{P}_n^{(0)} = \{k_1, k_2, \ldots, k_n\}$ and $\mathcal{P}_n^{(1)} = \{g_1, g_2, \ldots, g_n\}$, δ the constant of Lemma 3.9 (since A has Property (B1)), and $\mathcal{G}_1^n$ is the set of generators of G_1 of Definition 3.6 with respect to $\frac{\delta^{\mathrm{p}}}{r_n^2 \cdot 2^{n+1}}$, $\mathcal{F}_n$, $\mathcal{P}_n^{(0)}$, $\mathcal{P}_n^{(1)}$, and ι.
2. For any $a \in w_{n-1}^* \mathcal{F}_n w_{n-1}$,

   $$\left\|[u_n, a]\right\| \leqslant \frac{\delta^{\mathrm{p}}}{r_n^2 \cdot 2^{n+1}}$$

   where $w_n = u_1 \cdots u_n$, and moreover, the image of $\phi_n := h_{n+1}|_{H_n} - h_n$ lies inside $C(A)$ so that $\Lambda \circ \phi_n$ is well defined, and

   $$\mathrm{bott}_1\big(\big(w_{n-1}^{(r_n)}\big)^* z_i w_{n-1}^{(r_n)}, u_n\big) = \Lambda \circ \phi_n(g_i) \quad \text{and} \quad \mathrm{bott}_0\big(\big(w_{n-1}^{(r_n)}\big)^* p_i w_{n-1}^{(r_n)}, u_n\big) = 0$$

   for any $i = 1, \ldots, n$.

If $n = 1$, since $\rho_A(K_0(A))$ is dense in $\mathrm{Aff}(\mathrm{T}(A))$, by Lemma 4.1, there is a map $h_1 : \langle\{g_1\} \cup \mathcal{G}_1\rangle \to K_0(A)$ such that for any $x \in \{g_1\} \cup \mathcal{G}_1$,

$$\left|\psi(x) - \rho_A\big(h_1(x)\big)\right| < \frac{\delta_1 \delta}{2},$$

and a map $h_2 : \langle\{g_1, g_2\} \cup \mathcal{G}_2\rangle \to K_0(A)$ such that

$$\left|\psi(x) - \rho_A\big(h_2(x)\big)\right| \leqslant \frac{\delta_1 \delta}{2^2}$$

for any $x \in \{g_1, g_2\} \cup \mathcal{G}_2$. Concerning the function

$$\phi_1 = h_2|_{H_1} - h_1,$$

we have that $|\tau(\phi_1(g_1))| < \min\{\delta, \delta_1\}$ for any $\tau \in \mathrm{T}(A)$ and hence $\phi_1(g_1) \in C(A)$ by Lemma 3.9. By Lemma 3.5,

$$\left|\tau\big(\Lambda \circ \phi_1(x)\big)\right| = \left|\tau\big(\phi_1(x)\big)\right| < \delta_1$$

for any $\tau \in \mathrm{T}(A)$ and any $x \in \mathcal{G}_1$. Since A has Property (B2) associated with B and Δ_B, there is a unitary $u_1 \in \mathrm{U}(A)$ such that

$$\left\|[u_1, a]\right\| \leqslant \frac{\delta^{\mathrm{p}}}{r_1^2 \cdot 2^2} \quad \text{for all } a \in \mathcal{F}_1,$$

and

$$\mathrm{bott}_1(z_1, u_1) = \Lambda \circ \phi_1(g_1) \quad \text{and} \quad \mathrm{bott}_0(p_1, u_1) = 0.$$

Assume that we have constructed the maps $\{h_i : H_i \to K_0(A);\ i = 1, \ldots, n\}$ and unitaries $\{u_i;\ i = 1, \ldots, n-1\}$ satisfying the assertion above. By Lemma 4.1, there is a function $h_{n+1} : \langle\{g_1, \ldots, g_{n+1}\} \cup \mathcal{G}_1^n \cup \mathcal{G}_1^{n+1}\rangle \to K_0(A)$ such that for any $x \in \{g_1, \ldots, g_{n+1}\} \cup \mathcal{G}_1^n \cup \mathcal{G}_1^{n+1}$,

$$\left|\rho_A \circ h_{n+1}(x) - \psi(x)\right| < \frac{\delta_{n+1}\delta}{2^{n+1}}$$

where

$$\delta_{n+1} = \Delta_B\left(\frac{\delta^{\mathrm{p}}}{r_n^2 \cdot 2^{n+2}}, \mathcal{F}_{n+1}, \mathcal{P}_{n+1}^{(0)}, \mathcal{P}_{n+1}^{(1)}, \mathrm{Ad}(w_n) \circ \iota\right)$$

for $\mathcal{P}_{n+1}^{(0)} = \{k_1, k_2, \ldots, k_{n+1}\}$ and $\mathcal{P}_{n+1}^{(1)} = \{g_1, g_2, \ldots, g_{n+1}\}$.

Recall that $\phi_n = h_{n+1}|_{H_n} - h_n$. A direct calculation shows that for any $\tau \in \mathrm{T}(A)$,

$$\left|\tau(\Lambda \circ \phi_n)(x)\right| = \left|\tau\big(\phi_n(x)\big)\right| < \min\{\delta, \delta_{n+1}\}$$

for any $x \in \mathcal{G}_1^n$. Since A has Property (B2) associated with B and Δ_B, there is a unitary $u_n \in \mathrm{U}(A)$ such that

$$\left\| \left[u_n, w_{n-1}^* a w_{n-1}\right] \right\| \leqslant \frac{\delta^{\mathrm{p}}}{r_n^2 \cdot 2^{n+1}} \quad \text{for all } a \in \mathcal{F}_n,$$

and

$$\mathrm{bott}_1\big(\mathrm{Ad}(w_{n-1}) \circ \iota, u_n\big)\big|_{\mathcal{P}_n^{(1)}} = \Lambda \circ \phi_n \quad \text{and} \quad \mathrm{bott}_0\big(\mathrm{Ad}(w_{n-1}) \circ \iota, u_n\big)\big|_{\mathcal{P}_n^{(0)}} = 0.$$

This proves the assertion.

By Lemma 3.13, $\mathrm{Ad}(w_n)$ converges to a monomorphism $\alpha : B \to A$. Moreover, the extension $\eta_0(M_\alpha)$ is trivial, and $\eta_1(M_\alpha)$ is determined by the inductive limit of

$$\begin{array}{ccccccc}
0 \longrightarrow & K_0(A) & \longrightarrow & K_0(A) \oplus H_{n+1} & \longrightarrow & H_{n+1} & \longrightarrow 0 \\
 & \| & & \| \;\; \nwarrow^{\gamma_n} \quad \uparrow \iota_{n,n+1} & & \uparrow \iota_{n,n+1} & \\
0 \longrightarrow & K_0(A) & \longrightarrow & K_0(A) \oplus H_n & \longrightarrow & H_n & \longrightarrow 0
\end{array}$$

where $\gamma_n(g_i) = [\mathrm{p}(R^*(u_{n+1}, t), w_n^* z_i w_n)]$. However, since

$$\Lambda \circ \gamma_n(g_i) = \mathrm{bott}_1\big(\mathrm{Ad}(w_{n-1}) \circ \iota, u_n\big)(g_i) = \Lambda \circ \phi_n(g_i),$$

by the injectivity of Γ, we have that $\gamma_n = \phi_n$.

We assert that $\eta_1(M_\alpha)$ is also trivial. For any n, define a map $\theta_n' : H_n \to K_1(M_\alpha)$ by

$$\theta_n'(g) = \big(h_n(g), g\big).$$

We then have

$$\begin{aligned}
&\theta_{n+1}' \circ \iota_{n,n+1}(g) - \theta_n'(g) \\
&\quad = \big(h_{n+1} \circ \iota_{n,n+1}(g), \iota_{n,n+1}(g)\big) - \big(h_n(g) + \phi_n(g), \iota_{n,n+1}(g)\big) \\
&\quad = \big(h_{n+1} \circ \iota_{n,n+1}(g), \iota_{n,n+1}(g)\big) - \big(h_n(g) + \big(h_{n+1}\big(\iota_{n,n+1}(g)\big) - h_n(g)\big), \iota_{n,n+1}(g)\big) \\
&\quad = 0,
\end{aligned}$$

and hence the sequence (θ_n') defines a homomorphism $\theta_1 : K_1(B) \to K_1(M_\alpha)$. Moreover, since $\pi \circ \theta_1 = \mathrm{id}_{K_1(B)}$, the extension $\eta_1(M_\alpha)$ splits. Therefore, $[\alpha] = [\iota]$ in $KK(B, A)$.

Let us calculate the corresponding rotation map. Choose $g_i \in H_i$ with $[z_i] = g_i$. To simplify notation, without loss of generality, we will use α for $\alpha \otimes \mathrm{id}_{\mathrm{M}_r(\mathbb{C})}$ on $B \otimes \mathrm{M}_r(\mathbb{C})$ for any integer $r \geqslant 1$. Take a unitary path $v(t)$ in $\mathrm{M}_{2r_i}(A)$ from z_i to $\alpha(z_i)$ as follows:

$$v(t) = R^*(2t) \begin{pmatrix} 1 & 0 \\ 0 & (w_m^{(r(i))})^* \end{pmatrix} R(2t) \begin{pmatrix} z_i & 0 \\ 0 & 1 \end{pmatrix} R^*(2t) \begin{pmatrix} 1 & 0 \\ 0 & w_m^{(r(i))} \end{pmatrix} R(2t)$$

for $t \in [0, 1/2]$ and

$$v(t) = \big(w_m^{(r(i))}\big)^* z_i w_m^{(r(i))} \exp\big((2t-1)c_m\big)$$

for $t \in [1/2, 1]$, where

$$R(t) = \begin{pmatrix} \cos(\frac{\pi t}{2}) & \sin(\frac{\pi t}{2}) \\ -\sin(\frac{\pi t}{2}) & \cos(\frac{\pi t}{2}) \end{pmatrix}$$

and $c_m = \log((w_m^{(r(i))})^* z_i^* w_m^{(r(i))} \alpha(z_i))$.

Then, for any $\tau \in \mathrm{T}(A)$, we have

$$\begin{aligned}
&\frac{1}{2\pi i}\int_0^1 \tau\big(\dot{v}(t)v^*(t)\big)\,dt \\
&= \tau\big(\log\big(\big(w_m^{(r(i))}\big)^* z_i^* w_m^{(r(i))} \alpha(z_i)\big)\big) \\
&= \lim_{n\to\infty} \tau\big(\log\big(\big(w_m^{(r(i))}\big)^* z^* w_m^{(r(i))} \operatorname{Ad} w_n^{(r(i))}(z_i)\big)\big) \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \tau\big(\log\big(\big(\big(w_{m+k-1}^{(r(i))}\big)^* z_i^* w_{m+k-1}^{(r(i))}\big)\big(\big(w_{m+k}^{(r(i))}\big)^* z_i w_{m+k}^{(r(i))}\big)\big)\big) \\
&= \sum_{k=0}^{\infty} \tau\big(\log\big(\big(\big(w_{m+k-1}^{(r(i))}\big)^* z_i^* w_{m+k-1}^{(r(i))}\big)\big(\big(w_{m+k}^{(r(i))}\big)^* z_i w_{m+k}^{(r(i))}\big)\big)\big) \\
&= \sum_{k=0}^{\infty} \tau\big(\log\big(\big(\big(w_{m+k-1}^{(r(i))}\big)^* z_i^* w_{m+k-1}^{(r(i))}\big)\big(u_{m+k}^{(r(i))}\big(\big(w_{m+k-1}^{(r(i))}\big)^* z_i w_{m+k-1}^{(r(i))}\big) u_{m+k}^{(r(i))}\big)\big)\big) \\
&= \sum_{k=0}^{\infty} \tau\big(\mathrm{bott}_1\big(\big(w_{m+k-1}^{(r(i))}\big)^* z_i^* w_{m+k-1}^{(r(i))}, u_{m+k}\big)\big).
\end{aligned}$$

Then, by the choice of $\{u_n\}$, we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_0^1 \tau\big(\dot{v}(t)v^*(t)\big)\,dt &= \sum_{k=0}^{\infty} \tau\big(\Lambda \circ \phi_{m+k}(g_i)\big) \\
&= \sum_{k=0}^{\infty} \tau\big(\phi_{m+k}(g_i)\big) \\
&= \sum_{k=0}^{\infty} \tau\big(h_{m+k+1}(g_i) - h_{m+k}(g_i)\big) \\
&= \psi(g_i) - \tau\big(h_m(g_i)\big).
\end{aligned}$$

Thus,

$$\begin{aligned}
R_{\alpha_1}\big([v]\big) &= \psi(g_i) - \rho_A \circ h_m(g_i) \\
&= \psi\big(\big[\pi_0(v)\big]\big) + \rho_A\big(-h_m(g_i)\big)
\end{aligned}$$

$$= \psi([\pi_0(v)]) + \rho_A((0, g_i) - \theta'_m(g_i))$$
$$= \psi([\pi_0(v)]) + \rho_A([v] - \theta_1([\pi_0(v)])),$$

as desired. □

Corollary 4.3. *Let A be a unital simple* C*-*algebra with* $\mathrm{TR}(A) = 0$ *and let $B \subseteq A$ be an AH-algebra and denote by ι the inclusion map. For any $\psi \in \mathrm{Hom}(K_1(B), \mathrm{Aff}(\mathrm{T}(A)))$, there exists $\alpha \in \overline{\mathrm{Inn}}(B, A)$ such that there are maps $\theta_i : K_i(B) \to K_i(M_\alpha)$ with $\pi_0 \circ \theta_i = \mathrm{id}_{K_i(B)}$, $i = 0, 1$, and the rotation map $R_{\iota,\alpha} : K_1(M_\alpha) \to \mathrm{Aff}(\mathrm{T}(A))$ is given by*

$$R_{\iota,\alpha}(c) = \rho_A(c - \theta_1([\pi_0]_1(c))) + \psi([\pi_0]_1(c)) \quad \text{for all } c \in K_1(M_\alpha(A)).$$

In other words,

$$[\alpha] = [\iota]$$

in $KK(B, A)$ and the rotation map $R_{\iota,\alpha} : K_1(M_\alpha) \to \mathrm{Aff}(\mathrm{T}(A))$ is given by

$$R_{\iota,\alpha}(a, b) = \rho_A(a) + \psi(b)$$

for some identification of $K_1(M_\alpha)$ with $K_0(A) \oplus K_1(B)$.

Proof. It follows directly from Lemma 5.2 of [17], Theorem 4.2, and Proposition 3.18. □

Definition 4.4. Fix two unital C*-algebras A and B with $\mathrm{T}(A) \neq \emptyset$. Let $\mathcal{R}_0$ be the subset of $\mathrm{Hom}(K_1(B), \mathrm{Aff}(\mathrm{T}(A)))$ consisting of those homomorphisms $h \in \mathrm{Hom}(K_1(B), \mathrm{Aff}(\mathrm{T}(A)))$ such that there exists a homomorphism $d : K_1(B) \to K_0(A)$ such that

$$h = \rho_A \circ d.$$

Then $\mathcal{R}_0$ is clearly a subgroup of $\mathrm{Hom}(K_1(B), \mathrm{Aff}(\mathrm{T}(A)))$.

The following is a variation of Theorem 10.7 of [17].

Theorem 4.5. *Let C be a unital AH-algebra and let A be a unital separable simple* C*-*algebra with* $\mathrm{TR}(A) = 0$. *Suppose that $\phi, \psi : C \to A$ are two unital monomorphisms such that*

$$[\phi] = [\psi]$$

in $KK(C, A)$ and, for all $\tau \in \mathrm{T}(A)$,

$$\tau \circ \phi = \tau \circ \psi.$$

Suppose also that there exists a homomorphism $\theta \in \mathrm{Hom}(K_1(C), K_1(M_{\phi,\psi}))$ with $(\pi_0)_{*1} \circ \theta = \mathrm{id}_{K_1(C)}$ such that

$$R_{\phi,\psi} \circ \theta \in \mathcal{R}_0. \tag{4.1}$$

Then ϕ and ψ are asymptotically unitarily equivalent.

Proof. We may write

$$K_1(M_{\phi,\psi}) = K_0(A) \oplus \theta\bigl(K_1(C)\bigr).$$

Let $h = R_{\phi,\psi} \circ \theta$. If there is a homomorphism $d : K_1(C) \to K_0(A)$ such that $h = \rho_A \circ d$, define

$$\theta' = \theta - d.$$

Note that θ is a homomorphism, and $(\pi_0)_{*1} \circ \theta = \mathrm{id}_{K_1(C)}$. Then

$$R_{\phi,\psi} \circ \theta' = R_{\phi,\psi} \circ \theta - \rho_A \circ d = 0.$$

Since $[\phi] = [\psi]$ in $KK(C, A)$, there exists an element $\theta_0 \in \mathrm{Hom}_\Lambda(\underline{K}(C), \underline{K}(M_{\phi,\psi}))$ such that $[\pi_0] \circ [\theta_0] = [\mathrm{id}_{\underline{K}(C)}]$. Define $\theta_0' : K_i(C) \to K_i(M_{\phi,\psi})$ by $\theta_0'|_{K_1(C)} = \theta'$ and $\theta_0'|_{K_0(C)} = \theta_0|_{K_0(C)}$. By the UCT, there exists $\theta_0'' \in KL(C, M_{\phi,\psi})$ such that $\Gamma(\theta_0'') = \theta_0'$, where Γ is the map from $KL(C, M_{\phi,\psi})$ onto $\mathrm{Hom}(K_*(C), K_*(M_{\phi,\psi}))$. We will use the identification

$$KL(\mathrm{S}C, M_{\phi,\psi}) = \mathrm{Hom}_\Lambda\bigl(\underline{K}(\mathrm{S}C), \underline{K}(M_{\phi,\psi})\bigr)$$

(see [2]). Put

$$x_0 = [\pi_0] \circ \theta_0'' - [\mathrm{id}_{\underline{K}(C)}].$$

Then $\Gamma(x_0) = 0$. Define $\theta_1 = \theta_0'' - \theta_0 \circ x_0 \in \mathrm{Hom}_\Lambda(\underline{K}(C), \underline{K}(M_{\phi,\psi}))$. Then one computes that

$$[\pi_0] \circ \theta_1 = [\pi_0] \circ \theta_0'' - [\pi_0] \circ \theta_0 \circ x_0 \tag{4.2}$$

$$= \bigl([\mathrm{id}_{\underline{K}(C)}] + x_0\bigr) - [\mathrm{id}_{\underline{K}(C)}] \circ x_0 \tag{4.3}$$

$$= [\mathrm{id}_{\underline{K}(C)}] + x_0 - x_0 = [\mathrm{id}_{\underline{K}(C)}]. \tag{4.4}$$

Moreover,

$$\Gamma(\theta_1)|_{K_1(C)} = \theta'.$$

In particular,

$$R_{\phi,\psi} \circ (\theta_1)_{K_1(C)} = 0.$$

It follows from Theorem 10.7 of [17] that ϕ and ψ are asymptotically unitarily equivalent. □

Remark 4.6. In 4.5, the condition that such θ exists is also necessary. This follows from Theorem 9.1 of [17].

Definition 4.7. Let A be a unital C*-algebra, and let C be a unital separable C*-algebra. Denote by $\mathrm{Mon}^e_{asu}(C, A)$ the set of asymptotic unitary equivalence classes of unital monomorphisms.

Denote by $\boldsymbol{K}$ the map from $\mathrm{Mon}^e_{asu}(C, A)$ into $KK_e(C, A)^{++}$ defined by

$$\phi \mapsto [\phi] \quad \text{for all } \phi \in \mathrm{Mon}^e_{asu}(C, A).$$

Let $\kappa \in KK_e(C, A)^{++}$. Denote by $\langle\kappa\rangle$ the classes of $\phi \in \mathrm{Mon}^e_{asu}(C, A)$ such that $\boldsymbol{K}(\phi) = \kappa$.

Denote by $\widetilde{\boldsymbol{K}}$ the map from $\mathrm{Mon}^e_{asu}(C, A)$ into $KKT(C, A)^{++}$ defined by

$$\phi \mapsto \big([\phi], \phi_T\big) \quad \text{for all } \phi \in \mathrm{Mon}^e_{asu}(C, A).$$

Denote by $\langle\kappa, \lambda\rangle$ the classes of $\phi \in \mathrm{Mon}^e_{asu}(C, A)$ such that $\widetilde{\boldsymbol{K}}(\phi) = (\kappa, \lambda)$.

Theorem 4.8. *Let C be a unital AH-algebra and let A be a unital separable simple* C*-*algebra with* $\mathrm{TR}(A) = 0$. *Then the map $\widetilde{\boldsymbol{K}} : \mathrm{Mon}^e_{asu}(C, A) \to KKT(C, A)^{++}$ is surjective. Moreover, for each $(\kappa, \lambda) \in KKT(C, A)^{++}$, there exists a bijection*

$$\eta : \langle\kappa, \lambda\rangle \to \mathrm{Hom}\big(K_1(C), \mathrm{Aff}\big(\mathrm{T}(A)\big)\big)/\mathcal{R}_0.$$

Proof. It follows from 3.22 that $\widetilde{\boldsymbol{K}}$ is surjective.

Fix a pair $(\kappa, \lambda) \in KKT(C, A)^{++}$ and fix a unital monomorphism $\phi : C \to A$ such that $[\phi] = \kappa$ and $\phi_{\mathrm{T}} = \lambda$. For any homomorphism $\gamma \in \mathrm{Hom}(K_1(C), \mathrm{Aff}(\mathrm{T}(A)))$, it follows from 4.2 that there exists a unital monomorphism $\alpha \in \overline{\mathrm{Inn}}(\phi(C), A)$ with $[\alpha \circ \phi] = [\phi]$ in $KK(C, A)$ such that there exists a homomorphism $\theta : K_1(C) \to M_{\phi,\alpha\circ\phi}$ with $(\pi_0)_{*1} \circ \theta = \mathrm{id}_{K_1(C)}$ such that $R_{\phi,\alpha\circ\phi} \circ \theta = \gamma$. Let $\psi = \alpha \circ \phi$. Then $R_{\phi,\psi} \circ \theta = \gamma$. Note also since $\alpha \in \overline{\mathrm{Inn}}(\phi(C), A)$, $\psi_{\mathrm{T}} = \phi_{\mathrm{T}}$. In particular, $\widetilde{\boldsymbol{K}}(\psi) = \widetilde{\boldsymbol{K}}(\phi)$.

Suppose that $\theta' : K_1(A) \to K_1(M_{\phi,\psi})$ is such that $(\pi_0)_{*1} \circ \theta' = \mathrm{id}_{K_1(C)}$. Then

$$(\theta' - \theta)\big(K_1(C)\big) \subset K_0(A).$$

It follows that $R_{\phi,\psi} \circ \theta' - R_{\phi,\psi} \circ \theta \in \mathcal{R}_0$.

Thus we obtain a well-defined map $\eta : \langle[\phi], \phi_T\rangle \to \mathrm{Hom}(K_1(A), \mathrm{Aff}(\mathrm{T}(A)))/\mathcal{R}_0$. From what we have proved, the map η is surjective.

Suppose that $\phi_1, \phi_2 : C \to A$ are two unital monomorphisms such that $\phi_1, \phi_2 \in \langle[\phi], \phi_T\rangle$ and

$$R_{\phi,\phi_1} \circ \theta_1 - R_{\phi,\phi_2} \circ \theta_2 \in \mathcal{R}_0,$$

where $\theta_1 : K_1(C) \to K_1(M_{\phi,\phi_1})$ and $\theta_2 : K_1(C) \to K_1(M_{\phi,\phi_2})$ are homomorphisms such that $(\pi_0)_{*1} \circ \theta_1 = (\pi_0)_{*1} \circ \theta_2 = \mathrm{id}_{K_1(C)}$, respectively. Thus there is $\delta : K_1(C) \to K_0(A)$ such that

$$R_{\phi,\phi_2} \circ \theta_2 - R_{\phi,\phi_1} \circ \theta_1 = \rho_A \circ \delta.$$

Fix $z \in K_1(C)$ and let $u \in M_k(C)$ such that $[u] = z$ in $K_1(C)$. Let $U(\theta_1, z)(t) \in \mathrm{U}(\mathrm{M}_N(M_{\phi,\phi_1}))$ be a unitary represented by $\theta_1(z)$. We may assume that $N \geqslant k$. It is easy to see that $U(\theta_1, z)(t)$ is homotopic to a unitary $V(\theta_1, z)(t) \in \mathrm{U}(\mathrm{M}_N(M_{\phi,\phi_1}))$ with $V(\theta_1, z)(0) = \mathrm{diag}(\phi(u), 1_{N-k})$ and $V(\theta_2, z)(1) = \mathrm{diag}(\phi_1(u), 1_{N-k})$ (by enlarging the size of matrices if necessary). In particular, $[V(\theta_1, z)] = \theta_1(z)$ in $K_1(M_{\phi,\phi_1})$. Similarly, we may assume that there is a unitary $V(\theta_2, z) \in \mathrm{U}(\mathrm{M}_N(M_{\phi,\phi_2}))$ with $V(\theta_2, z)(0) = \mathrm{diag}(\phi(u), 1_{N-k})$ and $V(\theta_2, z)(1) = \mathrm{diag}(\phi_2(u), 1_{N-k})$ which is represented by $\theta_2(z)$ in $K_1(M_{\phi,\phi_2})$.

Now define

$$V(\theta_3, z)(t) = \begin{cases} V(\theta_1, z)(1 - 2t) & \text{for } t \in [0, 1/2], \\ V(\theta_2, z)(2t - 1) & \text{for } t \in (1/2, 1]. \end{cases} \tag{4.5}$$

Note that $V(\theta_3, z) \in \mathrm{U}(\mathrm{M}_N(M_{\phi_1,\phi_2}))$.

Define the homomorphism $\theta_3 : K_1(C) \to K_1(M_{\phi_1,\phi_2})$ by

$$\theta_3(z) = \left[V(\theta_3, z)\right] \quad \text{for all } z \in K_1(C).$$

It follows that

$$(\pi_0)_{*1} \circ \theta_3 = (\pi_0)_{*1} \circ \theta_1 = \mathrm{id}_{K_1(C)}.$$

Moreover, a direct calculation shows that

$$R_{\phi_1,\phi_2} \circ \theta_3 = R_{\phi,\phi_2} \circ \theta_2 - R_{\phi,\phi_1} \circ \theta_1 = \rho_A \circ \delta.$$

It follows from Theorem 4.5 that ψ_1 and ψ_2 are asymptotically unitarily equivalent. Therefore, η is one-to-one. □

Corollary 4.9. *Let C be a unital separable amenable simple* C^**-algebra with* $\mathrm{TR}(C) = 0$ *which satisfies the UCT and let A be a unital simple* C^**-algebra with* $\mathrm{TR}(A) = 0$. *Then the map* $\boldsymbol{K} : \mathrm{Mon}^e_{asu}(C, A) \to KK_e(C, A)^{++}$ *is surjective. Moreover, the map*

$$\eta : \langle[\phi]\rangle \to \mathrm{Hom}\big(K_1(C), \mathrm{Aff}\big(\mathrm{T}(A)\big)\big)/\mathcal{R}_0$$

is bijective.

Proof. It follows from 4.8 that it suffices to point out that in this case $\rho_C(K_0(C))$ is dense in $\mathrm{Aff}(\mathrm{T}(C))$ and $\mathrm{T}_{\mathfrak{f}}(C) = \mathrm{T}(C)$. □

Corollary 4.10. *Let A be a unital separable amenable simple* C^**-algebra with* $\mathrm{TR}(A) = 0$ *which satisfies the UCT. Then the map* $\boldsymbol{K} : \mathrm{Aut}(A) \to KK_e^{-1}(A, A)^{++}$ *is surjective. Moreover, the map*

$$\eta : \langle \mathrm{id}_A \rangle \to \mathrm{Hom}\big(K_1(A), \mathrm{Aff}\big(\mathrm{T}(A)\big)\big)/\mathcal{R}_0$$

is bijective.

Proof. This follows from 3.26 and 4.2 as in the proof of 4.8. □

5. Classification of simple C*-algebras

Denote by Q the UHF algebra with $K_0(Q) = \mathbb{Q}$ with $[1_Q] = 1$. If $\mathfrak{p}$ is a supernatural number, let $M_{\mathfrak{p}}$ denote the UHF-algebra associated with $\mathfrak{p}$.

Lemma 5.1. *Let B be a unital separable amenable simple* C^**-algebra such that $B \otimes Q$ has tracial rank zero. Let A be unital separable amenable simple* C^**-algebras with tracial rank zero satisfying the UCT, and let $\varphi_1, \varphi_2 : A \to B$ be two homomorphisms. Suppose that $\phi, \psi : A \to B \otimes Q$ are two unital monomorphisms such that*

$$[\phi] = [\psi] \quad \text{in } KK(A, B \otimes Q).$$

Suppose that ϕ induces an affine homeomorphism $\phi_T : \mathrm{T}(B \otimes Q) \to \mathrm{T}(A)$ by

$$\phi_T(\tau)(a) = \tau \circ \phi(a) \quad \textit{for all } a \in A_{s.a}$$

and for all $\tau \in \mathrm{T}(B \otimes Q)$. Then there exists an automorphism $\alpha \in \mathrm{Aut}(\phi(A), \phi(A))$ with $[\alpha] = [\mathrm{id}_{\phi(A)}]$ in $KK(\phi(A), \phi(A))$ such that $\alpha \circ \phi$ and ψ are strongly asymptotically unitarily equivalent.

Proof. The proof is exactly the same as that of the proof of Lemma 3.2 of [18] but we apply 4.2 above instead of 4.1 of [10] so that the restriction on $K_i(A)$ $(i = 0, 1)$ can be removed. □

Definition 5.2. Let C and A be two unital C*-algebras. Let $\phi, \psi : C \to A$ be two homomorphisms. Recall that ϕ and ψ are said to be strongly asymptotically unitarily equivalent if there exists a continuous path of unitaries $\{u(t): t \in [0, \infty)\}$ such that $u(0) = 1$ and

$$\lim_{t\to\infty} \mathrm{Ad}\, u(t) \circ \phi(c) = \psi(c) \quad \text{for all } c \in C.$$

Lemma 5.3. *(See [18, Theorem 3.4].) Let A and B be two unital separable amenable simple C*-algebras satisfying the UCT. Let $\mathfrak{p}$ and $\mathfrak{q}$ be supernatural numbers of infinite type such that $M_{\mathfrak{p}} \otimes M_{\mathfrak{q}} \cong Q$. Suppose that $A \otimes M_{\mathfrak{p}}$, $A \otimes M_{\mathfrak{q}}$, $B \otimes M_{\mathfrak{p}}$ and $B \otimes M_{\mathfrak{q}}$ have tracial rank zero.*

Let $\sigma_{\mathfrak{p}} : A \otimes M_{\mathfrak{p}} \to B \otimes M_{\mathfrak{p}}$ and $\rho_{\mathfrak{q}} : A \otimes M_{\mathfrak{q}} \to B \otimes M_{\mathfrak{q}}$ be two unital isomorphisms. Suppose that

$$[\sigma] = [\rho]$$

in $KK(A \otimes Q, B \otimes Q)$, where $\sigma = \sigma_{\mathfrak{p}} \otimes \mathrm{id}_{M_{\mathfrak{q}}}$ and $\rho = \rho_{\mathfrak{q}} \otimes \mathrm{id}_{M_{\mathfrak{p}}}$.

Then there is an automorphism $\alpha \in \mathrm{Aut}(\sigma_{\mathfrak{p}}(A \otimes M_{\mathfrak{p}}))$ such that

$$[\alpha \circ \sigma_{\mathfrak{p}}] = [\sigma_{\mathfrak{p}}]$$

in $KK(A \otimes M_{\mathfrak{p}}, B \otimes M_{\mathfrak{p}})$, and $\alpha \circ \sigma_{\mathfrak{p}} \otimes \mathrm{id}_{M_{\mathfrak{q}}}$ is strongly asymptotically unitarily equivalent to ρ.

Proof. It follows from 5.1 that there exists $\beta \in \mathrm{Aut}(B \otimes Q)$ such that $\beta \circ \sigma$ is strongly asymptotically unitarily equivalent to ρ. Moreover, $[\beta] = [\mathrm{id}_{B\otimes Q}]$ in $KK(B \otimes Q, B \otimes Q)$. Now consider two homomorphisms $\sigma_{\mathfrak{p}}$ and $\beta \circ \sigma_{\mathfrak{p}}$. One has

$$[\beta \circ \sigma_{\mathfrak{p}}] = [\sigma_{\mathfrak{p}}] \quad \text{in } KK(A \otimes M_{\mathfrak{p}}, B \otimes Q).$$

Since $\sigma_{\mathfrak{p}}$ is an isomorphism, it is easy to see that $\sigma_T : \mathrm{T}(B \otimes Q) \to \mathrm{T}(A \otimes M_{\mathfrak{p}})$ is an affine homeomorphism.

By applying 5.1 again, one obtains $\alpha \in \mathrm{Aut}(\sigma_{\mathfrak{p}}(A \otimes M_{\mathfrak{p}}))$ such that $[\alpha] = [\mathrm{id}_{\sigma_{\mathfrak{p}}}]$ in $KK(\sigma_{\mathfrak{p}}(A \otimes M_{\mathfrak{p}}), \sigma_{\mathfrak{p}}(A \otimes M_{\mathfrak{p}}))$ and $\alpha \circ \sigma_{\mathfrak{p}}$ is strongly asymptotically unitarily equivalent to $\beta \circ \sigma_{\mathfrak{p}}$.

Define $\beta \circ \sigma_{\mathfrak{p}} \otimes \mathrm{id}_{M_{\mathfrak{q}}} : A \otimes M_{\mathfrak{p}} \otimes M_{\mathfrak{q}} \to (B \otimes Q) \otimes M_{\mathfrak{q}}$. It is easy to see that $\beta \circ \sigma_{\mathfrak{p}} \otimes \mathrm{id}_{M_{\mathfrak{q}}}$ is strongly approximately unitarily equivalent to $\beta \circ \sigma$.

Note that $\sigma(A \otimes M_{\mathfrak{p}}) = B \otimes M_{\mathfrak{p}}$. Let $\sigma' = \alpha \circ \sigma_{\mathfrak{p}} \otimes \mathrm{id}_{M_{\mathfrak{q}}}$. It follows that σ' is strongly asymptotically unitarily equivalent to $\beta \circ \sigma$. Consequently σ' is strongly asymptotically unitarily equivalent to ρ. □

Theorem 5.4. *Let A and B be two unital separable amenable simple* C*-*algebras satisfying the UCT. Suppose that there is an isomorphism*

$$\kappa : \big(K_0(A), K_0(A)_+, [1_A], K_1(A)\big) \to \big(K_0(B), K_0(B)_+, [1_B], K_1(B)\big).$$

Suppose also that there is a pair of supernatural numbers $\mathfrak{p}$ and $\mathfrak{q}$ of infinite type which are relatively prime such that $M_{\mathfrak{p}} \otimes M_{\mathfrak{q}} \cong Q$ and

$$\mathrm{TR}(A \otimes M_{\mathfrak{p}}) = \mathrm{TR}(A \otimes M_{\mathfrak{q}}) = \mathrm{TR}(B \otimes M_{\mathfrak{p}}) = \mathrm{TR}(B \otimes M_{\mathfrak{q}}) = 0.$$

Then $A \otimes \mathcal{Z} \cong B \otimes \mathcal{Z}$.

Proof. The proof is exactly the same as that of Theorem 3.5 of [18] but we now apply 5.3 above instead of 3.4 of [18]. □

Corollary 5.5. *(See [26, Corollary 8.3].) Let A and B be two unital separable simple ASH-algebras whose projections separate traces which are $\mathcal{Z}$-stable. Suppose that*

$$\big(K_0(A), K_0(A)_+, [1_A], K_1(A)\big) \to \big(K_0(B), K_0(B)_+, [1_B], K_1(B)\big).$$

Then $A \cong B$.

Proof. As in the proof of 6.3 of [26], $A \otimes C$ and $B \otimes C$ have tracial rank zero for any unital UHF-algebra C. Thus Theorem 5.4 applies. □

Corollary 5.6. *Let A and B be two unital separable simple $\mathcal{Z}$-stable* C*-*algebras with unique tracial states which are inductive limits of type I* C*-*algebras. Suppose that*

$$\big(K_0(A), K_0(A)_+, [1_A], K_1(A)\big) \to \big(K_0(B), K_0(B)_+, [1_B], K_1(B)\big).$$

Then $A \cong B$.

Proof. For any UHF-algebra C, $A \otimes C$ is approximately divisible. Since A has a unique tracial state, so does $A \otimes C$. Therefore projections of $A \otimes C$ separate traces. It follows from [23] that $A \otimes C$ has real rank zero, stable rank one and weakly unperforated K_0. Moreover, $A \otimes C$ is also an inductive limit of type I C*-algebras. It follows from Theorem 4.15 and Proposition 5.4 of [15] that $\mathrm{TR}(A \otimes C) = 0$. Exactly the same argument shows that $\mathrm{TR}(B \otimes C) = 0$. It follows from Theorem 5.4 that $A \cong B$. □

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