Stable Invariant Lagrangian Subspaces: Factorization of Symmetric Rational Matrix Functions and Other Applications

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ABSTRACT

Several applications of earlier results by the authors concerning various notions of stability of invariant lagrangian subspaces are studied. The applications include stability of symmetric minimal factorizations of real symmetric rational matrix functions, stability of factorizations of certain classes of symmetric matrix polynomials, and stably well-posed matricial boundary value problems with symmetries. An interpretation of many stability results in terms of equilibria of Lie group actions on subspaces is given as well.

1. INTRODUCTION AND PRELIMINARIES

1.1. Introduction

In this paper we present some applications of the results obtained in the papers [21, 22]. The applications concern several topics. The first topic is
symmetric factorizations of real symmetric rational matrix functions (Section 2). For example, let \( W(\lambda) \) be an \( n \times n \) real rational matrix function such that

\[
W(\lambda) = \left[ W(-\lambda) \right]^T \geq 0
\]  

(1.1.1)

for all pure imaginary \( \lambda \) that are not poles of \( W(\lambda) \) (here and everywhere in the paper we use \( X \geq 0 \) to denote positive semidefinite Hermitian matrices \( X \), and the superscript \( T \) stands for the transpose). Such functions play a significant role in the theory of linear control systems with symmetries and circuit theory; see, e.g., [6, 7, 30, 1, 31. We consider then minimal factorizations of \( W(\lambda) \) of the form

\[
W(\lambda) = L(-\lambda)L(\lambda)
\]  

(1.1.2)

(the definition of minimal factorizations is given in Section 2) and characterize those factorizations (1.1.2) which are stable in a certain sense. By stability we mean that any rational real matrix function \( \tilde{W}(\lambda) \) with the property analogous in (1.1.1) and which is sufficiently close to \( W(\lambda) \) will have a minimal factorization

\[
\tilde{W}(\lambda) = \tilde{L}(-\lambda)^T\tilde{L}(\lambda)
\]  

(1.1.3)

with the factor \( \tilde{L}(\lambda) \) as close as we wish to \( L(\lambda) \). For the precise definitions of various notions of stability we refer the reader again to Section 2. We consider other symmetries as well, and also the special case of real symmetric matrix polynomials. The analysis of stability of symmetric factorizations of real rational matrix functions \( W(\lambda) \) with the property that \( W(\lambda) \geq 0 \) for all real \( \lambda \) that are not poles of \( W \) was done in [23].

Our second topic concerns certain matricial boundary value problems with symmetries (Section 3):

\[
T \frac{d\psi(t)}{dt} = -A\psi(t),
\]

\[
P\psi(0) = \varphi; \quad \psi(t) \text{ is bounded on } [0, \infty).
\]  

(1.1.4)

Here \( T \) and \( A \) are real matrices with \( T \) symmetric and invertible, \( P \) is a projection, and \( \varphi \in \text{Im } P \) is a given vector. Also, an inverse symmetry is assumed, i.e., there is a real matrix \( J \) such that \( J^2 = I, JT = -JT, JA = A^TJ^T \). The original motivation to study such boundary value problems comes from transport theory (see [31, 15, 19]; also [25] for a finite dimensional setup). We describe the stably well-posed problems of type (1.1.4), i.e., those that are
themselves well posed [by this we mean that for every \( \varphi \in \text{Im} \, P \) there is unique solution \( \psi(t) \)] and are such that every analogous problem with coefficients close to \( T, A, \) and \( P \) respectively is also well posed.

Finally, in Section 4 we offer an interpretation of many stability results in terms of equilibria of Lie group actions on subspaces.

The main technical tools of this paper as well as of [21–23] are canonical forms of pairs of real symmetric or skew-symmetric matrices. For the reader's convenience, we write down these canonical forms in the next subsection.

We conclude the introduction with some notation and conventions used throughout the paper. For a complex \( n \times n \) matrix \( X \) the partial multiplicities of \( X \) corresponding to its eigenvalue \( \lambda_0 \) are, by definition, the sizes of Jordan blocks with eigenvalue \( \lambda_0 \) that appear in the Jordan normal form of \( X \). This definition applies in particular to real \( n \times n \) matrices. Thus, for example, the matrix

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix}
\]

has two eigenvalues \( i \) and \(-i\) with partial multiplicities 1, 1 for each. The set of all eigenvalues of \( A \) is called the spectrum of \( A \) and will be denoted \( \sigma(A) \).

Given a real \( n \times n \) matrix \( A \) and its real eigenvalue \( \lambda_0 \), we call \( \text{Ker}(A - \lambda_0 I)^n \) the root subspace of \( A \) corresponding to \( \lambda_0 \). We shall often denote it by \( R(A, \lambda_0) \). For a pair of complex conjugate nonreal eigenvalues \( \lambda_0 - \mu \pm iv \) and \( \lambda_0 - \mu \pm iv \) of \( A \), the root subspace of \( A \) corresponding to the pair \( \{\lambda_0, \lambda_0\} \) is, by definition, \( \text{Ker}(A - \lambda_0 I)^n + \text{Ker}(A - \lambda_0 I)^n \) considered as a subspace in \( R^n \). We shall often denote it by \( R(A, \mu \pm iv) \). (Here and elsewhere \( R^n \) is the \( n \)-dimensional real vector space of column vectors.)

Given a symmetric part \( \sigma_0 \) of \( \sigma(A) \) (i.e., \( \lambda_0 \in \sigma_0 \) implies \( \overline{\lambda_0} \in \sigma_0 \)), the sum of the root subspaces of \( A \) corresponding to real eigenvalues or pairs of nonreal complex conjugate eigenvalues in \( \sigma_0 \) is called the spectral subspace of \( A \) corresponding to \( \sigma_0 \); it is a subspace in \( R^n \). The zero subspace is designated \((0)\). We use the notation \( e_k \) for the vector all whose coordinates (except for the \( k \)th) are zeros and whose \( k \)th coordinate is 1 (the dimension of \( e_k \) will be clear from the context). We use \( I_m \) (or \( I \)) to denote the \( m \times m \) identity matrix.

1.2. Canonical Forms

We describe here the canonical forms of certain pairs of matrices. First, we introduce some notation. For given \( \xi = \pm 1, \eta = \pm 1 \) let \( L_n(\xi, \eta) \) be the
class of all pairs \((A, H)\) of \(n \times n\) real matrices such that \(H\) is invertible, \(H^T = \xi H\), \(HA = \eta A^T H\). To avoid the trivial cases when \(L_n(\xi, \eta)\) is empty it will be assumed that \(n\) is even whenever \(\xi = -1\). Throughout the paper we use \(J_k(\lambda)\) to designate the lower triangular \(k \times k\) Jordan block with eigenvalue \(\lambda\):

\[
J_k(\lambda) = \begin{bmatrix}
\lambda & 0 & & & \\
1 & \lambda & 0 & & \\
& 1 & \ddots & \ddots & \\
& & \ddots & \ddots & \lambda \\
0 & & & 1 & \lambda
\end{bmatrix}
\]

For real numbers \(a, b, c, d\) also denote

\[
J_k[\begin{bmatrix} a & b \\ c & d \end{bmatrix}] = \begin{bmatrix}
a & b & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
c & d & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
1 & 0 & a & b & \cdots & 0 & 0 & 0 & 0 \\
0 & 1 & c & d & \cdots & 0 & 0 & 0 & 0 \\
& & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 & a & b \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & c & d
\end{bmatrix}
\]

the size of \(J_k[\begin{bmatrix} a & b \\ c & d \end{bmatrix}]\) is \(2k \times 2k\). The block diagonal matrix

\[
\begin{bmatrix}
Z_1 & 0 & \cdots & 0 \\
0 & Z_2 & \cdots & 0 \\
& & \ddots & \ddots \\
0 & 0 & \cdots & Z_p
\end{bmatrix}
\]

will be denoted \(Z_1 \oplus \cdots \oplus Z_p = \bigoplus_{i=1}^{p} Z_i\) or \(\text{diag}[Z_1, \ldots, Z_p]\). The block antidiagonal matrix

\[
\begin{bmatrix}
0 & 0 & \cdots & 0 & Z_1 \\
0 & 0 & \cdots & Z_2 & 0 \\
& & \ddots & \ddots & \ddots \\
Z_p & 0 & \cdots & 0 & 0
\end{bmatrix}
\]

will be denoted \(\text{antidiag}[Z_1, \ldots, Z_p]\).
All the following results on canonical forms can be found, e.g., in [5, 28, 29].

**Theorem 1.2.1.** Let \((A, H) \in L_\alpha(1, 1)\). Then there exists a real invertible matrix \(S\) such that the matrices \(A_1 = S^{-1}AS, H_1 = S^*HS\) have the following structure:

\[
A_1 = \bigoplus_{j=1}^{p} J_k(\lambda_j) \oplus \bigoplus_{j=1}^{q} J_{k_{p+j}}\left[ \begin{array}{cc} \lambda_{p+j} & \mu_{p+j} \\ -\mu_{p+j} & \lambda_{p+j} \end{array} \right],
\]

where \(\lambda_1, \ldots, \lambda_q\) are real numbers and \(\mu_{p+1}, \ldots, \mu_q\) are positive numbers;

\[
H_1 = \bigoplus_{j=1}^{p} \epsilon_j E_{k_j} \oplus \bigoplus_{j=1}^{q} E_{2k_{p+j}},
\]

where \(E_k = \text{antidiag}[1, 1, \ldots, 1]\) is a \(k \times k\) matrix, and \(\epsilon_j = \pm 1\) (for \(j = 1, \ldots, p\)). Moreover, the pairs of blocks \((J_k(\lambda_j), \epsilon_j E_{k_j})\) for \(j = 1, \ldots, p\) and

\[
\left( J_{k_j}\left[ \begin{array}{cc} \lambda_j & \mu_j \\ -\mu_j & \lambda_j \end{array} \right], E_{2k_{j}} \right)
\]

for \(j = p+1, \ldots, p+q\) are uniquely (up to permutations) determined by the pair \((A, H)\).

To describe the canonical pairs for \(L_n(-1, -1)\) and \(L_n(1, -1)\), it is convenient to introduce the matrices \(F_j = \text{antidiag}[1, -1, \ldots, (-1)^{j-1}]\) of size \(j \times j\) and \(G_j = \text{antidiag}[F_{j_1}^{-1}, -F_{j_2}^{-1}, \ldots, (-1)^{j-1}F_{j_2}^{-1}]\) of size \(2j \times 2j\). So \(F_j\) is symmetric for \(j\) odd and antisymmetric for even \(j\), while \(G_j\) is symmetric for all \(j\).

**Theorem 1.2.2.** Let \((A, H) \in L_n(-1, -1)\). Then there is a real invertible \(S\) such that \(S^{-1}AS\) and \(S^*HS\) are block diagonal matrices

\[
S^{-1}AS = \bigoplus_{j=1}^{q} A_j, \quad S^*HS = \bigoplus_{j=1}^{q} H_j
\] (1.2.1)
with the diagonal blocks \((A_i, H_i)\) of one of the following four types:

**Type 1.**

\[
A_i = \bigoplus_{j=1}^{p} J_{2n_j}(0) \oplus \bigoplus_{j=1}^{r} \left\{ \left(J_{2n_{p+j}}(0) \oplus \left[-J_{2n_{p+j}+1}(0)\right]^T\right) \right\},
\]

\[
H_i = \bigoplus_{j=1}^{p} \kappa_j F_{2n_j} \bigoplus_{j=1}^{r} \left[ \begin{array}{cc} 0 & I_{2n_{p+j}+1} \\ -I_{2n_{p+j}+1} & 0 \end{array} \right],
\]

where \(\kappa_j\) is 1 or \(-1\) (for \(j = 1, \ldots, p\)).

**Type 2.**

\[
A_i = \bigoplus_{j=1}^{p} \left\{ \left(J_n(a) \oplus \left[-J_n(a)\right]^T\right) \right\},
\]

\[
H_i = \bigoplus_{j=1}^{p} \left[ \begin{array}{cc} 0 & I_{n_j} \\ -I_{n_j} & 0 \end{array} \right],
\]

where \(a > 0\).

**Type 3.**

\[
A_i = \bigoplus_{j=1}^{p} J_{n_j} \left[ \begin{array}{cc} 0 & b \\ -b & 0 \end{array} \right] \quad (b > 0),
\]

\[
H_i = \bigoplus_{j=1}^{p} \kappa_j \text{antidiag}\left[ Z^{n_j}, -Z^{n_j}, \ldots, (-1)^{n_j-1}Z^{n_j} \right],
\]

where

\[
Z = \left[ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right]
\]

and \(\kappa_j\) is +1 or −1.

**Type 4.**

\[
A_i = \bigoplus_{j=1}^{p} \left\{ J_{n_j} \left[ \begin{array}{cc} a & b \\ -b & a \end{array} \right] \oplus \left[-J_{n_j} \left[ \begin{array}{cc} a & b \\ -b & a \end{array} \right]^T\right] \right\} \quad (a, b > 0),
\]

\[
H_i = \bigoplus_{j=1}^{p} \left[ \begin{array}{cc} 0 & I_{2n_j} \\ -I_{2n_j} & 0 \end{array} \right].
\]
The pairs of blocks \((A_1, H_1), \ldots, (A_q, H_q)\) in (1.2.1) are uniquely determined by \((A, H)\) up to a permutation.

**Theorem 1.2.3.** Given \((A, H) \in L_n(1, -1)\), there is an invertible real matrix \(S\) such that

\[
S^{-1}AS = A_1 \oplus \cdots \oplus A_q, \quad S^T HS = H_1 \oplus \cdots \oplus H_q, \quad (1.2.2)
\]

where the pairs of matrices \((A_i, H_i)\) are one of the following four types:

**Type 1.**

\[
A_i = \bigoplus_{j=1}^{p} J_{2n_j+1}(0) \oplus \bigoplus_{j=1}^{q} \left[ J_{n_{p+j}}(0) \oplus -J_{n_{p+j}}(0)^T \right],
\]

\[
H_i = \bigoplus_{j=1}^{p} \kappa_j F_{2n_j+1} \bigoplus_{j=1}^{q} \left[ \begin{array}{cc} 0 & I_{n_{p+j}} \\ I_{n_{p+j}} & 0 \end{array} \right],
\]

where \(n_{p+1}, \ldots, n_{p+q}\) are even integers and \(\kappa_1, \ldots, \kappa_p\) are \(\pm 1\).

**Type 2.**

\[
A_i = \bigoplus_{j=1}^{p} \left[ J_{n_j}(a) \oplus -J_{n_j}(a)^T \right],
\]

\[
H_i = \bigoplus_{j=1}^{p} \left[ \begin{array}{cc} 0 & I_{n_j} \\ I_{n_j} & 0 \end{array} \right],
\]

where \(a > 0\).

**Type 3.**

\[
A_i = \bigoplus_{j=1}^{p} J_{n_j} \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix},
\]

\[
H_i = \bigoplus_{j=1}^{p} \kappa_j G_{n_j},
\]
where \( b > 0 \) and \( \kappa_1, \ldots, \kappa_p \) are \( \pm 1 \);

\[
A_i = \bigoplus_{j=1}^{p} \left\{ J_{n_j} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \oplus - \left( J_{n_j} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \right)^T \right\},
\]

\[
H_i = \bigoplus_{j=1}^{p} \begin{bmatrix} 0 & I_{2n_j} \\ I_{2n_j} & 0 \end{bmatrix},
\]

where \( a, b > 0 \).

(Of course, the numbers \( a, b \) as well as the signs \( \kappa_j \) and the numbers \( p, q \) may be different for different pairs of matrices \( (A_i, H_i) \).)

Moreover the form (1.2.2) is uniquely determined by the matrices \( A \) and \( H \), up to simultaneous permutations of pairs of blocks \( (A_i, H_i) \).

Finally, we consider the class \( L_n(-1,1) \).

**Theorem 1.2.4.** Let \((A, H) \in L_n(-1,1)\); then there exists an invertible real matrix \( S \) such that \((S^{-1}AS, S^THS)\) is a block diagonal sum of blocks \((A_i, H_i)\) of the following types:

**Type 1.**

\[
A_i = \bigoplus_{j=1}^{p} \left[ J_{n_j}(a) \oplus J_{n_j}(a)^T \right],
\]

\[
H_i = \bigoplus_{j=1}^{p} \begin{bmatrix} 0 & I_{n_j} \\ -I_{n_j} & 0 \end{bmatrix},
\]

where \( a \) is real.

**Type 2.**

\[
A_i = \bigoplus_{j=1}^{p} \left( J_{n_j} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \oplus J_{n_j} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}^T \right) \quad (b > 0),
\]

\[
H_i = \bigoplus_{j=1}^{p} \begin{bmatrix} 0 & I_{2n_j} \\ -I_{2n_j} & 0 \end{bmatrix},
\]
Again, the pairs of blocks \((A_i, H_i)\) which appear in \((S^{-1}AS, S^T HS)\) are uniquely determined (up to permutation) by the pair \((A, H)\).

1.3. Invariant Lagrangian Subspaces and Their Stability

Given \((A, H) \in L_n(\xi, \eta)\), let \(J(A, H)\) be the class of all \(A\)-invariant \(H\)-lagrangian subspaces. In other words, a subspace \(M \subset \mathbb{R}^n\) belongs to the class \(J(A, H)\) if and only if \(Ax \in M\) for every \(x \in M\) (i.e., \(M\) is \(A\)-invariant), \(x^THy = 0\) for all \(x, y \in M\), and \(\dim M = n/2\) (i.e., \(M\) is \(H\)-lagrangian). The term "lagrangian" is usually used in the literature in the case \(\xi = -1\), but we shall extend this terminology also to the case \(\xi = 1\). Obviously, the evenness of \(n\) is necessary to ensure that \(J(A, H) \neq \emptyset\) (but not always sufficient). The results on existence of \(A\)-invariant \(H\)-lagrangian subspaces, as well as description of the sets \(J(A, H)\), are found in [21–23].

To study the stability of invariant lagrangian subspaces we need a metric on the set of subspaces in \(\mathbb{R}^n\). Such a standard metric is the gap, defined as follows: Given two subspaces \(M \subset \mathbb{R}^n\) and \(N \subset \mathbb{R}^n\), the gap \(\theta(M, N)\) is

\[
\theta(M, N) = \|P_M - P_N\|
\]

where \(P_M\) (respectively \(P_N\)) is the orthogonal projection on \(M\) (respectively \(N\)), understood with respect to the standard inner product \(\langle x, y \rangle = \sum_{i=1}^n x_i y_i\) for \(x = [x_1, x_2, \ldots, x_n]^T\), \(y = [y_1, y_2, \ldots, y_n]^T \in \mathbb{R}^n\). The norm \(\|\cdot\|\) is the matrix norm induced by the euclidean vector norm. We refer the reader to [2, 9, 12] for more information about the gap.

Let \((A, H) \in L_n(\xi, \eta)\). A subspace \(M \subset J(A, H)\) is called conditionally stable if for every \(\varepsilon > 0\) there is \(\delta > 0\) such that for any pair \((B, G) \in L_n(\xi, \eta)\) with \(J(B, G) \neq \emptyset\) and

\[
\|G - H\| + \|B - A\| < \delta
\]

there is \(M' \in J(B, G)\) satisfying

\[
\theta(M', M) < \varepsilon.
\]

If in the above definition "\(J(B, G) \neq \emptyset\) and" is omitted, we obtain the definition of unconditionally stable \(M\). A subspace \(M \in J(A, H)\) is called Lipschitz conditionally stable if there are positive constants \(\varepsilon\) and \(K\) such that for every pair \((B, G) \in L_n(\xi, \eta)\) with \(J(B, G) \neq \emptyset\) and

\[
\|G - H\| + \|B - A\| < \varepsilon
\]
there is $M' \in \mathcal{J}(B,G)$ satisfying
\[
\theta(M', M) \leq K(\|G - H\| + \|B - A\|).
\]

If "\(\mathcal{J}(B,G) \neq \emptyset\) and" is omitted, the subspace \(M\) will be called Lipschitz unconditionally stable.

These and other classes of stable invariant lagrangian subspaces were described in [21–23]. In the framework of complex matrices the stability of invariant subspaces with various definiteness properties with respect to \(H\) was studied in [24, 27, 10], and applications given [26, 27, 10].

2. FACTORIZATION OF REAL RATIONAL MATRIX FUNCTIONS WITH SYMMETRIES

2.1. General Framework

The subject matter of this section is real rational \(n \times n\) matrix functions \(W(\lambda)\). In other words, every entry in the \(n \times n\) matrix \(W(\lambda)\) is a quotient of two polynomials in the complex variable \(\lambda\) with real coefficients. It will be always assumed that the real rational matrix functions we deal with are regular, i.e., have not identically zero determinant. We will often also assume that \(W(\lambda)\) has no pole at infinity. Every such real rational matrix function \(W(\lambda)\) admits a representation

\[
W(\lambda) = D + C(\lambda I - A)^{-1}B, \tag{2.1.1}
\]

where \(C, A,\) and \(B\) are real matrices of sizes \(n \times p, p \times p,\) and \(p \times n,\) respectively, for some \(p.\) The matrix \(D = W(\infty)\) is determined uniquely by \(W(\lambda).\) Representations of the type (2.1.1) are called realizations of \(W(\lambda);\) we refer to the books [2, 11, 12, 17] for the theory of realizations and the various facts on realizations to be used later on. In particular, a realization (2.1.1) is called minimal if the size \(p\) is the smallest among all realizations. If (2.1.1) and

\[
W(\lambda) = D + \tilde{C}(\lambda I - \tilde{A})^{-1}\tilde{B}
\]

are two minimal realizations of the same \(W(\lambda),\) then there exists a unique real invertible matrix \(S\) such that

\[
\tilde{C} = CS, \quad \tilde{A} = S^{-1}AS, \quad \tilde{B} = S^{-1}B.
\]
We now assume that the real rational matrix function $W(\lambda)$ has certain symmetry properties. Precisely, we assume that for all $\lambda \in \mathbb{C}$ which are not poles of $W$ the equality

$$W(\lambda) = \xi [W(\eta \lambda)]^* = \xi [W(\eta \lambda)]^T \quad (2.1.2)$$

holds, where $\xi$ and $\eta$ are fixed numbers $\pm 1$. In realization terms [given the realization $(2.1.1)$] this means

$$D + C(\lambda I - A)^{-1}B = \xi D^T + \xi\eta B^T(\lambda I - \eta A^T)^{-1}C^T.$$

Hence

$$D = \xi D^T, \quad (2.1.3)$$

and by minimality there exists a unique invertible $H$ such that

$$HA = \eta A^TH, \quad HB = C^T, \quad \xi\eta B^TH = C. \quad (2.1.4)$$

Taking complex conjugates shows $H$ is real, and taking transposes we obtain

$$\xi\eta H^T = H. \quad (2.1.5)$$

In other words, $(A, H) \in L_n(\xi, \eta)$. If $D$ is invertible (a hypothesis we assume from now on), then putting

$$A^x = A - BD^{-1}C,$$

it is easily seen that also $(A^x, H) \in L_n(\xi, \eta)$. The importance of this observation is apparent from the formula for the inverse matrix function $W(\lambda)^{-1}$:

$$W(\lambda)^{-1} = D^{-1} - C(\lambda I - A^x)^{-1}BD^{-1}.$$ 

In the sequel we shall be studying factorizations of $W(\lambda)$, and the following proposition will be important for that purpose. We denote by $S \perp T$ the direct sum of subspaces $S, T \subset \mathbb{R}^n$; thus, $S \perp T = \mathbb{R}^n$ means that $S + T = \mathbb{R}^n$ and $S \cap T = (0)$. 
Proposition 2.1.1. Suppose $\xi = 1$, and let $M \in J(A, H)$ and $M^* \in J(A^*, H)$. Then $R^n = M + M^*$.

This can be seen by considering the complex subspaces $\hat{M} = \{x + iy \mid x, y \in M\}$, $\hat{M}^* = \{x + iy \mid x, y \in M^*\}$. Applying [20, Theorem 3.3] to these subspaces in $C^n$, we have $\hat{M} + \hat{M}^* = C^n$. But then $M + M^* = R^n$.

Next, we need the concepts of poles and zeros of a rational matrix function and their multiplicities. Let $W(A)$ be an $n \times n$ rational matrix function, generally with complex coefficients, and with $\det W(A) \neq 0$. Using the Smith form for rational matrix functions and its local variant (see, e.g., [9, 12]), it is easy to see that for any given $\lambda_0 \in C$ the function $W(A)$ can be written in the form

$$W(A) = E(A) \text{diag}((\lambda - \lambda_0)^{\alpha_1}, \ldots, (\lambda - \lambda_0)^{\alpha_n}) F(A),$$

where $E(A)$ and $F(A)$ are rational functions analytic and invertible at $\lambda_0$, and $\alpha_1 \geq \cdots \geq \alpha_n$ are integers. The integers $\{\alpha_j\}_{j=1}^n$ are uniquely determined by $W(A)$ and $\lambda_0$; the positive integers among them are called the partial zero multiplicities of $W(A)$ at $\lambda_0$, and the absolute values of the negative integers among $\{\alpha_j\}_{j=1}^n$ are called the partial pole multiplicities of $W(A)$ at $\lambda_0$. In general, we say that $\lambda_0$ is a pole of $W(A)$ if at least one of the $\alpha_j$'s is negative, and $\lambda_0$ is a zero of $W(A)$ if at least one of the $\alpha_j$'s is positive. This terminology is consistent with the more usual definition of a pole $W(A)$ as a pole of at least one of the entries of $W(A)$, and with the more usual definition of a zero of $W(A)$ as a pole of $W(A)^{-1}$. Note that for a real rational matrix function $W(\lambda)$, $\lambda_0$ is a zero (respectively pole) if and only if $\lambda_0$ is a zero (respectively pole), and the partial zero (respectively pole) multiplicities of $W(A)$ at $\lambda_0$ coincide with those at $\lambda_0$.

We turn now our attention to factorization. Let

$$W(A) = W_1(A)W_2(A),$$

where $W_1(A)$ and $W_2(A)$ are rational $n \times n$ matrix functions (generally, with complex coefficients). We regard the equality (2.1.6) as a factorization of the rational matrix function $W(A)$. The factorization (2.1.6) is said to be minimal if for any $\lambda_0 \in C$ the sums of the partial zero multiplicities of $W_1(A)$ at $\lambda_0$ and of $W_2(A)$ at $\lambda_0$ add up to the sum of the partial zero multiplicities of $W(A)$ at $\lambda_0$. (In fact, the definition of minimal factorization should involve the point at infinity as well, but since all our rational matrix functions will be assumed regular at infinity, the point at infinity is of no concern here.) One
can replace in this definition "zero" by "pole," which results in precisely the same concept of minimality.

For the real rational matrix functions $W(\lambda)$ with the symmetry properties (2.1.2) it is natural to consider minimal factorizations of the form

$$W(\lambda) = L(\eta\lambda)^T DL(\lambda),$$

where $D = W(\infty)$. It turns out that all such factorizations are described in terms of direct sums of invariant lagrangian subspaces. Recall that $J(A, H)$ denotes the set of all $A$-invariant $H$-lagrangian subspaces in $R^n$ (where $n$ is the size of $A$ and $H$).

**Theorem 2.1.2.** Let $W(\lambda) = D + C(\lambda I - A)^{-1} B$ be a minimal realization of the real rational matrix function $W(\lambda)$ satisfying (2.1.2). Let $M \in J(A, H)$, $M^x \in J(A^x, H)$, and suppose $R^n = M \oplus M^x$. Then $W$ admits a minimal factorization

$$W(\lambda) = L(\eta\lambda)^T DL(\lambda), \quad (2.1.7)$$

where

$$L(\lambda) = I + D^{-1} C\pi (\lambda I - \pi A\pi)^{-1} \pi B. \quad (2.1.8)$$

Here $\pi$ is the projection along $M$ onto $M^x$. Furthermore, the poles of $L(\eta\lambda)^T$ coincide with the eigenvalues of the restriction $A|_M$ of $A$ to its invariant subspace $M$, and the partial pole multiplicities of $L(\eta\lambda)^T$ at $\lambda_0$ are precisely the multiplicities of $\lambda_0$ as eigenvalue of $A|_M$. Also, the zeros of $L(\lambda)$ coincide with the eigenvalues of $A^x|_{M^x}$, and the partial zero multiplicities of $L(\lambda)$ at $\lambda_0$ are precisely the multiplicities of $\lambda_0$ as eigenvalue of $A^x|_{M^x}$.

Conversely, if $W$ admits a minimal factorization (2.1.7) with $L(\infty) = I$, then (2.1.8) holds for the projection $\pi$ along $M$ not $M^x$, for some choice of $M \in J(A, H)$ and $M^x \in J(A^x, H)$.

**Proof.** Let $\pi$ be as in the theorem. Then $W(\lambda)$ can be factorized as

$$W(\lambda) = K(\lambda) DL(\lambda),$$

where $L$ is as in the theorem and

$$K(\lambda) = I + C(I - \pi) [\lambda I - (I - \pi) A(I - \pi)]^{-1} (I - \pi) BD^{-1} \quad (2.1.9)$$

(see [2, Theorem 9.2]), and this is a minimal factorization.
Since $M^r = \text{Im} \pi$ and $M = \text{Ker} \pi$ are $H$-lagrangian, it is easy to check that $H\pi = (I - \pi^T)H$. Using (2.1.3) and (2.1.4), it is then straightforward to check that $L(\eta\lambda)^T = K(\lambda)$.

For the converse, we know from [2, Theorem 9.2] that any minimal factorization is connected with a supporting projection $\pi$ (i.e., $\text{Im} \pi$ is $A^x$-invariant, $\text{Ker} \pi$ is $A$-invariant), and the factors are given by (2.1.8) and (2.1.9). Since $K(\lambda) = L(\eta\lambda)^T$, using (2.1.4) it follows that

$$I + CH^{-1}\pi^T(\lambda I - \pi^THAH^{-1}\pi^T)^{-1}\pi^THBD^{-1} = I + C(I - \pi)[\lambda I - (I - \pi)A(I - \pi)]^{-1}(I - \pi)BD^{-1}.$$ 

Because both realizations are minimal, there is a unique invertible $S_1: \text{Im} \pi^T \to \text{Im}(I - \pi)$ such that

$$CH^{-1}\pi^T = C(I - \pi)S_1,$$

$$\pi^THAH^{-1}\pi^T = S_1^{-1}(I - \pi)A(I - \pi)S_1,$$

$$S_1^{-1}(I - \pi)B = \pi^THB.$$ 

Consequently, using also $A \text{ Ker} \pi \subset \text{Ker} \pi$ (i.e., $A^T\pi^T = \pi^TA^T\pi^T$), we obtain the equalities

$$CA^j(I - \pi)S_1 = CH^{-1}(\pi^THAH^{-1}\pi^T)^j = CA^jH^{-1}\pi^T$$

for $j = 0, \ldots$. As $\cap_{j=0}^\infty \text{Ker}(CA^j) = (0)$ [this is one of the properties of a minimal realization (2.1.1)], we get

$$(I - \pi)S_1 = S_1 = H^{-1}\pi^T.$$ 

Also

$$\pi^THA^xH^{-1}\pi^T = S_1^{-1}(I - \pi)A^x(I - \pi)S_1.$$ 

Now, since $A^x\pi = \pi A^x\pi$, we have

$$(I - \pi)A^x - (I - \pi)A^x(I - \pi).$$
and

$$\pi^T H A^s H^{-1} \pi^T = \pi^T H A^s H^{-1}.$$  

So the following equalities hold for $j = 0, 1, \ldots$:

$$\pi^T H A^{s+j} B = (\pi^T H A^s H^{-1} \pi^T)^j \pi^T H B = S_1^{-1}(I - \pi) A^{s+j} B.$$  

Use the property [ensured by the minimality of (2.1.1)] that zero is the only vector orthogonal to $\text{Im}(A^{s+j} B)$ for all $j = 0, 1, \ldots$ (here and everywhere in the paper we denote by $\text{Im} X$ the range of the matrix $X$, i.e., the column space of $X$). It follows that

$$\pi^T H = S_1^{-1}(I - \pi),$$

or

$$S_1 \pi^T = S_1 = (I - \pi) H^{-1}.$$  

So

$$S_1 = H^{-1} \pi^T = (I - \pi) H^{-1}.$$  

We conclude that $\text{Ker} \pi$ and $\text{Im} \pi$ are $H$-lagrangian, as required.  

Note that by Proposition 2.1.1 the hypothesis $R^n = M + M^*$ in Theorem 2.1.2 is automatically fulfilled for the case when $\xi = 1$.

The projection $\pi$ in Theorem 2.1.2 is called the supporting projection corresponding to the minimal factorization (2.1.7).

2.2. Symmetric Factorizations of Nonnegative Rational Matrix Functions and Their Stability

In this subsection we shall assume $\xi = 1$. We assume that the real $n \times n$ rational matrix function $W(\lambda)$ has value $I$ at infinity (i.e., $D = I$) and satisfies

$$W(\lambda) = W(-\lambda)^T \succeq 0$$ \hspace{1cm} (2.2.1)

for all $\lambda \in i \mathbb{R}$ not poles of $W$. In particular, $\eta = -1$ in the formula (2.1.2). For the case $\eta = +1$ and $W(\lambda) \succeq 0$ for $\lambda \in \mathbb{R}$ we refer to [23] for the full analysis of stability of symmetric factorizations. Note that if $\omega$ satisfies (2.2.1), then $\tilde{W}(\lambda) = W(i\lambda)$ is hermitian positive semidefinite for all real $\lambda$.  

For this reason we call a function $W$ satisfying (2.2.1) nonnegative. This
connection between $W$ and $\tilde{W}$ can be used to prove the following proposition.

Proposition 2.2.1. Let $W(\lambda) = I + C(\lambda I - A)^{-1}B$ be a minimal realiza-
tion of a real rational matrix function satisfying (2.2.1). Then for the canonical
forms of the pairs $(A, H)$ and $(A^*, H)$ in $L_o(-1, -1)$ as in Theorem 1.2.2 the
following holds. All partial multiplicities of $A$ and $A^*$ are pure imaginary
eigenvalues (including zero in case $A$ or $A^*$ is singular) are even, and the
signs $\kappa_j$ for the pair $(A, H)$ corresponding to these partial multiplicities are
all $+1$ for nonzero eigenvalues; the sign corresponding to a partial multipli-
city $m$ for the zero eigenvalue is $+1$ in case $m$ is divisible by 4, and $-1$ in
case $m$ is even but not divisible by 4. Further, the signs for the pair $(A^*, H)$
corresponding to partial multiplicities at nonzero eigenvalues are all $-1$; the
sign corresponding to a partial multiplicity $m$ for the zero eigenvalue is $-1$ in
case $m$ is divisible by 4, and $+1$ in case $m$ is even but not divisible by 4.

Conversely, suppose that $A$ and $A^*$ have only even partial multiplicities,
and the conditions on the signs of $(A, H)$ and $(A^*, H)$ as stated above are
satisfied. Then $W(\lambda)$ is nonnegative.

Proof. Using the observation that $\tilde{W}(\lambda) = W(i\lambda)$ is nonnegative for real
$\lambda$, together with Theorem 2.5 in [20] and Proposition 3.3 in [21], gives the
desired conclusion for the pair $(A, H)$. Next note that also $W(\lambda)^{-1}$ is
nonnegative for $\lambda \in i\mathbb{R}$. The pair in $L_o(-1, -1)$ corresponding to its poles is
$(A^*, -H)$. So the partial multiplicities of $A^*$ at pure imaginary eigenvalues
are even, and the signs corresponding to these partial multiplicities in the
canonical forms of $(A^*, -H)$ are all $+1$. Consequently, the signs correspond-
ing to $(A^*, H)$ are all $-1$.

For the converse, note that the assumptions imply that $J(A, H) \neq \emptyset$, $J(A^*, H) \neq \emptyset$. Take $M \in J(A, H)$ and $M^* \in J(A^*, H)$. From Proposition 2.1.1
and Theorem 2.1.2 it follows that $W(\lambda) = L(-\lambda)^TL(\lambda)$, i.e., $W(\lambda)$ is nonneg-
avtive.

Note that in [20, Proposition 4.2] the complex analogue of this result is
erroneously stated. (We are indebted to R. Vreugdenhil for pointing out this
error to us.) The correct statement is as follows.

Proposition 2.2.2. If $W(\lambda) = I + C(\lambda I - A)^{-1}B$ is a minimal realiza-
tion for a function $W$ which is nonnegative for real $\lambda$ that are not poles of
$W(\lambda)$, and $H$ is such that

$$H = H^* \text{ is invertible}, \quad HA = A^*H, \quad HB = C^*,$$
then the signs in the sign characteristic of \((A, H)\) are all \(+1\)'s and the signs in the sign characteristic of \((A', H)\) are all \(-1\)'s.

In Proposition 4.2 in [20] it was stated that the latter signs are \(+1\)'s also.

Let \(W(\lambda)\) be nonnegative, with minimal realization \(W(\lambda) = I + C(\lambda I - A)^{-1}B\). Suppose the factorization

\[
W(\lambda) = L(-\lambda)^T L(\lambda)
\]

(2.2.2)

where \(L(\infty) = I\) is minimal, and \(L(\lambda)\) has a minimal realization \(L(\lambda) = I + C_i(\lambda I - A_i)^{-1}B_i\). This factorization will be called conditionally stable if for each \(\varepsilon > 0\) there is \(\delta > 0\) such that any nonnegative rational matrix function \(W'(\lambda) = I + C(\lambda I - A')^{-1}B'\) with

\[
\|A - A'\| + \|B - B'\| + \|C - C'\| < \delta
\]

(2.2.3)

admits a minimal factorization

\[
W'(\lambda) = L'(\lambda)^T L'(\lambda),
\]

(2.2.4)

where \(L'(\infty) = I\) and \(L'(\lambda)\) has a minimal realization \(L'(\lambda) = I + C_i(\lambda I - A_i)^{-1}B_i\) with

\[
\|A_1 - A'_1\| + \|B_1 - B'_1\| + \|C_1 - C'_1\| < \varepsilon.
\]

(2.2.5)

If we drop the requirement that \(W(\lambda)\) will be nonnegative, and just require that (2.2.4), (2.2.5) hold for any rational \(W'(\lambda)\) with \(W'(\lambda) = W'(-\lambda)^T\) and for which (2.2.3) holds, then the factorization (2.2.2) is called unconditionally stable.

Note that if the realization \(W(\lambda) = I + C(\lambda I - A)^{-1}B\) is minimal and \(\delta > 0\) sufficiently small, then the realization \(W'(\lambda) = I + C(\lambda I - A')^{-1}B'\) is also minimal. This follows from an alternative description of a minimal realization in terms of the controllability (null kernel property) of the pair \((C, A)\) and the observability (full range property) of the pair \((A, B)\) (see, e.g., [12]).

The next theorem describes the relation between the notions just introduced and stability properties of certain subspaces.
**Theorem 2.2.3.** Let \( W(\lambda) = L(-\lambda)^T L(\lambda) \) be a minimal factorization of \( W(\lambda) = I + C(\lambda I - A)^{-1}B \). This factorization is (un)conditionally stable if and only if for the corresponding supporting projection \( \pi \) the subspaces \( \text{Ker} \, \pi = M \) and \( \text{Im} \, \pi = M^\pi \) are (un)conditionally stable in \( J(A, H) \) and \( J(A^\pi, H) \) respectively.

**Proof.** First we consider the case of conditional stability. Suppose \( M \) and \( M^\pi \) are conditionally stable, and let \( W'(\lambda) = I + C'(\lambda I - A')^{-1}B' \) be a nonnegative rational matrix function for which (2.2.3) holds. If \( \delta > 0 \) is small enough, the above realization for \( W'(\lambda) \) will be minimal. Let \( H' \) be the matrix for which the analogue of (2.1.4) holds for this realization of \( W' \). Then \( J(A', H') \neq \emptyset \) and \( J(A^\pi', H') \neq \emptyset \) because of Proposition 2.2.1. Since \( \text{Im} \, \pi = M^\pi \) and \( \text{Ker} \, \pi = M \) are conditionally stable, there exist \( M_1 \) and \( M^\pi_1 \) in \( J(A', H') \), and \( J(A^\pi', H') \) respectively, such that \( \theta(M, M_1) < \varepsilon \) and \( \theta(M^\pi, M^\pi_1) > \varepsilon \) for \( \delta \) small enough. By Proposition 2.1.1, \( R'' = M_1 + M^\pi_1 \). Let \( \pi_1 \) be the projection onto \( M^\pi_1 \) along \( M_1 \). Then \( \|\pi - \pi_1\| < C \varepsilon \), where the constant \( C > 0 \) is independent of \( \pi_1 \). From (2.1.7) it follows easily that this implies the conditional stability of the factorization \( W(\lambda) = L(-\lambda)^T L(\lambda) \) (cf. Lemma 8.9 in [2]).

Conversely, assume the factorization (2.2.2) is conditionally stable. Let \( (A', H') \in L_n(-1, -1) \) with \( ||A - A'|| + ||H - H'|| < \delta \). It follows from Proposition 2.2.1 above and from Theorem 6.5 in [24] that for \( \delta \) small enough all partial multiplicities of \( A' \) at pure imaginary eigenvalues are even and all signs in the canonical form of \( (A', H') \) corresponding to the partial multiplicities at nonzero eigenvalues are \( +1 \), and for the partial multiplicities at zero (if any) they are \( +1 \) (respectively, \( -1 \)) in case the partial multiplicity is divisible by \( 4 \) (respectively, is even but not divisible by \( 4 \)). In particular \( J(A', H') \neq \emptyset \).

Consider for \( \alpha < 1 \), \( \alpha \) close to one, the rational matrix function

\[
W_\alpha(\lambda) = I + \alpha C(\lambda I - A)^{-1} B \alpha = \alpha^2 W(\lambda) + (1 - \alpha^2) I.
\]

Clearly \( W_\alpha(\lambda) > 0 \) for \( \lambda \in i \mathbb{R} \), so \( A - \alpha^2 BC \) has no pure imaginary eigenvalues. Choose \( \delta_1, 0 < \delta_1 < \delta \), so small that for any \( A' \) with \( ||H - H'|| + ||A - A'|| < \delta_1 \), \( (A', H') \in L_n(-1, -1) \), and \( (A', B) \) controllable, we have also that \( A' - \alpha^2 BB^T H' \) has no pure imaginary eigenvalues, and the number of eigenvalues (counting multiplicities) of \( A' - \alpha^2 BB^T H' \) in the open right half plane is \( n/2 \).

Construct

\[
W'(\lambda) = I + \alpha B^T H' (\lambda I - A')^{-1} B \alpha.
\] (2.2.6)
From the remarks just made and Proposition 2.2.1 one sees that $W'(\lambda) > 0$ for $\lambda \in i \mathbb{R}$. Since the factorization $W(\lambda) = L(-\lambda)^T L(\lambda)$ is conditionally stable, it follows that for $\alpha$ close enough to one and $\delta_1$ close enough to zero there is a supporting projection $\pi'$ for the minimal realization (2.2.6) of $W'(\lambda)$ such that $\text{Im} \pi' \in J(A' - \alpha^2 BB^TH', H')$, $\text{Ker} \pi \in J(A', H')$, and $\|\pi - \pi'\|$ is as small as one wishes. (See also Lemma 8.9 in [2].) It follows that $\theta(\text{Ker} \pi, \text{Ker} \pi') \leq \|\pi - \pi'\|$ can be made as small as one wishes. Hence $\text{Ker} \pi$ is conditionally stable in $J(A, H)$. In a similar way one shows that $\text{Im} \pi$ is conditionally stable in $J(A^*, H)$.

Next consider the case of unconditional stability. Suppose $M = \text{Ker} \pi$ and $M^* = \text{Im} \pi$ are unconditionally stable. A similar argument to that in the first paragraph of the proof shows that the factorization (2.2.2) is unconditionally stable.

Now suppose that the factorization (2.2.2) is unconditionally stable. Let $(A', H') \in L_*(-1, -1)$ be such that $\|A - A'\| + \|H - H'\| < \delta$, and consider the rational matrix function

$$W'(A) = I + BTH'(\lambda I - A')^{-1}B.$$  \hfill (2.2.7)

Clearly $W'(-\lambda)^T = W'(\lambda)$, and for $\delta$ small enough $(A', B)$ is controllable, so the realization (2.2.7) for $W'$ is minimal for small $\delta$. Since the factorization (2.2.2) is unconditionally stable, the function (2.2.7) admits a minimal factorization $W'(\lambda) = L'(-\lambda)^T L'(\lambda)$ such that for the corresponding supporting projection $\pi'$ we have $\|\pi - \pi'\|$ as small as we wish. In particular, it follows then that $\text{Ker} \pi' \in J(A', H')$ (by Proposition 2.2.1), and that $\theta(\text{Ker} \pi, \text{Ker} \pi')$ can be as small as one wishes. So $\text{Ker} \pi$ is unconditionally stable in $J(A, H)$. Likewise one proves that $\text{Im} \pi$ is unconditionally stable in $J(A^*, H)$.

We remark here that the proof for the complex case given in [26] is incomplete, as the incorrect statement in [20, Proposition 4.21 was used there. A correct proof for the statements in [26, Theorem 2.5] can be given precisely as in the above proof of Theorem 2.2.3.

An immediate consequence of Theorem 2.2.3 and our earlier results on stability of invariant lagrangian subspaces in [21] (in particular Theorem 3.4 there) is the following theorem.

**Theorem 2.2.4.** Let there be given a nonnegative rational matrix function $W(\lambda)$ with $W(\infty) = I$.

(i) There is an unconditionally stable minimal factorization of the type

$$W(\lambda) = L(-\lambda)^T L(\lambda),$$  \hfill (2.2.8)
where $L(\lambda)$ is a rational real matrix function with $L(\infty) = 1$, if and only if $W$ has no pure imaginary poles and zeros.

(ii) There always exists a conditionally stable factorization of the type (2.2.8). The minimal factorization (2.2.8) is (un)conditionally stable (assuming (i) holds for the case of unconditional stability) if and only if the following conditions hold:

(a) For every nonzero real pole (zero) $\lambda_0$ of $W(\lambda)$ of geometric multiplicity bigger than one, exactly one of the numbers $\lambda_0, -\lambda_0$ is a pole (zero) of $L(\lambda)$.

(b) For every nonzero real pole (zero) $\lambda_0$ of $W(\lambda)$ of geometric multiplicity one and even algebraic multiplicity, the algebraic multiplicity of $\lambda_0$ as a pole (zero) of $L(\lambda)$ is even.

(c) For every nonreal, non-pure-imaginary pole (zero) $a + ib$ of $W(\lambda)$ of geometric multiplicity bigger than one, exactly one of the numbers $a + ib, -a + ib$ is a pole (zero) of $L(\lambda)$. In that case $a - ib$ or $-a - ib$, respectively, is also a pole (zero) of $L(\lambda)$.

Proof. (i) follows by combining Theorem 2.2.3 above with Theorem 3.4(ii) in [21]. One obtains (ii) by noting that from Proposition 2.2.1 and Theorem 2.2.2, it follows that the conditions in Theorem 3.4(i) in [21] are fulfilled. Part (iii) is then an immediate consequence of part (iii) of Theorem 3.4 in [21].

As a corollary to the previous theorem we have the following.

**Corollary 2.25.** Suppose $W(\lambda)$ is a nonnegative rational matrix function with only pure imaginary poles and zeros. Then there exists a unique factorization $W(\lambda) = L(-\lambda)^T L(\lambda)$, and this factorization is conditionally stable.

We continue by introducing some more concepts of stability for the factorization (2.2.2). The factorization (2.2.2) is called *unconditionally Lipschitz stable* if there exist positive constants $K$ and $\delta$ such that any rational matrix function $W'(\lambda) = W'(\lambda)^T$ with minimal realization $W'(\lambda) = I + C'(\lambda - A')^{-1}B'$ satisfying (2.2.3), i.e.,

$$\|A - A'\| + \|B - B'\| + \|C - C'\| < \delta,$$

admits a minimal factorization (2.2.4) where $L'(\lambda)$ has a minimal realization.
\[ L'() = I + C_i'(\lambda - A_i')^{-1}B_i' \] with

\[ \|A_1 - A_1'\| + \|B_1 - B_1'\| + \|C_1 - C_1'\| \leq K \cdot (\|A - A'\| + \|B - B'\| + \|C - C'\|). \]

If we require this to be the case only for nonnegative functions \( W'(A) \) satisfying (2.2.3), the factorization (2.2.2) is called conditionally Lipschitz stable. The proof of the next theorem is essentially the same as the proof of Theorems 2.2.3, 2.2.4.

**Theorem 2.2.6.** Let \( W(\lambda) \) be given as in Theorem 2.2.4. Then the following statements are equivalent:

(i) there is a conditionally Lipschitz stable minimal factorization of the type

\[ W(\lambda) = L(-\lambda)^T L(\lambda) \quad (2.2.9) \]

where \( L(\lambda) \) is a real rational matrix function with \( L(\infty) = I \);

(ii) there exists an unconditionally Lipschitz stable minimal factorization of the type (2.2.9);

(iii) \( W \) has no pure imaginary poles and zeros.

In case these statements hold, the minimal factorization (2.2.9) is Lipschitz stable (conditional or unconditional) if and only if one (or both) of the following two equivalent statements is true:

(iv) \( L(\lambda) \) and \( L(-\lambda)^T \) have no common poles and zeros,

(v) The supporting projection \( \pi \) corresponding to the factorization (2.2.9) has the property that \( \text{Im} \pi \) is Lipschitz stable as an element of \( J(A^+, H) \) and \( \text{Ker} \pi \) is Lipschitz stable as an element of \( J(A, H) \).

Note that in case \( W \) has no pure imaginary poles and zeros, in particular the minimal factorization \( W(\lambda) = L(-\lambda)^T L(\lambda) \), with \( L \) having all its poles and zeros in \( \text{Re} \lambda > 0 \), is Lipschitz stable.

2.3. Symmetric Stable Factorization of Nonnegative Matrix Polynomials

In this section we study the stability of symmetric factorizations for nonnegative \( m \times m \) matrix polynomials

\[ L(\lambda) = \lambda^n L_n + \lambda^{n-1} L_{n-1} + \cdots + L_0. \quad (2.3.1) \]
We assume $L(-\lambda)^T = L(\lambda)$, the coefficients $L_i$ are real $m \times m$ matrices for $0 \leq i \leq n$, and $L(\lambda)$ is nonnegative, i.e., $\langle L(\lambda)x, x \rangle \geq 0$ for all pure imaginary $\lambda$ and $x \in \mathbb{R}^m$. In that case $n$ is even, say $n = 2k$. Henceforth we shall assume $L_n = (-1)^k I$. We are interested in factorizations of the type

$$L(\lambda) = M(-\lambda)^T M(\lambda),$$

(2.3.2)

where $M(\lambda) = \lambda^k I + \lambda^{k-1} M_{k-1} + \cdots + M_0$ is a monic matrix polynomial of degree $k = n/2$. Analogous factorizations concerning matrix polynomials nonnegative (or, more generally, hermitian) on the real line were studied in [18, 8, 11], and their stability properties were studied in [26].

Introduce the companion matrix of $L$,

$$C_L = \begin{bmatrix}
0 & I_m & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & I_m & 0 \\
0 & \cdots & 0 & I_m & \vdots \\
(-1)^{k+1}L_0 & \cdots & \cdots & (-1)^{k+1}L_{n-1}
\end{bmatrix},$$

and also the following matrix:

$$B_L = \begin{bmatrix}
L_1 & L_2 & L_3 & \cdots & \cdots & L_{n-1} & I_m \\
-L_2 & -L_3 & \cdots & \cdots & -L_{n-1} & -I_m & 0 \\
L_3 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
L_{n-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
-I_m & 0 & \cdots & \cdots & \cdots & \cdots & 0
\end{bmatrix}.$$

The matrices $C_L$ and $B_L$ are of size $nm \times nm$. Since $L(-\lambda)^T = L(\lambda)$, we have

$$L_j = (-1)^j L_j^T \quad (0 \leq j \leq n),$$
as one easily checks. So $B_L = -B_L^T$. Further, a computation shows that

$$B_L C_L = \begin{bmatrix} (-1)^k L_0 & 0 & 0 & \cdots & 0 \\ 0 & -L_2 & -L_3 & \cdots & -L_{n-1} & -I_m \\ 0 & L_3 & L_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & L_{n-1} & I_m & \cdots & 0 \\ 0 & -I_m & 0 & \cdots & 0 \end{bmatrix}$$

So

$$B_L C_L = (B_L C_L)^T = -C_L^T B_L, \quad \text{i.e.,} \quad (C_L, B_L) \in L_{nm}(-1, -1).$$

Recall that factorizations of the type $L(\lambda) = M_1(\lambda)M_2(\lambda)$, where $M_1$ and $M_2$ are $m \times m$ matrix polynomials and $M_2$ is monic (i.e., with leading coefficient 1) and of degree $l$, are in one to one correspondence with the $C_L$-invariant subspaces $\Lambda \subset \mathbb{C}^n$ such that the restriction $[I_{nl} \ 0]_\Lambda: \Lambda \rightarrow \mathbb{C}^{nl}$ of $[I_{nl} \ 0]$ to the subspace $\Lambda$ is invertible (as a linear transformation); see [13, 9]. Such a subspace $\Lambda$ will be called a *supporting subspace*. The factor $M_2(\lambda)$ is given by

$$M_2(\lambda) = \lambda^l I_m - [I_{nl} \ 0](C_L|_\Lambda)^l (\nu_1 + \lambda \nu_2 + \cdots + \lambda^{l-1} \nu_1), \quad (2.3.3)$$

where

$$[\nu_1, \ldots, \nu_l] = ([I_{nl} \ 0]|_\Lambda)^{-1}. \quad (2.3.4)$$

(see [13, 9]). In the next lemma we characterize the supporting subspaces that correspond to factorizations of the type (2.3.2). Recall that by $J(C_L, B_L)$ we denote the set of all $C_L$-invariant $B_L$-lagrangian subspaces.

**Lemma 2.3.1.** The supporting subspace $\Lambda$ corresponds to a factorization $L(\lambda) = M(\lambda)^TM(\lambda)$ with degree $M = k$ if and only if $\Lambda \in J(C_L, B_L)$. Moreover, any subspace $\Lambda \in J(C_L, B_L)$ is supporting.

**Proof.** Use the fact that $L(i\lambda)$ is Hermitian, and in fact positive semidefinite, for real $\lambda$. The result then easily follows from Chapter 11 in [9].
Next, we characterize nonnegativity of $L(\lambda)$ in terms of partial multiplicities of $C_L$ and the signs in the canonical form of $(C_L, B_L)$.

**Lemma 2.3.2.** The matrix polynomial $L(\lambda)$ is nonnegative for $\lambda \in \mathbb{R}$ if and only if $L$ has only even partial multiplicities at pure imaginary eigenvalues (including zero), the signs in the canonical form of $(C_L, B_L)$ corresponding to partial multiplicities of nonzero eigenvalues are all $+1$, and the signs corresponding to a partial multiplicity $m$ at the zero eigenvalue is $+1$ if $m$ is divisible by 4, and $-1$ if $m$ is even but not divisible by 4.

**Proof.** Note that the matrix polynomial $\tilde{L}(\lambda) = L(i\lambda)$ is hermitian and positive semidefinite for $\lambda \in \mathbb{R}$. Thus we have

$$\tilde{L}(\lambda) = \lambda^n I + (i)^{n-1}L_{n-1}\lambda^{n-1} + \cdots + L_0.$$  

Put

$$\tilde{C}_L = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & I_m \\ -L_0 & -iL_1 & \cdots & \cdots & -i^{n-1}L_{n-1} \end{bmatrix},$$

the companion matrix of $\tilde{L}$, and also introduce

$$\tilde{B}_L = \begin{bmatrix} iL_1 & -L_2 & -iL_3 & \cdots & i^{n-1}L_{n-1} & I_m \\ -L_2 & -iL_3 & \cdots & \vdots & \vdots & 0 \\ -iL_3 & \cdots & \cdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\ i^{n-1}L_{n-1} & \cdots & \cdots & \cdots & \cdots & 0 \\ I_m & 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}.$$  

Note that $\tilde{B}_L$ is hermitian and invertible. Then $\tilde{C}_L$ is $\tilde{B}_L$-self-adjoint, and since $\tilde{L}$ is positive semidefinite, the partial multiplicities of $\tilde{C}_L$ at its real eigenvalues are even, and the signs in the sign characteristic of $(\tilde{C}_L, \tilde{B}_L)$ are all $+1$. (See Chapter 12 in [9].)
Now take $S = \text{diag}(iI_m, i^2I_m, \ldots, i^{2k}I_m)$. Then $S^* = S^{-1}$, and one checks that

$$S^*LSS^* = iB_L, \quad SCS^* = iC_L.$$ 

Thus, the partial multiplicities of $C_L$ at its pure imaginary eigenvalues are all even. The statement on the signs now also follows from [21, Proposition 3.3].

Conversely, suppose the conditions in the theorem hold. Then $J(C_L, B_L) \neq \varnothing$. Then Lemma 2.3.1 implies that $L$ is nonnegative. 

We now give the main theorem of this section, which states a sufficient condition for the stability of the factorization (2.3.2) under nonnegative perturbations. We shall say that factorization (2.3.2) is conditionally stable if for every $\varepsilon > 0$ there is a $\delta > 0$ such that every nonnegative matrix polynomial $L'(\lambda) = (-1)^k\lambda^nI + \lambda^{n-1}L'_{n-1} + \cdots + L'_0$ satisfying $\sum_{j=0}^{n-1}||L_j - L'_j|| < \delta$ has a factorization $L'(\lambda) = M'(\lambda)^T M'(\lambda)$, where $M'(\lambda) = \lambda^kI + \lambda^{k-1}M'_k + \cdots + M'_0$ with $\sum_{j=0}^{k-1}||M_j - M'_j|| < \varepsilon$ (here $n = 2k$). Dropping the condition that $L'(\lambda)$ must be nonnegative from the previous definition, we obtain the definition of unconditional stability.

The definition of a zero of a rational matrix function given in Section 2.1 applies in particular for matrix polynomials. Thus, $\lambda_0$ is a zero of a matrix polynomial $L(\lambda)$ if and only if $\det L(\lambda_0) = 0$, and we will use the notion of partial zero multiplicities as well. The geometric multiplicity of zero $\lambda_0$ is defined as the number of partial zero multiplicities corresponding to $\lambda_0$.

**Theorem 2.3.3.** Let $L(\lambda)$ be an $m \times m$ nonnegative matrix polynomial of even degree $n$ with leading coefficient $L_n = (-1)^kI$, where $k = n/2$.

(i) If $\det L(\lambda)$ has no zeros on the imaginary line, then there exists an unconditionally stable factorization of the type

$$L(\lambda) = M(-\lambda)^T M(\lambda), \quad (2.3.5)$$

where $M(\lambda)$ is a monic matrix polynomial.

(ii) There always exists a conditionally stable factorization of the type (2.3.5).

(iii) Suppose the conditions (a)–(c) below hold for the factorization (2.3.5). Then this factorization is conditionally or unconditionally stable, depending on whether or not $\det L(\lambda)$ has zeros on the imaginary line.

(a) For every real zero $\lambda_0 \neq 0$ of $L$ of geometric multiplicity larger than one, precisely one of the numbers $\lambda_0, -\lambda_0$ is a zero of $M$. 


(b) For every real zero $\lambda_0 \neq 0$ of $L$ of geometric multiplicity one and even partial multiplicity, the partial multiplicity of $\lambda_0$ as a zero of $M$ is even (it turns out that if $\lambda_0$ as a zero of $L$ has geometric multiplicity one, then $\lambda_0$ has geometric multiplicity one as a zero of $M$, so the partial multiplicity of $\lambda_0$ as a zero of $M$ is unambiguously defined).

(c) For every nonreal, non-pure-imaginary zero $a + ib$ of $L$ of geometric multiplicity larger than one, precisely one of the numbers $a + ib$, $-a + ib$ is a zero of $M$.

Proof. Suppose (i) is satisfied. Then there exists an unconditionally stable invariant lagrangian subspace in $J(C_L, B_L)$, since $C_L$ has no pure imaginary eigenvalues in that case. A conditionally stable element in $J(C_L, B_L)$ always exists, by Lemma 2.3.2.

Now suppose (a)-(c) hold. Then the subspace $\Lambda \in J(C_L, B_L)$ corresponding to the factorization $L(\lambda) = M(-\lambda)^T M(\lambda)$ is conditionally or unconditionally stable (depending on whether or not $L$ has zeros on the imaginary line). Then the theorem follows easily from the definition of stable factorizations (conditional or unconditional), taking into account (2.3.3) and (2.3.4).

We conjecture that conditions (a)-(c) are also necessary for the (un)conditional stability of the factorization (2.3.5).

Of particular interest are factorizations (2.3.5) where every zero of $\det M(\lambda)$ has nonnegative real part.

**Corollary 2.3.4.** Let $L(\lambda)$ be as in Theorem 2.3.3. Then $L(\lambda)$ admits a unique factorization

$$L(\lambda) = M(-\lambda)^T M(\lambda)$$

where $M(\lambda)$ is a monic matrix polynomial such that all zeros of $\det M(\lambda)$ have nonnegative real parts. Moreover, this factorization is conditionally stable, and in case $\det L(\lambda)$ has no zeros on the imaginary axis it is also unconditionally stable.

2.4. Stable Factorizations of Rational Matrix Functions with Constant Zero Signature

In this section we shall consider factorizations of the type (2.1.7) for the cases when $\xi = -1$. In both these cases ($\eta = 1$ or $\eta = -1$) we have $D = -D^T$ [cf. (2.1.2)], where $W(\lambda) = D + C[A - A]^{-1}B$. It follows that $iD$ is complex hermitian, and its signature (i.e., the difference between the
number of positive eigenvalues and the number of negative eigenvalues, counted with multiplicities) is zero. Put $\tilde{W}(\lambda) = iW(i^{1-\eta})$. Then $\tilde{W}(\lambda)$ is hermitian for real $\lambda$, and in case $W(\lambda)$ factorizes as in (2.1.7), i.e.

$$W(\lambda) = L(\eta\lambda)^T DL(\lambda), \quad (2.4.1)$$

we have

$$\tilde{W}(\lambda) = [\tilde{L}(\lambda)]^*(iD)\tilde{L}(\lambda). \quad (2.4.2)$$

As a consequence the signature of $\tilde{W}(\lambda)$ is zero for every real $\lambda$ which is not a zero or a pole of $W$. For this reason we shall say that $W(\lambda)$ has constant signature zero. In fact we define this notion independently of the factorization (2.4.1) as follows: Let $W(\lambda)$ be real rational $n \times n$ matrix function, analytic and invertible at infinity and satisfying

$$W(\lambda) = -[W(\eta\lambda)]^T \quad (2.4.3)$$

for $\lambda$ not a pole of $W$, where $\eta = \pm 1$. Further, let

$$W(\lambda) = D + C(\lambda I - A)^{-1}B \quad (2.4.4)$$

be a minimal realization for $W(\lambda)$, and let $H = -\eta H^T$ be the unique real invertible matrix such that

$$HA = \eta A^TH, \quad HB = C^T, \quad -\eta B^TH = C$$

[cf. (2.1.4)]. Then $W(\lambda)$ is said to have constant signature zero if the hermitian matrix $\tilde{W}(\lambda) = iW(i^{1-\eta})$ has signature zero for every real $\lambda$ which is neither a pole nor a zero of $\tilde{W}$, and moreover, $J(A, H) \neq \emptyset$ and $J(A^T, H) \neq \emptyset$. It is easy to see that this definition does not depend on the choice of the minimal realization (2.4.4). The requirement that $J(A, H) \neq \emptyset$ could be expressed as $\tilde{W}$ having signature zero with respect to poles and zeros separately (compare with the polynomial case studied in [14, 4]).

Observe that $\tilde{W}(\lambda)$ may have signature zero for all real $\lambda$ (not zeros or poles of $\tilde{W}$) but the sets $J(A, H)$ and $J(A^T, H)$ be nevertheless empty, as the
following example shows:

\[
\hat{W}(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left( \lambda I - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} \lambda(\lambda - 1)^{-1} & 0 \\ 0 & \lambda^{-1}(1 - \lambda) \end{bmatrix}
\]

Clearly, \(\hat{W}(\lambda)\) has constant signature zero for \(\lambda \in \mathbb{R} \setminus \{0, 1\}\), but both

\[
J\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \quad \text{and} \quad J\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)
\]

are empty.

Theorem 1.2.4 shows that every rational matrix function \(W(\lambda)\) which is analytic and invertible at infinity and satisfies (2.4.3) with \(n = 1\) has constant signature zero.

We shall consider stability of minimal factorizations of the form (2.4.1) for functions \(W\) which have constant signature zero. As was proved in Theorem 2.1.2, such minimal factorizations are in one-one correspondence with pairs of subspaces \(M \in J(A, H), M^* \in J(A^*, H)\) such that \(M \perp M^* = C^n\).

The minimal factorization (2.4.1), where \(L(\lambda) = I + C_i(\lambda I - A_i)^{-1}B_1\) is a minimal realization for \(L\), is called conditionally stable if given \(\varepsilon > 0\) there exists a \(\delta > 0\) with the following property: Any \(W'(\lambda) = D + C'(\lambda I - A')^{-1}B'\) such that \(W'(\lambda) = -[W'(\eta\lambda)]^T\) (\(\lambda\) is not a pole of \(W'\)), \(W'(\lambda)\) has constant signature zero, and

\[
\|A - A'\| + \|B - B'\| + \|C - C'\| < \delta
\]

admits a minimal factorization

\[
W'(\lambda) = L'(\eta\lambda)^TDL'(\lambda)
\]

with \(L'(\lambda) = I + C_i'(\lambda I - A_i)^{-1}B_i'\) and

\[
\|A_1 - A_1'\| + \|B_1 - B_1'\| + \|C_1 - C_1'\| < \varepsilon.
\]

If we drop the condition that \(W'(\lambda)\) has constant signature zero, we obtain the definition of unconditional stability. In particular, for the case \(\eta = 1\) the
notions of conditional and unconditional stability coincide, and we refer to them simply as stability.

Next we define the concepts of conditional and unconditional strong stability for the factorization (2.4.1). Let $W(\lambda) = D + C(\lambda I - A)^{-1}B$ and $I(\lambda) = I + C_{1}(\lambda I - A_{1})^{-1}B_{1}$ satisfy (2.4.1). This factorization is called conditionally strongly stable if the following holds. Let

$$W_{p}(\lambda) = D + C_{p}(\lambda I - A_{p})^{-1}B_{p}, \quad p = 1, 2, \ldots,$$

be a sequence of rational matrix functions such that for all $p$ we have $W_{p}(\lambda) = -[W_{p}(\eta \lambda)]^{T}$ ($\lambda$ is not a pole of $W_{p}$), $W_{p}(\lambda)$ has constant signature zero, and moreover,

$$\lim_{p \to \infty} \left( \|A - A_{p}\| + \|B - B_{p}\| + \|C - C_{p}\| \right) = 0.$$

Let

$$W_{p}(\lambda) = L_{p}(\eta \lambda)^{T}DL_{p}(\lambda), \quad p = 1, 2, \ldots, \quad (2.4.5)$$

be minimal factorizations with $L_{p}(\lambda) = I + C_{1p}(\lambda I - A_{1p})^{-1}B_{1p}$ such that for any open set $\mathfrak{t}$ whose boundary does not contain zeros or poles of $W$ we have

$$\#(\text{poles of } L_{p} \text{ in } \mathfrak{t}) = \#(\text{poles of } L \text{ in } \mathfrak{t}),$$

$$\#(\text{zeros of } L_{p} \text{ in } \mathfrak{t}) = \#(\text{zeros of } L \text{ in } \mathfrak{t}). \quad \ (2.4.6)$$

Then

$$\|A_{1} - A_{1p}\| + \|B_{1} - B_{1p}\| + \|C_{1} - C_{1p}\| \to 0 \quad (p \to \infty). \quad (2.4.7)$$

(Here, as usual, the number of poles of a rational matrix function in a certain set is counted with multiplicities, and the same holds for the number of zeros.) In other words, as soon as the functions $L_{p}$ in (2.4.5) have a chance of being close to $L$, in the sense that (2.4.6) is satisfied, they do indeed converge to $L$ in the sense of (2.4.7). If we drop the requirement that $W_{p}$
have constant signature zero, we obtain the definition of unconditional strong stability.

We now consider first the case \( \eta = -1 \).

**Lemma 2.4.1.** Let \( W(\lambda) = L(-\lambda)^TDL(\lambda) \) be a minimal factorization for \( W(\lambda) = D + C(\lambda - A)^{-1}B \), with corresponding supporting projection \( \pi \). If \( M^\tau = \text{Im} \pi \) and \( M = \text{Ker} \pi \) are (un)conditionally (strongly) stable in \( J(A^\tau, H) \) and \( J(A, H) \), respectively, then the factorization \( W(\lambda) = L(-\lambda)^TDL(\lambda) \) is (un)conditionally (strongly) stable.

**Proof.** We only prove the lemma for the case of conditional stability. The proof for the other cases is analogous. So assume \( M \) and \( M^\tau \) are conditionally stable in \( J(A, H) \) and \( J(A^\tau, H) \), respectively. Let \( W'(\lambda) = D + C' + \lambda I_n - A'^{-1}B' \) be a rational matrix function with constant signature zero, and with \( \|A - A'\| + \|B - B'\| + \|C - C'\| < \delta \). For \( \delta \) small enough this realization for \( W' \) will be minimal. Let \( H' \) be such that \( H' = H'^T, \quad H'A' = A'^TH', \quad H'B' = C'^T \). Since \( W' \) has constant signature zero, by assumption \( J(A', H') \neq \emptyset \) and \( J(A'^\tau, H') \neq \emptyset \). The conditional stability of \( M \) and \( M^\tau \) then implies the existence of \( M_1 \in J(A', H') \) and \( M_1^\tau \in J(A'^\tau, H') \) such that \( \theta(M, M_1) < \epsilon \) and \( \theta(M^\tau, M_1^\tau) < \epsilon \) for \( \delta \) small enough. Since \( M + M^\tau = \mathbb{R}^n \), we can take \( \epsilon \) so small that \( \theta(M, M_1) < \epsilon \) and \( \theta(M^\tau, M_1^\tau) < \epsilon \) implies \( M_1 + M_1^\tau = \mathbb{R}^n \). Let \( \pi_1 \) be the projection onto \( M_1^\tau \) along \( M_1 \). Then for some constant \( \kappa > 0 \) independent of \( \pi \), we have \( \|\pi - \pi_1\| < \kappa \epsilon \). Clearly this implies the conditional stability of the factorization (2.4.1). \( \square \)

In case \( W(\lambda) \) is analytic and invertible at zero, the previous lemma admits a converse for the cases of unconditional stability.

**Lemma 2.4.2.** Suppose \( W(\lambda) = D + C(\lambda I - A)^{-1}B \) has no pole or zero at zero. If the minimal factorization

\[
W(\lambda) = L(-\lambda)^TDL(\lambda)
\]

is unconditionally (strongly) stable, then the subspaces \( \text{Im} \pi \) and \( \text{Ker} \pi \), where \( \pi \) is the supporting projection of the factorization above, are unconditionally (strongly) stable in \( J(A^\tau, H) \) and \( J(A, H) \), respectively.

**Proof.** Take \( (A', H') \in L_n(-1, -1) \) with \( \|A - A'\| + \|H - H'\| \) small. Let

\[
W'(\lambda) = D + B^TH'(\lambda I - A')^{-1}B.
\] (2.4.8)
Since the factorization $W(\lambda) = L(-\lambda)^T DL(\lambda)$ is unconditionally stable, $W'(\lambda)$ has a minimal factorization

$$W'(\lambda) = L'(-\lambda)^T DL'(\lambda). \quad (2.4.9)$$

In particular it follows from Theorem 2.2.2 that $J(A', H') \neq \emptyset$ and $J(A^x, H') \neq \emptyset$. Since $A$ and $A^x$ are invertible by assumption, it follows from $J(A', H') \neq \emptyset$, $J(A^x, H) \neq \emptyset$ for any $A', H'$ close to $A, H$, respectively, that $\sigma(A) \cap i\mathbb{R} = \emptyset$ (as one easily sees by considering the canonical form). Since $W'$ factorizes as (2.4.9) with $L'$ close to $L$, there is a supporting projection $\pi'$ close to $\pi$. Hence Ker $\pi$ is unconditional (strongly) stable in $J(A, H)$. An analogous argument shows that $\operatorname{Im} \pi$ is unconditional (strongly) stable in $J(A^x, H)$.

With Lemmas 2.4.1 and 2.4.2 we can now give sufficient conditions for the existence of a factorization (2.4.1) which is (un)conditionally (strongly) stable. In case of unconditional stability these sufficient conditions are also necessary, provided $W(\lambda)$ is analytic and invertible at zero. In all cases we give a description of the stable factorizations.

**Theorem 2.4.3.** Let $W(\lambda) = D + C(\lambda I - A)^{-1}B$ be a minimal realization for a rational matrix function $W(\lambda)$ with $W(\lambda) = -[W(-\lambda)]^T$ ($\lambda$ not a pole of $W$) and having constant signature zero. Assume $W$ has no pole or zero at zero. Then the following holds:

(i) If the partial pole and zero multiplicities of $W$ at pure imaginary eigenvalues are even and the signs $\kappa_j$ in the canonical form of $(A, H)$ (respectively, $(A^x, H)$) corresponding to a pair of pure imaginary eigenvalues $\pm ib$ are all the same, then there exists a conditionally strongly stable factorization of the type (2.4.1).

(ii) Assume the conditions in (i) are satisfied. Assume further that conditions (a)–(c) below hold for the factorization $W(\lambda) = L(-\lambda)^T DL(\lambda)$. Then this factorization is conditionally strongly stable.

(a) For every nonzero real pole (zero) $\lambda_0$ of $W(\lambda)$ of geometric multiplicity larger than one, exactly one of the numbers $\lambda_0$, $-\lambda_0$ is a pole (zero) of $L(\lambda)$.

(b) For every real pole (zero) $\lambda_0$ of $W(\lambda)$ of geometric multiplicity one and even algebraic multiplicity, the algebraic multiplicity of $\lambda_0$ as a pole (zero) of $L(\lambda)$ is even.
(c) For every nonreal, non-pure-imaginary pole (zero) $a + ib$ of $W(\lambda)$ of geometric multiplicity larger than one, exactly one of the numbers $a + ib$, $-a + ib$ is a pole (zero) of $L(\lambda)$. In that case $a - ib$ or $-a - ib$, respectively, is also a pole (zero) of $L(\lambda)$.

(iii) The minimal factorization $W(\lambda) = L(-\lambda)^T DL(\lambda)$ is unconditionally stable if and only if $W$ has no pure imaginary poles or zeros and conditions (a)-(c) in (ii) hold. In this case every unconditionally stable factorization is unconditionally strongly stable.

Proof. Using Lemmas 2.4.1 and 2.4.2, the theorem is a direct consequence of Theorem 1.4 in [22].

We do not develop the notion of strong stability (conditional and uncondi-
tional) for symmetric minimal factorizations of the rational matrix functions $W(\lambda)$ with symmetry of the type $W(\lambda) = [W(\eta \lambda)]^T$, because in this case every (un)conditionally stable minimal factorization is automatically strongly (un)conditionally stable. Also, it turns out that for the functions $W(\lambda)$ with the symmetry $W(\lambda) = -W(\lambda)^T$ (which we consider below), strongly stable symmetric minimal factorizations never exist.

Next we consider the case $\eta = +1$. In this case $W(\lambda) = -W(\lambda)^T$ and we are interested in minimal factorizations of the type

$$W(\lambda) = L(\lambda)^T DL(\lambda). \quad (2.4.10)$$

As $J(A, H)$ is always nonempty for $(A, H) \in L_{\pi}(-1, 1)$ (cf. Theorem 1.2.4; also [22, Section 2.2]), in the following lemma the conditional and unconditional stability of $\text{Im } \pi$ and $\text{Ker } \pi$ are the same and will be referred to as stability.

Lemma 2.4.4. Let $W(\lambda) = D + C(\lambda I - A)^{-1}B$ be a minimal realization for a rational matrix function with $W(\lambda) = -W(\lambda)^T$, and assume $W(\lambda) = L(\lambda)^T DL(\lambda)$ is a minimal factorization, where $I(\lambda)$ has a minimal realization $I(\lambda) = I + C_0(\lambda I - A_1)^{-1}B_1$. Let $\pi$ be the corresponding supporting projection. Then the factorization is (un)conditionally stable if and only if the subspaces $\text{Im } \pi$ and $\text{Ker } \pi$ are (un)conditionally stable in $J(A^\pi, H)$ and $J(A, H)$, respectively. (Recall that $H = -H^T$ is the unique invertible real matrix such that $HA = A^T H$, $HB = C^T$, $-B^T H = C$.)
Proof. Suppose first the factorization is (un)conditionally stable. Let $(A', H') \in L_n((-1, +1))$ be such that $\|A - A'\| + \|H - H'\|$ is small. Construct

$$W'(\lambda) = D + B^T H' (\lambda I - A')^{-1} B.$$ 

One easily checks that $W'(\lambda) = - W'(\lambda)^T$. As $J(A', H') \neq \emptyset$ and $J(A'^*, H') \neq \emptyset$, automatically $W'(\lambda)$ has constant signature zero. As the factorization (2.4.10) is (un)conditionally stable, $W'(\lambda)$ has a minimal factorization

$$W'(\lambda) = I'(\lambda)^T D I'(\lambda)$$

with $\|\pi - \pi_1\|$ small, where $\pi_1$ is the supporting projection corresponding to this factorization. Hence Ker $\pi$ is (un)conditionally stable in $J(A, H)$. A similar argument shows that Ker $\pi$ is (un)conditionally stable in $J(A', H')$. The converse is proved as in Lemma 2.4.1. 

Before applying Lemma 2.4.4, let us recall from Theorem 2.1 in [22] the necessary and sufficient conditions for $M \in J(A, H)$ to be stable. We may assume $(A, H)$ is in the canonical form (as in Theorem 1.2.4)

$$A = \sum_{j=1}^{p} \left[ J_{n_j}(a_j) \oplus J_{n_j}(a_j)^T \right]$$

$$H = \bigoplus_{j=1}^{p} \left[ 0_{n_j} \oplus I_{n_j} \right] \oplus \bigoplus_{j=p+1}^{q} \left[ 0_{I_{2n_j}} \oplus I_{2n_j} \right].$$

There exists a stable $M \in J(A, H)$ if and only if the geometric multiplicity at each given eigenvalue of $A$ is two. In that case $M \in J(A, H)$ is stable if and only if conditions (a)-(d) below hold [we let $m_j = n_j/2$ if $n_j$ is even, and let $m_j = (n_j - 1)/2$ if $n_j$ is odd; $e_{jk}$ stands for the $k$th unit coordinate vector corresponding to the $j$th pair of blocks in (2.4.11)].

(a) $M \cap R(A, a_j) = \text{Ker}(A - a_j I)^{m_j}$ whenever $1 \leq j \leq p$ and $n_j$ is even;

(b) $M \cap R(\Lambda, a_j) = \text{Ker}(A - a_j I)^{m_j} \oplus \text{span}(ae_{j(n_j+1)} + \beta e_{j(n_j+m_j)})$ for some real $\alpha, \beta$ not both zero, whenever $1 \leq j \leq p$ and $n_j$ is odd;
(c) $M \cap R(A, a_j \pm ib_j) = \text{span}\{e_{jn+1}, \ldots, e_{qn}\}$ whenever $p + 1 \leq j \leq q$ and $n_j$ is even;
(d) $M \cap R(A, a_j \pm ib_j) = \text{span}\{e_{jn+1}, \ldots, e_{qn}, \alpha e_{jn+1} + \beta e_{qn-j}, \alpha e_{jn-1} + \beta e_{qn-j}\}$ for some real $\alpha, \beta$ not both zero, whenever $p + 1 \leq j \leq q$ and $n_j$ is odd.

Applying this fact in conjunction with Lemma 2.4.4, we arrive at the following result.

**Theorem 2.4.5.** Let $W(\lambda) = D + C(\lambda I - A)^{-1}B$ be a minimal realization for a rational matrix function with $W(A) = -W(A)^T$, and assume

$$W(\lambda) = L(\lambda)^TDL(\lambda)$$

(2.4.12)

is a minimal factorization with corresponding supporting projection $\pi$. The following statements are equivalent:

(i) the factorization (2.4.11) is conditionally stable,
(ii) the factorization (2.4.11) is unconditionally stable,
(iii) the geometric multiplicity of $W$ at each of its poles and zeros is two, and the conditions (a)-(d) above hold for $\text{Im} \, \pi$ as an element if $J(A^\pi, H)$ and for $\text{Ker} \, \pi$ as an element of $J(A, H)$.

We shall now consider Lipschitz stability. Let $W(\lambda)$ be a real rational matrix function with $W(\lambda) = -W(\eta\lambda)^T$ ($\eta = \pm 1$) and with minimal realization

$$W(\lambda) = D + C(\lambda I - A)^{-1}B.$$

A minimal factorization $W(\lambda) = L(\eta\lambda)^TDL(\lambda)$ where $L(\lambda)$ has a minimal realization $L(\lambda) = I + C_1(\lambda I - A_1)^{-1}B_1$ is called Lipschitz conditionally stable if there are $K > 0$, $\varepsilon > 0$ such that any $W'(\lambda) = D + C'(\lambda I - A')^{-1}B'$ with $W'(\lambda) = -W'(\lambda)^T$, a constant signature zero, and

$$\|A - A'\| + \|B - B'\| + \|C - C'\| < \varepsilon$$

admits a minimal factorization $W'(\lambda) = L'(\eta\lambda)^TDL'(\lambda)$ with $L'(\lambda) = I + C_1'(\lambda I - A_1')^{-1}B_1'$ that satisfies $\|A_1 - A_1'\| + \|B_1 - B_1'\| + \|C_1 - C_1'\| \leq K(\|A - A'\| + \|B - B'\| + \|C - C'\|)$. If we drop the requirement that $W'$ have constant signature zero, we obtain the definition of Lipschitz unconditional stable factorization. Again, in the case $\eta = 1$ the two notions of Lipschitz stable factorization coincide.
Lemma 2.4.6.

(i) Suppose $\eta = -1$. If for the supporting projection $\pi$ corresponding to the minimal factorization $W(\lambda) = L(-\lambda)^TDL(\lambda)$ we have that the subspaces $\text{Im } \pi$ and $\text{Ker } \pi$ are Lipschitz (un)conditionally stable, then the factorization is Lipschitz (un)conditionally stable. Conversely, if the factorization is Lipschitz unconditionally stable, then $\text{Im } \pi$ and $\text{Ker } \pi$ are Lipschitz unconditionally stable as elements of $J(A^T,H)$ and $J(A,H)$, respectively.

(ii) Suppose $\eta = 1$. Again let $\pi$ be the supporting projection corresponding to the minimal factorization $W(\lambda) = L(A)TDL(A)$. Then this factorization is Lipschitz (un)conditionally stable if and only if $\text{Im } \pi$ and $\text{Ker } \pi$ are Lipschitz (un)conditionally stable in $J(A^T,H)$ and $J(A,H)$, respectively.

Proof. The proof is analogous to the proofs of Lemmas 2.4.1, 2.4.2, and 2.4.4. We can apply now the corresponding theorems in [22].

Theorem 2.4.7. Suppose $\eta = -1$, and let $W(\lambda) = D + C(\lambda I - A)^{-1}B$ be a minimal realization for a function $W(\lambda)$ with constant signature zero. Assume $W$ has no pole or zero at zero. Let $W(\lambda) = L(-\lambda)^TDL(\lambda)$ be a minimal factorization with the corresponding supporting projection $\pi$. This factorization is unconditionally Lipschitz stable if and only if $W$ has no pure imaginary poles and zeros and $L(\lambda)$ and $L(-\lambda)^T$ have no common poles and zeros. In this case the factorization is also conditionally Lipschitz stable.

Proof. Suppose the factorization $W(\lambda) = L(-\lambda)^TDL(\lambda)$ is unconditionally Lipschitz stable. Then $\text{Im } \pi$ and $\text{Ker } \pi$ are unconditionally Lipschitz stable in $J(A^T,H)$ and $J(A,H)$, respectively, by Lemma 2.4.6. So they are certainly unconditionally stable. It then follows from Theorem 1.4 in [21] that $W$ has no poles and zeros on the imaginary axis. Further, Theorem 1.5 in [21] implies that $\text{Im } \pi$ and $\text{Ker } \pi$ are sums of root subspaces of $A^T$ and $A$, respectively. This immediately gives that $L(\lambda)$ and $L(-\lambda)^T$ have no common poles and zeros. In this case the subspaces $\text{Im } \pi$ and $\text{Ker } \pi$ are also conditionally Lipschitz stable, and hence the factorization is also conditionally Lipschitz stable by Lemma 2.4.6.

Theorem 2.4.8. Suppose $\eta = 1$, let $W(\lambda) = D + C(\lambda I - A)^{-1}B$ be a minimal realization for a function with constant signature zero, and let $W(\lambda) = L(\lambda)^TDL(\lambda)$ be a minimal factorization with corresponding supporting projection $\pi$. This factorization is (un)conditionally Lipschitz stable if and
only if the geometric multiplicity of $W$ at each of its poles and zeros is two and the partial multiplicities of $W$ at its poles and zeros are all one. In this case any minimal factorization $W(\lambda) = L(\lambda)^TDL(\lambda)$ is both unconditionally and conditionally Lipschitz stable.

**Proof.** Suppose the factorization is Lipschitz stable (unconditional or conditional). Then according to Lemma 2.4.6 the subspaces $\text{Im } \pi$ and $\text{Ker } \pi$ are Lipschitz stable. Theorem 2.9 in [21] then implies the statement concerning the poles and zeros of $W$. Conversely, suppose the geometric multiplicities of $W$ at each of its poles and zeros is two and the partial multiplicities are all one. In that case every element in $J(A, H)$ and every element in $J(A^*, H^*)$ is Lipschitz stable by Theorem 2.9 in [21]. It follows from Lemma 2.4.6 that any minimal factorization $W(\lambda) = L(\lambda)^TDL(\lambda)$ is Lipschitz stable.

3. **MATRICAL BOUNDARY VALUE PROBLEMS WITH INVERSE SYMMETRY**

Let $T$ and $A$ be real $n \times n$ matrices with symmetric and invertible $T$. Consider the following matricial boundary value problem:

$$
T \frac{d\psi(t)}{dt} = -A\psi(t);
$$

$$
P\psi(0) = \varphi; \quad \psi(t) \text{ is bounded on } 0 < t < \infty.
$$

(3.1)

Here $P : \mathbb{R}^n \to \mathbb{R}^n$ is a known projection, $\varphi \in \text{Im } P$ is given, and the unknown function $\psi(t)$ (0 $\leq$ $t$ $<$ $\infty$) takes values in $\mathbb{R}^n$. Such boundary value problems are commonplace in transport theory (see, e.g., [31, 15]), where they are considered mostly in the infinite dimensional framework. Typically, $\psi(t)$ takes values in the Hilbert space of square summable Lebesgue measurable functions on $[-1, 1]$, $T$ is the operator of multiplication by $t$, $A$ is a constant plus a certain self-adjoint integral operator, and $P$ is the orthogonal projection on this Hilbert space.

We shall be interested mainly in the existence of a unique solution for every $\varphi \in \text{Im } P$. In such a case we say that the problem (3.1) is *well posed*. Define $M$ to be the direct sum of the $T^{-1}A$-invariant root subspaces corresponding to the eigenvalues of $T^{-1}A$ in the open right half plane and the subspaces of the form $\text{Ker}(T^{-1}A - \lambda I)$, where $\lambda$ is a pure imaginary or
zero eigenvalue (if such exists) of $T^{-1}A$. The subspace $M$ will be called the indicator subspace. Clearly, $M \subseteq \mathbb{R}^n$. We then have the following easily proven fact:

**Proposition 3.1.** The system (3.1) is well posed if and only if the map $P|_M : M \to \text{Im } P$ is one to one and onto, where $M$ is the indicator subspace.

We introduce now an additional symmetry into the problem. A real $n \times n$ matrix $J$ is called an inverse symmetry for the problem (3.1) if

$$J^2 = I, \quad JT = - TJ^T, \quad JA = A^TJ^T. \quad (3.2)$$

In particular, $TJ$ is skew symmetric, so in order for an inverse symmetry to exist, the size $n$ of the matrices must be even (this will be assumed from now on in this section). The notion of inverse symmetry is natural from the point of view of transport theory (see [19, 15], where $J$ is often assumed to possess the additional property $J = J^T$; in the infinite dimensional framework described above, $J$ is given by $J\varphi(t) = \varphi(-t)$, $\varphi \in L_2[-1, 1]$.

The equalities (3.2) imply

$$(JT)(T^{-1}A) = -(T^{-1}A)^T(JT).$$

So $(T^{-1}A, JT) \in L_n(-1, -1)$. Conversely, if a matrix $J$ with $J^2 = I$ such that $(T^{-1}A, JT) \in L_n(-1, -1)$, then $J$ is an inverse symmetry for (3.1).

Observe that the problem (3.1) with the inverse symmetry (3.2) transforms to a problem of the same type under the transformation $T \to S^TTS$, $A \to S^TAS$, $P \to S^{-1}PS$ with the inverse symmetry $J \to S^TJS^{-1}$ (here $S$ is a real invertible matrix). This observation allows us to assume in the proofs, without loss of generality, that the pair $(T^{-1}A, JT)$ is in the canonical form.

We say that (3.1) is stably well posed if (3.1) is well posed and any nearby problem with analogous symmetries is also well posed. Formally this means the following: There exists $\varepsilon > 0$ such that every problem

$$\tilde{T} \frac{d\tilde{\psi}(t)}{dt} = - \tilde{A}\tilde{\psi}(t),$$

$$\tilde{P}\tilde{\psi}(0) = \tilde{\varphi}, \quad \tilde{\psi}(t) \text{ is bounded on } 0 < t < \infty, \quad (3.3)$$

with $\tilde{T} = T^T$, $\tilde{A}$ real, and $\tilde{P}$ a projection, and with the inverse symmetry $\tilde{J}$, is well-posed provided

$$||\tilde{T} - T|| + ||\tilde{A} - A|| + ||\tilde{P} - P|| + ||\tilde{J} - J|| < \varepsilon.$$
Observe that for $\varepsilon > 0$ small enough $\tilde{T}$ must be invertible and $\dim \text{Im} \tilde{P} = \dim \text{Im} P$.

We now state and prove the main result of this section.

**Theorem 3.2.** The following statements are equivalent:

(i) The problem (3.1) with the inverse symmetry $J$ is stably well posed.

(ii) There exists $\varepsilon > 0$ such that every problem

$$\tilde{T} \frac{d\tilde{\psi}(t)}{dt} = -\tilde{A}\tilde{\psi}(t),$$

$$P\tilde{\psi}(0) = \varphi; \quad \tilde{\psi}(t) \text{ is bounded on } 0 \leq t < \infty$$

is well posed provided $J\tilde{A} = \tilde{A}'J^T$ and $\|\tilde{A} - A\| < \varepsilon$.

(iii) $A$ is invertible, and each pure imaginary eigenvalue (if any) of $T^{-1}A$ has algebraic multiplicity equal to the geometric multiplicity with all signs $\kappa_j$ the same in the canonical form of $(T^{-1}A, JT)$.

The following proposition will be handy in the proof of Theorem 3.2.

**Proposition 3.3.** Let $(A, H) \in L_n(-1, -1)$. Then the only $A$-invariant $H$-neutral subspace (in $\mathbb{R}^n$) is the zero subspace if and only if all eigenvalues of $A$ are pure imaginary nonzero, for each eigenvalue $ib \ (b \neq 0)$ its geometric multiplicity coincides with its algebraic multiplicity, and for each pair of eigenvalues $\pm ib \ (b > 0)$ the signs $\kappa_j$ in the canonical form of $(A, H)$ are all equal.

**Proof.** If at least one of the conditions of the proposition concerning the spectral structure of $(A, H)$ does not hold, then the canonical form and Theorem 3.1 of [21] imply easily the existence of a nonzero $A$-invariant $H$-neutral subspace.

Conversely, assume all the specified conditions on the spectral structure of $(A, H)$ are satisfied. We prove that any nonzero $A$-invariant subspace is not $H$-neutral. Without loss of generality assume that $\sigma(A) = \{ \pm ib \}$ for some fixed $b > 0$ and that $(A, H)$ is in the canonical form

$$A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix},$$

$$H = \pm \left( \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right).$$
Let $x = (x_1, y_1, x_2, y_2, \ldots, x_{n/2}, y_{n/2})^T$ be a nonzero vector belonging to an $A$-invariant subspace $M$. Then also $y = (y_1, -x_1, y_2, -x_2, \ldots, y_{n/2}, -x_{n/2})^T$ belongs to $M$. But $x^THy = \pm \sum_{i=1}^{n/2} (x_i^2 + y_i^2) \neq 0$.

**Proof of Theorem 3.2.** We shall use in the proof the following well-known facts (see, e.g., [12] for the proofs):

(a) If $Q$ is a projection and $Q|_N : N \to \text{Im} Q$ is one to one and onto for some subspace $N$, then for any projection $Q'$ sufficiently close to $Q$ and any subspace $N'$ with $\theta(N', N)$ sufficiently small, the map $Q'|_N : N' \to \text{Im} Q'$ is again one to one and onto.

(b) Let $A$ be an $n \times n$ (real) matrix, and let $M (\subset \mathbb{R}^n)$ be an $A$-invariant subspace; write

$$M = M \cap N_1 + \cdots + M \cap N_p,$$

where $N_1, \ldots, N_p$ are all the root subspaces of $A$ corresponding either to real eigenvalues or to pairs of conjugate nonreal eigenvalues. Then there is a constant $K > 0$ such that for any $n \times n$ matrix $B$ sufficiently close to $A$ and any $B$-invariant subspace $M_B$ the inequalities

$$K^{-1}\theta(M_B, M) \leq \theta(M_B \cap N_{iB}, M \cap N_i) \leq K\theta(M_B, M), \quad i = 1, \ldots, p,$$

hold. Here $N_{iB}$ is the sum of the root subspaces of $B$ corresponding to the eigenvalues of $B$ that are close to the real eigenvalue or to the pair of conjugate nonreal eigenvalues of $A$ to which $N_i$ corresponds.

We prove first the equivalence of (i) and (iii). In view of fact (b) and Proposition 3.1 we have to consider only the case when

$$\sigma(T^{-1}A) = \{ \pm ib \}$$

for some $b \geq 0$. Assume first $b > 0$.

Suppose that (iii) holds. Then by Proposition 3.3 there is no nonzero $T^{-1}A$-invariant $JT$-neutral subspace. The same property should be true for real matrices $\tilde{A}$ and $\tilde{T} = \tilde{T}^T$ close to $A$ and $T$, respectively, and any inverse symmetry $\tilde{J}$ close to $J$ such that $(\tilde{T}^{-1}\tilde{A}, \tilde{J}\tilde{T}) \in L(-1, -1)$. Indeed, if this were not true, one could find sequences $\tilde{A}_m \to A$, $\tilde{T}_m \to T$, $\tilde{J}_m \to J$ ($m \to \infty$) such that there is a nonzero $\tilde{T}_m^{-1}\tilde{A}_m$-invariant $\tilde{J}_m\tilde{T}_m$-neutral subspace $N_m$. Any partial limit (which exists because of the compactness of the set of subspaces in $\mathbb{R}^n$) of the sequence $\{N_{m; m-1}\}$ would be nonzero $T^{-1}A$-invariant $JT$-neu-
tral, a contradiction with Proposition 3.3. In particular, by Proposition 3.3, for any triple of matrices \((\hat{\mathbf{A}}, \hat{T}, \hat{f})\) with the required symmetries which is sufficiently close to \((\mathbf{A}, \mathbf{T}, \mathbf{f})\), the spectrum of \(\hat{T}^{-1} \hat{\mathbf{A}}\) is purely imaginary and for each of its eigenvalues the geometric multiplicity coincides with the algebraic multiplicity. Hence the dimension of the indicator subspace remains constant, and by Proposition 3.1 the stable well-posedness follows.

Suppose now (still assuming \(b > 0\)) that (iii) does not hold. So

\[
T^{-1} \mathbf{A} = \bigoplus_{j=1}^{p} J_{n_j} \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix},
\]

\[
\mathbf{J} \mathbf{T} = \bigoplus_{j=1}^{p} \kappa_j \text{antidiag}(F^{n_j}, -F^{n_j}, \ldots, (-1)^{n_j-1}F^{n_j}),
\]

where \(\kappa_j\) are +1 or −1,

\[
\mathbf{F} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},
\]

and either at least one of the \(n_j\)'s is bigger than one, or all \(n_j\)'s are equal to 1 but not all \(\kappa_j\)'s are the same. We produce a small perturbation \(\mathbf{Z}\) of \(T^{-1} \mathbf{A}\) such that \((\mathbf{Z}, \mathbf{J} \mathbf{T}) \in L(-1, -1)\) and the indicator subspace \(\tilde{\mathbf{M}}\) for the problem

\[
T \psi' = -\hat{\mathbf{A}} \psi, \quad P \psi(0) = \varphi, \quad \psi \text{ bounded for } t \geq 0,
\]

where \(\hat{\mathbf{A}} = T \mathbf{Z}\), has the property that

\[
\dim \tilde{\mathbf{M}} = 2p.
\]

[The right-hand side of (3.5) is just the dimension of the indicator subspace for the original problem (3.1), (3.2).] This will show that (i) does not hold.

If one of the \(n_j\)'s (say, \(n_1\)) is even, then let \(\mathbf{Z}\) be the matrix obtained from \(T^{-1} \mathbf{A}\) by replacing the middle \(4 \times 4\) block in

\[
J_{n_1} \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}
\]
with

\[
\begin{bmatrix}
0 & b & -\varepsilon^2 & 0 \\
-b & 0 & 0 & -\varepsilon^2 \\
1 & 0 & 0 & b \\
0 & 1 & -b & 0
\end{bmatrix}.
\]

where \( \varepsilon \neq 0 \) is close to zero. It turns out (cf. the proof of Theorem 3.4 in [21]) that \((Z, JT) \in L(-1, -1)\) and that \( \pm i(b - \varepsilon) \) and \( \pm i(b + \varepsilon) \) are eigenvalues of \( Z \) with algebraic multiplicity one. Consequently, \( \dim \tilde{M} \geq 2(p + 1) \). If \( n_1 \) is odd \( \geq 3 \), then let \( Z \) be obtained from \( T^{-1}A \) by replacing the middle \( 2 \times 2 \) block in

\[
J_n\begin{pmatrix}
0 & b \\
-b & 0
\end{pmatrix},
\]

with

\[
\begin{bmatrix}
0 & b + \varepsilon \\
-b - \varepsilon & 0
\end{bmatrix}, \quad \varepsilon \neq 0.
\]

One checks that \((Z, JT) \in L(-1, -1)\) and that \( \dim \tilde{M} \geq 2(p + 1) \). It remains to consider the case when all \( n_j \)'s are equal to 1. For simplicity of notation assume \( p = 2 \); then \( \kappa_1 = -\kappa_2 \). Let

\[
Z = \begin{bmatrix}
0 & b & \varepsilon & 0 \\
-b & 0 & 0 & \varepsilon \\
\varepsilon & 0 & 0 & b \\
0 & \varepsilon & -b & 0
\end{bmatrix},
\]

where \( \varepsilon \) is a real number close to zero. One verifies that \((Z, JT) \in L(-1, -1)\), and the eigenvalues of \( Z \) are \( ib + \varepsilon, ib - \varepsilon \), and their complex conjugates. So \( \dim \tilde{M} = 2 \) (for \( \varepsilon \neq 0 \) and close to zero) but \( \dim M = 4 \), and we are done in this case.

Consider now the case when \( \sigma(T^{-1}A) = \{0\} \). We have to show that the problem \((3.1)\) is not stably well posed. Again, we produce a small perturbation \( Z \) of \( T^{-1}A \) such that \((Z, JT) \in L(-1, -1)\) and the indicator subspace
\[ \tilde{M} \text{ for the problem} \]
\[
T\psi' = - (T\mathcal{Z})\psi, \quad P\psi(0) = \varphi, \quad \psi \text{ bounded for } t \geq 0
\]
satisfies \( \dim \tilde{M} \neq \dim M \). Here \( M \) is the indicator subspace for the original problem. We take \( T^{-1}A \) and \( JT \) in the canonical form (as in Theorem 1.1.2):

\[
T^{-1}A = \bigoplus_{j=1}^{p} J_{2n_j}(0) \oplus \bigoplus_{j=1}^{q} \left\{ J_{2n_{p,j}+1}(0) \oplus \left[ - J_{2n_{p,j}+1}(0) \right] \right\}^T, \quad (3.6)
\]

\[
JT = \bigoplus_{j=1}^{p} \kappa_j F_{2n_j} \oplus \bigoplus_{j=1}^{q} \begin{bmatrix} 0 & I_{2n_{p,j}+1} \\ -I_{2n_{p,j}+1} & 0 \end{bmatrix}. \quad (3.7)
\]

So

\[
\dim M = p + 2q.
\]

If \( q \geq 1 \), then replacing the block \( J_{2n_{p+1}}(0) \oplus [ - J_{2n_{p+1}}(0) ]^T \) by \( J_{2n_{p+1}}(\varepsilon) \oplus [ - J_{2n_{p+1}}(\varepsilon) ]^T \), where \( \varepsilon > 0 \) is close to zero, we obtain the matrix \( Z \) such that \((Z, JT) \in L(-1, -1)\), and

\[
\dim \tilde{M} = p + 2(q - 1) + 1 = p + 2q - 1
\]

for the corresponding indicator space \( \tilde{M} \). If one of \( n_1, \ldots, n_p \), say \( n_1 \), is bigger than 1, let \( Z \) be obtained from \( T^{-1}A \) by adding the matrix with \( \varepsilon \) in the \((1,1)\) position, \(- \varepsilon \) in the \((2n_1, 2n_1)\) position, and zeros elsewhere. One verifies that for \( \varepsilon \neq 0 \)

\[
\dim \tilde{M} = p + 2q + 1.
\]

Finally, if \( n_1 = 1 \), then add to \( T^{-1}A \) a negative number close to zero in the \((1,2)\) position to obtain \( Z \).

This concludes the proof of (i) \( \Leftrightarrow \) (iii) in Theorem 3.2. A careful analysis of the proof reveals that (ii) \( \Rightarrow \) (iii) is proved as well (arguing by contradiction). As (i) \( \Rightarrow \) (ii) is obvious, the proof of Theorem 3.2 is completed.
4. STABLE EQUILIBRIA OF ONE PARAMETER SUBGROUPS OF CLASSICAL LIE GROUPS

In this section we propose a completely different point of view on the stability of invariant subspaces.

Let $G$ be a Lie group of $n \times n$ matrices (generally, with complex entries). The group $G$ acts naturally on the set of all subspaces in $\mathbb{C}^n$ (or $\mathbb{R}^n$, if the matrices in $G$ are real): if $g \in G$, and $M \subset \mathbb{C}^n$ is a subspace, then $g(M) = \{gx \mid x \in M\}$. Consider the one dimensional subgroups $\mathcal{J}_a = \{e^{ta}\}_{t \in \mathbb{R}}$ in $G$, where $a \in L$, the Lie algebra of $G$. A subspace $M \subset \mathbb{C}^n$ (or $\mathbb{R}^n$) is called an equilibrium of $\mathcal{J}_a$ if the action of $g \in \mathcal{J}_a$ leaves $M$ invariant, i.e., $g(M) = M$ for every $g \in \mathcal{J}_a$. Equivalently, $M$ is $a$-invariant. An equilibrium $M$ of $\mathcal{J}_a$ will be called stable if any nearby one dimensional subgroup has an equilibrium arbitrarily close to $M$. Formally, this means that given $\epsilon > 0$ there exists $\delta > 0$ such that for every $b \in L$ with $\|b - a\| < \delta$ there is an equilibrium $M'$ of $\mathcal{J}_b$ with the property that $\theta(M', M) < \epsilon$. If there exist constants $E, K > 0$ such that for every $b \in L$ with $\|b - a\| < E$ there is an equilibrium $M'$ of $\mathcal{J}_b$ with

$$\theta(M', M) \leq K\|b - a\|,$$

then we say that $M$ is Lipschitz stable. Thus, stability (or Lipschitz stability) of $M$ as an equilibrium is the same as stability (or Lipschitz stability) of $M$ as an $a$-invariant subspace, when the perturbations are restricted to the Lie algebra $L$. So many results on stability of invariant subspaces can be interpreted as results on stability of equilibria. We give just two examples of such interpretation.

Let $G = \text{SO}(p, p)$ be the group of $2p \times 2p$ real matrices $g$ with determinant 1 such that

$$g^T \begin{bmatrix} -I_p & 0 \\ 0 & I_p \end{bmatrix} g = \begin{bmatrix} -I_p & 0 \\ 0 & I_p \end{bmatrix}.$$

The corresponding Lie algebra $L$ consists of all $2p \times 2p$ real matrices

$$X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix}$$

such that $X_1$ and $X_3$ are skew symmetric and $X_2$ is arbitrary (see, e.g., [16]).
Denoting \( H = ( - I_p) \oplus I_p \), we see that \( X \in L \) if and only if \( (X, H) \in L(1, - 1) \). We invoke now Theorems 1.4 and 1.5 of [22] and obtain the following results:

**Theorem 4.1.**

(a) There exists an \( H \)-lagrangian subspace \( M \) which is a stable equilibrium of \( \mathcal{S}_X \) if and only if \( X \) has no pure imaginary or zero eigenvalues.

(b) If \( \sigma(X) \cap i\mathbb{R} = \emptyset \), then an \( H \)-lagrangian \( X \)-invariant subspace \( M \) is a stable equilibrium of \( \mathcal{S}_X \) if and only if all the following conditions hold:

(i) \( M \cap R(X; \lambda) \) is either (0) or \( R(X; \lambda) \) whenever \( \lambda \) is a positive eigenvalue of \( X \) with \( \dim \text{Ker}(X - \lambda I) < 1 \);

(ii) \( M \cap R(X; \lambda) \) is an arbitrary even dimensional subspace of \( R(X; \lambda) \) whenever \( \lambda \) is a positive eigenvalue of \( X \) with \( \dim \text{Ker}(X - \lambda I) = 1 \) and with even algebraic multiplicity;

(iii) \( M \cap R(X; a \pm ib) \) is either (0) or \( R(X; a \pm ib) \) whenever \( a \pm ib (a > 0, b \neq 0) \) are eigenvalues of \( X \) with geometric multiplicity at least 2 (i.e., there are at least two linearly independent eigenvectors of \( X \) in \( \mathbb{C}^{2p} \) corresponding to each eigenvalue \( a \pm ib \) and \( a - ib \)).

(c) If \( \sigma(X) \cap i\mathbb{R} = \emptyset \), then an \( H \)-lagrangian \( X \)-invariant subspace \( M \) is a Lipschitz stable equilibrium of \( \mathcal{S}_X \) if and only if \( M \) is spectral, i.e., \( M \cap R(X; \lambda) \) is either (0) or \( R(X; \lambda) \) for every positive eigenvalue \( \lambda \) of \( X \) and \( M \cap R(X; a \pm ib) \) is either (0) or \( R(X; a \pm ib) \) for every pair of nonreal eigenvalues \( a \pm ib (a > 0) \) of \( X \).

Two comments are in order. Firstly, because of the symmetry inherent in the requirement that \( M \) be \( H \)-lagrangian, the conditions (i)–(iii) concerning \( R(X; \lambda) \) with negative \( \lambda \) or \( R(X; a \pm ib) \) with \( a < 0 \) follow from the corresponding conditions (i)–(iii) as stated (i.e., with \( \lambda > 0 \) and \( a > 0 \)). Secondly, Theorem 1.4 in [22] provides criteria for the existence and description of stable and Lipschitz stable equilibria within the class of \( H \)-lagrangian subspaces. However, it is not difficult to see that the same result applies also in the case when general (i.e., not necessarily \( H \)-lagrangian) equilibria of a perturbed one parameter subgroup are considered.

Our second example is the group \( G = \text{Sp}(n) \) of all \( 2n \times 2n \) real matrices \( g \) such that

\[
g^T K g = K,
\]

where

\[
K = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}
\]
The corresponding Lie algebra is (see, e.g., [16])

\[
L = \left\{ \begin{bmatrix} X_1 & X_2 \\ X_3 & -X_1^T \end{bmatrix} \middle| X_{ij} \text{ real matrices, } X_2 = X_2^T, X_3 = X_3^T \right\}.
\]

So \( X \in L \) if and only if \((X, K) \in L_2(-1, -1)\). Now appeal to Theorems 3.4 and 3.6 in [21]:

**Theorem 4.2.**

(a) There exists a \( K \)-lagrangian stable equilibrium of \( \mathcal{E}_X \) if and only if \( \sigma(X) \cap \mathbb{R} = \emptyset \). In this case a \( K \)-lagrangian \( X \)-invariant subspace \( M \) is a stable equilibrium for \( \mathcal{E}_X \) if and only if conditions (i)–(iii) of Theorem 4.1 hold.

(b) If \( \sigma(X) \cap \mathbb{R} = \emptyset \), then a \( K \)-lagrangian subspace \( M \) is a Lipschitz stable equilibrium if and only if \( M \) is spectral.

The two comments made after Theorem 4.1 apply here as well.

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