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On the Jordan block structure of images of some unipotent elements in modular irreducible representations of the classical algebraic groups ☆

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Abstract

Images of root elements in p-restricted irreducible representations of the classical algebraic groups over a field of characteristic $p > 0$ and images of regular unipotent elements of naturally embedded subgroups of type A_2 in such representations of groups of type A_n with $n > 2$ and $p > 2$ are investigated. Let $\omega = \sum_{i=1}^{n} m_i\omega_i$ be the highest weight of a representation under consideration. If ω is locally small with respect to p in a certain sense, the sizes of all Jordan blocks (without multiplicities) in the images of root elements are found, except the case of the groups of type B_n and C_2 and short roots where all such sizes congruent to $m_i + 1$ modulo 2 are determined with the ith simple root being short; for $p > 2$ and $n > 3$, all odd dimensions of such blocks for groups of type A_n and regular unipotent elements of naturally embedded subgroups of type A_2 are found. Here the class of locally small weights with respect to p depends upon the type of a group and upon elements considered. For root elements in a group of type A_n, the weight ω is locally small if $m_i + m_{i+1} < p - 1$ for some i. For root elements in other classical groups, the definitions of the relevant classes are more complicated and depend upon the root length; however, in all these cases locally small weights are determined in terms of certain linear functions of their values on two simple roots linked at the Dynkin diagram of a group. For groups of type A_n with $n > 3$ and regular unipotent elements of naturally embedded A_2-subgroups, the weight ω is locally small if $m_i + m_{i+1} + m_{i+2} + m_{i+3} < p - 2$ for some i with $i < n - 2$. For arbitrary p-restricted

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representations, the presence of blocks of certain sizes in the images of elements indicated above is established.

Keywords: Classical algebraic groups; Representations; Unipotent elements; Jordan blocks

1. Introduction

Let G be a classical simply connected algebraic group of rank $n > 1$ over an algebraically closed field K of characteristic $p > 0$. Denote by ω_i, $1 \leqslant i \leqslant n$, the fundamental weights of G labeled as in Bourbaki's book [1]. For a unipotent element $u \in G$ and a rational representation ϕ of G denote by $\mathrm{J}_\phi(u)$ the set of sizes of blocks (without their multiplicities) of the canonical Jordan form of $\phi(u)$. In what follows, Irr_p is the set of irreducible p-restricted representations of G, $\omega(\phi)$ is the highest weight of ϕ.

For an irreducible representation ϕ of G with p-restricted $\omega(\phi) = \sum_{i=1}^{n} m_i\omega_i$ locally small with respect to p in a certain sense, the sets $\mathrm{J}_\phi(u)$ are completely determined for root elements u except the case of the groups of types B_n and C_2 and short roots α_i where all elements in $\mathrm{J}_\phi(u)$ congruent to $m_i + 1$ modulo 2 are found; for $p > 2$, $n > 3$, $G = A_n(K)$, a regular unipotent element u in a naturally embedded subgroup of type A_2, and a p-restricted representation ϕ with a locally small highest weight, all odd block sizes in $\mathrm{J}_\phi(u)$ are determined. Recall that a dominant weight $\sum_{i=1}^{n} m_i\omega_i$ is p-restricted if all $m_i < p$. Root elements are assumed to be nonunity elements of root subgroups. Such elements are called long or short if they are associated with long or short roots, respectively. The notion of a locally small weight depends upon G and a problem considered (root elements or regular unipotent elements in a subgroup of type A_2) and will be precisely defined later. In the majority of cases considered, it occurs that $\mathrm{J}_\phi(u)$ contains all a priori possible blocks.

In characteristic 0, each unipotent element is contained in a Zariski closed subgroup of type A_1. So, the complete reducibility of representations of semisimple groups and well-known properties of A_1-modules imply that $\mathrm{J}_\phi(u)$ coincides with the set of the composition factor dimensions for the restriction of a representation ϕ to an A_1-subgroup containing u. For the classical groups and naturally embedded subgroups of type A_1, these factors can be deduced from the classical branching rules, this yields $\mathrm{J}_\phi(u)$ for a representation ϕ and root elements u (see [9, Theorem 1] for details). In [7,8] the sets $\mathrm{J}_\phi(u)$ were found for any unipotent elements of groups of type A_2, A_3, and C_2. In principle, these sets are determined by the weight multiplicities of ϕ and the labeled Dynkin diagram of u. However, for arbitrary classical groups and unipotent elements, we see no approach to an explicit description of $\mathrm{J}_\phi(u)$ even in characteristic 0. For regular unipotent elements, this problem is equivalent to a strong refinement of well-known Dynkin's theorem [5] on the spindle property of the weight systems of irreducible representations. Namely, one needs to find out which of the inequalities for the sums of the weight multiplicities at fixed levels given by Dynkin's theorem are strict and which of them are in fact equalities.

In the case of characteristic p the situation is still more complicated. Here only elements of order p can be embedded into subgroups of type A_1, and restrictions of irreducible representations of G to such subgroups are in general not completely reducible. Hence the composition factors of these restrictions do not determine the Jordan block structure. Moreover, it is substantially more difficult to find these factors since the weight multiplicities are unknown. So only partial results on the Jordan block structure can be expected. Naturally, for a unipotent element u the degree of the minimal polynomial of $\phi(u)$ is equal to the size of the biggest block in $\mathrm{J}_\phi(u)$. In [15] the minimal polynomials of unipotent elements of order p in irreducible representations of semisimple algebraic groups in characteristic p were found. Tiep and Zalesskii in [17, Theorem 2.20] described the irreducible representations ϕ of the simple algebraic groups in characteristic $p > 3$ where $\mathrm{J}_\phi(u) \subseteq \{1, p-1, p\}$ for root elements u. If ϕ is a p-restricted representation with this property, $p > 5$ or $G \neq G_2(K)$, then ϕ is the basic Steinberg representation with highest weight $\sum_{i=1}^n (p-1)\omega_i$. This description is crucial for the classification of the irreducible complex representations of finite groups of Lie type in characteristic p unramified above p and remaining irreducible after the reduction modulo p obtained in [17, Theorem 1.2]. Some results on the presence of specific blocks in $\mathrm{J}_\phi(u)$ for root elements u and $n = 2$ were obtained in [17, Section 2.3] as well. Information on the Jordan block structure of unipotent elements in representations of algebraic groups can be useful for investigating recognition problems for representations and linear groups and constructing recognition algorithms for these purposes. Such results can be easily transferred to finite groups of Lie type (in particular, this was done in [17]) which extends the field of their potential applications.

We need some more notation to state the principal results. Denote by $\langle \omega, \beta \rangle$ the value of a weight ω on a root β. Throughout the text, $\mathbb{N}$ is the set of nonnegative integers. If $u = x_\alpha(t)$ is a root element, then $\alpha_{m,u}$ is the maximal root of the same length as α. It follows from [1, Tables I–IV] that

$$\langle \omega, \alpha_{m,u} \rangle = \begin{cases} \sum_{i=1}^{n} m_i, & G = A_n(K) \text{ or } G = C_n(K) \text{ and } \alpha \text{ is long;} \\ m_1 + m_n + \sum_{i=2}^{n-1} 2m_i, & G = B_n(K) \text{ and } \alpha \text{ is long;} \\ m_n + \sum_{i=1}^{n-1} 2m_i, & G = B_n(K) \text{ and } \alpha \text{ is short;} \\ m_1 + \sum_{i=2}^{n} 2m_i, & G = C_n(K) \text{ and } \alpha \text{ is short;} \\ m_1 + m_{n-1} + m_n + \sum_{i=2}^{n-2} 2m_i, & G = D_n(K). \end{cases} \tag{1}$$

For $\phi \in \mathrm{Irr}_p$, set $m_\phi(u) = \min(\langle \omega(\phi), \alpha_{m,u} \rangle + 1, p)$, for nonnegative integers a and b with $a \leqslant b$, put $\mathbb{N}_a^b = \{i \in \mathbb{N} \mid a \leqslant i \leqslant b\}$. In what follows we assume that $p > 2$ and $n > 2$ if $G = B_n(K)$ and that $n > 3$ for $G = D_n(K)$.

Definition 1. Let $n > 1$ and $\omega = \sum_{i=1}^{n} m_i\omega_i$ be a dominant p-restricted weight of G. For $G = A_n(K)$ or $D_n(K)$, the weight ω is locally p-small if $m_i + m_{i+1} < p - 1$ for some i with $i < n - 1$, or $G = A_n(K)$ and $m_{n-1} + m_n < p - 1$, or $G = D_n(K)$ and $m_{n-2} + m_n < p - 1$. The weight ω is locally p-small of type I if $G = B_n(K)$ and either $m_i + m_{i+1} < p - 1$ for some $i < n - 1$, or $2m_{n-1} + m_n < p - 2$, or $G = C_n(K)$ and $m_{n-1} + 2m_n < p - 2$, and ω is locally p-small of type II if $G = B_n(K)$ and $2m_{n-1} + m_n < p$ or $G = C_n(K)$ and either $m_i + m_{i+1} < p - 1$ for some $i < n - 1$, or $m_{n-1} + 2m_n < p - 1$.

Throughout the text we assume that $\phi \in \mathrm{Irr}_p$ and $\omega = \omega(\phi) = \sum_{i=1}^{n} m_i\omega_i$. The following theorem holds.

Theorem 2. *Let $n > 2$, $\phi \in \mathrm{Irr}_p$, and $u \in G$ be a root element. Assume that ω is locally p-small for $G = A_n(K)$ or $D_n(K)$. For $G = B_n(K)$ or $C_n(K)$ assume that ω is locally p-small of type I if u is long and ω is locally p-small of type II if u is short. Then $\mathrm{J}_\phi(u) = \mathbb{N}_1^{m_\phi(u)}$, except the case where $G = B_n(K)$ and u is short. In the exceptional case, the set*

$$J = \{k \mid 1 \leqslant k \leqslant m_\phi(u),\ k \equiv m_n + 1 \ (\mathrm{mod}\ 2)\} \subseteq \mathrm{J}_\phi(u),$$

$$\mathrm{J}_\phi(u) = J \quad \textit{if}\ m_\phi(u) = \langle\omega, \alpha_{m,u}\rangle + 1, \quad \textit{and} \quad p \in \mathrm{J}_\phi(u) \quad \textit{if}\ m_\phi(u) = p.$$

For $n = 2$, Definition 1 seems somewhat inappropriate, but the picture is similar. It is more convenient to consider this case separately. This is done in Proposition 3.

Proposition 3. *Let $n = 2$. Assume that $u = x_i(t)$ with $t \neq 0$. Set $m = \langle\omega, \alpha_{m,u}\rangle$. Then the following holds.*

(i) *For $G = A_2(K)$, we have $\mathrm{J}_\phi(u) = \mathbb{N}_1^{m+1}$ if $m_1 + m_2 < p - 1$, or $m_1 + m_2 = p - 1$ and $m_1m_2 = 0$, and $\mathrm{J}_\phi(u) = \mathbb{N}^{p}_{\min(p-m_1, p-m_2)}$ if $m_1, m_2 < p - 1 = m_1 + m_2$.*

(ii) *For $G = C_2(K)$ and $i = 2$, we have $\mathrm{J}_\phi(u) = \mathbb{N}_1^{m+1}$ if $m_1 + 2m_2 < p - 2$, or $m_1 + m_2 = p - 2$, or $m_1 + m_2 = p - 1$ and $p > 2$; $\mathrm{J}_\phi(u) = \mathbb{N}^{m+1}_{p-m_1-m_2-1}$ if $m_1 + m_2 + 3 \leqslant p < m_1 + 2m_2 + 3$; and $\mathrm{J}_\phi(u) = \{2\}$ if $m_1 = 0$, $m_2 = 1$, $p = 2$.*

(iii) *For $G = C_2(K)$ and $i = 1$, we have $\mathrm{J}_\phi(u) = \{j \in \mathbb{N}_1^{m+1} \mid j \equiv m_1 + 1 \ (\mathrm{mod}\ 2)\}$ if $m_1 + 2m_2 < p$.*

In the cases (i)–(iii), *if the relevant assumptions hold, then V_ω is a completely reducible module and $\mathrm{Irr}_{H(j)}\,\omega = \{a \mid a + 1 \in \mathrm{J}_\phi(u)\}$. In case* (ii), *if $m_1 + 2m_2 < p - 2$, and in case* (iii), *we have $\mathrm{Irr}_{H(i)}\,\omega = \mathrm{Irr}(W_\omega|H(i))$ (here W_ω is the Weyl module, see Section* 2).

In all cases, one can guarantee that certain integers belong to $\mathrm{J}_\phi(u)$.

Proposition 4. *Let $n > 1$. For a root element $u = x_\alpha(t)$, set*

$$c_\phi(u) = \min(m_i \mid \alpha_i \textit{ and } \alpha \textit{ are of the same length}).$$

Then $\mathbb{N}_{c_\phi(u)+1}^{m_\phi(u)} \subseteq \mathrm{J}_\phi(u)$, *except the case where* $G = B_n(K)$ *or* $C_2(K)$ *and* α *is short. In the exceptional case,* $\mathrm{J}_\phi(u)$ *contains all* $a \in \mathbb{N}$ *with* $a \equiv m_i + 1 \pmod 2$ *for the short root* α_i *and* $p \in \mathrm{J}_\phi(u)$ *if* $m_\phi(u) = p$.

The following example shows that for $G = C_n(K)$ and long roots the assumptions in Theorem 2 cannot be weakened. Assume that $G = C_n(K)$, $p > 2$, $\omega = \omega_{n-1} + \frac{p-3}{2}\omega_n$ or $\frac{p-1}{2}\omega_n$, and u is a long root element. Then it is actually proved in [18] that

$$\mathrm{J}_\phi(u) = \left\{\frac{p-1}{2}, \frac{p+1}{2}\right\}.$$

Proposition 3 yields that these assumptions cannot be weakened and for $G = A_2(K)$.

For $G = A_n(K)$, $p > 2$, and $n > 3$, another class of unipotent elements has been considered as well.

Theorem 5. *Let* $n > 3$, $p > 2$, *and* $G = A_n(K)$. *Assume that* u *is a regular unipotent element of a naturally embedded subgroup of type* A_2. *Set* $g = \min(2m_1 + \cdots + 2m_n + 1, p)$ *and* $I = \{k \in \mathbb{N}_1^g \mid k \equiv 1 \pmod 2\}$. *If* $m_i + m_{i+1} + m_{i+2} + m_{i+3} < p - 2$ *for some* $i < n-2$, *then* $I \subseteq \mathrm{J}_\phi(u) \subseteq \mathbb{N}_1^g$. *Furthermore, if in this situation* $2m_1 + \cdots + 2m_n + 1 \leqslant p$, *then* $\mathrm{J}_\phi(u) = I$.

For $G = A_n(K)$, $n > 2$, and a wide class of representations $\phi \in \mathrm{Irr}_p$, one can show the presence of certain integers in $\mathrm{J}_\phi(u)$.

Proposition 6. *Let* $G = A_n(K)$, p *and* u *be as in Theorem* 5. *Assume that* $m_i + m_{i+1} \leqslant (p-1)/2$ *for some* i. *Set* $m = \min_i(m_i + m_{i+1})$ *and* $M = \min((p-1)/2, \sum_{j=1}^n m_j)$. *Then* $2k + 1 \in \mathrm{J}_\phi(u)$ *for* $k \in \mathbb{N}_m^M$.

It is well known that $\mathrm{J}_\phi(u) = \{p\}$ for every element x of order p if ϕ is a basic Steinberg representation since in this case the restriction of ϕ to the relevant nontwisted Chevalley group G_p over the field of order p is a projective representation and the conjugacy class of x in G meets G_p (see, for instance, [17, Lemma 2.32]).

1.1. On the proofs of the main results

The general plan is as follows. First root elements in groups of rank 2 are handled (Proposition 3). Here the arguments are based on the description of the composition factors in the restrictions of relevant representations to naturally embedded subgroups of type A_1 [9, Theorem 2]. It occurs that for locally p-small weights these restrictions are completely reducible. We apply results of [15] on the minimal polynomials of elements of order p in irreducible representations of the classical groups and well-known facts on representations of the group $A_1(K)$ to get an upper bound for $\mathrm{J}_\phi(u)$. For Theorem 2, the following principal scheme is used. Fix i and j such that α_i and α_j are adjoint roots on the Dynkin diagram of G and the coefficients m_i and m_j satisfy the relevant assumptions in Definition 1.

Assume that the root α_i has the same length as the root with which a root element under consideration is associated. Denote by H the subgroup generated by the root subgroups associated with the roots α_i and $-\alpha_i$ and by S the subgroup generated by such subgroups associated with the roots α_i, $-\alpha_i$, α_j, and $-\alpha_j$. We have $S \cong A_2(K)$ or $C_2(K)$. Then, for a module V affording a representation considered, a decomposition

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_{l-1} \oplus V_l \tag{2}$$

with some special properties is constructed. Here all V_j are sums of weight subspaces of V and are KH-modules as well, and V_1 is the S-module generated by a highest weight vector. It occurs that V_j are completely reducible p-restricted H-modules for $j < l$. Using Smith' theorem (see Proposition 7) and Proposition 3, we conclude that for our root element u the set $\mathrm{J}_{V_1}(u)$ consists of all integers in some interval starting with 1 or all such integers of a fixed parity. Then we analyze the weight structure of H-modules V_j and use well-known facts on the representations of the group $A_1(K)$ to show that the restrictions of u to V_j with $2 \leqslant j < l$ yield other required block sizes. In some cases, for groups of types B, C, and D, we can simplify the general scheme applying Smith' theorem and results proven for type A.

For Theorem 5, the approach is quite similar, but here we fix a quadruple i, $i+1$, $i+2$, $i+3$ with $m_i + m_{i+1} + m_{i+2} + m_{i+3} < p-2$, replace S with the subgroup S_1 of type A_4 generated by the root subgroups associated with the roots $\pm\alpha_k$, $i \leqslant k \leqslant i+3$, and H by a subgroup H_1 of type A_2 in S_1 generated by certain root subgroups. Next, a decomposition similar to the decomposition (2) is constructed. In this case, results on the composition factors of restrictions of certain representations to naturally embedded subgroups of type A_2 [11, Theorem 1.3] are applied to find the Jordan block structure of u on V_1. Here we get all odd integers from some interval as the block sizes. An element u is embedded into a subgroup Π of type A_1 that lies in H_1. The restriction of weights from a maximal torus of H_1 to that of Π is considered. All V_j with $j < l$ are completely reducible Π-modules with p-restricted irreducible components of odd dimensions.

The proofs of Propositions 4 and 6 are similar to those of Theorems 2 and 5, respectively, but easier. Here one does not try to find small block sizes and hence there is no necessity to consider a bigger subgroup S or S_1.

2. Notation and preliminary results

In what follows $\mathbb{Z}$ is the set of all integers, $\mathbb{C}$ is the field of complex numbers, $L(\Gamma)$, $\mathfrak{X}(\Gamma)$, $W(\Gamma)$, and $R(\Gamma)$ are the Lie algebra, the weight system, the Weyl group, and the root system of a simple algebraic group Γ, respectively. We fix a base $\alpha_1, \ldots, \alpha_n$ in $R(G)$ and consider the fundamental weights with respect to this base. All modules considered are assumed to be rational and finite-dimensional. For a G-module V and a Zariski closed semisimple subgroup $S \subseteq G$ the symbols $\mathfrak{X}(V)$, V^μ, μ_S, $V|S$, and $\operatorname{Irr} V|S$ denote the set of all weights of V, the weight subspace of a weight $\mu \in \mathfrak{X}(G)$ in V, the restriction of a weight μ to S, the restriction of V to S, and the set of composition factors of $V|S$ (without multiplicities), respectively. The set of weights of the group $A_1(K)$ is canonically identified with $\mathbb{Z}$ and that of dominant weights with $\mathbb{N}$. If $S = A_1(K)$, we identify $V_i \in \operatorname{Irr} V|S$ with i and write $\operatorname{Irr} V|S \subseteq \mathbb{N}$. Set $\operatorname{Irr}_S \lambda = \operatorname{Irr} V_\lambda|S$. For $\alpha \in R(G)$, $t \in K$, $k \in \mathbb{N}$, the symbols $x_\alpha(t)$, X_α, $\mathcal{X}_\alpha$, and $X_{\alpha,k}$ denote the root elements of G and $L(G)$, the root subgroup of G associated with α, and the element of the hyperalgebra of $L(G)$ associated with the pair (α, k), respectively. For $k < p$ one has $X_{\alpha,k} = (X_\alpha)^k/k!$. If $\alpha = \pm\alpha_i$, we write $x_{\pm i}(t)$, $X_{\pm i}$, $\mathcal{X}_{\pm i}$, and $X_{\pm i,k}$. For positive roots $\beta_1, \ldots, \beta_j$ of G, let $H(\beta_1, \ldots, \beta_j)$ be the subgroup generated by the groups $\mathcal{X}_{\beta_1}, \ldots, \mathcal{X}_{\beta_j}$ and $\mathcal{X}_{-\beta_1}, \ldots, \mathcal{X}_{-\beta_j}$. In all cases where subgroups of this form are considered, the roots $\beta_1, \ldots, \beta_j$ are chosen such that they constitute a base of the root system of $H(\beta_1, \ldots, \beta_j)$. In this situation, the fundamental weights of $H(\beta_1, \ldots, \beta_j)$ are determined with respect to this base. Set $H(i_1, \ldots, i_k) = H(\alpha_{i_1}, \ldots, \alpha_{i_k})$. For a G-module V, the set $\mathrm{J}_V(u)$ is defined such as $\mathrm{J}_\phi(u)$. For a dominant weight $\mu \in \mathfrak{X}(\Gamma)$, let V_μ and W_μ be the irreducible and Weyl modules with highest weight μ, respectively. It is always clear from the context what group is meant. For any $\mu \in \mathfrak{X}(V_\omega)$, we have $\mu = \omega - \sum_{i=1}^n b_i\alpha_i$, $b_i \in \mathbb{N}$ [16, Theorem 39]; in this situation, set $b_i(\mu) = b_i$. If an irreducible G-module is fixed, the symbol v^+ is used to denote a fixed nonzero highest weight vector in V.

The following facts are heavily used in the proofs of the main results.

Proposition 7 (Smith [14]). *Let $S = H(i_1, \ldots, i_k) \subseteq G$, then $KSv^+ \subseteq V_\lambda$ is an irreducible S-module with highest weight λ_S and a direct summand of the S-module V_λ.*

Lemma 8 (Seitz [13, 1.5]). *Let V be a G-module and $v \in V \setminus \{0\}$ be a vector of weight λ. Assume that $\langle \lambda, \alpha \rangle = m < p$ for a root α of G and that $\mathcal{X}_\alpha$ fixes v. Then $X_{-\alpha,k}v \neq 0$ for $0 \leqslant k \leqslant m$.*

Lemma 9. *Let V be an $A_1(K)$-module and $|a| < p$ for all $a \in \mathfrak{X}(V)$. Then V is completely reducible.*

Proof. Recall that in this case the Weyl modules W_a are irreducible for nonnegative $a \in (X)(V)$ (see, for instance, [6, Chapter II, 2.16]). Now the lemma follows from [6, Chapter II, Proposition 2.14]. $\square$

Lemma 10. *Let $\Gamma = A_1(K)$, $a < p$, and V_a be an irreducible Γ-module. Then $\mathrm{J}_{V_a}(u) = \{a + 1\}$ for a nonunity unipotent element $u \in \Gamma$.*

Proof. This follows immediately from the well-known description of p-restricted Γ-modules (see, for instance, Steinberg [16, §12]). $\square$

Proposition 11. *For a root unipotent element $u \in G$, the degree of the minimal polynomial of $\phi(u)$ is equal to $m_\phi(u)$.*

Proof. The proposition follows from the formulae for the minimal polynomials of elements of order p [15, Theorem 1.1, Proposition 1.3, and Algorithm 1.4]. $\square$

Lemma 12. *Let Δ be a group, $u \in \Delta$, and $V = U_1 \oplus \cdots \oplus U_t$ be a direct sum of Δ-modules. Then $\mathrm{J}_V(u) = \bigcup_{l=1}^t \mathrm{J}_{U_l}(u)$.*

Proof. This is obvious. $\square$

Corollary 13. *Let $\Gamma = A_1(K)$, $u \in \Gamma$ be a nonunity unipotent element, and let V be a Γ-module with the maximal weight $a < p$. Then $a + 1 \in \mathrm{J}_V(u) \subseteq \mathbb{N}_1^{a+1}$. If $b \equiv a \pmod 2$ for all $b \in \mathfrak{X}(V)$, then $x \equiv a + 1 \pmod 2$ for all $x \in \mathrm{J}_V(u)$.*

Proof. By Lemma 9, V is completely reducible. Hence $V = V_a \oplus V'$ and V' is a direct sum of irreducible Γ-modules V_c with $c \leqslant a$ and $c \in \mathfrak{X}(V)$. By Lemma 10, $\mathrm{J}_{V_d}(u) = \{d + 1\}$ for $d < p$. It remains to apply Lemma 12. $\square$

Corollary 14. *Let V be an irreducible G-module, $S = H(i_1, \ldots, i_k)$, and $W = KSv^+ \subseteq V$. Assume that $u \in S$. Then $\mathrm{J}_W(u) \subseteq \mathrm{J}_V(u)$.*

Proof. By Proposition 7, W is a direct summand of V. Now apply Lemma 12. $\square$

Corollary 15. *Let $V = V_\omega$. Fix i with $1 \leqslant i \leqslant n$. Assume that $\mathfrak{X}(V) = \bigcup_{l=1}^{s} \mathfrak{X}_l$ where $\mathfrak{X}_l$ are such that for $k \neq l$ and for any $\mu \in \mathfrak{X}_k$, $\nu \in \mathfrak{X}_l$ there exists $j \neq i$ with $b_j(\mu) \neq b_j(\nu)$. Let $U_l = \sum_{\mu \in \mathfrak{X}_l} V^\mu$. Then U_l is an $H(i)$-module and $\mathrm{J}_V(u) = \bigcup_{l=1}^{s} \mathrm{J}_{U_l}(u)$ for a root unipotent element $u \in H(i)$.*

Proof. To show that U_l is an $H(i)$-module, it suffices to prove that $x_{\pm i}(t)V^\mu \subseteq U_l$ for $\mu \in \mathfrak{X}_l$. Let $v \in V^\mu$. By [16, Lemma 72], $x_i(t)v = v + \sum_{r=1}^{\infty} t^r v_r$ with $v_r \in V^{\mu + r\alpha_i}$. For $j \neq i$, we have $b_j(\mu + r\alpha_i) = b_j(\mu)$. Hence all $v_r \in U_l$ and so $x_i(t)v \in U_l$. Similarly, $x_{-i}(t)v \in U_l$. Now the assertion follows from Lemma 12. $\square$

Lemma 16. *Let $V = V_\nu$. Fix $i, j \in \mathbb{N}_1^n$. Assume that $n > 2$, $\mu_1, \ldots, \mu_k \in \mathfrak{X}(V)$, $b_i(\mu_s) = 0$, $\langle \mu_s, \alpha_i \rangle \neq \langle \mu_t, \alpha_i \rangle$ for $s \neq t$, and that for each s with $1 \leqslant s \leqslant k$ there exists $f \neq j$ such that $b_f(\mu_s) \neq 0$. Construct the subsets $\mathfrak{X}_1, \ldots, \mathfrak{X}_{k+2} \subseteq \mathfrak{X}(V)$ as follows: $\mathfrak{X}_1 = \{\lambda \in \mathfrak{X}(V) \mid b_h(\lambda) = 0$ for $h \neq i, j\}$, $\mathfrak{X}_l = \{\lambda \in \mathfrak{X}(V) \mid \lambda = \mu_{l-1} - r\alpha_i,\ r \in \mathbb{N}\}$ for $2 \leqslant l \leqslant k + 1$, and $\mathfrak{X}_{k+2} = \mathfrak{X}(V) \setminus (\bigcup_{l=1}^{k+1} \mathfrak{X}_l)$. Then the subsets $\mathfrak{X}_1, \ldots, \mathfrak{X}_{k+2}$ satisfy the assumptions of Corollary 15 with respect to i.*

Proof. It is clear that $\mu_s + c\alpha_i \notin \mathfrak{X}(V)$ for $c > 0$. Now one easily observes that $\mathfrak{X}_l = \{\lambda \in \mathfrak{X}(V) \mid b_f(\lambda) = b_f(\mu_{l-1})$ for $f \neq i\}$ if $2 \leqslant l \leqslant k + 1$. Since $\langle \mu_{s-1}, \alpha_i \rangle \neq \langle \mu_{t-1}, \alpha_i \rangle$ for $s \neq t$ with $2 \leqslant s, t \leqslant k + 1$, we conclude that for each pair (λ, ν) with $\lambda \in \mathfrak{X}_s$ and $\nu \in \mathfrak{X}_t$ there exists $g \neq i$ with $b_g(\lambda) \neq b_g(\nu)$. Now the assertion of the lemma follows from the assumptions on μ_l and the construction of $\mathfrak{X}_s$. $\square$

Corollary 17. *In the assumptions of Lemma 16, suppose that $\langle \mu_{l-1}, \alpha_i \rangle < p$ for $2 \leqslant l \leqslant k + 1$. Construct the sets U_l, $1 \leqslant l \leqslant k + 2$, as in Corollary 15. Let $u \in H(i)$ be a root element. Then $1 + \langle \mu_l, \alpha_i \rangle \in \mathrm{J}_\phi(u)$ for $2 \leqslant l \leqslant k + 1$. In particular, if $\mathbb{N}_1^a \subseteq \mathrm{J}_{U_1}(u)$ and $\langle \mu_{l-1}, \alpha_i \rangle = a + l - 2 < p$ for $2 \leqslant l \leqslant k + 1$, then $\mathbb{N}_1^{a+k} \subseteq \mathrm{J}_\phi(u)$.*

Proof. Set $\langle \mu_{l-1}, \alpha_i \rangle = a_l$. Observe that $\langle \mu, \alpha_i \rangle \leqslant a_l < p$ for all $\mu \in \mathfrak{X}_l$, $2 \leqslant l \leqslant k + 1$. Now apply Lemmas 10 and 12 and Corollary 13. $\square$

3. Root elements

Proof of Proposition 3. The composition factors of $V_\omega|H(i)$ are found in [9, Theorem 2]. Lemma 9 yields the complete reducibility of $V_\omega|H(i)$. Apply Lemma 10 to complete the proof. □

Proof of Theorem 2. Set $V = V_\omega$. Recall that all root elements associated with roots of the same length are conjugate in G. Since the maximal integer in $\mathrm{J}_\phi(u)$ is equal to the degree of the minimal polynomial of $\phi(u)$, Proposition 11 implies that $\mathrm{J}_\phi(u) \subseteq \mathbb{N}_1^{m_\phi(u)}$. Hence in all cases, where the theorem asserts that $\mathrm{J}_\phi(u) = \mathbb{N}_1^{m_\phi(u)}$, it remains to prove that $\mathbb{N}_1^{m_\phi(u)} \subseteq \mathrm{J}_\phi(u)$.

Until the end of the proof of the theorem, if $S = H(i_1, \dots, i_k)$, we set $W = W_S = KSv^+$. Put $\mathfrak{X}_1 = \{\mu \in \mathfrak{X}(V) \mid V^\mu \subseteq W\}$. The proof is based on Results 13–17. We either find a relevant subgroup $S = H(i_1, \dots, i_k)$ containing u and show that $\mathrm{J}_W(u)$ contains all required block sizes, or choose such S and weights $\mu_1, \dots, \mu_k$ satisfying the assumptions of Corollary 17 and apply Results 13 and 15–17. Observe that if $S = H(i_1, i_2) \cong A_2(K)$, $u \in S$, and $m_S = m_{i_1} + m_{i_2} < p - 1$, then Proposition 3(i) and Corollary 14 yield that

$$\mathbb{N}_1^{m_S+1} \subseteq \mathrm{J}_V(u). \tag{3}$$

These arguments are used in the relevant cases for all types of groups to obtain relatively small blocks.

Case 1. Let $G = A_n(K)$ and $m_i + m_{i+1} < p - 1$. Set $M = m_\phi(u) - m_i - m_{i+1} - 1$ and $S = H(i, i+1)$. By formula (3), $\mathbb{N}_1^{m_i+m_{i+1}+1} \subseteq \mathrm{J}_V(u)$. If $\omega = m_i\omega_i + m_{i+1}\omega_{i+1}$, we are done. Otherwise, $M > 0$. If $l \in \mathbb{N}_1^M$ and $l \leqslant \sum_{j=1}^{i-1} m_j$, there exist $s \leqslant i - 1$ and $b \leqslant m_s$ such that $l = b + \sum_{j=s+1}^{i-1} m_j$ (the latter sum is 0 if $s = i - 1$). Put $\mu_l = \omega - b\alpha_s - (b + m_{s+1})\alpha_{s+1} - \cdots - (b + m_{s+1} + \cdots + m_{i-1})\alpha_{i-1} - m_{i+1}\alpha_{i+1}$. If $\sum_{j=1}^{i-1} m_j < l \leqslant M$, there exist $t > i + 1$ and $c \leqslant m_t$ such that $l = c + \sum_{j=1}^{i-1} m_j + \sum_{j=i+2}^{t-1} m_j$ (the first sum is 0 if $i = 1$ and the second one is 0 if $t = i + 2$). Now set $\mu_l = \omega - c\alpha_t - (c + m_{t-1})\alpha_{t-1} - \cdots - (c + m_{t-1} + \cdots + m_{i+1})\alpha_{i+1} - m_1\alpha_1 - (m_1 + m_2)\alpha_2 - \cdots - (m_1 + \cdots + m_{i-1})\alpha_{i-1}$. Observe that in the first case μ_l lies in the same $W(G)$-orbit with $\omega - b\alpha_s$ and in the second one with $\omega - c\alpha_t$. As $\omega - b\alpha_s$ and $\omega - c\alpha_t \in \mathfrak{X}(V)$ by Lemma 8, $\mu_l \in \mathfrak{X}(V)$ in all cases. Now construct the subsets $\mathfrak{X}_2, \dots, \mathfrak{X}_{M+1}, \mathfrak{X}_{M+2}$ using the weights μ_l as in Corollary 17 and apply that corollary.

Case 2. Let $G = B_n(K)$ and u be a long root element. First assume that $m_1 + m_2 < p - 1$. Set $i = 1$ and $S = H(1, 2)$ and apply formula (3) to conclude that $\mathbb{N}_1^{m_1+m_2+1} \subseteq \mathrm{J}_V(u)$. If $\omega = m_1\omega_1$, the required assertion is proved. Otherwise $m_1 + m_2 < m_\phi(u) - 1$. In this case, set $M = m_\phi(u) - m_1 - m_2 - 1$. Formula (1) shows that $M \leqslant m_2 + 2m_3 + \cdots + 2m_{n-1} + m_n$. By [10, Item (b) of Corollary III.2], for each $k \in \mathbb{N}_1^{2m_2+2m_3+\cdots+2m_{n-1}+m_n}$ the set $\mathfrak{X}(V)$ contains a weight μ with $b_1(\mu) = 0$ and $b_2(\mu) = k$. Hence, for each $l \in \mathbb{N}_1^M$, there exists $\mu_l \in \mathfrak{X}(V)$ with $b_1(\mu_l) = 0$ and $b_2(\mu_l) = m_2 + l$. Observe that $\langle \mu_l, \alpha_1 \rangle = m_1 + m_2 +$

$l < p$. Using the weights μ_l, construct the subsets $\mathfrak{X}_2, \ldots, \mathfrak{X}_{M+1}, \mathfrak{X}_{M+2} \subseteq \mathfrak{X}(V)$ as in Corollary 17. Now our assertion follows from Corollary 17.

Next, assume that $m_1 + m_2 \geqslant p - 1$, but $m_i + m_{i+1} < p - 1$ for some $i < n - 1$. Then $\sum_{j=1}^{n-1} m_j \geqslant p - 1$. Set $S = H(1, \ldots, n-1)$. By Proposition 7, W is an irreducible S-module with highest weight $\sum_{j=1}^{n-1} m_j \omega_j$. The arguments of Case 1 yield $\mathrm{J}_W(u) = \mathbb{N}_1^p$. Apply Corollary 14 to complete the proof in this case.

Finally, suppose that $m_i + m_{i+1} \geqslant p - 1$ for each $i < n - 1$ and $2m_{n-1} + m_n < p - 2$. Set $S = H(n-1, n)$, $H = H(n-1)$, and assume that $u \in H$. By Proposition 3(ii) and Corollary 14, $N_1^{m_{n-1}} \subseteq \mathrm{J}_W(u)$. Set $q = p - 1 - m_{n-1}$. According to our assumptions, $m_{n-2} \geqslant q > 0$. Hence, for $l \in \mathbb{N}_1^q$, the weight $\mu_l = \omega - l\alpha_{n-2} \in \mathfrak{X}(V)$ by Lemma 8. We have $\langle \mu_l, \alpha_{n-1} \rangle \leqslant p - 1$. Now Corollary 17 completes the proof.

Case 3. Next, let $G = B_n(K)$ and u be a short root element. Set $F = 2m_{n-1} + m_n$ and $S = H(n-1, n)$ and assume that $u \in H(n)$. Since $F < p$, by Propositions 3(iii) and 7, W is a completely reducible $H(n)$-module and $\mathrm{J}_W(u) = \{k \in \mathbb{N}_1^{F+1} \mid k \equiv 1 + m_n \pmod 2\}$. If $m_j = 0$, for $j < m_{n-1}$, we have $F = m_\phi(u) - 1$ and Corollary 14 forces $J \subseteq \mathrm{J}_\phi(u)$. Otherwise, set $M = m_\phi(u) - 1 - F$. Observe that $M > 1$. By formula (1), $M \leqslant 2\sum_{j=1}^{n-2} m_j$. By [10, Item (c) of Corollary III.2], for $k \in \mathbb{N}_1^{m_1+\cdots+m_{n-1}}$ there exists a weight $\lambda \in \mathfrak{X}(V)$ with $b_n(\lambda) = 0$ and $b_{n-1}(\lambda) = k$. Hence, for even $l \in \mathbb{N}_1^M$, there exists $\mu_l \in \mathfrak{X}(V)$ such that $b_n(\mu_l) = 0$ and $\langle \mu_l, \alpha_n \rangle = 2m_{n-1} + m_n + l$. Observe that $b_{n-2}(\mu_l) \neq 0$ and $\langle \mu_l, \alpha_n \rangle < p$ for all l considered. Denote by $2t$ the maximal even $l \in \mathbb{N}_1^M$. For $2 \leqslant v \leqslant t + 1$, set $\mathfrak{X}_v = \{\lambda \in \mathfrak{X}(V) \mid \lambda = \mu_{2v-2} - b\alpha_n,\ b \in \mathbb{N}\}$. Put $\mathfrak{X}_{t+2} = \mathfrak{X}(V) \setminus (\bigcup_{j=1}^{t+1} \mathfrak{X}_j)$. Now Corollaries 14 and 17 imply that $J \subseteq \mathrm{J}_\phi(u)$. If $\langle \omega, \alpha_{m,u} \rangle < p$, $V|H(n)$ is completely reducible by Lemma 9. One easily observes that $\langle \lambda, \alpha_n \rangle \equiv m_n \pmod 2$ for all $\lambda \in \mathfrak{X}(V)$. Hence all block sizes in $\mathrm{J}_\phi(u)$ are of the same parity and $\mathrm{J}_\phi(u) = J$ by Corollary 13. Otherwise, the degree of the minimal polynomial of $\phi(u)$ is equal to p by Proposition 11. Hence $p \in \mathrm{J}_\phi(u)$. This completes the proof for $G = B_n(K)$.

Case 4. Now let $G = C_n(K)$ and u be a long root element. We have $m_{n-1} + 2m_n < p - 2$. Put $H = H(n)$ and $S = H(n-1, n)$ and assume that $u \in H$. By Proposition 3(ii), $\mathbb{N}_1^{m_{n-1}+m_n+1} \subseteq \mathrm{J}_W(u)$. If $m_j = 0$ for $j < n - 1$, Corollary 14 yields the claim. Otherwise, set $M = m_\phi(u) - m_{n-1} - m_n - 1$ and observe that $M > 0$. Formula (1) yields $M \leqslant \sum_{j=1}^{n-2} m_j$. By [10, Item (c) of Corollary III.2], for each $b \in \mathbb{N}_1^{m_1+\cdots+m_{n-1}}$, the set $\mathfrak{X}(V)$ contains a weight μ with $b_n(\mu) = 0$ and $b_{n-1}(\mu) = b$. Hence, for each $l \in \mathbb{N}_1^M$, there exists $\mu_l \in \mathfrak{X}(V)$ with $b_n(\mu_l) = 0$ and $b_{n-1}(\mu_l) = m_{n-1} + l$. Considering the orbit of μ_l under the action of $W(G)$, one easily concludes that $b_{n-2}(\mu_l) \neq 0$. Observe that $\langle \mu_l, \alpha_n \rangle \leqslant m_\phi(u) - 1 < p$. To complete the proof, apply Corollary 17.

Case 5. Next, let $G = C_n(K)$ and u be a short root element. If $m_i + m_{i+1} < p - 1$ for some $i < n - 1$, proceed as for long root elements of $B_n(K)$ in the similar case. In this situation if $m_1 + m_2 < p - 1$, the existence of weights $\mu_l \in \mathfrak{X}(V)$ with $1 \leqslant l \leqslant m_\phi(u) - m_1 - m_2 - 1$ such that $b_1(\mu_l) = 0$ and $b_2(\mu_l) = m_2 + l$ is required. This existence follows from [10, Item (d) of Corollary III.2] which asserts that for each $k \leqslant 2\sum_{j=2}^{n} m_j$ the set $\mathfrak{X}(V)$ contains a weight μ with $b_1(\mu) = 0$ and $b_2(\mu) = k$.

Now assume that $m_i + m_{i+1} \geqslant p-1$ for all $i < n-1$ and $m_{n-1} + 2m_n < p-1$. Set $H = H(n-1)$ and $S = H(n-1, n)$. By Propositions 7 and 3(iii), W is a completely reducible H-module and $\mathrm{J}_W(u) = \{k \in \mathbb{N}_1^{m_{n-1}+2m_n+1} \mid k \equiv m_{n-1}+1 \pmod 2\}$. Since $m_{n-2} + m_{n-1} \geqslant p-1$, we have $m_{n-2} \neq 0$. Therefore $v = X_{-(n-2)}v^+ \neq 0$ by Lemma 8. Set $W_1 = KSv$. It is clear that $\mathcal{X}_{n-1}$ and $\mathcal{X}_n$ fix v. Hence W_1 is an indecomposable S-module with highest weight $\tau = (m_{n-1}+1)\omega_1 + m_n\omega_2$. By [6, Lemma 2.13(b)], W_1 is a quotient of W_τ and V_τ is a quotient of W_1. Hence $\operatorname{Irr} V_\tau|H \subseteq \operatorname{Irr} W_1|H \subseteq \operatorname{Irr} W_\tau|H$. Proposition 3(iii) implies that

$$\operatorname{Irr} V_\tau|H = \operatorname{Irr} W_\tau|H = \bigl\{V_c \bigm| c \in \mathbb{N}_0^{m_{n-1}+2m_n+1},\ c \equiv m_{n-1}+1 \pmod 2\bigr\}. \tag{4}$$

Set $\mathfrak{X}_2 = \{\lambda \in \mathfrak{X}(V) \mid b_{n-2}(\lambda) = 1,\ b_j(\lambda) = 0 \text{ for } j < n-2\}$ and $U_2 = \bigoplus_{\lambda \in \mathfrak{X}_2} V^\lambda$. Obviously, U_2 is an S-module. We claim that $\mathfrak{X}_2 = \{\lambda \in \mathfrak{X}(V) \mid V^\lambda \cap W_1 \neq 0\}$. It is clear that $\mu \in \mathfrak{X}_2$ if $V^\mu \cap W_1 \neq 0$. If $\lambda \in \mathfrak{X}_2$, we have $\lambda = \omega - \alpha_{n-2} - b_{n-1}(\lambda)\alpha_{n-1} - b_n(\lambda)\alpha_n$. It suffices to prove that $\lambda' = \tau - b_{n-1}(\lambda)\alpha_{n-1} - b_n(\lambda)\alpha_n \in \mathfrak{X}(W_1)$ (as an S-module). Acting by $W(S)$, we can assume that $\langle \lambda', \alpha_i \rangle \geqslant 0$ for $i = n-1$ and n. By the Premet theorem [12], $\mathfrak{X}(V_\tau)$ coincides with the weight system of the irreducible complex representation of the group $C_2(\mathbb{C})$ with highest weight τ. Now [2, Chapter VIII, Proposition 7.5] that concerns the weight systems of complex representations implies that $\lambda' \in \mathfrak{X}(V_\tau) = \mathfrak{X}(W_1)$, as desired. Our claim on $\mathfrak{X}_2$ just proven yields that $\langle \lambda, \alpha_{n-1} \rangle \leqslant m_{n-1} + 2m_n + 1 < p$ for $\lambda \in \mathfrak{X}_2$ as this holds for $\mu \in \mathfrak{X}(V_\tau)$. Now Lemma 9 forces that U_2 is a completely reducible H-module. By Proposition 3(i) and formula (4), $\{k \in \mathbb{N}_1^{m_{n-1}+2m_n+2} \mid k \equiv m_{n-1} \pmod 2\} \subseteq \mathrm{J}_{U_2}(u)$. If $m_{n-1} + 2m_n = p-2$, set $\mathfrak{X}_3 = \mathfrak{X}(V) \setminus (\mathfrak{X}_1 \cup \mathfrak{X}_2)$. Otherwise, put $M = p - 1 - m_{n-1}$ and observe that $M > 1$. Since $m_{n-2} + m_{n-1} \geqslant p-1$, we have $m_{n-2} \geqslant M$. Hence, for each $l \in \mathbb{N}_2^M$, the weight $\mu_l = \omega - l\alpha_{n-2} \in \mathfrak{X}(V)$ by Lemma 8. For $l \in \mathbb{N}_2^M$, set $\mathfrak{X}_{l+1} = \{\lambda \in \mathfrak{X}(V) \mid \lambda = \mu_l - k\alpha_{n-1},\ k \in \mathbb{N}\}$. Put $\mathfrak{X}_{M+2} = (V) \setminus (\bigcup_{j=1}^{M+1} \mathfrak{X}_j)$. The construction of the subsets $\mathfrak{X}_t$, $1 \leqslant t \leqslant M+2$, yields that they satisfy the assumptions of Corollary 15 with respect to $n-1$. Observe that $\langle \mu_l, \alpha_{n-1} \rangle = l + m_{n-1} \leqslant M < p$. Now apply Corollaries 14 and 15 and the arguments of the proof of Corollary 17 to complete the proof for $G = C_n(K)$.

Case 6. Finally, let $G = D_n(K)$. Using the graph automorphism of G interchanging α_{n-1} and α_n if necessary, one can assume that $m_{n-1} \geqslant m_n$. First, suppose that $m_{n-2} + m_n < p-1$. Put $H = H(n)$ and $S = H(n-2, n)$. By formula (3), $\mathbb{N}_1^{m_{n-2}+m_n+1} \subseteq \mathrm{J}_V(u)$. If $\omega = m_n\omega_n$, we are done. Otherwise, set $M = m_\phi(u) - m_{n-2} - m_n - 1$ and observe that $0 < M \leqslant m_1 + 2m_2 + \cdots + 2m_{n-3} + m_{n-2} + m_{n-1}$. By [10, Item (e) of Corollary III.2], for each $k \leqslant m_1 + 2m_2 + \cdots + 2m_{n-2} + m_{n-1}$, the set $\mathfrak{X}(V)$ contains a weight λ with $b_n(\lambda) = 0$ and $b_{n-2}(\lambda) = k$. Hence, for $l \in \mathbb{N}_1^M$, there exists $\mu_l \in \mathfrak{X}(V)$ with $b_n(\mu_l) = 0$ and $b_{n-2}(\mu_l) = m_{n-2} + l$. Considering the orbit of μ_l under the action of $W(G)$, one easily concludes that $b_j(\mu_l) \neq 0$ for some $j \neq n-2$. It is clear that $\langle \mu_l, \alpha_n \rangle < p$. Now apply Corollary 17 to complete the proof in this case.

Next, assume that $m_{n-2} + m_n \geqslant p-1$. Then $m_{n-2} + m_{n-1} \geqslant p-1$ as well. Therefore our assumptions yield that $m_i + m_{i+1} < p-1$ for some $i < n-2$. Set $S = H(1, \dots, n-1)$. Naturally, $S \cong A_{n-1}(K)$. By Proposition 7, W is an irreducible S-module with highest weight $\sum_{j=1}^{n-1} m_j\omega_j$. Observe that $\sum_{j=1}^{n-1} m_j \geqslant p-1$. Now the arguments in Case 1 yield that $\mathrm{J}_W(u) = \mathbb{N}_1^p$. It remains to apply Corollary 14. The theorem is proved. $\square$

Proof of Proposition 4. The proof is based on Results 14, 16, and 17 and is quite similar to that of Theorem 2. We emphasize that here n can be equal to 2. The arguments below include this case as well. Let u be such as in the assertion of the proposition. Fix i with $m_i = c_\phi(u)$. As before, set $V = V_\omega$. If $G = A_n(K)$ or $D_n(K)$, or $i < n$ with $n > 2$, there exists $j \leqslant n$ such that α_i and α_j are linked on the Dynkin diagram of G and have the same length. If $m_i + m_j < p-1$, the assertion of the proposition follows from Theorem 2. Hence assume that $m_i + m_j \geqslant p-1$. Set $M = p-1-m_i$. Then $M \leqslant m_j$. For $l \in \mathbb{N}_0^M$, put $\mu_l = \omega - l\alpha_j$ and $\mathfrak{X}_l = \{\lambda \in \mathfrak{X}(V) \mid \lambda = \mu_l - k\alpha_i, k \in \mathbb{N}\}$. Set $\mathfrak{X}_{M+2} = \mathfrak{X}(V) \setminus (\bigcup_{j=1}^{M+1} \mathfrak{X}_j)$. Observe that $\langle \mu_l, \alpha_i \rangle < p$. Then complete the proof as in Case 1 of Theorem 2 using Lemma 16 and Corollary 17.

Now, let $G = B_n(K)$ or $C_n(K)$ and $i = n$. Set $\mathfrak{X}_1 = \{\omega - k\alpha_i \in \mathfrak{X}(V)\}$ and $M = m_\phi(u) - m_i - 1$. If $M = 0$ or α_i is short and $M = 1$, the result follows from Lemma 10 and Corollary 14. Hence assume that $M > 0$ and $M > 1$ if α_i is short. In the latter case, denote by $2s$ the maximal even integer in $\mathbb{N}_1^M$. Arguing as in Case 3 of the proof of Theorem 2, for $G = B_n(K)$ and every $l \in \mathbb{N}$ construct a weight $\nu_l \in \mathfrak{X}(V)$ with $b_n(\nu_l) = 0$ and $\langle \nu_l, \alpha_n \rangle = 2l + m_n$. Since $M \leqslant 2m_2$ for $G = C_2(K)$ and $i = 1$, Lemma 8 shows that such weights ν_l exist in this case as well. For $G = C_n(K)$ and $i = n$, argue as in Case 4 of the proof cited above and for each $l \in \mathbb{N}_1^M$ construct a weight $\nu_l \in \mathfrak{X}(V)$ with $b_n(\nu_l) = 0$ and $\langle \nu_l, \alpha_n \rangle = l + m_n$. Then complete the proof for all three cases considered in this paragraph using the schemes proposed in Cases 3 and 4 of the proof of Theorem 2 with the weights ν_l instead of μ_l. In this case, we do not need to consider $b_{n-1}(\nu_l)$. $\square$

4. Regular unipotent elements of a subgroup of type A_2

Lemma 18. *Let $p > 2$, $\Gamma = A_2(K)$, $\mu = a_1\omega_1 + a_2\omega_2$ be a dominant weight of Γ, and $V = V_\mu$. Assume that $\Pi \subseteq \Gamma$ is a Zariski closed simple subgroup of type A_1 containing a regular unipotent element. Then $2a_1 + 2a_2 \in \operatorname{Irr} V|\Pi$. If $a = \lambda_\Pi$ for $\lambda \in \mathfrak{X}(V)$, then a is even and $a \leqslant 2a_1 + 2a_2$.*

Proof. The existence of such subgroup Π is well known and follows, for instance, from the construction of the irreducible representation of $A_1(K)$ with highest weight 2. It is also well known (see, for instance, [4, Chapter 5]) that all the labels on the labelled Dynkin diagram of a regular unipotent element are equal to 2 and hence there exist maximal tori $T_\Pi \subseteq \Pi$ and $T \subseteq \Gamma$ such that $T_\Pi \subseteq T$ and the homomorphism $\tau : \mathfrak{X}(\Gamma) \to \mathbb{Z}$ determined by the restriction of weights from T to T_Π maps α_1 and α_2 to 2. Since $\omega_1 = (2\alpha_1 + \alpha_2)/3$ and $\omega_2 = (\alpha_1 + 2\alpha_2)/3$ [1, Table I], this forces $\tau(\omega_i) = 2$ for $i = 1, 2$. Therefore $\tau(\lambda)$ is even for any $\lambda \in \mathfrak{X}(V)$ and $\tau(\lambda) \leqslant \tau(\mu) = 2a_1 + 2a_2$. This implies the second part of the assertion of the lemma. Furthermore, one can see that a nonzero vector of V^μ generates an indecomposable Π-module with highest weight $2a_1 + 2a_2$ and therefore $2a_1 + 2a_2 \in \operatorname{Irr} V|\Pi$. This completes the proof. $\square$

For $\lambda = a_1\omega_1 + a_2\omega_2 \in \mathfrak{X}(A_2(K))$, set $s(\lambda) = a_1 + a_2$.

Lemma 19. *Let $n > 3$ and $G = A_n(K)$. Assume that $\mu = a_1\omega_1 + \cdots + a_n\omega_n$ is a dominant weight of G and $a_1 + \cdots + a_n < p - 2$. Set $V = V_\mu$ and $S = H(i, i+1) \subseteq G$. Then*

$$\operatorname{Irr} V|S = \left\{ V_\lambda \;\middle|\; \lambda = x_1\omega_1 + x_2\omega_2,\ x_1 \leqslant \sum_{i=1}^{n-1} a_i,\ x_2 \leqslant \sum_{i=2}^{n} a_i,\ x_1 + x_2 \leqslant \sum_{i=1}^{n} a_i \right\},$$

and $V|S$ is completely reducible for $n = 4$. In any case, $s(\lambda) \leqslant a_1 + \cdots + a_n$ if $V_\lambda \in \operatorname{Irr} V|S$.

Proof. The factors are described in [11, Theorem 1.3]. The complete reducibility of the restriction for $n = 4$ follows from [3, Theorem 6.2]. □

Proof of Theorem 5. Set $V = V_\omega$, $S = H(i, i+1, i+2, i+3)$, $H = H(\alpha_i, \alpha_{i+1} + \alpha_{i+2} + \alpha_{i+3})$, $W = KSv^+$, $L = m_i + m_{i+1} + m_{i+2} + m_{i+3}$, and $M = \min(m_1 + \cdots + m_n, (p-1)/2)$.

One can assume that u is a regular unipotent element of H. Let $\Pi \subseteq H$ be a Zariski closed subgroup of type A_1 containing u (the existence of such subgroup was discussed in the proof of Lemma 18). By Lemma 19, $s(\lambda) \leqslant m_1 + \cdots + m_n$ if $V_\lambda \in \operatorname{Irr} V|H$. Now Lemma 18 implies that $a \leqslant 2\sum_{j=1}^{n} m_j$ for each $a \in \operatorname{Irr} V|\Pi$. Since $|u| = p$, this forces $\mathrm{J}_V(u) \subseteq \mathbb{N}_1^g$. Furthermore, by Proposition 7 and Lemma 19, W is a completely reducible H-module with irreducible components V_λ with highest weights $\lambda = a_1\omega_1 + a_2\omega_2$, where (a_1, a_2) runs over all pairs of integers with $a_1 \leqslant m_i + m_{i+1} + m_{i+2}$, $a_2 \leqslant m_{i+1} + m_{i+2} + m_{i+3}$, and $a_1 + a_2 \leqslant L$. Hence, for each $a \in \mathbb{N}_0^L$, the set $\operatorname{Irr} W|H$ contains a factor V_λ with $s(\lambda) = a$. By Lemma 18, $2a \in \operatorname{Irr}(V_\lambda|\Pi)$ and $b \leqslant 2a$ for each $b \in \mathfrak{X}(V_\lambda|\Pi)$. Now Lemma 9 implies that V_λ is a completely reducible Π-module if $s(\lambda) \leqslant (p-1)/2$. Set $L_1 = \min(L, (p-1)/2)$. Corollary 13 and Lemma 12 yield that $2j + 1 \in \mathrm{J}_W(u)$ for $j \in \mathbb{N}_0^{L_1}$. If $M \leqslant L$, Corollary 14 forces that $I \subseteq \mathrm{J}_V(u)$. Now assume that $M > L$. Set $\mathfrak{X}_1 = \{\lambda \in \mathfrak{X}(V) \mid V^\lambda \subseteq W\}$ and $M' = M - L$. Let $l \in \mathbb{N}_1^{M'}$. First suppose that $l \leqslant \sum_{j=1}^{i-1} m_j$. Then there exist $s < i$ and $b \leqslant m_s$ such that $l = b + \sum_{j=s+1}^{i-1} m_j$ (the latter sum is 0 if $s = i-1$). Set $\mu_l = \omega - b\alpha_s - (b + m_{s+1})\alpha_{s+1} - \cdots - (b + m_{s+1} + \cdots + m_{i-1})\alpha_{i-1}$. Now assume that $\sum_{j=1}^{i-1} m_j < l \leqslant M'$. Then there exist $t > i+3$ and $c \leqslant m_t$ such that $l = c + \sum_{j=1}^{i-1} m_j + \sum_{j=i+4}^{t-1} m_j$ (the first sum is 0 if $i = 1$ and the second one is 0 if $t = i+4$). In this case, put $\mu_l = \omega - c\alpha_t - (c + m_{t-1})\alpha_{t-1} - \cdots - (c + m_{t-1} + \cdots + m_{i+4})\alpha_{i+4} - m_1\alpha_1 - (m_1 + m_2)\alpha_2 - \cdots - (m_1 + \cdots + m_{i-1})\alpha_{i-1}$ (if $i = 1$, the last term in this formula is $k\alpha_5$ with $k \in \mathbb{N}$). Using Lemma 8 and arguing as in Case 1 of the proof of Theorem 2, one can deduce that $\mu_l \in \mathfrak{X}(V)$ for all $l \in \mathbb{N}_1^{M'}$. Now set $\mathfrak{X}_{l+1} = \{\lambda \in \mathfrak{X}(V) \mid \lambda = \mu_l - \sum_{j=i}^{i+3} b_j\alpha_j\}$ for $l \in \mathbb{N}_1^{M'}$, $\mathfrak{X}_{M'+2} = \mathfrak{X}(V) \setminus (\bigcup_{k=1}^{M'+1} \mathfrak{X}_k)$, and $U_k = \sum_{\mu \in \mathfrak{X}_k} V^\mu$. Arguing as in the proof of Corollary 15, one easily observes that $V = \bigoplus_{k=1}^{M'+2} U_k$, U_k are S-modules and hence H-modules and Π-modules. We claim that $s(\mu_H) \leqslant L + l \leqslant (p-1)/2$ for each $\mu \in \mathfrak{X}(U_{l+1})$ (the weight system of the S-module U_{l+1}), $1 \leqslant l \leqslant M'$. Indeed, set $\nu = \alpha_i + \alpha_{i+1} + \alpha_{i+2} + \alpha_{i+3}$ and observe that ν is a root of H and that $s(\mu_H) = \langle \mu, \nu \rangle$. Since $\langle \alpha_j, \nu \rangle \geqslant 0$ for all $i \leqslant j \leqslant i+3$, we have $\langle \mu, \nu \rangle \leqslant \langle \mu_l, \nu \rangle = L + l$, which yields the claim. As $\langle \mu_l, \nu \rangle = L + l$, we have $s((\mu_l)_H) = L + l$. Now Lemma 18 and Corollary 13 imply that $d < p$ for all $d \in \operatorname{Irr}(U_k|\Pi)$, $2(L + l) \in \operatorname{Irr}(U_{l+1}|\Pi)$, and $2(L + l) + 1 \in \mathrm{J}_{U_{l+1}}(u)$. Now apply Lemma 12 and conclude that $I \subseteq \mathrm{J}_V(u)$.

If $\sum_{j=1}^{n} m_j \leqslant (p - 1)/2$, we have $\langle \mu, \nu \rangle \leqslant (p - 1)/2$ for each $\mu \in \mathfrak{X}(V)$ since $\langle \mu, \alpha \rangle \leqslant \langle \omega, \alpha_{m,u} \rangle = \sum_{j=1}^{n} m_j$ for each root α. Hence $s(\mu_H) \leqslant (p - 1)/2$ for all such μ. Now Lemma 18 yields that $a \leqslant p - 1$ for all $a \in \mathfrak{X}(V|\Pi)$ and that in our case all weights of $V|\Pi$ are even integers $\leqslant 2\sum_{j=1}^{n} m_j$. It remains to apply Corollary 13. The theorem is proved. $\square$

Proof of Proposition 6. The proof is similar to that of Theorem 5. Keep the notation V and U_l. Fix i with $m_i + m_{i+1} = m$. Set $S = H(i, i + 1)$ and $U_1 = KSv^+$. Then $S \cong A_2(K)$. We can assume that $u \in S$. Fix an A_1-subgroup Π containing u as in Theorem 5. For $\lambda \in \mathfrak{X}(S)$, define $s(\lambda)$ as before. By Proposition 7, U_1 is an irreducible S-module with highest weight $m_i\omega_1 + m_{i+1}\omega_2$. Now Lemmas 9 and 18 imply that U_1 is a completely reducible Π-module and has an irreducible Π-component with highest weight $2m$. Therefore $2m + 1 \in \mathrm{J}_{U_1}(u)$ by Lemma 10. If $m = M$, we are done. Hence assume that $M > m$ and put $M_1 = M - m$. Set $\alpha = \alpha_i + \alpha_{i+1}$. One easily observes that $\langle \mu, \alpha_i \rangle + \langle \mu, \alpha_{i+1} \rangle = \langle \mu, \alpha \rangle = s(\mu_S)$ for $\mu \in \mathfrak{X}(V)$. Arguing as in the proof of Theorem 5, for each $l \in \mathbb{N}_1^{M_1}$ we construct a weight $\mu_l \in \mathfrak{X}(V)$ such that $b_i(\mu_l) = b_{i+1}(\mu_l) = 0$ and $\langle \mu_l, \alpha \rangle = m + l$.

Next, set

$$\mathfrak{X}_1 = \{\lambda \in \mathfrak{X}(V) \mid V^\lambda \subseteq U_1\}, \qquad \mathfrak{X}_k = \{\lambda \in \mathfrak{X}(V) \mid \lambda = \mu_{k-1} - c\alpha_i - d\alpha_{i+1}\}$$

for $2 \leqslant k \leqslant M_1 + 1$, and $\mathfrak{X}_{M_1+2} = \mathfrak{X}(V) \setminus (\bigcup_{j=1}^{M_1+1} \mathfrak{X}_j)$. For $2 \leqslant k \leqslant M_1 + 2$ put $U_k = \sum_{\mu \in \mathfrak{X}_k} V^\mu$. Arguing as in the proof of Corollary 15, we can conclude that U_k is an S-module and $V = \bigoplus_{j=1}^{M_1+2} U_k$. If $\lambda \in \mathfrak{X}_k$ with $2 \leqslant k \leqslant M_1 + 1$, we have $\langle \lambda, \alpha \rangle \leqslant \langle \mu_{k-1}, \alpha \rangle = m + k - 1 \leqslant (p - 1)/2$. Hence $s(\lambda_S) \leqslant (p - 1)/2$ and $\lambda_\Pi \leqslant p$ by Lemma 18. Now Lemmas 9 and 18 and Corollary 13 imply that U_k is a completely reducible Π-module with the maximal weight $2(m + k - 1)$ and hence $1 + 2(m + k - 1) \in \mathrm{J}_{U_k}(u)$. Lemma 12 completes the proof. $\square$

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