Tauberian theorems for double sequences that are statistically summable \((C, 1, 1)\) ♠

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Abstract

Let \((x_{jk}: j, k = 0, 1, 2, \ldots)\) be a double sequence of real or complex numbers, and set

\[
\sigma_{mn} := (m + 1)^{-1}(n + 1)^{-1} \sum_{j=0}^{m} \sum_{k=0}^{n} x_{jk} \quad \text{for } m, n = 0, 1, 2, \ldots.
\]

We give necessary and sufficient conditions, under which

\[
\text{st-lim } \sigma_{mn} = \xi \quad \text{implies } \text{st-lim } x_{jk} = \xi,
\]

where \(\xi\) is a finite number. These Tauberian conditions are one-sided when the \(x_{jk}\) are real numbers, and they are two-sided when the \(x_{jk}\) are complex numbers. In particular, these Tauberian conditions are clearly satisfied if \((x_{jk})\) is statistically slowly decreasing in the case of real sequences or if \((x_{jk})\) is statistically slowly oscillating in the case of complex sequences.

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1. Introduction and background

The concept of statistical convergence of ordinary (single) sequences was introduced by Fast [2] in 1951. Basic properties of statistical convergence were proved, among others, by Schoenberg [10], Fridy [3], Connor [1] (and see references in these papers).

We extended in [8] the concept of statistical convergence from single to multiple sequences and proved some basic results. A double sequence $(x_{jk} : j, k = 0, 1, 2, \ldots )$ of (real or complex) numbers is said to be statistically convergent to some number $\xi$, in symbol:

\[ \text{st-lim } x_{jk} = \xi, \text{ if for every } \varepsilon > 0, \lim_{m,n \to \infty} (m + 1)^{-1}(n + 1)^{-1} \left| \left\{ j \leq m \text{ and } k \leq n : |x_{jk} - \xi| \geq \varepsilon \right\} \right| = 0, \]

where by $j \leq m$ and $k \leq n$ we mean $j = 0, 1, \ldots, m$ and $k = 0, 1, \ldots, n$; by $|S|$ we denote the cardinality of the set $S \subset \mathbb{N}^2$; and $m$ and $n$ tend to infinity independently of one another.

It is plain that statistical convergence enjoys the property of additivity and homogeneity. Furthermore, ordinary convergence (in Pringsheim’s sense) implies statistical convergence to the same limit.

As usual, the (first) arithmetic mean $\sigma_{mn}$ of a double sequence $(x_{jk})$ is defined by

\[ \sigma_{mn} := (m + 1)^{-1}(n + 1)^{-1} \sum_{j=0}^{m} \sum_{k=0}^{n} x_{jk}, \quad m, n = 0, 1, 2, \ldots \]

We say that $(x_{jk})$ is statistically summable $(C, 1, 1)$ to some number $\xi$ if $\text{st-lim } \sigma_{mn} = \xi$.

(The letter ‘C’ comes from the name ‘Cesàro.’)

We claim that if a double sequence $(x_{jk})$ is bounded, then

\[ \text{st-lim } x_{jk} = \xi \implies \text{st-lim } \sigma_{mn} = \xi. \]

In fact, assume $|x_{jk}| \leq H$ for all $j$ and $k$, where $H$ is a constant. Clearly, $|\xi| \leq H$, as well. Given an $\varepsilon > 0$, by (1.1) we have

\[ \lim_{m,n \to \infty} (m + 1)^{-1}(n + 1)^{-1} |A_{mn}| = 0, \]

where $A_{mn} := \left\{ j \leq m \text{ and } k \leq n : |x_{jk} - \xi| \geq \varepsilon \right\}$.

Let

\[ B_{mn} := \{ j \leq m \text{ and } k \leq n \}\setminus A_{mn}. \]

By (1.2), we may write that

\[ \sigma_{mn} - \xi = (m + 1)^{-1}(n + 1)^{-1} \left\{ \sum_{(j,k) \in A_{mn}} + \sum_{(j,k) \in B_{mn}} \right\} (x_{j,k} - \xi), \]

whence it follows that

\[ |\sigma_{mn} - \xi| \leq 2H(m + 1)^{-1}(n + 1)^{-1} |A_{mn}| + \varepsilon < 2\varepsilon, \]

if both $m$ and $n$ are large enough, say $m, n > n_0(\varepsilon)$. Thus, we have
\[(M + 1)^{-1}(N + 1)^{-1} \left| \{m \leq M \text{ and } n \leq N : |\sigma_{mn} - \xi| \geq 2\varepsilon \} \right| \leq n_0(N + 1)^{-1} + n_0(M + 1)^{-1}, \quad M, N > n_0.\]

This proves that for every \(\varepsilon > 0\),
\[
\lim_{M,N \to \infty} (M + 1)^{-1}(N + 1)^{-1} \left| \{m \leq M \text{ and } n \leq N : |\sigma_{mn} - \xi| \geq 2\varepsilon \} \right| = 0.
\]

2. New results

The converse implication in (1.3) is not true in general, even in the case of ordinary convergence. Our main goal is to find conditions under which
\[
\text{st-lim} \sigma_{mn} = \xi \quad \text{implies} \quad \text{st-lim} x_{jk} = \xi. \tag{2.1}
\]

First, we consider double sequences \((x_{jk})\) of real numbers and formulate one-sided necessary and sufficient Tauberian conditions.

**Theorem 1.** For a double sequence \((x_{jk})\) of real numbers implication (2.1) holds if and only if the following two conditions are satisfied:

\[
\inf_{\lambda > 1} \limsup_{M,N \to \infty} (M + 1)^{-1}(N + 1)^{-1} \left| \{m \leq M \text{ and } n \leq N : \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} (x_{jk} - x_{mn}) \leq \varepsilon \} \right| = 0 \tag{2.2}
\]

and

\[
\inf_{0 < \lambda < 1} \limsup_{M,N \to \infty} (M + 1)^{-1}(N + 1)^{-1} \left| \{m \leq M \text{ and } n \leq N : \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} (x_{mn} - x_{jk}) \leq \varepsilon \} \right| = 0, \tag{2.3}
\]

where by \(\lambda_m\) we denote the integral part of the product \(\lambda m\), in symbol \(\lambda_m := [\lambda m]\).

**Remark 1.** It turns out from the proof of Theorem 1 (see in Section 3 below) that even more is true: If \((x_{jk})\) is statistically summable \((C, 1, 1)\) to a finite number and conditions (2.2) and (2.3) are satisfied, then we necessarily have
\[
\text{st-lim} \left(\lambda_m - m\right)^{-1} \left(\lambda_n - n\right)^{-1} \sum_{j=\lambda_m+1}^{\lambda_m} \sum_{k=\lambda_n+1}^{\lambda_n} (x_{jk} - x_{mn}) = 0 \tag{2.4}
\]

for every \(\lambda > 1\), and
st-lim(m − \lambda m)^{-1}(n − \lambda n)^{-1} \sum_{j=\lambda m+1}^{m} \sum_{k=\lambda n+1}^{n} (x_{mn} - x_{jk}) = 0 \quad (2.5)

for every 0 < \lambda < 1.

Following Schmidt [9] (see [7] in the case of single sequences), we say that a double sequence (x_{jk}) is statistically slowly decreasing with respect to the first index if, for every \epsilon > 0,

\inf_{\lambda > 1} \limsup_{M,N \to \infty} (M + 1)^{-1} (N + 1)^{-1} \left\{ m \leq M \text{ and } n \leq N : \min_{m < j \leq \lambda m} (x_{jn} - x_{mn}) \leq -\epsilon \right\} = 0. \quad (2.6)

We say that (x_{jk}) is statistically slowly decreasing in the strong sense with respect to the first index if (2.6) is satisfied with

\min_{m < j \leq \lambda m} (x_{jk} - x_{mk}) \quad \text{in place of} \quad \min_{m < j \leq \lambda m} (x_{jn} - x_{mn}). \quad (2.6')

Analogously, we say that (x_{jk}) is statistically slowly decreasing with respect to the second index if, for every \epsilon > 0,

\inf_{\lambda > 1} \limsup_{M,N \to \infty} (M + 1)^{-1} (N + 1)^{-1} \left\{ m \leq M \text{ and } n \leq N : \min_{n < k \leq \lambda n} (x_{mk} - x_{mn}) \leq -\epsilon \right\} = 0, \quad (2.7)

and (x_{jk}) is said to enjoy this property in the strong sense if (2.7) is satisfied with

\min_{m < j \leq \lambda m} (x_{jk} - x_{jn}) \quad \text{in place of} \quad \min_{n < k \leq \lambda n} (x_{mk} - x_{mn}). \quad (2.7')

Remark 2. Adopting the reasoning from [7, Remark 3] to this case, it is not difficult to check that (2.6) implies

\inf_{0 < \lambda < 1} \limsup_{M,N \to \infty} (M + 1)^{-1} (N + 1)^{-1} \left\{ m \leq M \text{ and } n \leq N : \min_{\lambda m < j \leq m} (x_{mn} - x_{jn}) \leq -\epsilon \right\} = 0,

and vice versa. The same equivalence holds in the case of (2.7) and in the cases where (2.6) and (2.7) are meant in the strong sense (that is, in the cases when (2.6) is modified by (2.6'), and (2.7) is modified by (2.7')). Now, taking into account that for the expression in (2.2), we have

\left(\lambda_m - \lambda_m\right)^{-1}\left(\lambda_n - \lambda_n\right)^{-1} \sum_{j=\lambda m+1}^{\lambda m} \sum_{k=\lambda n+1}^{\lambda n} (x_{jk} - x_{mn})

\geq \min_{m < j \leq \lambda m} (x_{jk} - x_{mk}) + \min_{n < k \leq \lambda n} (x_{mk} - x_{mn})
Corollary 1. For a double sequence \((x_{jk})\) of real numbers implication \((2.1)\) holds if \((x_{jk})\) is statistically slowly decreasing with respect to both indices and, in addition, in the strong sense with respect to one of the indices.

We say that a double sequence \((x_{jk})\) satisfies a Tauberian condition of Landau type with respect to the first index if there exist constants \(n_0 > 0\) and \(H > 0\) such that
\[
j(x_{jn} - x_{j-1,n}) \geq -H \quad \text{whenever} \quad j, n > n_0. \tag{2.8}
\]
(See [6] in the case of single sequences.) We claim that condition \((2.6)\) even in the strong sense follows from \((2.8)\). Indeed, if \(n_0 \leq m < j \leq \lambda m\) and \(k > n_0\), then we have
\[
x_{jk} - x_{mk} = \sum_{\ell=m+1}^{j} (x_{\ell k} - x_{\ell-1,k}) \geq -H \sum_{\ell=m+1}^{j} \frac{1}{\ell} \geq -H \ln \frac{j}{m} \geq -H \ln \lambda.
\]
If we choose \(\lambda\) so that \(1 < \lambda \leq \exp(\varepsilon/H)\), then \(-H \ln \lambda \geq -\varepsilon\), and consequently, the set
\[
\{n_0 < m \leq M \text{ and } n_0 < n \leq N : \min_{m \leq j \leq \lambda m, n < k \leq \lambda n} (x_{jk} - x_{mk}) \leq -\varepsilon\}
\]
is empty. Thus, condition \((2.6)\) is trivially satisfied.

A Tauberian condition of Landau type with respect to the second index is defined analogously as follows:
\[
k(x_{mk} - x_{m,k-1}) \geq -H \quad \text{whenever} \quad m, k > n_0. \tag{2.9}
\]
Corollary 2. For a double sequence \((x_{jk})\) of real numbers implication \((2.1)\) holds if conditions \((2.8)\) and \((2.9)\) are satisfied.

In particular, if the double sequence \((x_{jk})\) is nondecreasing with respect to each of the indices, that is, if for all \(j, k\) large enough,
\[
x_{j-1,k} \leq x_{jk} \quad \text{and} \quad x_{j,k-1} \leq x_{jk},
\]
then conditions \((2.8)\) and \((2.9)\) hold trivially for every \(H > 0\). Consequently, for such a sequence statistical convergence follows from statistical summability \((C, 1, 1)\).

Motivated by an analogous result of Fridy and Khan [4, Theorem 2.3] in the case of a single sequence, we formulate the following

Conjecture. If a double sequence \((x_{jk})\) is such that conditions \((2.8)\) and \((2.9)\) hold for some constants \(n_0\) and \(H > 0\), then the ordinary convergence (in Pringsheim’s sense) of \((x_{jk})\) follows from its statistical summability \((C, 1, 1)\).

In the rest of this section, we consider double sequences \((x_{jk})\) of complex numbers. In Theorem 2 we give two-sided Tauberian conditions, each of which is necessary and sufficient in order that statistical convergence follow from statistical summability \((C, 1, 1)\).
Theorem 2. For a double sequence \((x_{jk})\) of complex numbers implication (2.1) holds if and only if one of the following two conditions is satisfied: for every \(\varepsilon > 0\), either

\[
\inf_{\lambda > 1} \limsup_{M,N \to \infty} (M + 1)^{-1} (N + 1)^{-1} \left\{ m \leq M \text{ and } n \leq N:\right.
\]

\[
\left(\lambda_m - m\right)^{-1} \left(\lambda_n - n\right)^{-1} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} (x_{jk} - x_{mn}) \geq \varepsilon \right\} = 0, \tag{2.10}
\]

or

\[
\inf_{0 < \lambda < 1} \limsup_{M,N \to \infty} (M + 1)^{-1} (N + 1)^{-1} \left\{ m \leq M \text{ and } n \leq N:\right.
\]

\[
\left(m - \lambda_m\right)^{-1} \left(n - \lambda_n\right)^{-1} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} (x_{mn} - x_{jk}) \geq \varepsilon \right\} = 0. \tag{2.11}
\]

Remark 3. Even more is true: If \((x_{jk})\) is statistically summable \((C, 1, 1)\) to a finite number and one of conditions (2.10) and (2.11) is satisfied, then we necessarily have (2.4) for all \(\lambda > 1\), and (2.5) for all \(0 < \lambda < 1\).

We can draw similar corollaries from Theorem 2 as we did it in the case of Theorem 1. Following Hardy [5], a double sequence \((x_{jk})\) of complex numbers is said to be \textit{statistically slowly oscillating} with respect to the first index if, for every \(\varepsilon > 0\),

\[
\inf_{\lambda > 1} \limsup_{M,N \to \infty} (M + 1)^{-1} (N + 1)^{-1} \left\{ m \leq M \text{ and } n \leq N:\right.
\]

\[
\max_{m < j \leq \lambda_m} \left| x_{jn} - x_{mn} \right| \geq \varepsilon \right\} = 0. \tag{2.12}
\]

We say that \((x_{jk})\) is \textit{statistically slowly oscillating in the strong sense} with respect to the first index if (2.12) is satisfied with

\[
\max_{m < j \leq \lambda_m} \left| x_{jn} - x_{mn} \right| \text{ in place of } \max_{m < j \leq \lambda_m} \left| x_{jn} - x_{mn} \right|.
\]

The statistically slow oscillation property with respect to the second index is defined analogously.

Remark 4. Similarly to Remark 2, condition (2.12) is equivalent to the following one: for every \(\varepsilon > 0\),

\[
\inf_{0 < \lambda < 1} \limsup_{M,N \to \infty} (M + 1)^{-1} (N + 1)^{-1} \left\{ m \leq M \text{ and } n \leq N:\right.
\]

\[
\max_{\lambda_m < j \leq m} \left| x_{mn} - x_{jn} \right| \geq \varepsilon \right\} = 0,
\]

and an analogous equivalence holds in the strong sense, as well.
Corollary 3. For a double sequence \((x_{jk})\) of complex numbers implication (2.1) holds if \((x_{jk})\) is statistically slowly oscillating with respect to both indices and, in addition, in the strong sense with respect to one of the indices.

Condition (2.12) is satisfied even in the strong sense (cf. (2.8)) if there exist constants \(n_0 > 0\) and \(H > 0\) such that

\[ j |x_{jn} - x_{j-1,n}| \leq H \quad \text{whenever } j, n > n_0. \]  
(2.13)

The symmetric counterpart of (2.13) is the following (cf. (2.9)):

\[ k |x_{mk} - x_{m,k-1}| \leq H \quad \text{whenever } m, k > n_0. \]  
(2.14)

Corollary 4. For a double sequence \((x_{jk})\) of complex numbers implication (2.1) holds if conditions (2.13) and (2.14) are satisfied.

Conditions (2.13) and (2.14) may be called two-sided Tauberian conditions of Hardy type (see [5] in the case of single sequences).

Before the proofs, we note that the above results are the extensions of those in [7] from single to double sequences.

3. Proofs

We begin with two lemmas.

Lemma 1. If a double sequence \((x_{jk})\) is statistically summable \((C, 1, 1)\) to a finite number \(\xi\), then for every \(\lambda > 0\),

\[
\text{st-lim} \sigma_{\lambda_m, \lambda_n} = \xi, \quad \text{where } \lambda_m := \lfloor \lambda m \rfloor.
\]

The proof can be carried out in the same way as the corresponding lemma is proved in the case of a single sequence (see [7, Lemma 2]).

The following lemma on the so-called ‘moving rectangular averages’ may be useful in other contexts, as well.

Lemma 2. If a double sequence \((x_{jk})\) is statistically summable \((C, 1, 1)\) to a finite number \(\xi\), then for every \(\lambda > 1\),

\[
\text{st-lim}(\lambda_m - m)^{-1}(\lambda_n - n)^{-1} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} x_{jk} = \xi,
\]  
(3.1)

and for every \(0 < \lambda < 1\),

\[
\text{st-lim}(m - \lambda_m)^{-1}(n - \lambda_n)^{-1} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} x_{jk} = \xi.
\]  
(3.2)
Proof. Case $\lambda > 1$. An easy exercise (relying only on the definition of $\sigma_{mn}$) to show that if $\lambda > 1$ and $m, n$ are large enough in the sense that $\lambda^m > m$ and $\lambda^n > n$, then

$$
\frac{1}{(\lambda^m - m)(\lambda^n - n)} \sum_{j=m+1}^{\lambda^m} \sum_{k=n+1}^{\lambda^n} x_{jk} = \sigma_{mn} + \frac{\lambda^m + 1}{\lambda_n - n}(\sigma_{m,\lambda_n} - \sigma_{mn}) + \frac{\lambda^m + 1}{\lambda^m - m}(\sigma_{\lambda_m,n} - \sigma_{mn})
$$

$$
+ \frac{(\lambda^m + 1)(\lambda_n + 1)}{(\lambda^m - m)(\lambda^n - n)}(\sigma_{\lambda_m,\lambda_n} - \sigma_{m,\lambda_n} - \sigma_{\lambda_m,n} + \sigma_{mn}).
$$

(3.3)

Now, (3.1) follows from the statistical summability $(C, 1, 1)$ of $(x_{jk})$, Lemma 1, and the fact that for large enough $m$,

$$
\frac{\lambda^m + 1}{\lambda^m - m} \leq \frac{2\lambda}{\lambda - 1}.
$$

(3.4)

Case $0 < \lambda < 1$. This time, we make use of the following representation:

$$
\frac{1}{(m - \lambda_m)(n - \lambda_n)} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} x_{jk} = \sigma_{mn} + \frac{\lambda_m + 1}{n - \lambda_n}(\sigma_{mn} - \sigma_{m,\lambda_n}) + \frac{\lambda_m + 1}{m - \lambda_m}(\sigma_{mn} - \sigma_{\lambda_m,n})
$$

$$
+ \frac{(\lambda_m + 1)(\lambda_n + 1)}{(m - \lambda_m)(n - \lambda_n)}(\sigma_{\lambda_m,\lambda_n} - \sigma_{m,\lambda_n} - \sigma_{\lambda_m,n} + \sigma_{mn}),
$$

(3.5)

and inequality

$$
\frac{\lambda_m + 1}{m - \lambda_m} \leq \frac{2\lambda}{\lambda - 1},
$$

(3.6)

provided $m$ and $n$ are large enough. 

Proof of Theorem 1. Necessity. Assume that $(x_{jk})$ is both statistically convergent and statistically summable $(C, 1, 1)$ to the same number. Applying Lemma 2 yields (2.4) for every $\lambda > 1$, and (2.5) for every $0 < \lambda < 1$.

Sufficiency. Assume that $(x_{jk})$ is statistically summable $(C, 1, 1)$, and conditions (2.2) and (2.3) are satisfied. In order to prove that $(x_{jk})$ is statistically convergent to the same number, it is enough to prove that

$$
st\text{-lim}(x_{mn} - \sigma_{mn}) = 0.
$$

(3.7)

First, we consider the case $\lambda > 1$. It follows from (3.3) that

$$
x_{mn} - \sigma_{mn} = \frac{\lambda_n + 1}{\lambda_n - n}(\sigma_{m,\lambda_n} - \sigma_{mn}) + \frac{\lambda_m + 1}{\lambda^m - m}(\sigma_{\lambda_m,n} - \sigma_{mn})
$$

$$
+ \frac{(\lambda_m + 1)(\lambda_n + 1)}{(\lambda^m - m)(\lambda^n - n)}(\sigma_{\lambda_m,\lambda_n} - \sigma_{m,\lambda_n} - \sigma_{\lambda_m,n} + \sigma_{mn})
$$

$$
- \frac{1}{(\lambda^m - m)(\lambda^n - n)} \sum_{j=m+1}^{\lambda^m} \sum_{k=n+1}^{\lambda^n} (x_{jk} - x_{mn}),
$$

(3.8)
whence, for any \( \varepsilon > 0 \),
\[
\{ m \leq M \text{ and } n \leq N : x_{mn} - \sigma_{mn} \geq \varepsilon \}
\subseteq \left\{ m \leq M \text{ and } n \leq N : \frac{\lambda_n + 1}{\lambda_m - n} (\sigma_{m, \lambda_n} - \sigma_{mn}) + \frac{\lambda_m + 1}{\lambda_n - m} (\sigma_{m, \lambda_n} - \sigma_{mn}) + \frac{(\lambda_m + 1)(\lambda_n + 1)}{(\lambda_m - m)(\lambda_n - n)} (\sigma_{m, \lambda_n} - \sigma_{m, \lambda_n} + \sigma_{mn}) \leq \frac{\varepsilon}{2} \right\}
\cup \left\{ m \leq M \text{ and } n \leq N : \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} (x_{jk} - x_{mn}) \leq -\frac{\varepsilon}{2} \right\}
= : A_{MN}(\varepsilon) \cup B_{MN}(\varepsilon),
\tag{3.9}
\]
say. By virtue of Lemma 1 and (3.4), we have for every \( \varepsilon > 0 \),
\[
\lim_{M,N \to \infty} (M+1)^{-1}(N+1)^{-1} \left| A_{MN}(\varepsilon) \right| = 0.
\tag{3.10}
\]
On the other hand, given any \( \delta > 0 \), by (2.2) there exists some \( \lambda > 1 \) such that
\[
\limsup_{M,N \to \infty} (M+1)^{-1}(N+1)^{-1} \left| B_{MN}(\varepsilon) \right| \leq \delta.
\tag{3.11}
\]
Combining (3.9)–(3.11) gives
\[
\limsup_{M,N \to \infty} (M+1)^{-1}(N+1)^{-1} \left| \{ m \leq M \text{ and } n \leq N : x_{mn} - \sigma_{mn} \geq \varepsilon \} \right| \leq \delta.
\]
Since \( \delta > 0 \) is arbitrary, we have for every \( \varepsilon > 0 \),
\[
\lim_{M,N \to \infty} (M+1)^{-1}(N+1)^{-1} \left| \{ m \leq M \text{ and } n \leq N : x_{mn} - \sigma_{mn} \geq \varepsilon \} \right| = 0.
\tag{3.12}
\]
Second, we consider the case \( 0 < \lambda < 1 \). It follows from (3.5) that
\[
x_{mn} - \sigma_{mn} = \frac{\lambda_n + 1}{n - \lambda_n} (\sigma_{mn} - \sigma_{m, \lambda_n}) + \frac{\lambda_m + 1}{m - \lambda_m} (\sigma_{m, \lambda_n} - \sigma_{mn}) + \frac{(\lambda_m + 1)(\lambda_n + 1)}{(m - \lambda_m)(n - \lambda_n)} (\sigma_{mn} - \sigma_{m, \lambda_n} - \sigma_{m, \lambda_n} + \sigma_{mn}) + \frac{1}{(m - \lambda_m)(n - \lambda_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} (x_{mn} - x_{jk}),
\tag{3.13}
\]
whence, for any \( \varepsilon > 0 \),
\[
\{ m \leq M \text{ and } n \leq N : x_{mn} - \sigma_{mn} \leq -\varepsilon \}
\subseteq \left\{ m \leq M \text{ and } n \leq N : \frac{\lambda_n + 1}{n - \lambda_n} (\sigma_{mn} - \sigma_{m, \lambda_n}) + \frac{\lambda_m + 1}{m - \lambda_m} (\sigma_{mn} - \sigma_{m, \lambda_n}) + \frac{(\lambda_m + 1)(\lambda_n + 1)}{(m - \lambda_m)(n - \lambda_n)} (\sigma_{mn} - \sigma_{m, \lambda_n} - \sigma_{m, \lambda_n} + \sigma_{mn}) \leq -\frac{\varepsilon}{2} \right\}
\cup \left\{ m \leq M \text{ and } n \leq N : \frac{1}{(m - \lambda_m)(n - \lambda_n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} (x_{mn} - x_{jk}) \leq -\frac{\varepsilon}{2} \right\}.
Using the same argument as in the case of \( \lambda > 1 \), by virtue of Lemma 1, (2.3) and (3.6), we conclude for every \( \varepsilon > 0 \),

\[
\lim_{M,N \to \infty} (M + 1)^{-1}(N + 1)^{-1} \left\{ |m \leq M \text{ and } n \leq N: x_{mn} - \sigma_{mn} \leq -\varepsilon \right\} = 0. \tag{3.14}
\]

Combining (3.12) and (3.14) yields for every \( \varepsilon > 0 \),

\[
\lim_{M,N \to \infty} (M + 1)^{-1}(N + 1)^{-1} \left\{ |m \leq M \text{ and } n \leq N: \varepsilon \geq |x_{mn} - \sigma_{mn}| \right\} = 0.
\]

This proves (3.7), and the proof of Theorem 1 is complete. \( \Box \)

**Proof of Theorem 2.** **Necessity.** It is identical with the justification of the necessity part in Theorem 1.

**Sufficiency.** Assume that \((x_{jk})\) is statistically summable \((C, 1, 1)\), and one of the conditions (2.10) and (2.11) is satisfied. In order to prove that \((x_{jk})\) is statistically convergent to the same number, again it is enough to prove (3.7).

Let \( \varepsilon > 0 \) be given. In case \( \lambda > 1 \), by (3.8) we have

\[
\left\{ m \leq M \text{ and } n \leq N: \left| \frac{\lambda_{m} + 1}{\lambda_{n} - n}(\sigma_{m,\lambda_{n}} - \sigma_{mn}) + \frac{\lambda_{m} + 1}{\lambda_{m} - m}(\sigma_{m,n} - \sigma_{mn}) \right| \geq \varepsilon \right\}
\]

\[
\cup \left\{ m \leq M \text{ and } n \leq N: \frac{1}{(\lambda_{m} - m)(\lambda_{n} - n)} \sum_{j=m+1}^{\lambda_{m}} \sum_{k=n+1}^{\lambda_{n}} (x_{jk} - x_{mn}) \geq \varepsilon \right\}
\]

\[
= : A_{MN}^{(1)}(\varepsilon) \cup B_{MN}^{(1)}(\varepsilon), \tag{3.15}
\]

say; while in case \( 0 < \lambda < 1 \), by (3.13) we have

\[
\left\{ m \leq M \text{ and } n \leq N: \left| \frac{\lambda_{m} + 1}{\lambda_{n} - n}(\sigma_{mn} - \sigma_{m,\lambda_{n}}) + \frac{\lambda_{m} + 1}{m - \lambda_{m}}(\sigma_{mn} - \sigma_{mn}) \right| \geq \varepsilon \right\}
\]

\[
\cup \left\{ m \leq M \text{ and } n \leq N: \frac{1}{(m - \lambda_{m})(n - \lambda_{n})} \sum_{j=m+1}^{m} \sum_{k=n+1}^{n} (x_{mn} - x_{jk}) \geq \varepsilon \right\}
\]

\[
= : A_{MN}^{(2)}(\varepsilon) \cup B_{MN}^{(2)}(\varepsilon), \tag{3.16}
\]

say.

Given \( \delta > 0 \), by (2.10) there exists some \( \lambda > 1 \) such that

\[
\limsup_{M,N \to \infty} (M + 1)^{-1}(N + 1)^{-1} \left| B_{MN}^{(1)}(\varepsilon) \right| \leq \delta.
\]
or by (2.11) there exists some $0 < \lambda < 1$ such that
\[
\limsup_{M,N \to \infty} (M + 1)^{-1} (N + 1)^{-1} |B^{(2)}_{MN}(\varepsilon)| \leq \delta.
\]
By (3.15), (3.16) and Lemma 2, in either case we conclude that
\[
\limsup_{M,N \to \infty} (M + 1)^{-1} (N + 1)^{-1} \left| \left\{ m \leq M \text{ and } n \leq N : |x_{mn} - \sigma_{mn}| \geq \varepsilon \right\} \right| \leq \delta.
\]
Since $\delta > 0$ is arbitrary, it follows that for every $\varepsilon > 0$,
\[
\lim_{M,N \to \infty} (M + 1)^{-1} (N + 1)^{-1} \left| \left\{ m \leq M \text{ and } n \leq N : |x_{mn} - \sigma_{mn}| \geq \varepsilon \right\} \right| = 0.
\]
This proves (3.7) and the proof of Theorem 2 is complete. \(\square\)

References