On the Complexity of $H$-Coloring

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Let $H$ be a fixed graph, whose vertices are referred to as 'colors.' An $H$-coloring of a graph $G$ is an assignment of 'colors' to the vertices of $G$ such that adjacent vertices of $G$ obtain adjacent 'colors.' (An $H$-coloring of $G$ is just a homomorphism $G \to H$.) The following $H$-coloring problem has been the object of recent interest:

**Instance:** A graph $G$.

**Question:** Is it possible to $H$-color the graph $G$?

$H$-colorings generalize traditional graph colorings, and are of interest in the study of grammar interpretations. Several authors have studied the complexity of the $H$-coloring problem for various (families of) fixed graphs $H$. Since there is an easy $H$-colorability test when $H$ is bipartite, and since all other examples of the $H$-colorability problem that were treated (complete graphs, odd cycles, complements of odd cycles, Kneser graphs, etc.) turned out to be NP-complete, the natural conjecture, formulated in several sources (including David Johnson's NP-completeness column), asserts that the $H$-coloring problem is NP-complete for any non-bipartite graph $H$. We give a proof of this conjecture.

**INTRODUCTION**

Graph coloring problems arise in various contexts of both applied and theoretical natures [5, 12, 16, 17]. At the same time, $k$-colorability is one of the basic NP-complete problems. In fact, it is considered 'harder' than other typical NP-complete problems: It is believed that (unless P = NP) there does not exist a polynomial approximation algorithm guaranteed to color any graph with at most $c$ times the minimum number of colors,
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for any constant $c$. This has only been proved for small constants $c$ [4, 6]. Moreover, any known polynomial coloring algorithm uses $\Omega(n(\log \log n)^2/(\log n)^2)$ colors on some 3-colorable graph with $n$ vertices [21]. This apparent difficulty of the graph coloring problem is not well understood, and it is reflected also in the more general $H$-coloring problem studied here. The complexity of the $H$-coloring problem was investigated by several authors [1, 2, 11, 15, 18, 19], but only special cases were settled; in particular, there is a simple $H$-colorability test when $H$ is bipartite, and the problem is NP-complete when $H$ is a complete graph, an odd cycle, or a member of a few other very restricted families [1, 11, 18, 19]. We prove that the $H$-coloring problem is NP-complete for any non-bipartite graph $H$. This was conjectured in [18]; cf. also [19] and [13]. Our proof is interesting not so much for the reductions we use, which are similar to those previously used, but rather for the intricate interplay of the various graphs, some quite complex, which must be employed in these reductions. These complications may help to explain why the problem had previously resisted solution.

Let $G$ and $H$ be graphs. A homomorphism $f: G \rightarrow H$ is a mapping $f$ of $V(G)$ to $V(H)$ such that $f(g),$ $f(g')$ are adjacent vertices of $H$ whenever $g,$ $g'$ are adjacent vertices of $G$. Since a homomorphism $c: G \rightarrow K_n$ is just an $n$-coloring of $G$, the term $H$-coloring of $G$ has been employed to describe a homomorphism $G \rightarrow H$. Homomorphisms and $H$-colorings have been studied in various contexts [1-3, 7-10, 13-15, 17-20]; in particular, for their relation to grammars and interpretations, in [17]. Here we study the $H$-coloring problem, i.e., the decision problem “Is a given graph $G$ $H$-colorable?” Clearly, each $H$-coloring problem is in the class NP. It is easy to see that if $H$ is a bipartite graph then $G$ is $H$-colorable if and only if $G$ is 2-colorable. For some non-bipartite graphs $H$ the $H$-coloring problem is NP-complete. Obviously, this is the case of $K_n$-coloring; moreover, $C_{2k+1}$-coloring is NP-complete according to [18, 19], where several other NP-completness results of this type were obtained. (Also see [1, 2, 7, 11-15, 20].)

**Theorem 1.** If $H$ is bipartite then the $H$-coloring problem is in P.
If $H$ is not bipartite then the $H$-coloring problem is NP-complete.

**The Reductions**

A. The Indicator Construction

Let $I$ be a fixed graph, and let $i$ and $j$ be distinct vertices of $I$ such that some automorphism of $I$ maps $i$ to $j$ and $j$ to $i$. The indicator construction (with respect to $(I, i, j)$) transforms a given graph $H$ into the graph $H^*$
defined to have the same vertex set as $H$ and to have as the edge set all pairs $hh'$ for which there is a homomorphism of $I$ to $H$ taking $i$ to $h$ and $j$ to $h'$ (cf. Fig. 1). Because of our assumption on $I$, the edges of $H^*$ will be undirected.

**Lemma 1.** If the $H^*$-coloring problem is NP-complete, then so is the $H$-coloring problem.

(In applying Lemma 1 we need to be careful to ensure that $H^*$ has no loops, i.e., that no homomorphism of $I$ to $H$ can map $i$ and $j$ to the same vertex. Otherwise the $H^*$-coloring problem will not be NP-complete: if $H^*$ has a loop then any $G$ admits an $H^*$-coloring—map all vertices of $G$ to the vertex with a loop.)

**Proof.** Given a graph $G$, let $*G$ be the graph obtained from $G$ by replacing each edge $gg'$ by a disjoint copy of $I$, identifying $i$ with $g$ and $j$ with $g'$. It is now easy to see from the definitions that there is a homomorphism $*G \to H$ if and only if there is a homomorphism $G \to H^*$.

Before introducing the next construction we need to review the following concepts [9, 10]: If $H$ is a subgraph of $H'$, then a *retraction* of $H'$ to $H$ is a homomorphism $r: H' \to H$ such that $r(h) = h$ for all vertices $h$ of $H$. A graph is a *core* (or minimal graph [3]) if it does not admit a retraction to a proper subgraph; equivalently, $H$ is a core if it does not admit a homomorphism to a proper subgraph. It is easy to see [10, 3] that every graph $H'$ contains a unique (up to isomorphism) subgraph $H$ which is a core and admits a retraction $r: H' \to H$; we call $H$ the core of $H'$. Note that if $H$ is a core of $H'$, then there are homomorphisms $H \to H'$ (the inclusion) and $H' \to H$ (a retraction); thus $G$ is $H'$-colorable if and only if it is $H$-colorable. This allows us to restrict our attention to cores $H$. (The core of a bipartite graph $H$ is $K_2$; the core of a graph $H$ with loops is one loop.)
Now it follows that in both cases testing for $H$-colorability is easy. In particular, this proves the first half of Theorem 1.)

B. The Sub-indicator Construction

Let $J$ be a fixed graph, with specified vertices $j$ and $k_1, k_2, ..., k_t$. The sub-indicator construction (with respect to $J, j, k_1, k_2, ..., k_t$) transforms a given core $H$ with $t$ specified vertices $h_1, h_2, ..., h_t$, to its subgraph $H^\sim$ induced by the vertex set $V^\sim$ defined as follows: Let $W$ be the graph obtained from the disjoint union of $J$ and $H$ by identifying each $k_i$ with the corresponding $h_i$ ($i = 1, 2, ..., t$). A vertex $v$ of $H$ belongs to $V^\sim$ just if there is a retraction of $W$ to $H$ which maps the vertex $j$ to $v$. (Cf. Fig. 2.)

**Lemma 2.** Let $H$ be a core. If the $H^\sim$-coloring problem is NP-complete then so is the $H$-coloring problem.

**Proof.** Given a graph $G$, we define $\sim G$ as the graph obtained from the disjoint union of $G$, $H$, and $|V(G)|$ copies of $J$, by identifying, for every $i = 1, 2, ..., t$, the vertex $k_i$ (in each copy of $J$) with the vertex $h_i$ of $H$, and identifying each vertex $g$ of $G$ with the vertex $j$ in the $g$th copy of $J$. If there is a homomorphism $\sim G \rightarrow H$, then the copy of $H$ which is a subgraph of $\sim G$ must map onto $H$, because $H$ is a core. It is then easy to see that there is a homomorphism $G \rightarrow H^\sim$. The converse, that the existence of a homomorphism $G \rightarrow H^\sim$ implies the existence of a homomorphism $\sim G \rightarrow H$, is also easy to see.

C. The Edge-sub-indicator Construction

Let $J$ be a fixed graph with a specified edge $jj'$ and $t$ specified vertices $k_1, k_2, ..., k_t$, such that some automorphism of $J$ keeps each vertex $k_i$ fixed while exchanging the vertices $j$ and $j'$. The edge-sub-indicator construction
transforms a given core $H$ with $t$ specified vertices $h_1, h_2, ..., h_t$ into its subgraph $H^*$ determined by those edges $hh'$ of $H$ which are images of the edge $jj'$ under retractions of $W$ (defined as in B) to $H$ (cf. Fig. 3). Note that because of our assumption on $J$, the edges of $H^*$ are again undirected.

**Lemma 3.** Let $H$ be a core. If the $H^*$-coloring problem is NP-complete then so is the $H$-coloring problem.

**Proof.** Given a graph $G$ we let $^*G$ denote the graph obtained from the disjoint union of $G$, $H$, and $|E(G)|$ copies of $J$ by identifying, for each $i = 1, 2, ..., t$, the vertex $k_i$ (in each copy of $J$) with the vertex $h_i$ of $H$, and identifying each edge $e = gg'$ of $G$ with the edge $jj'$ in the $i$th copy of $J$. (Because of the symmetry of $J$, it does not matter whether $g$ gets identified with $j$ and $g'$ with $j'$, or the other way round.) As before, there is a homomorphism $^*G 	o H$ if and only if there is a homomorphism $G 	o H^*$.

Special cases of the first two constructions have been used in [18, 19]. The third construction is somewhat more cumbersome, but is crucial for our proof.

To prove the NP-completeness of the $H$-coloring problem for a particular non-bipartite $H$, we may appeal to an indicator construction and reduce the problem to proving the NP-completeness of the $H^*$-coloring problem; we shall always choose $I, i, j$ in such a way that $H^*$, in addition to being undirected, has no loops, contains all the edges of $H$, and at least one more edge. Or we may appeal to a sub-indicator construction, and reduce the problem to proving the NP-completeness of the $H^*$-coloring problem; we shall always choose $J, j, k_i$ so that $H^*$ is still non-bipartite, but has fewer vertices than $H$. (We cannot use the edge-sub-indicator construction by itself because it reduces the number of edges and thus counteracts the effect of the indicator construction. However, we shall

![Fig. 3. Example edge-sub-indicator construction.](image)
only be using it when it can be immediately followed by a sub-indicator construction.)

Hence, let $H$ be a non-bipartite graph for which the $H$-coloring problem is not NP-complete and such that the $H'$-coloring problem is NP-complete for any non-bipartite $H'$

(1) with fewer vertices than $H$, or

(2) with the same number of vertices as $H$, but with more edges.

Clearly, if the Theorem does not hold then such an $H$ must exist. Moreover, since each $K_r$-coloring problem is NP-complete (when $r \geq 3$), $H$ has $n > 3$ vertices and $m < \binom{n}{2}$ edges. We shall proceed to derive a number of structural properties of the graph $H$, which will eventually imply that it cannot exist, thereby proving the Theorem. It follows from our earlier remarks that $H$ is a core.

**The Structure of Triangles**

Our first goal is to prove that each edge of $H$ belongs to a unique triangle. We do this in a sequence of steps:

(A1) *H contains a triangle.* Indeed, suppose that the shortest odd cycle $C$ of $H$ has $k$ vertices, $k \geq 5$. Consider the indicator construction where the indicator $I$ is a path of length three with endpoints $i$ and $j$ (as in Fig. 1). It transforms $H$ into the graph $H^*$ which is undirected (by the obvious symmetry of $I$), has no loops (because $H$ has no triangles), contains all edges of $H$ (it is easy to visualise how to "fold" $I$ onto an arbitrary edge of $H$), and also contains some chords of $C$ which were not present in $H$ (because $k \geq 5$). According to our assumption (2) the $H^*$-coloring problem is NP-complete; by Lemma 1, the $H$-coloring problem is also NP-complete, contrary to assumption.

(A2) *$H$ contains no $K_4$.* Otherwise we can use the sub-indicator $J = K_3$ with one endpoint $j$ and the other $k_1$ (as in Fig. 2), and with $h_1$ being any vertex of $H$ which belongs to a $K_4$. The transformed graph $H^*$ does not contain $h_1$, but does contain a triangle in its neighborhood. Thus $H^*$ is a non-bipartite graph with fewer vertices than $H$; this again contradicts our assumptions and Lemma 2.

(A3) *Each vertex of $H$ belongs to a triangle.* Consider the sub-indicator construction with the disconnected sub-indicator $J$ of Fig. 4 (the choice of

![Fig. 4. A disconnected sub-indicator $J$.](image-url)
$h_1$ is irrelevant): $H^-$ consists of those vertices of $H$ which belong to a triangle. By (A1) $H^-$ is non-empty and non-bipartite. If it were smaller than $H$ we would obtain a contradiction with (1) and Lemma 2.

(A4) *Any two vertices of $H$ have a common neighbor.* Use as sub-indicator a path of length two with endpoints $j$ and $k_1$. If the vertices $u$ and $v$ of $H$ have no common neighbour then setting $h_1 = u$ results in $H^-$ which does not contain $v$. Since $u$ lies in a triangle by (A3), $H^-$ is non-bipartite and we obtain a contradiction with (1) and Lemma 2.

(A5) *There is no homomorphism $S \to H$* (the graph $S$ is given in Fig. 5). If such a homomorphism exists, let $u', v', \ldots$, be the images of the vertices $u, v, \ldots$, of $S$ in $H$. The sub-indicator $J$ of Fig. 5, with $h_1 = u'$ and $h_2 = v'$, yields a graph $H^-$ which contains the triangle $a'b'c'$ but does not contain the vertex $w'$ (by (A2)). We obtain a contradiction as before. (We are grateful to Emo Welzl for this sub-indicator.)

(A6) *$H$ contains no $K_4^-$.* The indicator construction with the indicator $I$ of Fig. 6 transforms $H$ into an $H^*$ which is undirected (by the symmetry of $I$), has no loops (by A5), and contains all edges of $H$ (by A4). (There is a 3-coloring of $I$ with $i$ and $j$ being colored by different colors.) We will now show that if $H$ contains a $K_4^-$ then $H^*$ has more edges than $H$, contrary to (2) and Lemma 1: Suppose $K_4^-$ is a subgraph of $H$ with $u$ and $v$ as above. Since $H$ is a core, the neighborhoods of $u$ and $v$ cannot be
the same—thus some vertex $w$ of $H$ is adjacent to (say) $v$ but not $u$. It is easy to construct a homomorphism $I \rightarrow H$ taking $i$ to $u$ and $j$ to $w$ (using (A3)). Thus $uv$ is an edge of $H^*$, but not of $H$, contrary to (2) and Lemma 1.

(A7) Each edge of $H$ belongs to a unique triangle. This now follows from (A4) and (A6).

In particular, the graph spanned by the neighbors of any vertex of $H$ is a union of disjoint edges. Our next objective is to investigate the interconnections among the triangles of $H$.

(A8) In $H$, any triangle $abc$ and edge $cc'$ ($c' \neq a, b, c$) are contained in a subgraph $T$. (The graph $T$ is defined in Fig. 7.) The sub-indicator construction with the sub-indicator $J$ of Fig. 7, applied to $H$ with $h_1 = a$ and $h_2 = b$ results in an $H^*$ containing the triangle $abc$; thus $H^*$ is non-bipartite. If $c'$ is a vertex of $H^*$, then it is the image of $j$ under some retraction of $W$, and (using (A6) to see that all depicted vertices are distinct) we conclude that $T$ is a subgraph of $H$. Otherwise $H^*$ has fewer vertices than $H$, contrary to (1) and Lemma 1.

Let $U$ be the graph defined in Fig. 8.
For any homomorphism $U \to H$, the images of $i$ and $j$ are adjacent in $H$. Consider the indicator construction with the indicator $U$ of Fig. 8. The fact that $H^*$ has no loops follows easily from (A5). Moreover, $H^*$ contains all edges of $H$: Indeed, any edge of $H$ belongs to a triangle, by (A7), and $I = U$ admits a homomorphism $f$ onto a triangle, with $f(i) \neq f(j)$. If there were a homomorphism $U \to H$ with the images of $i$ and $j$ non-adjacent, then $H^*$ would have strictly more edges than $H$, contrary to (2) and Lemma 1.

In $H$, any two triangles $abc$, $ab'c'$ are contained in a subgraph $P$ (from Fig. 9). We apply (A8) to $ab'c'$ and $ab$ to obtain the triangle $bdf$ with the additional edges $b'd$, $c'f$. Two applications of (A7) yield the two triangles $b'de$, $c'fg$. Finally three applications of (A9) imply the edges $ce$, $cg$, and $eg$. (Throughout, we appeal to (A6) to verify that all depicted vertices are distinct.)

**The Structure of Squares**

From now on we base our considerations on a fixed vertex $r$, chosen to be a vertex of maximum degree in $H$. By (A7), the neighborhood of $r$ consists of $k \geq 2$ (say) disjoint edges $a_1a'_1, a_2a'_2, \ldots, a_ka'_k$. Let $R$ denote the subgraph of $H$ induced by the remaining vertices $V(H) - \{r, a_1, a'_1, \ldots, a_k, a'_k\}$; according to (A4), each vertex $x$ of $R$ is adjacent to some $a_i$. By (A7), each edge $uv$ of $R$ belongs to a triangle $uvw$; if $w = a_i$, we label the edge $uv$ by $a_i$. (Of course, the whole triangle $uvw$ could belong to $R$, in which case none of the edges $uw$, $uw$, $vw$ would be labelled.) If $v$ in $R$ is adjacent to some $a_i$, then the edge $a_iv$ lies in a triangle $a_iuvw$ where $w$ is also in $R$; hence $v$ is incident with an edge labelled $a_i$. Note that (A6) implies that each edge obtains at most one label, and that two edges of the same label cannot intersect or have two of their endpoints adjacent. We shall state this as follows:
(B1) In any path of length at most 3, no two edges have the same label.

For the same reason, no vertex can be incident with both an edge labelled \( a_i \) and an edge labelled \( a'_i \).

For any \( i \neq j \) we can apply (A10) to the two triangles \( ra_ia'_i \) and \( ra_ja'_j \) to conclude that there is in \( R \) a four-cycle with edges consecutively labelled \( a_i, a_j, a'_i, a'_j \) (cf. Fig. 10).

Such a four-cycle will be called a square; there may, of course, be four-cycles in \( H \) (or even \( R \)) which are not "squares". There are at least \( \binom{4}{2} \) squares, and they may intersect. Their structure is analyzed in this section, and it leads to a proof of Theorem 1.

(B2) The squares are edge-disjoint. If two squares intersect in an edge, it must be some squares \( a_ia_ja'_ia'_j \) and \( a_ia_ma'_ia'_m \) (because each edge has at most one label)—cf. Fig. 11. The indicator \( I \) from Fig. 11 admits an automorphism exchanging \( i \) and \( j \), has a 3-coloring in which \( i \) and \( j \) obtain different colors, but has no 3-coloring in which \( i \) and \( j \) are given the same color. These facts imply that the graph \( H^* \) obtained from \( H \) by the indicator construction with this \( I \) is undirected, has no loops (any homomorphism \( I \to H \) identifying \( i \) and \( j \) would have to map \( I \) to a triangle, i.e., be a 3-coloring, because of (A6)), and contains all edges of \( H \)
(because each of them lies in a triangle). Since there exists a homomorphism \( I \to H \) taking \( c \) to \( a' \), \( i \) to \( u \), and \( j \) to \( v \), \( H^* \) also contains the edge \( uv \), which did not belong to \( H \) (by (B1)). As always, this contradicts (2) and Lemma 1.

(B3) \( H \) is \( 2k \)-regular. According to (A4) \( H \) is connected; hence it will suffice to prove that if \( r \) has the maximum degree \( 2k \), then all of its neighbors also have degree \( 2k \). The vertex \( a_i \) (\( i = 1, 2, \ldots, k \)) lies in a triangle with the opposite edge labelled \( a_j \); there are at least \( k - 1 \) such edges—one for each \( j \neq i \), arising from the square \( a_i a_j a'_i a'_j \)—and by (B2) they are all distinct. Moreover, \( a_i \) also lies in the triangle \( ra_i a'_i \); hence the degree of \( a_i \) is also \( 2k \).

It follows from the same proof that each edge \( a_i x \), for \( x \) in \( R \), meets an edge labelled by \( a_i \), that each labelled edge belongs to a square (thus to a unique square), and that \( H \) contains exactly one square labelled \( a_i a_j a'_i a'_j \) for each \( i \neq j \).

(B4) Each vertex of \( R \) belongs to a square. This follows from the preceding remarks: each vertex \( x \) of \( R \) must be adjacent to some \( a_i \) (or \( a'_i \)) by (A4), and hence is incident with a labelled edge, and thus with a square.

(B5) Each component of \( R^L \) is complete bipartite. Consider the 23-point indicator \( I \) of Fig. 12. The symmetry condition with respect to \( i \) and \( j \) is obviously satisfied. Furthermore no homomorphism \( f: I \to H \) can identify vertices \( i \) and \( j \). Otherwise the images of \( i, u, \) and \( v \) form a triangle as illustrated in Fig. 13. Therefore \( f(t_1) = f(v), f(t_2) = f(i), \) and \( f(t_3) = f(u) \) by (A7); this implies that \( f(i), f(u), f(v), f(t_1) \) form a \( K_4 \), contrary to (A2). Since \( I \) admits a 3-coloring in which \( i \) and \( j \) obtain different colors, it follows that every edge of \( H \) belongs to \( H^* \). If \( H^* \) has more edges than \( H \), we have a contradiction with (2) and Lemma 1; hence we may assume that for any homomorphism \( f: I \to H, f(i) \) and \( f(j) \) are adjacent in \( H \).

Consider any path of length three in \( R^L \); say edges \( b_i b_u \) labelled \( a_x \) (and contained in some square \( S_1 \)), \( b_u b_x \) labelled \( a_y \) (and contained in a square \( S_2 \)), and \( b_x b_j \) labelled \( a_z \) (and contained in a square \( S_3 \)): It is easy

![Fig. 12. The 23-point indicator I.](image-url)
to construct a homomorphism $f: I \rightarrow H$ taking $i$ to $b_i$, $u$ to $b_u$, $v$ to $b_v$, $j$ to $b_j$, $t_x$ to $a_x$, $t_y$ to $a_y$, $t_z$ to $a_z$, and $t_t$ to $r$. (This can be done whether or not the squares $S_1$, $S_2$, and $S_3$ are disjoint.) Thus $b_i b_j$ is an edge of $H$. We now show that $b_i b_j$ is labelled, i.e., an edge of $R^L$. By (A7), the edge $b_i b_j$ belongs to a unique triangle whose third vertex is some $c$. Then the graph $U$ from Fig. 8 admits a homomorphism to $H$ taking $i$ to $r$, $j$ to $c$, $u$ to $b_u$ and $v$ to $b_v$ (cf. Fig. 14). Consequently, $c$ is adjacent to $r$ in $H$, i.e., $c$ is some $a_q$; thus $b_i b_j$ is labelled by $a_q$. We have shown that in $R^L$ every 3-path is in a 4-cycle. Since by (A7) there are no triangles in $R^L$, (B5) follows.

We conclude from (B1) and (B5) that each label $a_i$ (or $a'_i$) occurs at most once in a component of $R^L$. It also follows from (B5) and (A7) that an unlabelled edge of $R$ joins two vertices of different components of $R^L$. Note that the unique triangle containing an unlabelled edge of $R$ has both other edges unlabelled (and in $R$).

(B6) Suppose $xyz$ is a triangle in $R$ with all three edges unlabelled, $x$ is incident with an edge labelled $a_i$, $z$ is incident with an edge labelled $a_j$, and $yv$ is any edge of $R$ such that $v$ is incident with an edge labelled $a'_i$. Then $v$ is also incident with an edge labelled $a_j$. (Most often we shall be applying (B6) in situations where $yv$ itself is labelled by $a'_i$.) The claim follows from (A9) applied to the homomorphism $U \rightarrow H$ taking $u$ to $x$, $v$ to $a_i$, $i$ to $v$, and $j$ to $a_j$ (cf. Fig. 15).

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**FIGURE 13**

**FIGURE 14**

**FIGURE 15**
If \( uv \) is an unlabelled edge of \( R \), and if \( u \) is incident with an edge labelled \( a_i \) while \( v \) is incident with an edge labelled \( a'_j \), we mark the edge \( uv \) by the index \( i \). Note that each edge obtains at most one mark: If the unlabelled edge \( uv \) of \( R \) obtained both marks \( i \) and \( j \), then consider the third vertex \( w \) on the unique triangle \( uvw \) with unlabelled edges. According to (B4) it is incident with a labelled edge, say, of label \( a_m \). If \( vv_1 \) and \( vv_2 \) are the edges with labels \( a_i \) (or \( a'_i \)) and \( a_j \) (or \( a'_j \)) at \( v \), then, according to (B6), both \( v_1 \) and \( v_2 \) are incident with an edge labelled \( a_m \), contradicting the fact that each component of \( R^\sim \) is bipartite and contains at most one edge of any label.

(B7) Every unlabelled edge of \( R \) is marked by exactly one index \( i \). It only remains to verify that each unlabelled edge has at least one mark. Consider the edge-sub-indicator \( J \) of Fig. 16. We choose \( h_1 \) to be \( r \). By the apparent symmetry of \( jj' \), the graph \( H^\sim \) is again undirected. It is easy to see that \( H^\sim \) contains all edges \( rai, rai', \) all edges \( aiai', \) all edges \( aiv \) with \( v \) in \( R \), and (by the remarks following (B3)) precisely those edges of \( R \) for which there exists an index \( i \) such that one endpoint is incident with an edge labelled \( a_i \) and the other endpoint is incident with an edge labelled \( a'_i \). Since we have already observed that each labelled edge belongs to a square, it follows that \( H^\sim \) contains all labelled edges; it also contains all marked (unlabelled) edges. Suppose there was in \( R \) an unlabelled edge \( uv \) without a mark; since every edge of \( H \) belongs to a unique triangle, there exists a vertex \( w \) in \( R \) with unlabelled edges \( uw, vw \). We know that \( uv \) is not an edge of \( H^\sim \). At this point we cannot appeal to our assumptions because \( H^\sim \) has neither fewer vertices nor more edges than \( H \). Therefore we let \( K = H^\sim, h_1 = w \), and consider the sub-indicator \( J \) which is a path of length two with endpoints \( k_1 \) and \( j \). Note that \( u \) does not belong to \( K^\sim \) because \( u \) and \( w \) have exactly one common neighbor, \( v \), in \( H \) by (A7), and \( uv \) is not an edge of \( K = H^\sim \). Moreover, \( w \) is incident with a labelled edge, by (B4); say, some \( wx \) is labelled \( a_i \). Then \( wxa_i \) is a triangle in \( K^\sim \), so that \( K^\sim \) is not bipartite. Since \( K^\sim \) is a non-bipartite graph with fewer vertices than \( H \), the \( K^\sim \)-coloring problem is \( NP \)-complete by assumption (1); hence the \( K \)-coloring problem (i.e., the \( H^\sim \)-coloring problem) is \( NP \)-complete by Lemma 3, and the \( H \)-coloring problem is \( NP \)-complete by Lemma 2. This
contradiction establishes that each unlabelled edge of $R$ is marked by some index $i$.

We conclude that in $R$ there are labelled edges, forming the graph $R^L$ which consists of complete bipartite components, and marked edges, forming edge-disjoint triangles each of which joins three different components of $R^L$.

(B8) $R^L$ has a component $K$ isomorphic to $K_{2,k}$. Consider a vertex $v$ of $R$, which has the maximum degree in $R^L$. If the degree is $k$, i.e., if $v$ is incident with $k$ labelled edges, then there is at $v$ precisely one label from each pair $\{a_i, a'_i\}$. It now follows easily from (B5) and the first remark following it that the component of $R^L$ containing $v$ is isomorphic to $K_{2,k}$. On the other hand, if the degree of $v$ in $R^L$ is less than $k$, then $v$ lies in a triangle $vxy$ with marked edges. Suppose that the edge $xy$ is marked by $i$. Then, according to (B1), $v$ is not incident with any edges labelled $a_i$ or $a'_i$. Let $xw$ be labelled by $a_i$ and $yw'$ by $a'_i$. Thus (B6) implies that both $w$ and $w'$ have degree in $R^L$ greater than $v$, contrary to our hypothesis.

We shall assume from now on that the special component $K = K_{2,k}$ has all edges with primed labels incident with the same vertex of degree $k$ (cf. Fig. 17). Clearly, this involves no loss of generality, as we may rename the neighbours of $r$ accordingly.

We shall call a vertex of $R$ positive (respectively negative) if it is not incident with any edge labelled with a primed label (respectively unprimed label). A vertex which is neither positive nor negative shall be called mixed. Thus the special component $K$ as shown in Fig. 17 has one positive vertex, one negative vertex, and $k$ mixed vertices.

(B9) If $v$ is a mixed vertex of $K$, incident with edges labelled $a_i$ and $a'_i$, then $v$ is adjacent to an endpoint of each edge labelled $a'_i$ or $a_j$. This is obvious for the edge labelled $a'_i$ in $K$—it lies in the square containing $v$. Suppose that $v$ is not adjacent to either endpoint of an edge $u_1u_2$ labelled $a'_i$ which belongs to some component $C$ of $R^L$. According to (A4), there exist vertices $w_1$ adjacent to $v$ and $u_1$, and $w_2$ adjacent to $v$ and $u_2$. Moreover,
If \( w_t \) (\( t = 1 \) or \( 2 \)) belongs to \( K \), then it would be one of the two neighbors of \( v \); it could not be incident with an edge labelled \( a_i \) because of (B1). This leaves only one possible neighbor of \( v \), say \( w_t \) (\( t = 1, 2 \)) as shown in Fig. 18. Since \( u_t v \) is not in a component of \( R^t \), \( u_t w_t \) is unlabelled, and it belongs to a unique triangle \( u_t w_t z \) with unlabelled edges. By (B4), the vertex \( z \), completing the triangle with unlabelled edges containing the edge \( w_t u_t \), belongs to at least two labelled edges, and by (B1) their labels cannot be \( a_i \). This yields a contradiction with (B6), at the vertex \( v \).

However, neither \( w_t \) (\( t = 1, 2 \)) can belong to a component of \( R^t \) different from \( C \) and \( K \). Otherwise there is a triangle \( w_t u_t z \) with three unlabelled edges of \( R \). According to (B6), any label occurring at an edge incident to \( z \) also occurs at an edge incident to \( v \). Hence \( z \) would have to be incident only with edges labelled \( a_i \) and \( a_j \), and so lie in a unique square, labelled \( a_i a_j a_j a_i \). Since there is only one square with these labels in \( H \), this would mean that \( z = v \), which contradicts our assumption that \( v \) is not adjacent to \( u_t \).

Suppose \( w_t \) (\( t = 1, 2 \)) belongs to \( C \), and let the edge \( w_t u_t \) be labelled \( a_s \). There is a homomorphism \( U \rightarrow H \) taking \( u \) to \( r \), \( v \) to \( a_s \), \( i \) to \( v \), and \( j \) to \( u_{3-t} \), (consider the triangles \( ra_i a'_i \) and \( a_j w_t u_t \) as shown in Fig. 19). Hence by (A9) \( v \) is adjacent to \( u_{3-t} \), contrary to our assumption.

\[ \text{Fig. 18. } t = 1 \text{ or } 2. \]

\[ \text{Fig. 19} \]
The last place for a vertex \( w_t \) \( (t = 1, 2) \) is among the vertices adjacent to \( r \); because \( v \) is only incident with edges labelled \( a_i \) and \( a_j \), and because \( u \), cannot be incident with an edge labelled \( a_i \) (already being incident with an edge labelled \( a'_j \) ), this would mean that \( w_t = a_j \). Thus it is not possible for both \( w_1 \) and \( w_2 \) to be adjacent to \( r \).

(B10) Two mixed vertices cannot be adjacent. We first prove that a mixed vertex of \( K \) is not adjacent to another mixed vertex: Obviously, a mixed vertex \( u \) of \( K \) is not adjacent to another mixed vertex of \( K \). Thus we may restrict our attention to unlabelled edges: Let \( uv \) be an unlabelled edge, and let \( uv \) and \( uw \) also be unlabelled. Assume that \( u \) is incident with edges labelled \( a_i \) and \( a'_j \); then one of \( uv, uw \) is marked by \( i \) and the other by \( j \). Without loss of generality, let \( v \) be incident with an edge labelled \( a_i \), and \( w \) with an edge labelled \( a'_j \) (thus \( uv \) is marked by \( j \) and \( uw \) by \( i \)). Now (B6) implies that the label of any edge incident with \( v \) also occurs at the unique positive vertex of \( K \), and the label of any edge incident with \( w \) also occurs at the unique negative vertex of \( K \). Therefore \( v \) is a positive vertex, and \( w \) a negative vertex.

Now we prove that two arbitrary mixed vertices cannot be adjacent by a labelled edge. Otherwise, let \( ai \) be the label of such an edge. Since \( K \) contains all labels, there is a mixed vertex \( v \) of \( K \) which is incident with an edge labelled \( a'_i \). Then \( v \) must be adjacent to one of the endpoints of the edge labelled \( a_i \) by (B9), which contradicts the first paragraph of the present case. Next we consider two mixed vertices \( u \) and \( v \) adjacent by an unlabelled edge, belonging to a triangle \( uuv \) with unlabelled edges. Suppose that the edge \( uv \) is marked by \( i \), namely that some edge \( ux \) is labelled \( a_i \), and some edge \( vy \) by \( a'_j \). Then by what we have just observed \( x \) and \( y \) cannot be mixed, and hence \( x \) is positive and \( y \) negative. By (B6) any label of an edge incident with \( w \) also occurs at \( x \) and at \( y \). This is only possible if \( x \) is not incident with any labelled edges, contrary to (B4).

Conclusion of the Proof. We now show that \( H \) is 3-colorable. This will show that the core of \( H \) is \( K_3 \), and hence \( H \)-coloring is NP-complete, contrary to our assumptions. Thus the Theorem will be proved. Color the vertex \( r \) as well as all mixed vertices by color 1; color all positive vertices of \( R \) as well as all primed neighbours of \( r \) by color 2; and color all negative vertices of \( R \) and all unprimed neighbours of \( r \) by color 3. We now show that this is a legal coloring. According to (B10), two vertices of color 1 cannot be adjacent. Moreover, two vertices of color 2 (respectively color 3) also cannot be adjacent. This is implied by the following remarks: Two positive (respectively negative) vertices of \( R \) cannot be adjacent. If they were adjacent by a labelled edge, then such an edge could not be part of a square, contradicting a remark made after (B3). They also could not be
adjacent by an unlabelled edge, as it easily follows from (B10) and (B7) that any triangle of unlabelled edges joins a mixed vertex, a positive vertex, and a negative vertex.

**Remark.** The situation is less clear for directed graphs. Even a conjecture anticipating which $H$-coloring problems are polynomial and which are NP-complete does not suggest itself. Only a few results are known [2, 18]. There are some simple digraphs $H$ (paths, cycles, transitive tournaments, etc.) for which polynomial $H$-coloring algorithms exist [2, 18]. Typically, they make use of results of the following type: There is a homomorphism $D \to H$ if and only if there is no homomorphism $H' \to D$ (for some fixed digraph $H'$, depending on $H$) [2, 7, 14, 20]. (These results may be viewed as proving that $H$-colorability is in NP $\cap$ coNP, and are, in some sense, prototype results of this type; this line of study is pursued in [14, 20].) There are also a few classes of digraphs $H$ with NP-complete $H$-colorability problems [18]. We note in passing that many NP-complete $H$-coloring problems may be produced by using the construction (of *$G$) from the proof of Lemma 1, with a suitable choice of the indicator $I$. Specifically, let $(I, i, j)$ be a digraph indicator such that for graphs $G$ and $H$, there is a homomorphism $G \to H$ if and only if there is a homomorphism of digraphs *$G \to *H$. Such indicators are called 'strongly rigid' in the terminology of [8]; they can be constructed to satisfy many additional properties—assuring for example that *$H$ is an acyclic, or even balanced, digraph. (A digraph is acyclic if it has no directed cycles; it is balanced if it has the same number of forward and backward arcs on any cycle.) In any event, if $D = *H$ for such an $I$ and a non-bipartite $H$, then the $D$-coloring problem is also NP-complete. Thus there are balanced (and hence also acyclic) digraphs $H$ for which the $H$-coloring problem is NP-complete. Acyclic digraphs $H$ with NP-complete $H$-coloring problems were also constructed by S. Burr and by W. Gutjahr and E. Welzl (personal communications).

Finally, it should be mentioned that any digraph $H$, such that the 'symmetric part' $H_S$ of $H$ (all pairs $uv$ of vertices for which both $uv$ and $vu$ are arcs of $H$) is a non-bipartite graph, also results in an NP-complete $H$-coloring problem. This is an easy corollary of Theorem 1 and the observation that a graph $G$ admits a homomorphism $G \to H_S$ if and only if it (viewed as a digraph) admits a homomorphism $G \to H$. It follows easily from this that almost every digraph $H$ gives an NP-complete $H$-coloring problem.

Since the first version of this paper (Simon Fraser School of Computing Science Technical Report TR-86-4), there has been some progress on the problem of coloring by directed graphs. In particular, in a paper to appear in the *SIAM J. Discrete Math.*, J. Bang-Jensen, the first author, and G. MacGillivray prove that for semicomplete graphs $H$ (and in particular
for tournaments) the presence of two directed cycles makes the problem NP-complete, and otherwise it is polynomial.

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