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ON THE EXISTENCE OF FREE MODELS IN ABSTRACT ALGEBRAIC INSTITUTIONS*

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Abstract. To provide a formal framework for discussing specifications of abstract data types we restrict the notion of institution due to Goguen and Burstall (1984) which formalises the concept of a logical system for writing specifications, and deal with *abstract algebraic institutions.* These are institutions equipped with a notion of submodel which satisfy a number of technical conditions. Our main results concern the problem of the existence of free constructions in abstract algebraic institutions. We generalise a characterisation of algebraic specification languages that guarantee the existence of reachable initial models for any consistent set of axioms given by Mahr and Makowsky (1984). Then the more general problem of the existence of free functors (left adjoints to forgetful functors) for any theory morphism is analysed. We give a construction of a free model of a theory over a model of a subtheory (with respect to an arbitrary theory morphism) which requires only the existence of initial models. This yields a characterisation of strongly liberal abstract algebraic institutions. We also show how to specialise there characterisation results for the partial algebras.

Contents

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1. Introduction

An abstract data type may be specified by giving a signature and a set of axioms over this signature, which describes a class of many-sorted algebras that satisfy the axioms. Which formulae are actually accepted as axioms and what it means for an algebra to satisfy an axiom is determined by the *logical system* we use for our specifications.

The pioneering papers [1, 25, 27] used equational logic for this purpose (i.e., the only axioms involved were equations with the standard notion of satisfaction). Nowadays, however, examples of logical systems in use include first-order logic (with and without equality), Horn-clause logic, higher-order logic, infinitary logic, temporal logic and many others. Note that all these logical systems may be (and actually are) considered with or without predicates, admitting partiality of operations or not. This leads to different concepts of signature and of model, perhaps even more obvious in examples like polymorphic signatures, order-sorted signatures, continuous algebras or error algebras.

The informal notion of logical system has been formalised by Goguen and Burstall [21], who introduced for this purpose the notion of *institution* (which generalises the ideas of 'abstract model theory' [8]). An institution consists of a collection of 'abstract signatures' together with, for any 'signature' Σ, a set of Σ-sentences, a collection of Σ-models and a satisfaction relation between Σ-models and Σ-sentences. The only 'semantic' requirement ('satisfaction condition') is that, when we change signatures, the induced translations of sentences and models preserve the satisfaction relation. This satisfaction condition expresses the intentional independence of the meaning of specifications from the actual notation.

Specifications given in some standard institutions (e.g., in first-order logic) often are *loose*, i.e., admit many nonisomorphic models. These loose specifications may be very useful, but sometimes we want to use some fixed (up to isomorphism) data type as, for example, natural numbers or truth values. This requires some mechanism for imposing additional constraints on the models admitted by a specification.

The most widely accepted choice at this point is to require initiality (cf. [23] for an extensive treatment of this notion). In this approach, from among all possible models of a set of axioms, we choose as an acceptable realisation of the abstract data type only the unique (up to isomorphism) initial model. For some often-used institutions (e.g., equational logic) this turns out to be equivalent to two requirements:

- 'no junk': every data object can be constructed using only the operations in the signature,
- 'no confusion': two data objects are identified if and only if they can be proved equal using the axioms.

Unfortunately, not every class of models contains an initial model. Thus, if one wants to avoid proving the existence of initial models for each specification separately (see, e.g., [15, 42] where some results supporting such an approach are given), one has to use an institution that guarantees the existence of initial models of any consistent set of axioms. It is well known that, for example, equational logic has this property but first-order logic does not.

Mahr and Makowsky [29] (cf. also [37]) proved that the class of models of any specification in an algebraic specification language contains an initial algebra that satisfies the above-mentioned 'no junk' condition if and only if every formula in this language is expressible as a set of universal Horn formulae with possibly infinitely many premises. The key to this result was the characterisation of, respectively, free classes (due to Mal'cev [31]) and implicational classes ([3, 7]).

However, the initial algebra is not always the intended realisation of an abstract data type. Quite often we want some parts of a data type to be interpreted loosely—and some others to be interpreted in a standard 'initial' way *given an interpretation of these 'loose' parts.* In other words, we require that some part of a model must be a 'free extension' of some other part. This may be formally expressed using 'initially restricting algebraic theories' [34] or, more generally, data constraints as introduced in [13] (cf. also [17, 21]).

A characterisation of algebraic institutions which guarantee existence of such a free extension satisfying an appropriately generalised 'no junk' condition was given in [37]. The key result there was a construction of these free extensions which only requires the existence of initial models. Note that this is not a trivial result. Although it is well known that a free object is an initial object in an appropriate comma category, to prove its existence we still have to define (or rather 'code') this comma category in our underlying institution.

The importance of these characterisations was limited by the quite restrictive requirement of *algebraicity*, which means that only standard algebraic signatures and standard (total) algebras were considered (without fixing, however, the form of formulae).

However, one of the key results, the characterisation of implicational classes, was given in [3, 7] in a much more general categorical framework of a category of 'models' satisfying only rather mild requirements (we briefly review this result in Section 8).

In this paper we present an attempt to formulate the other results in a similarly general categorical framework. This leads to a characterisation of *abstract algebraic institutions* which admit free constructions, i.e., which guarantee the existence of free extensions (satisfying the generalised 'no junk' condition) of models along an arbitrary theory morphism.

By an abstract algebraic institution we mean (Section 3) an institution equipped with a notion of submodel and quotient model. This amounts to the requirement that for every signature Σ the category of Σ-models has a factorisation system. Moreover, we require that every ground variety w.r.t. this factorisation system is definable in the institution and that the institution satisfies the 'abstractness condition' (the satisfaction relation identifies all isomorphic models). Finally, we assume that the institution guarantees the existence of a diagram (in the sense of model theory) for any model. Some other restrictions are purely technical.

In Section 4 we prove that Mal'cev's characterisation of free classes holds in abstract algebraic institutions, which allows us to generalise the result due to Mahr and Makowsky. Section 5 presents a construction of a free extension of a model of a theory along an arbitrary theory morphism in an abstract algebraic institution based on the existence of initial models. We use this result in Section 6 to give a characterisation of abstract algebraic institutions which guarantee the existence of such free extensions satisfying a generalised 'no junk' condition. In Section 7 we point out that the requirement of liberality (cf. [21]) may perhaps be slightly too restrictive. Section 8 briefly reviews the characterisation of implicational classes [3, 7, 33] and applies this characterisation to reformulate the results presented in Sections 4 and 6. Finally, in Section 9 we specialise our result to give a characterisation of abstract algebraic institutions of *partial algebras* in which free reachable constructions exist. This also serves as an extensive example which illustrates the notions introduced and the results proved in the previous sections. Section 10 contains a brief summary of our results.

Throughout this paper we assume the reader to be familiar with basic notions of category theory (although not necessarily with any deep results). See [6, 26, 28] for the standard definitions of, e.g., category, functor, pushout, pullback, limit, colimit, cocontinuity etc, which we omit here.

2. Preliminaries

In this section, mainly to fix the notation, we briefly review basic notions, definitions, and facts used in the rest of this paper.

The central notion investigated in this paper is that of initiality: for an arbitrary category $\boldsymbol{K}$, an object $A \in |\boldsymbol{K}|$ is *initial* in a class of objects $K \subseteq |\boldsymbol{K}|$ if $A \in K$ and for any $B \in K$ there is exactly one morphism from A to B. Dually, an object $A \in |\boldsymbol{K}|$ is said to be *terminal* in a class of objects $K \subseteq |\boldsymbol{K}|$ if $A \in K$ and for any $B \in K$ there is exactly one morphism from B to A (despite its importance in algebraic specification (cf., e.g., [41]) we are not going to investigate the latter notion here in detail; we just use it in some proofs).

An *algebraic signature* is a pair $\langle S, \Omega\rangle$, where S is a set (of sort names) and Ω is a family of sets $\{\Omega_{w,s}\}_{w\in S^*, s\in S}$ (of operation names). We write $f: w \to s$ to denote $w \in S^*$, $s \in S$, $f \in \Omega_{w,s}$. An *algebraic signature morphism* $\sigma: \langle S, \Omega\rangle \to \langle S', \Omega'\rangle$ is a pair $\langle \sigma_{\text{sorts}}, \sigma_{\text{opns}}\rangle$, where $\sigma_{\text{sorts}}: S \to S'$ and σ_{opns} is a family of maps $\{\sigma_{w,s}: \Omega_{w,s} \to \Omega'_{\sigma^*(w),\sigma(s)}\}_{w\in S^*, s\in S}$, where $\sigma^*(s1, \ldots, sn)$ denotes $\sigma_{\text{sorts}}(s1), \ldots, \sigma_{\text{sorts}}(sn)$, for $s1, \ldots, sn \in S$. We will write $\sigma(s)$ for $\sigma_{\text{sorts}}(s)$, $\sigma(w)$ for $\sigma^*(w)$, and $\sigma(f)$ for $\sigma_{w,s}(f)$, where $f \in \Omega_{w,s}$.

The category of algebraic signatures ***AlgSig*** has algebraic signatures as objects and algebraic signature morphisms as morphisms; the composition of morphisms is the composition of their corresponding components as functions. (This obviously forms a category.)

Let $\Sigma = \langle S, \Omega\rangle$ be an algebraic signature.

A (*total*) *Σ-algebra A* consists of an S-indexed family of carrier sets $|A| = \{|A|_s\}_{s\in S}$ and for each $f: s1, \ldots, sn \to s$ a function $f_A: |A|_{s1} \times \cdots \times |A|_{sn} \to |A|_s$. For all Σ-algebras A and B, a *Σ-homomorphism* from A to B, $h: A \to B$, is a family of functions $\{h_s: |A|_s \to |B|_s\}_{s\in S}$ such that, for any $f: s1, \ldots, sn \to s$ and $a_1 \in |A|_{s1}, \ldots, a_n \in |A|_{sn}$, $h_s(f_A(a_1, \ldots, a_n)) = f_B(h_{s1}(a_1), \ldots, h_{sn}(a_n))$.

The category of (total) Σ-algebras $\boldsymbol{Alg}(\Sigma)$ has Σ-algebras as objects and Σ-homomorphisms as morphisms; the composition of homomorphisms is the composition of their corresponding components as functions. (This obviously forms a category.)

For any algebraic signature morphism $\sigma: \Sigma \to \Sigma'$ and Σ'-algebra A, the *σ-reduct* of A is the Σ-algebra $A|_\sigma$, defined by $|A|_\sigma|_s = |A|_{\sigma(s)}$, for $s \in S$ and $f_{A|_\sigma} = \sigma(f)_A$, for $f: w \to s$ in Σ. Similarly, for any Σ'-homomorphism $h: A \to B$ the σ-reduct of h is the Σ-homomorphism $h|_\sigma: A|_\sigma \to B|_\sigma$ defined by $(h|_\sigma)_s = h_{\sigma(s)}$, for $s \in S$. The mappings $A \mapsto A|_\sigma$, $h \mapsto h|_\sigma$ form a functor from $\boldsymbol{Alg}(\Sigma')$ to $\boldsymbol{Alg}(\Sigma)$, which we sometimes denote by $\boldsymbol{Alg}(\sigma)$. It is easy to see that we have in fact defined a (contravariant) functor $\boldsymbol{Alg}: \boldsymbol{AlgSig}^{\mathrm{op}} \to \boldsymbol{Cat}$ (where $\boldsymbol{Cat}$ is the category of all categories).[1]

It is well known (cf., e.g., [1]) that the category $\boldsymbol{Alg}(\Sigma)$ has an initial algebra T_Σ, which is (up to isomorphism) the algebra of *ground Σ-terms*. ($\boldsymbol{Alg}(\Sigma)$ also contains a terminal object, which is a Σ-algebra with exactly one element of every sort.) For any Σ-algebra A, for any ground Σ-term $t \in |T_\Sigma|_s$, $s \in S$, the unique Σ-homomorphism $h^A: T_\Sigma \to A$ determines the *value* $t_A = h^A_s(t)$ of t in the algebra A.

By a *ground Σ-equation* we mean any pair $\langle t, t' \rangle$ (written in the form $t = t'$) of ground Σ-terms of the same sort. We say that a Σ-algebra A *satisfies* a ground Σ-equation $t = t'$ if $t_A = t'_A$.

Let $\sigma: \Sigma \to \Sigma'$ be an algebraic signature morphism. The unique Σ-homomorphism from T_Σ to $T_{\Sigma'}|_\sigma$ determines a translation of ground Σ-terms to ground Σ'-terms; it maps any ground Σ-term t to the ground Σ'-term $\sigma(t)$ which results from t by substituting all the names of operations from Σ by their counterparts (given by σ) from Σ'. This, in turn, determines a translation of ground Σ-equations to ground Σ'-equations which maps any ground Σ-equation $t = t'$ to $\sigma(t) = \sigma(t')$. Notice that this again defines a functor from the category of algebraic signatures to the category of (discrete) categories (in fact, to the category of sets here).

In the theory of abstract data types it is often convenient to consider only algebras which are reachable, or generated by the empty set (cf., e.g., the 'generation principle' in [9]). A Σ-algebra A is called *reachable* if it satisfies the 'no junk' condition, i.e., if any element of A is the value in A of a ground Σ-term. This may be reformulated using the notion of a subalgebra (cf., e.g., [24]): for any Σ-algebra A, a Σ-subalgebra of A is a Σ-algebra B such that $|B| \subseteq |A|$ and the operations of A coincide with those of B on $|B|$. Since we are interested in characterising objects only up to

[1] Of course, we want $\boldsymbol{Cat}$ to be the category of 'large' categories (i.e., categories with *classes* of objects and morphisms) and so the collection of objects of $\boldsymbol{Cat}$ cannot be a class, which may raise some questions about set-theoretic foundations of this work. We do not discuss this point here, and we disregard other such foundational issues in this paper; we refer to, e.g., [26, 28] for a detailed discussion of proposed solutions to this problem.

isomorphism, we slightly generalise this standard formulation and say that B is a subalgebra of A if there is an *injective* Σ-homomorphism from B to A. A 'dual' notion is that of a quotient: a Σ-algebra B is a quotient of a Σ-algebra A if there is a *surjective* Σ-homomorphism from A to B. Now, a Σ-algebra A is reachable if it has no proper subalgebra (i.e., any subalgebra of A is isomorphic to A), or equivalently, if it is a quotient of the algebra of ground Σ-terms.

In this formulation, the above definition may be used to introduce a notion of reachability in an arbitrary category. What we need, however, is an appropriate generalisation of the concept of injective and, respectively, surjective homomorphisms. This is given when the category is equipped with a factorisation system (cf., e.g., [26]).

Let $\boldsymbol{K}$ be an arbitrary category.

By a *factorisation system* for $\boldsymbol{K}$ we mean a pair $\langle \boldsymbol{E}, \boldsymbol{M}\rangle$ such that:
- $\boldsymbol{E}$ is a class of epimorphisms in $\boldsymbol{K}$, $\boldsymbol{M}$ is a class of monomorphisms in $\boldsymbol{K}$,
- $\boldsymbol{E}$ and $\boldsymbol{M}$ are closed under composition and contain all isomorphisms in $\boldsymbol{K}$,
- every morphism in $\boldsymbol{K}$ has $\langle \boldsymbol{E}, \boldsymbol{M}\rangle$-factorisation, i.e., for any morphism f there are $e_f \in \boldsymbol{E}$ and $m_f \in \boldsymbol{M}$ such that $f = e_f ; m_f$,
- the $\langle \boldsymbol{E}, \boldsymbol{M}\rangle$-factorisations are unique up to isomorphism, i.e., for any $e1, e2 \in \boldsymbol{E}$ and $m1, m2 \in \boldsymbol{M}$, if $e1 ; m1 = e2 ; m2$, then there is an isomorphism i such that $e1 ; i = e2$ and $i ; m2 = m1$.

Notational remark

Throughout the paper the composition in any category is denoted by ; (semicolon) and written in the diagrammatic order. Identities are denoted by id (with indices, if necessary).

It is well known that $\boldsymbol{E}$ and $\boldsymbol{M}$ unambiguously determine one another, i.e., if $\langle \boldsymbol{E}1, \boldsymbol{M}\rangle$ and $\langle \boldsymbol{E}2, \boldsymbol{M}\rangle$ (respectively, $\langle \boldsymbol{E}, \boldsymbol{M}1\rangle$ and $\langle \boldsymbol{E}, \boldsymbol{M}2\rangle$) are factorisation systems for $\boldsymbol{K}$, then $\boldsymbol{E}1 = \boldsymbol{E}2$ (respectively, $\boldsymbol{M}1 = \boldsymbol{M}2$). In fact, $\boldsymbol{E}$ may be defined as the class of all epimorphisms for which the diagonal fill-in lemma (Lemma 2.1 below) holds. Note, however, that such a definition does not guarantee that $\langle \boldsymbol{E}, \boldsymbol{M}\rangle$ is a factorisation system.

For the rest of this section let us fix an arbitrary category $\boldsymbol{K}$ with a factorisation system $\langle \boldsymbol{E}, \boldsymbol{M}\rangle$. Sometimes we refer to elements of $\boldsymbol{E}$ and $\boldsymbol{M}$ as *factorisation epimorphisms* and *monomorphisms*, respectively. We assume that $\boldsymbol{K}$ is $\boldsymbol{E}$-co-well-powered, i.e. (see [26, Definition 17.15]), for every object $A \in |\boldsymbol{K}|$ there is a *set* of factorisation epimorphisms $E \subseteq \boldsymbol{E}$ with domain A such that for *every* $e \in \boldsymbol{E}$ with domain A there are an $e' \in E$ and an isomorphism i such that $e = e' ; i$. Moreover, we assume that $\boldsymbol{K}$ has an initial object Λ and all products (of sets of objects). For any object $A \in |\boldsymbol{K}|$ the unique morphism from Λ to A is denoted by h^A.

A standard example of a category with a factorisation system which satisfies all the assumptions listed above is the category $\boldsymbol{Alg}(\Sigma)$ of Σ-algebras (for any algebraic signature Σ) with the class of all surjective Σ-homomorphisms as factorisation epimorphisms and the class of all injective Σ-homomorphisms are factorisation monomorphisms.

2.1. Lemma (diagonal fill-in lemma). *For any morphisms* $f1, f2$, e, *and* m, *if* $f1; m = e; f2$, $e \in \boldsymbol{E}$ *and* $m \in \boldsymbol{M}$, *then there is a morphism* g *such that* $e; g = f1$ *and* $g; m = f2$.

2.2. Fact. *If* $e \in \boldsymbol{E}$ *and* $e;f \in \boldsymbol{M}$, *for some* f, *then* e *is an isomorphism. If* $m \in \boldsymbol{M}$ *and* $f; m \in \boldsymbol{E}$, *for some* f, *then* m *is an isomorphism.*

Proof. Let $e \in \boldsymbol{E}$ and $e;f \in \boldsymbol{M}$. Let $f = e1; m1$ with $e1 \in \boldsymbol{E}$ and $m1 \in \boldsymbol{M}$. Thus, $(e; e1); m1 \in \boldsymbol{M}$. By the uniqueness of factorisations, $e; e1$ is an isomorphism, which proves that e is an isomorphism itself since it is an epimorphism and a coretraction [26, Proposition 6.15]. The second part of this fact is obvious by duality. □

Now, for any object $A \in |\boldsymbol{K}|$, by a *subobject* of A we mean any object $B \in |\boldsymbol{K}|$ together with a morphism $m: B \to A$ such that $m \in \boldsymbol{M}$. Similarly, by a *quotient* of A we mean any object B together with a morphism $e: A \to B$ such that $e \in \boldsymbol{E}$. Notice that, in the category of Σ-algebras with the usual factorisation system (defined above), subobjects and quotients may be identified with, respectively, subalgebras and quotient algebras, as expected.

We say that an object $A \in |\boldsymbol{K}|$ is *reachable* if it has no proper subobject, i.e., if every morphism $m \in \boldsymbol{M}$ with codomain A is an isomorphism. Again, in the category of Σ-algebras this gives the same notion of reachability as defined previously.

We say that a class $K \subseteq |\boldsymbol{K}|$ of objects of $\boldsymbol{K}$ is *closed under*:

- *isomorphism* if, for any isomorphism i, if the domain of i belongs to K, then so does its codomain,
- *products* if, for any set $F \subseteq K$, the product of F belongs to K,
- *nonempty products* if for any nonempty set $F \subseteq K$, the product of F belongs to K,
- *subobjects* ('submodels') if, for any morphism $m \in \boldsymbol{M}$, if the codomain of m belongs to K, then so does its domain,
- *quotients* ('homomorphic images') if, for any morphism $e \in \boldsymbol{E}$, if the domain of e belongs to K, then so does its codomain,
- '*extensions*' if, for any morphism f, if the domain of f belongs to K, then so does its codomain.

Throughout the rest of this paper we assume that all classes of objects of any category discussed here are closed under isomorphism.

For any object $A \in |\boldsymbol{K}|$, $\boldsymbol{Ext}(A)$ denotes the least class of objects in $\boldsymbol{K}$ which contains A and is closed under extensions, i.e., $B \in \boldsymbol{Ext}(A)$ if and only if there is a morphism from A to B.

A class $K \subseteq |\boldsymbol{K}|$ is called a *variety* (respectively, *strict quasi-variety*, *quasi-variety*) if it is closed under quotients, subobjects, and products (respectively, under subobjects and products, under subobjects and nonempty products). $K \subseteq |\boldsymbol{K}|$ is called a *ground variety* if it is of the form $\boldsymbol{Ext}(A)$ for some reachable object $A \in |\boldsymbol{K}|$.

The following may be viewed as an analysis of the basic properties of reachable objects in an arbitrary category with a factorisation system. Although the standard algebraic versions of these facts are well known in the folklore of the theory of algebraic specifications, it is worth noting that their formulations (and proofs) in the abstract framework of an arbitrary category seem to be more intuitive and simple.

2.3. Fact. (1) *$A \in |\boldsymbol{K}|$ is reachable iff $h^A \in \boldsymbol{E}$.*

(2) *If $A \in |\boldsymbol{K}|$ is reachable, then, for every $B \in |\boldsymbol{K}|$, there is at most one morphism from A to B.*

(3) *If $A, B \in |\boldsymbol{K}|$, B is reachable, and $f: A \to B$, then $f \in \boldsymbol{E}$.*

(4) *Every object $A \in |\boldsymbol{K}|$ has a unique (up to isomorphism) reachable subobject.*

(5) *If $A \in |\boldsymbol{K}|$ is reachable, then it is initial in $\boldsymbol{Ext}(A)$.*

(6) *If $A \in |\boldsymbol{K}|$ is reachable, then $\boldsymbol{Ext}(A)$ is closed under products, subobjects, and quotients, i.e., any ground variety is a variety.*

Proof. (1) ($\Rightarrow$): Let $h^A = e;m$, for some $e \in \boldsymbol{E}$ and $m \in \boldsymbol{M}$. By hypothesis, m is an isomorphism, which proves that $h^A \in \boldsymbol{E}$.

($\Leftarrow$): Consider $m: B \to A$, $m \in \boldsymbol{M}$. Obviously, $h^B; m = h^A \in \boldsymbol{E}$. Thus, by Fact 2.2, m is an isomorphism.

(2) Let $f1, f2: A \to B$. By initiality of Λ we have $h^A; f1 = h^B = h^A; f2$. Thus, since, by (1), h^A is an epimorphism, $f1 = f2$.

(3) Let $f = e1; m1$, with $e1 \in \boldsymbol{E}$ and $m1 \in \boldsymbol{M}$. Since B is reachable, $m1$ is an isomorphism and so $f = e1; m1 \in \boldsymbol{E}$.

(4) Let $h^A = e; m$, with $e \in \boldsymbol{E}$ and $m \in \boldsymbol{M}$. Then the domain of m is reachable (by (1)) and is a subobject of A. Its uniqueness follows from the uniqueness of factorisations (and (1)).

(5) Obvious by (2) and the definition of $\boldsymbol{Ext}$.

(6) Closure of $\boldsymbol{Ext}(A)$ under products and quotients is obvious. Let $f: A \to B$, $m: C \to B$, with $m \in \boldsymbol{M}$. By initiality of Λ we have $h^A; f = h^C; m$, with $h^A \in \boldsymbol{E}$ (by (1)) and $m \in \boldsymbol{M}$. Thus, by Lemma 2.1 there is a morphism from A to C, which completes the proof. □

2.4. Lemma. *Any nonempty quasi-variety has a reachable initial object.*

Proof. Let $K \subseteq |\boldsymbol{K}|$ be a nonempty class closed under nonempty products and subobjects. Let K_r be a *set* of reachable elements of K such that every reachable element of K is isomorphic to an element of K_r (it exists since K is $\boldsymbol{E}_\Sigma$-co-well-powered and reachable objects are quotients of Λ). Now, the reachable subobject of the product of K_r is a reachable initial object in K: it exists since every object has a reachable subobject (Fact 2.3(4)), it belongs to K since K is a quasi-variety (and K_r is nonempty), there is a morphism from it to any element of K (by the construction and Fact 2.3(4)) and this morphism is unique (by Fact 2.3(2)). □

3. Abstract algebraic institutions

Following [21] we introduce institutions to formalise the notion of a logical system for writing specifications. An institution consists of a collection of signatures together with, for any signature Σ, collections of Σ-sentences and of Σ-models and a satisfaction relation between Σ-models and Σ-sentences. Note that signatures are arbitrary abstract objects in this approach, not necessarily the usual algebraic signatures used in many standard approaches to algebraic specification (see, e.g., [1]). The only 'semantic' requirement is that when we change signatures, the induced translations of sentences and models preserve the satisfaction relation. This condition expresses the intended independence of the meaning of a specification from the actual notation.

The work of Barwise [8] on abstract model theory, although focussed on purely model-theoretic problems, is similar in intent to the theory of institutions. Note, however, that the notions used there and the conditions they must satisfy are more restrictive and rule out some of the examples we would like to deal with in the theory of specifications.

3.1. Definition ([21]). An *institution* INS consists of:
- a category $\mathbf{Sign}_{\mathrm{INS}}$ (of signatures),
- a functor $\mathbf{Sen}_{\mathrm{INS}}:\mathbf{Sign}_{\mathrm{INS}}\to\mathbf{Cat}$ such that, for any signature Σ, $\mathbf{Sen}_{\mathrm{INS}}(\Sigma)$ is a discrete category. $\mathbf{Sen}_{\mathrm{INS}}$ gives for any signature Σ the class of *Σ-sentences* and for any signature morphism $\sigma:\Sigma\to\Sigma'$ the function $\mathbf{Sen}_{\mathrm{INS}}(\sigma):\mathbf{Sen}_{\mathrm{INS}}(\Sigma)\to \mathbf{Sen}_{\mathrm{INS}}(\Sigma')$ translating Σ-sentences to Σ'-sentences,
- a functor $\mathbf{Mod}_{\mathrm{INS}}:\mathbf{Sign}^{\mathrm{op}}_{\mathrm{INS}}\to\mathbf{Cat}$. $\mathbf{Mod}_{\mathrm{INS}}$ gives for any signature Σ the category of *Σ-models* and for any signature morphism $\sigma:\Sigma\to\Sigma'$ the *σ-reduct functor* $\mathbf{Mod}_{\mathrm{INS}}(\sigma):\mathbf{Mod}_{\mathrm{INS}}(\Sigma')\to\mathbf{Mod}_{\mathrm{INS}}(\Sigma)$ translating Σ'-models to Σ-models,
- a satisfaction relation $\models_{\Sigma,\mathrm{INS}}\subseteq|\mathbf{Mod}_{\mathrm{INS}}(\Sigma)|\times|\mathbf{Sen}_{\mathrm{INS}}(\Sigma)|$ for each signature Σ,

such that the following 'satisfaction condition' holds:

For any signature morphism $\sigma:\Sigma\to\Sigma'$ the translations $\mathbf{Mod}_{\mathrm{INS}}(\sigma)$ of models and $\mathbf{Sen}_{\mathrm{INS}}(\sigma)$ of sentences preserve the satisfaction relation, i.e., for any $\varphi\in|\mathbf{Sen}_{\mathrm{INS}}(\Sigma)|$ and $M'\in|\mathbf{Mod}_{\mathrm{INS}}(\Sigma')|$,

$$M'\models_{\Sigma',\mathrm{INS}}\mathbf{Sen}_{\mathrm{INS}}(\sigma)(\varphi)\ \text{ iff }\ \mathbf{Mod}_{\mathrm{INS}}(\sigma)(M')\models_{\Sigma,\mathrm{INS}}\varphi.$$

We can use the definitions from Section 2 and give a very simple example of an institution. We define the institution of ground equations GEQ as follows:
- $\mathbf{Sign}_{\mathrm{GEQ}}$ is the category of algebraic signatures $\mathbf{AlgSig}$.
- For any algebraic signature Σ, $\mathbf{Sen}_{\mathrm{GEQ}}(\Sigma)$ is the set of all ground Σ-equations; for any algebraic signature morphism $\sigma:\Sigma\to\Sigma'$, $\mathbf{Sen}_{\mathrm{GEQ}}(\sigma)$ maps any ground Σ-equation $t=t'$ to the ground Σ'-equation $\sigma(t)=\sigma(t')$.
- For any algebraic signature Σ, $\mathbf{Mod}_{\mathrm{GEQ}}(\Sigma)$ is $\mathbf{Alg}(\Sigma)$; for any algebraic signature morphism $\sigma:\Sigma\to\Sigma'$, $\mathbf{Mod}_{\mathrm{GEQ}}(\sigma)$ is the functor $\_|_{\sigma}:\mathbf{Alg}(\Sigma')\to\mathbf{Alg}(\Sigma)$.

- For any algebraic signature Σ, $\models_{\Sigma,\mathrm{GEQ}}$ is the satisfaction relation as defined in Section 2.

It is easy to check that GEQ is an institution (the satisfaction condition is a special case of [13, Satisfaction Lemma]).

Notational conventions:
- We omit subscripts (INS, Σ) whenever possible.
- For any signature morphism $\sigma: \Sigma \to \Sigma'$, $\boldsymbol{Sen}(\sigma)$ is denoted just by σ and $\boldsymbol{Mod}(\sigma)$ is denoted by $\_|_\sigma$ (i.e., for $\varphi \in |\boldsymbol{Sen}(\Sigma)|$, $\sigma(\varphi)$ stands for $\boldsymbol{Sen}(\sigma)(\varphi)$, and, e.g., for $M' \in |\boldsymbol{Mod}(\Sigma')|$, $M'|_\sigma$ stands for $\boldsymbol{Mod}(\sigma)(M')$).
- For $\Phi \subseteq |\boldsymbol{Sen}(\Sigma)|$ and $K \subseteq |\boldsymbol{Mod}(\Sigma)|$, we write $K \models \Phi$ with the obvious meaning.
- For any signature Σ and $\Phi \subseteq |\boldsymbol{Sen}(\Sigma)|$, $\mathrm{Mod}(\Phi)$ denotes the collection of all Σ-models M that satisfy Φ (i.e., such that $M \models \Phi$).

For any signature Σ, morphisms of the category of Σ-models are called *Σ-morphisms.* We identify any class K of Σ-models with the full subcategory of $\boldsymbol{Mod}(\Sigma)$ with objects K. We say that a class of Σ-models K is *definable* if there is a set of Σ-sentences $\Phi \subseteq |\boldsymbol{Sen}(\Sigma)|$ such that K consists of exactly those Σ-models that satisfy Φ, i.e., $K = \mathrm{Mod}(\Phi)$. For any signature morphism $\sigma: \Sigma \to \Sigma'$ by a *σ-expansion* of a Σ-model M we mean any Σ'-model M' such that $M'|_\sigma = M$. Similarly, by a *σ-expansion* of a Σ-morphism f we mean any Σ'-morphism f' such that $f'|_\sigma = f$.

However, the very elegant and extremely general framework of an arbitrary institution is too general for our purposes. For example, the institution GEQ of ground equations, which is a typical example of a logical system used in the theory of algebraic specifications, has a number of properties which are useful in this context but which do not have to hold in an arbitrary institution (as they are not required by the definition). First of all, GEQ comes naturally equipped with factorisation systems. As mentioned in Section 2, for any algebraic signature Σ, the category $\boldsymbol{Alg}(\Sigma)$ of Σ-algebras has a factorisation system formed by the class $\boldsymbol{E}_\Sigma$ of all surjective Σ-homomorphisms as factorisation epimorphisms and the class $\boldsymbol{M}_\Sigma$ of all injective Σ-homomorphisms as factorisation monomorphisms. Moreover:

(1) The category ***AlgSig*** is finitely cocomplete (see [22, Proposition 5]). Note that the initial algebraic signature is the one with no sorts (and, hence, no operations either). Moreover, the category of algebras over this signature contains exactly one (empty) algebra and exactly one (empty) homomorphism, and so it is a terminal object in the category ***Cat*** of all categories. A more general property holds as well: $\boldsymbol{Mod}_{\mathrm{GEQ}}: \boldsymbol{AlgSig}^{\mathrm{op}} \to \boldsymbol{Cat}$ translates finite colimits in ***AlgSig*** to finite limits in ***Cat*** (see [10]).

(2) For any algebraic signature Σ, the category $\boldsymbol{Alg}(\Sigma)$ has an initial object (the algebra of ground Σ-terms) and all products (of sets of Σ-algebras). The product of a set of Σ-algebras is just the Cartesian product with the operations defined componentwise. Moreover, although the collection of all Σ-algebras forms a proper class, any Σ-algebra has up to isomorphism only a *set* of quotients.

(3) For any algebraic signature morphism $\sigma: \Sigma \to \Sigma'$, the σ-reduct functor preserves subalgebras (i.e., injectivity of homomorphisms is preserved by the reduct functors). Similarly, the reduct functors preserve products of algebras.

(4) Ground equations do not allow to distinguish between isomorphic algebras, i.e., all isomorphic algebras satisfy exactly the same equations.

(5) For any algebraic signature Σ, for any reachable Σ-algebra A there is a Σ-homomorphism from A to a Σ-algebra B if and only if B satisfies all the ground equations which hold in A.

(6) We can use the method of diagrams (in the sense of model theory, cf. [16]): for any algebraic signature Σ, for any Σ-algebra A, we can form an algebraic signature $\Sigma(A)$ (called the diagram signature for A) which is the extension of Σ by a constant of the appropriate sort for each element of $|A|$. Then:

(a) A has a natural expansion to a $\Sigma(A)$-algebra $E(A)$, where the new constants are interpreted as the corresponding elements of $|A|$. Obviously, $E(A)$ is reachable.

(b) For any Σ-algebra B, any Σ-homomorphism $h: A \to B$ determines an expansion of B to a $\Sigma(A)$-algebra $E_h(B)$ where the new constants are interpreted as values of h on the corresponding elements of $|A|$. Moreover, this expansion does not depend on the decomposition of h, i.e., for any Σ-algebra C and Σ-homomorphisms $h1: A \to C$ and $h2: C \to B$ such that $h = h1; h2$, $h2$ (or more precisely, its underlying map) is a $\Sigma(A)$-homomorphism from $E_{h1}(C)$ to $E_h(B)$.

(c) Intuitively, the expansions described above do not introduce more structure than necessary to make A reachable.

Guided by these properties we restrict the notion of institution and only deal with *abstract algebraic institutions*, which are institutions with factorisation systems subject to several technical conditions.

3.2. Definition. An *abstract algebraic institution* is an institution INS together with, for any signature Σ, a factorisation system $\langle \boldsymbol{E}_\Sigma, \boldsymbol{M}_\Sigma \rangle$ for $\boldsymbol{Mod}(\Sigma)$ such that the following conditions hold:

(1) The category of signatures is finitely cocomplete and ***Mod*** preserves finite colimits (i.e., ***Mod*** translates finite colimits in ***Sign*** to limits in ***Cat***).

(2) For any signature Σ, the category $\boldsymbol{Mod}(\Sigma)$ of Σ-models has an initial object and all products (of sets of models). Moreover, it is $\boldsymbol{E}_\Sigma$-co-well-powered.

(3) For any signature morphism $\sigma: \Sigma \to \Sigma'$ the σ-reduct functor preserves submodels (i.e., for any $m' \in \boldsymbol{M}_{\Sigma'}$, $m'|_\sigma \in \boldsymbol{M}_\Sigma$) and products.

(4) (*Abstraction condition*): For any signature Σ, $A, B \in |\boldsymbol{Mod}(\Sigma)|$ and $\varphi \in |\boldsymbol{Sen}(\Sigma)|$, if A and B are isomorphic, then $A \models \varphi$ iff $B \models \varphi$.

(5) (*Definability of ground varieties*): For any signature Σ, any ground variety of Σ-models is definable.

(6) (*Existence of diagrams*): For any signature Σ and model $M \in |\boldsymbol{Mod}(\Sigma)|$ there is a signature $\Sigma(M)$ and a signature morphism $\iota: \Sigma \to \Sigma(M)$ such that:

(a) M has a reachable ι-expansion $E(M)$.

(b) For any Σ-morphism $f: M \to N$ there is a unique ι-expansion of N, $E_f(N)$, such that f has a (unique, by Fact 2.3(2)) ι-expansion from $E(M)$ to $E_f(N)$. Moreover, for any Σ-morphisms $f: M \to N1$ and $h: N1 \to N2$, h has a unique ι-expansion, denoted by $E(h)$, from $E_f(N1)$ to $E_{f;h}(N2)$.

(c) For models 'containing' $E(M)$, the ι-reduct functor preserves quotients, i.e., for any factorisation epimorphism $e \in \boldsymbol{E}_{\Sigma(M)}$ with domain in $\boldsymbol{Ext}(E(M))$, $e|_\iota \in \boldsymbol{E}_\Sigma$ as well.

If this is the case we call $\Sigma(M)$ the *diagram signature* for M with the signature inclusion ι and we call $E(M)$ the *diagram expansion* of M.

By the basis of an abstract algebraic institution we mean the triple $\langle \boldsymbol{Sign}, \boldsymbol{Mod}, \{\langle \boldsymbol{E}_\Sigma, \boldsymbol{M}_\Sigma \rangle\}_{\Sigma \in |\boldsymbol{Sign}|} \rangle$.

Discussion. The above requirements may seem to be rather restrictive. We feel, however, that they are quite natural and, moreover, they are satisfied in a number of standard institutions such as, for example, standard algebraic institutions (i.e., the institutions with the same basis as GEQ, see [37]), institutions of partial algebras (see, e.g., [36], and also Section 6), and also for order-sorted [20] and polymorphic [32] signatures, error [18, 19] and continuous ([2], also [40]) algebras. We hope that this should be easy to see by comparison with the list of properties of the standard algebraic institution of ground equations above. Perhaps the only nontrivial requirement is that of the existence of diagram signatures—however, in all the institutions mentioned above they may be formed in exactly the same way as in the standard algebraic case.

Condition (1) is quite a standard requirement which appears whenever the institution is supposed to provide some tools for 'putting things together' (cf., e.g., [13, 17, 36]). The following lemma is a consequence of our assumption that the functor ***Mod*** translates pushouts in ***Sign*** to pullbacks in ***Cat*** (we omit a simple proof based on the construction of pullbacks in ***Cat***).

3.3. Lemma. *If*

$$\begin{array}{ccc} \Sigma' & \xrightarrow{\sigma'} & \Sigma 1' \\ {\scriptstyle\iota}\uparrow & & \uparrow{\scriptstyle\iota'} \\ \Sigma & \xrightarrow[\sigma]{} & \Sigma 1 \end{array}$$

*is a pushout in **Sign**, then, for any two models $M1 \in |\boldsymbol{Mod}(\Sigma 1)|$ and $M' \in |\boldsymbol{Mod}(\Sigma')|$ such that $M1|_\sigma = M'|_\iota$, there is a unique model $M1' \in |\boldsymbol{Mod}(\Sigma 1')|$ such that $M1'|_{\sigma'} = M'$ and $M1'|_{\iota'} = M1$. Moreover, for any two morphisms $f1$ in $\boldsymbol{Mod}(\Sigma 1)$ and f' in $\boldsymbol{Mod}(\Sigma')$*

such that $f1|_{\sigma} = f'|_{\iota}$, *there is a unique morphism* $f1'$ *in* $\mathbf{Mod}(\Sigma 1')$ *such that* $f1'|_{\sigma'} = f'$ *and* $f1'|_{\iota'} = f1$.

The existence of factorisation systems together with (2) and (3) provide an institution with notions of submodel and quotient model which are necessary to formulate our results. Condition (4) just says that we want to define and consider models only up to isomorphism. Condition (5) guarantees that abstract algebraic institutions have a certain minimal specification power. In the standard algebraic case it reduces to the requirement of expressibility of ground equations. Finally, (6) guarantees that in abstract algebraic institutions we can use the method of diagrams (in the sense of, e.g., [16]). This corresponds to the requirement in [29] that an algebraic specification language must be 'rich enough'.

(Note that (b) is equivalent to the requirement that the ι-reduct functor is an isomorphism of the comma categories (cf. [26, Definition 4.18]) $\langle E(M), \mathbf{Mod}(\Sigma(M))\rangle$ and $\langle M, \mathbf{Mod}(\Sigma)\rangle$. In fact, all our results remain correct if we weaken this assumption and only require these two comma categories to be equivalent.)

The uniqueness of $E(h)$ required in (b) implies the following fact (notation from Definition 3.2).

3.4. Fact. *For models 'containing'* $E(M)$ *the ι-reduct functor reflects isomorphisms, i.e., for any* $\Sigma(M)$*-morphism f with domain in* $\mathbf{Ext}(E(M))$, *if* $f|_{\iota}$ *is an isomorphism, then so is f.*

Let INS be an abstract algebraic institution.

By a *specification* in INS we mean a pair $\langle \Sigma, \Phi\rangle$, where Σ is a signature and Φ is a set of Σ-sentences. Note, however, that when dealing with a specification we can use not only the properties explicitly stated in Φ but also all their logical consequences, i.e., sentences that hold in any model of the specification. By a *theory* we mean a specification in which the set of sentences already contains all its logical consequences. This can be stated more formally as follows: for any signature Σ and $K \subseteq |\mathbf{Mod}(\Sigma)|$ let $\mathrm{Sen}(K)$ denote the set of all Σ-sentences that hold in K, i.e., $\mathrm{Sen}(K) = \{\varphi \in |\mathbf{Sen}(\Sigma)| \mid K \models \varphi\}$. A theory is a specification $\langle \Sigma, \Phi\rangle$, where $\Phi = \mathrm{Sen}(\mathrm{Mod}(\Phi))$. Obviously, any specification $\langle \Sigma, \Phi\rangle$ *induces* the smallest theory which contains it, namely $\langle \Sigma, \mathrm{Sen}(\mathrm{Mod}(\Phi))\rangle$. If $T = \langle \Sigma, \Phi\rangle$ is a theory, we use the notation $\mathrm{Mod}(T)$ for the collection of all T-models, i.e., all Σ-models that satisfy Φ. For any signature Σ, by the *empty Σ-theory* we mean the theory consisting of all trivial Σ-sentences, i.e., the theory $\langle \Sigma, \mathrm{Sen}(|\mathbf{Mod}(\Sigma)|)\rangle$.

For any two theories $T1 = \langle \Sigma 1, \Phi 1\rangle$ and $T2 = \langle \Sigma 2, \Phi 2\rangle$, by a *theory morphism* from $T1$ to $T2$, $\sigma : T1 \to T2$, we mean a signature morphism $\sigma : \Sigma 1 \to \Sigma 2$ such that $\sigma(\varphi) \in \Phi 2$ for any $\varphi \in \Phi 1$.

Note that if $\sigma : T1 \to T2$ is a theory morphism, then the σ-reduct functor $\_|_{\sigma}$ translates $T2$-models to $T1$-models, $\_|_{\sigma} : \mathrm{Mod}(T2) \to \mathrm{Mod}(T1)$.

4. Initiality

It is often the case that from among all admissible models of a theory we would like to pick out only the initial one(s).

Recall that a Σ-model A is *initial* in a class of Σ-models K if $A \in K$ and for any $B \in K$ there is exactly one Σ-morphism from A to B. Any two initial models of a class K are isomorphic, which justifies the use of the expression 'the initial model'. By the initial model of a theory we mean the initial model in the class of all models of this theory.

In general, the class of models of a theory need not contain an initial model. However, there are abstract algebraic institutions in which any theory has an initial model. In this section we try to characterise these institutions.

4.1. Definition. We say that an institution *admits initial semantics* if any nonempty class of models definable in this institution contains an initial model. We say that an abstract algebraic institution *strongly* admits initial semantics if any nonempty class of models definable in it contains an initial model which is reachable.

Mahr and Makowsky [29] gave a complete characterisation of standard algebraic institutions (called algebraic specification languages there) which strongly admit initial semantics[2] (see also [37]). In fact, this characterisation is a consequence of a characterisation of so-called free classes of algebras due to Mal'cev [31] and a special case of a characterisation of quasi-varieties (see [3, 7, 33], we restate this characterisation here as Theorem 8.2).

In this section we show that the characterisation analogous to Mahr and Makowsky's result holds for arbitrary, *abstract* algebraic institutions. Note that, since the characterisation of quasi-varieties is already given in this general framework, we only have half the job to do—it is enough to generalise Mal'cev's theorem.

Let us fix an arbitrary abstract algebraic institution INS.

For any signature Σ, we say that a class $K \subseteq |\mathbf{Mod}(\Sigma)|$ of Σ-models is *free* if, for any signature morphism $\sigma : \Sigma \to \Sigma'$ and ground variety V of Σ'-models, if $K|_\sigma^{-1} \cap V$ is nonempty, then it contains a reachable initial model, where $K|_\sigma^{-1}$ denotes the class of all Σ'-models A such that $A|_\sigma \in K$. (Although this notion is not directly connected with freeness in the sense of category theory, we keep Mal'cev's terminology.)

Using Lemma 2.4 we can prove the following abstract version of Mal'cev's theorem (cf. [31]).

4.2. Theorem. *For any signature Σ, a class of Σ-models is free if and only if it is a quasi-variety.*

[2] There is a slight mistake in the exact formulation of this result in [29], though.

Proof. ($\Leftarrow$): Let K be a quasi-variety of Σ-models, $\sigma: \Sigma \to \Sigma'$ be a signature morphism, and V be a ground variety of Σ'-models. Since K is a quasi-variety and the σ-reduct functor preserves submodels and products, $K|_\sigma^{-1}$ is a quasi-variety. Hence, by Fact 2.3(6), $K|_\sigma^{-1} \cap V$ is a quasi-variety as well and so, by Lemma 2.4, it has a reachable initial model provided that it is nonempty.

($\Rightarrow$): Let K be a free class of Σ-models. We have to show that it is closed under subobjects and nonempty products.

Let $A \in K$. Consider an arbitrary subobject of A, $m: B \to A$, where $m \in \boldsymbol{M}_\Sigma$. Let $\Sigma(B)$ be a diagram signature for B with the signature inclusion $\iota: \Sigma \to \Sigma(B)$ and let $E(B)$ be the diagram expansion of B. Consider the class $K|_\iota^{-1} \cap \boldsymbol{Ext}(E(B))$. First, observe that it is nonempty, since it contains $E_m(A)$ (the ι-expansion of A defined by m—see Definition 3.2). Then, since $\boldsymbol{Ext}(E(B))$ is a ground variety, $K|_\iota^{-1} \cap \boldsymbol{Ext}(E(B))$ has a reachable initial model, say M. We prove that $M|_\iota$ is isomorphic to B.

By definition of $\boldsymbol{Ext}(E(B))$, there exists a $\Sigma(B)$-morphism $f: E(B) \to M$. Moreover, by Fact 2.3(3), $f \in \boldsymbol{E}_{\Sigma(B)}$. Hence, by our assumptions, $f|_\iota: B \to M|_\iota$, with $f|_\iota \in \boldsymbol{E}_\Sigma$.

On the other hand, by the initiality of M and the fact that $E_m(A)$ belongs to $K|_\iota^{-1} \cap \boldsymbol{Ext}(E(B))$, there is (a unique) $\Sigma(B)$-morphism $h: M \to E_m(A)$. Then, by the initiality of $E(B)$ in $\boldsymbol{Ext}(E(B))$ and the definition of $E_m(A)$, $f; h = E(m): E(B) \to E_m(A)$, and so, $f|_\iota; h|_\iota = m \in \boldsymbol{M}_\Sigma$. Thus, by Fact 2.2, $f|_\iota$ is an isomorphism, which proves that K is closed under submodels, as $M|_\iota \in K$.

Now, about products: let $A_\beta \in K$ for $\beta < \alpha$, $\alpha > 0$. Let B be the product of $\langle A_\beta \rangle_{\beta<\alpha}$ with projections $\pi_\beta: B \to A_\beta$. We have to prove that $B \in K$.

Let again $\Sigma(B)$ be a diagram signature for B with the signature inclusion $\iota: \Sigma \to \Sigma(B)$ and let $E(B)$ be the diagram expansion of B. Consider $K|_\iota^{-1} \cap \boldsymbol{Ext}(E(B))$. First, note that, for $\beta < \alpha$, $E_{\pi_\beta}(A_\beta) \in K|_\iota^{-1} \cap \boldsymbol{Ext}(E(B))$. Thus, $K|_\iota^{-1} \cap \boldsymbol{Ext}(E(B))$ has a reachable initial model, say M. We prove that $M|_\iota$ is isomorphic to B.

Let $f: E(B) \to M$ (it exists since $M \in \boldsymbol{Ext}(E(B))$). By Fact 2.3(3), $f \in \boldsymbol{E}_{\Sigma(B)}$. Hence, by our assumptions, $f|_\iota: B \to M|_\iota$, with $f|_\iota \in \boldsymbol{E}_\Sigma$.

Then, by the initiality of M and the fact that, for $\beta < \alpha$, $E_{\pi_\beta}(A_\beta)$ belongs to $K|_\iota^{-1} \cap \boldsymbol{Ext}(E(B))$, for $\beta < \alpha$ there is (a unique) $\Sigma(B)$-morphism $h_\beta: M \to E_{\pi_\beta}(A_\beta)$. Thus, by the definition of a product, there exists a unique Σ-morphism $g: M|_\iota \to B$ such that, for $\beta < \alpha$, $g; \pi_\beta = h_\beta|_\iota$. Moreover, by the initiality of $E(B)$ in $\boldsymbol{Ext}(E(B))$ and the definition of $E_{\pi_\beta}(A_\beta)$, for $\beta < \alpha$, $(f; h_\beta)|_\iota = \pi_\beta$. Now, for $\beta < \alpha$, $(f|_\iota; g); \pi_\beta = f|_\iota; h_\beta|_\iota = \pi_\beta$. By the definition of product this proves that $f|_\iota; g$ is an isomorphism. Thus, by Fact 2.2, $f|_\iota$ is an isomorphism, which completes the proof that K is closed under nonempty products and so the proof of the theorem as well. □

4.3. Lemma. *An abstract algebraic institution strongly admits initial semantics if and only if every class definable in it is free.*

Proof. ($\Leftarrow$): Assume that every definable class is free. Let K be a nonempty definable class of Σ-models. Obviously, $K = K \cap |\boldsymbol{Mod}(\Sigma)|$ and since $|\boldsymbol{Mod}(\Sigma)| = \boldsymbol{Ext}(\Lambda_\Sigma)$ (Λ_Σ is an initial Σ-model) is a ground variety, K has a reachable initial model.

($\Rightarrow$): Let K be a definable class of Σ-models, i.e., $K = \mathrm{Mod}(\Phi)$ for some $\Phi \subseteq |\boldsymbol{Sen}(\Sigma)|$. K is closed under isomorphism by the abstractness condition (Definition 3.2). Let $\sigma: \Sigma \to \Sigma'$ be a signature morphism and V be a ground variety of Σ'-models. By the definability of ground varieties, $V = \mathrm{Mod}(\Phi')$, for some $\Phi' \subseteq |\boldsymbol{Sen}(\Sigma')|$. Let $\sigma(\Phi) = \{\sigma(\varphi) \mid \varphi \in \Phi\}$. By the satisfaction condition, $K|_\sigma^{-1} = \mathrm{Mod}(\sigma(\Phi))$. Hence, $K|_\sigma^{-1} \cap V = \mathrm{Mod}(\sigma(\Phi) \cup \Phi')$ is definable in the underlying institution, and thus has a reachable initial model (provided that it is nonempty). $\square$

Theorem 4.2 and Lemma 4.3 imply the following theorem.

4.4. Theorem. *An abstract algebraic institution strongly admits initial semantics if and only if every class definable in it is a quasi-variety.*

5. Basic construction

As we have already mentioned, the choice of the initial model of a theory may be too restrictive. What we often need are models in which only some parts of a theory are interpreted in some standard way relative to some other parts which may be interpreted loosely. A formal definition of these more general constraints refers to a standard 'free' extension of a model of a subtheory to a model of the whole theory (cf. [13, 17, 21], see also Section 7). We show here that this free extension may always be constructed as a reduct of an initial model of a 'bigger' theory. In the next section we apply this construction and use Theorem 4.4 to characterise abstract algebraic institutions in which such free extensions always exist.

Let INS be an abstract algebraic institution, fixed throughout this section.

Let $T1 = \langle \Sigma 1, \Phi 1 \rangle$ and $T2 = \langle \Sigma 2, \Phi 2 \rangle$ be theories and $\sigma: T1 \to T2$ be a theory morphism.

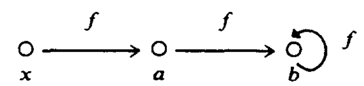
For any $A \in \mathrm{Mod}(T1)$, by a *σ-free model over A* we mean a model $\boldsymbol{F}_\sigma(A) \in \mathrm{Mod}(T2)$ together with a $\Sigma 1$-morphism $\eta_A: A \to \boldsymbol{F}_\sigma(A)|_\sigma$ such that the following 'universality condition' holds: for any $B \in \mathrm{Mod}(T2)$ and $\Sigma 1$-morphism $h: A \to B|_\sigma$, there is a unique $\Sigma 2$-morphism $h^{\#}: \boldsymbol{F}_\sigma(A) \to B$ such that $\eta_A ; h^{\#}|_\sigma = h$.

Note that if a σ-free model over A exists for any $A \in \mathrm{Mod}(T1)$, then the mappings $A \mapsto \boldsymbol{F}_\sigma(A)$ and $A \mapsto \eta_A$ determine a functor $\boldsymbol{F}_\sigma: \mathrm{Mod}(T1) \to \mathrm{Mod}(T2)$ which is left adjoint to the σ-reduct functor $\_|_\sigma: \mathrm{Mod}(T2) \to \mathrm{Mod}(T1)$; η is the unit of the adjunction and its counit ε is determined by $\varepsilon_B = (\mathrm{id}_{B|_\sigma})^{\#}$ for $B \in \mathrm{Mod}(T2)$ (cf. [28, Theorem IV.1.2]).

Let us consider (in the framework of the standard algebraic institution GEQ of ground equations) a very simple example. Let $\Sigma 1$ be an algebraic signature with exactly one sort, one unary operation f and two constants a, b; let $\Sigma 2$ be $\Sigma 1$ with an additional constant c. Then, let $T1$ be the empty $\Sigma 1$-theory and let $T2$ be the theory with signature $\Sigma 2$ induced by equations $f(a)=c$ and $a=b$. Obviously, the inclusion σ of $\Sigma 1$ into $\Sigma 2$ is a theory morphism $\sigma: T1 \to T2$.

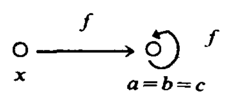
Now, consider the following $\Sigma 1$-algebra A:

A:



The σ-free model over A is the following $\Sigma 2$-algebra:

$\boldsymbol{F}_\sigma(A)$:



with the obvious unit $\Sigma 1$-homomorphism.

Informally, to construct $\boldsymbol{F}_\sigma(A)$ one may extend the algebraic signature $\Sigma 2$ by a constant for each element of $|A|$ (it is, in fact, sufficient to add constants for elements which are not values of ground terms) and then consider an initial model of a theory $T3$ which results by adding to the theory $T2$ all the equations which hold in (the diagram expansion of) A. In our example, $T3$ is essentially induced by $T2$ and $f(d)=a, f(a)=b, f(b)=b$, where d is the new constant corresponding to the element x. $\boldsymbol{F}_\sigma(A)$ is the initial model of $T3$ viewed as a $\Sigma 2$-algebra. This intuition leads to a construction used in the proof of the following theorem.

5.1. Theorem. *Let $T1$ and $T2$ be theories, $\sigma: T1 \to T2$ be a theory morphism and $A \in \mathrm{Mod}(T1)$. Then there exists a theory $T3$ such that the σ-free model over A exists if and only if $T3$ has an initial model.*

Proof. We give an explicit construction of the theory $T3$.

Let $T1 = \langle \Sigma 1, \Phi 1 \rangle$ and $T2 = \langle \Sigma 2, \Phi 2 \rangle$.

Let $\Sigma 1(A)$ be a diagram signature for A with the signature inclusion $\iota: \Sigma 1 \to \Sigma 1(A)$ and let $E(A)$ be the diagram expansion of A.

By the *(positive) diagram* of A we mean the family $\Delta^+(A)$ of all $\Sigma 1(A)$-sentences that hold in $\boldsymbol{Ext}(E(A))$. Note that since $\boldsymbol{Ext}(E(A))$ is a ground variety, by our assumptions it is definable and so $\boldsymbol{Ext}(E(A)) = \mathrm{Mod}(\Delta^+(A))$.

Now, let

$$\begin{array}{ccc} \Sigma 1(A) & \xrightarrow{\sigma'} & \Sigma 2(\sigma(A)) \\ {\scriptstyle \iota}\uparrow & & \uparrow{\scriptstyle \iota'} \\ \Sigma 1 & \xrightarrow[\sigma]{} & \Sigma 2 \end{array}$$

be a pushout in the category of signatures.

By the σ-image of the diagram of A we mean $\sigma'(\Delta^+(A))$, the diagram of A translated by σ', i.e., $\sigma'(\Delta^+(A)) = \{\sigma'(\delta) \mid \delta \in \Delta^+(A)\}$.

For any $B \in \mathrm{Mod}(T2)$ and $\Sigma 1$-morphism $h: A \to B|_{\sigma}$, by the *h-expansion* of B we mean the $\Sigma 2(\sigma(A))$-model $E_h^{\sigma}(B)$ defined by $E_h^{\sigma}(B)|_{\iota'} = B$ (i.e., $E_h^{\sigma}(B)$ is a ι'-expansion of B) and $E_h^{\sigma}(B)|_{\sigma'} = E_h(B|_{\sigma})$, where $E_h(B|_{\sigma})$ is the ι-expansion of $B|_{\sigma}$ defined by h (see Definition 3.2). Lemma 3.3 guarantees that $E_h^{\sigma}(B)$ is well-defined.

Now, let $T3 = \langle \Sigma 2(\sigma(A)), \Phi 3\rangle$ be the $\Sigma 2(\sigma(A))$-theory induced by $\iota'(\Phi 2) \cup \sigma'(\Delta^+(A))$.

We need the following technical lemmas.

5.2. Lemma. *For any* $B \in \mathrm{Mod}(T2)$ *and* $\Sigma 1$*-morphism* $h: A \to B|_{\sigma}$, $E_h^{\sigma}(B) \in \mathrm{Mod}(T3)$.

Proof. It is sufficient to prove that $E_h^{\sigma}(B) \models \sigma'(\Delta^+(A))$. By the satisfaction condition this is equivalent to $E_h^{\sigma}(B)|_{\sigma'} \models \Delta^+(A)$, which is obvious since $E_h^{\sigma}(B)|_{\sigma'} = E_h(B|_{\sigma}) \in$ ***Ext***$(E(A))$ and $\Delta^+(A)$ holds in ***Ext***$(E(A))$ by definition. □

5.3. Lemma. *Let* $C \in \mathrm{Mod}(T3)$. *Then* $C|_{\iota} \in \mathrm{Mod}(T2)$ *and, moreover, there is a unique* $\Sigma 1$*-morphism* $h: A \to (C|_{\iota'})|_{\sigma}$ *such that* $C = E_h^{\sigma}(C|_{\iota'})$.

Proof. The first part easily follows from the satisfaction condition.

To prove the second part, note that by the satisfaction condition $C|_{\sigma'} \models \Delta^+(A)$, and so $C|_{\sigma'} \in$ ***Ext***$(E(A))$. Hence, by Fact 2.3(5), there is a unique $\Sigma 1(A)$-morphism $E(h): E(A) \to C|_{\sigma'}$. Thus, since $(C|_{\sigma'})|_{\iota} = (C|_{\iota'})|_{\sigma}$, for $h = E(h)|_{\iota}$, $C|_{\sigma'} = E_h((C|_{\iota'})|_{\sigma})$ (by definition) and so $C = E_h^{\sigma}(C|_{\iota'})$. The uniqueness of h follows from the uniqueness of $E(h)$: suppose that there is another $h': A \to (C|_{\iota'})|_{\sigma}$ such that $C = E_{h'}^{\sigma}(C|_{\iota'})$. By definition, h' is a ι-reduct of a morphism from $E(A)$ to $E_{h'}((C|_{\iota'})|_{\sigma})$. But, $E_{h'}((C|_{\iota'})|_{\sigma}) = C|_{\sigma'} = E_h((C|_{\iota'})|_{\sigma})$, and so $h' = E(h)|_{\iota} = h$. □

5.4. Lemma. *If* $B1$, $B2 \in \mathrm{Mod}(T2)$, *and* $h1: A \to B1|_{\sigma}$ *and* $h2: A \to B2|_{\sigma}$ *are* $\Sigma 1$*-morphisms, then there is a* 1-1 *correspondence between* $\Sigma 2(\sigma(A))$*-morphisms from* $E_{h1}^{\sigma}(B1)$ *to* $E_{h2}^{\sigma}(B2)$ *and* $\Sigma 2$*-morphisms* $h: B1 \to B2$ *such that* $h1; h|_{\sigma} = h2$.

Proof. The correspondence is given by the ι'-reduct functor $\_|_{\iota'}$. To see this, first note that for any $\Sigma 2(\sigma(A))$-morphism $f: E_{h1}^{\sigma}(B1) \to E_{h2}^{\sigma}(B2)$, $h1; (f|_{\iota'})|_{\sigma} = h2$, since, by Fact 2.3(2), $E(h1); f|_{\sigma'} = E(h2)$ (recall that $E(h1): E(A) \to E_{h1}(B1|_{\sigma})$ is a ι-expansion of $h1$ and similarly for $E(h2)$) and of course $(f|_{\sigma'})|_{\iota} = (f|_{\iota'})|_{\sigma}$.

Now, let $h: B1 \to B2$ and $h1; (h|_{\sigma}) = h2$. By our assumptions there is a unique $E(h|_{\sigma}): E_{h1}(B1|_{\sigma}) \to E_{h2}(B2|_{\sigma})$ such that $E(h|_{\sigma})|_{\iota} = h|_{\sigma}$. Thus, by Lemma 3.3, there is $\Sigma 2(\sigma(A))$-morphism $f: E_{h1}^{\sigma}(B1) \to E_{h2}^{\sigma}(B2)$ such that $f|_{\iota'} = h$, which proves that the ι'-reduct functor is surjective here. The uniqueness of f (and, hence, the injectivity of the ι'-reduct functor) follows from the uniqueness of $E(h|_{\sigma})$ such that $E(h|_{\sigma})|_{\iota} = h|_{\sigma}$ by Lemma 3.3. □

Proof of Theorem 5.1 (*continued*). (⇒): Let $\langle \boldsymbol{F}_{\sigma}(A), \eta_A\rangle$ be a σ-free model over A. By definition, $\boldsymbol{F}_{\sigma}(A) \in \mathrm{Mod}(T2)$ and η_A is a $\Sigma 1$-morphism, $\eta_A: A \to \boldsymbol{F}_{\sigma}(A)|_{\sigma}$. We show that $E^{\sigma}_{\eta_A}(F_\sigma(A))$ is initial in Mod($T3$): it belongs to Mod($T3$) by Lemma 5.2 and its initiality follows from the universality condition for $\langle F_\sigma(A), \eta_A\rangle$. Namely, consider $C \in$ Mod($T3$). By Lemma 5.3 there is a unique $\Sigma 1$-morphism $h : A \to (C|_{\iota'})|_\sigma$ such that $C = E^{\sigma}_{h}(C|_{\iota'})$. Hence, by the universality condition there is exactly one $\Sigma 2$-morphism $h^{\#}$ from $F_\sigma(A) = E^{\sigma}_{\eta_A}(F_\sigma(A))|_{\iota'}$ (by definition) to $C|_{\iota'}$ such that $\eta_A ; h^{\#}|_\sigma = h$. Thus, by Lemma 5.4 there is exactly one $\Sigma 2(\sigma(A))$-morphism from $E^{\sigma}_{\eta_A}(F_\sigma(A))$ to $E^{\sigma}_{h}(C|_{\iota'}) = C$ (by definition) which completes the proof of the 'only if' part.

($\Leftarrow$): Let C be initial in Mod($T3$). Define $\langle F_\sigma(A), \eta_A\rangle$ by $F_\sigma(A) = C|_{\iota'}$ and $C = E^{\sigma}_{\eta_A}(C|_{\iota'})$. By Lemma 5.3, $F_\sigma(A) \in$ Mod($T2$) and η_A are well-defined. The universality condition follows from the initiality of C. Namely, consider $B \in$ Mod($T2$) and a $\Sigma 1$-morphism $h : A \to B|_\sigma$. By Lemma 5.2, $E^{\sigma}_{h}(B) \in$ Mod($T3$). Hence, by the initiality of C, there is exactly one $\Sigma 2(\sigma(A))$-morphism from $C = E^{\sigma}_{\eta_A}(F_\sigma(A))$ (by definition) to $E^{\sigma}_{h}(B)$. Thus, by Lemma 5.4, there is exactly one $\Sigma 2$-morphism $h^{\#}$ from $F_\sigma(A)$ to $E^{\sigma}_{h}(B)|_{\iota'} = B$ (by definition) such that $\eta_A ; (h^{\#}|_\sigma) = h$, which proves that $\langle F_\sigma(A), \eta_A\rangle$ really is a σ-free model over A. □

6. Liberality

6.1. Definition ([21]). We say that an institution is *liberal* if for any two theories $T1$ and $T2$ and theory morphism $\sigma: T1 \to T2$ there exists a σ-free model over any model of $T1$, or equivalently if the σ-reduct functor $\_|_\sigma$: Mod($T2$) $\to$ Mod($T1$) has a left adjoint.

The construction given in the previous section (Theorem 5.1) shows that to prove liberality of an institution it is 'almost' enough to prove that the institution admits initial semantics. However, additionally we have to ensure that the institution *guarantees satisfiability*, i.e., that any theory in this institution is satisfiable (has a model).

6.2. Corollary. *An abstract algebraic institution is liberal iff it admits initial semantics and guarantees satisfiability.*

Proof. The 'if' part is obvious by Theorem 5.1. To prove the 'only if' part, note that for any theory T the initial model in Mod(T) is just a ι-free model over the only $\Sigma_\emptyset$-model, where $\Sigma_\emptyset$ is the 'empty signature' (the initial object in ***Sign***) and ι is the 'inclusion' of the empty theory over the empty signature into T. □

6.3. Lemma. *If an abstract algebraic institution guarantees satisfiability and every class of models definable in it is closed under submodels, then any definable class of models contains a model which is terminal in the category of models of the given signature.*

Proof. Let Φ be a set of sentences $\Phi \subseteq |\boldsymbol{Sen}(\Sigma)|$ for some signature Σ and let $\mathbf{1}$ be a terminal model in $\boldsymbol{Mod}(\Sigma)$. Let $\Sigma(\mathbf{1})$ be a diagram signature for $\mathbf{1}$ with the signature inclusion $\iota: \Sigma \to \Sigma(\mathbf{1})$. Let $\mathbf{1}_{\Sigma(\mathbf{1})}$ be a terminal model in $\boldsymbol{Mod}(\Sigma(\mathbf{1}))$. By definition, there is a $\Sigma(\mathbf{1})$-morphism $h: E(\mathbf{1}) \to \mathbf{1}_{\Sigma(\mathbf{1})}$. Now, since reduct functors preserve products, $\mathbf{1}_{\Sigma(\mathbf{1})}|_\iota$ is terminal in $\boldsymbol{Mod}(\Sigma)$, and hence $h|_\iota: \mathbf{1} \to \mathbf{1}_{\Sigma(\mathbf{1})}|_\iota$ is an isomorphism. By our assumptions, the ι-reduct functor reflects isomorphisms in $\boldsymbol{Ext}(E(\mathbf{1}))$ (Fact 3.4), so h itself is an isomorphism in $\boldsymbol{Mod}(\Sigma(\mathbf{1}))$ and $E(\mathbf{1})$ is a terminal $\Sigma(\mathbf{1})$-model. This implies that $E(\mathbf{1})$ is a submodel of any element of $\boldsymbol{Ext}(E(\mathbf{1}))$: let $f: E(\mathbf{1}) \to A$, $f = e; m$, with $e \in \boldsymbol{E}_{\Sigma(\mathbf{1})}$ and $m \in \boldsymbol{M}_{\Sigma(\mathbf{1})}$. Since $E(\mathbf{1})$ is terminal, there is an $h: A \to E(\mathbf{1})$ and, moreover, $f; h = \mathrm{id}$. Thus, $e; (m; h) = \mathrm{id}$ and so, by Fact 2.2, e is an isomorphism, which proves that $f \in \boldsymbol{M}_{\Sigma(\mathbf{1})}$.

Now, since $E(\mathbf{1})$ is reachable, $\boldsymbol{Ext}(E(\mathbf{1}))$ is a ground variety and so it is definable, say $\boldsymbol{Ext}(E(\mathbf{1})) = \mathrm{Mod}(\Phi_T)$. Let $\Psi = \iota(\Phi) \cup \Phi_T$. Since the institution guarantees satisfiability, Ψ has a model, say $A \models \Psi$. By the construction, $E(\mathbf{1})$ is a submodel of A and since every definable class is closed under submodels, $E(\mathbf{1}) \models \iota(\Phi)$ which by the satisfaction condition is equivalent to $E(\mathbf{1})|_\iota \models \Phi$. Thus, $\mathbf{1} = E(\mathbf{1})|_\iota$ satisfies Φ as well, which completes the proof. □

Theorem 4.4 and Corollary 6.2 together with Lemma 6.3 almost give us a characterisation of liberal abstract algebraic institutions. We have, however, to formulate a generalised 'no junk' condition.

Let $\sigma: \Sigma 1 \to \Sigma 2$ be a signature morphism. We say that a model $A \in |\boldsymbol{Mod}(\Sigma 2)|$ is *σ-reachable* if, for any $m \in \boldsymbol{M}_{\Sigma 2}$ with codomain A, if $m|_\sigma$ is an isomorphism, then so is m (i.e., A has no proper submodels with an isomorphic σ-reduct). In the standard algebraic case this means exactly that A is generated by its σ-part, i.e., A has no proper subalgebra with the same σ-reduct.

6.4. Fact. *A Σ-model is reachable if and only if it is ι-reachable, where $\iota: \Sigma_\emptyset \to \Sigma$ is the unique signature morphism from the initial signature to Σ.*

Proof. Since $\boldsymbol{Mod}$ preserves colimits, $\boldsymbol{Mod}(\Sigma_\emptyset)$ is a terminal object in $\boldsymbol{Cat}$, i.e., a category with exactly one object and exactly one morphism. Thus, for every Σ-morphism m, $m|_\iota$ is an isomorphism. □

The above notion of σ-reachability was used in the context of an arbitrary institution in [36] and is based on the definition given in [17] in the standard algebraic framework. Alternatively, as suggested in [21], we may define that in a liberal institution a model A is σ-reachable if it is a 'natural' quotient of σ-free expansion of $A|_\sigma$. In the standard algebraic case this means that every element of A is the value in A of a term built using operations from $\Sigma 2$ and elements of the σ-reduct of A (as constants of the appropriate sort).

6.5. Fact. *If the σ-reduct functor has a left adjoint, say $\boldsymbol{F}_\sigma$, then $A \in |\boldsymbol{Mod}(\Sigma)|$ is σ-reachable iff the counit morphism $\varepsilon_A = (\mathrm{id}_{A|_\sigma})^{\#}: \boldsymbol{F}_\sigma(A|_\sigma) \to A$* belongs to $\boldsymbol{E}_{\Sigma 2}$.

Proof. ($\Rightarrow$): Let A be σ-reachable. Consider a factorisation $\varepsilon_A = e\,;m$, where $e \in \boldsymbol{E}_{\Sigma 2}$ and $m \in \boldsymbol{M}_{\Sigma 2}$. Now, $\eta_{A|_\sigma}; e|_\sigma; m|_\sigma = \mathrm{id}$, and so, since $m|_\sigma \in \boldsymbol{M}_{\Sigma 1}$ (the reduct functors preserve submodels), $m|_\sigma$ is an isomorphism (by Fact 2.2). Thus, by σ-reachability of A, m is an isomorphism as well, hence $\varepsilon_A \in \boldsymbol{E}_{\Sigma 2}$.

($\Leftarrow$): Suppose that $\varepsilon_A \in \boldsymbol{E}_{\Sigma 2}$. Let $m: B \to A$, $m \in \boldsymbol{M}_{\Sigma 2}$ with $m|_\sigma$ an isomorphism. Define $f: \boldsymbol{F}_\sigma(A|_\sigma) \to B$ by $f = ((m|_\sigma)^{-1})^{\#}$. Now, $\eta_{A|_\sigma}; (f; m)|_\sigma = (\eta_{A|_\sigma}; f|_\sigma); m|_\sigma = (m|_\sigma)^{-1}; m|_\sigma = \mathrm{id}_{A|_\sigma}$. By the freeness of $\boldsymbol{F}_\sigma(A|_\sigma)$, this implies that $f; m = \varepsilon_A \in \boldsymbol{E}_{\Sigma 2}$, which (again by Fact 2.2) proves that m is an isomorphism. □

Note that the above proof does not depend on most of the requirements listed in Definition 3.2 (the only used requirement is that the reduct functors preserve submodels). Thus, Fact 6.5 shows that under rather mild assumptions about factorisation systems for categories of models the two notions of σ-reachability mentioned above are equivalent.

6.6. Definition. An abstract algebraic institution is called *strongly liberal* if for any two theories $T1 = \langle \Sigma 1, \Phi 1\rangle$ and $T2 = \langle \Sigma 2, \Phi 2\rangle$ and theory morphism $\sigma: T1 \to T2$, for any $A \in \mathrm{Mod}(T1)$ there is a σ-free model over A, $\langle \boldsymbol{F}_\sigma(A), \eta_A\rangle$, such that $\boldsymbol{F}_\sigma(A)$ is σ-reachable.

6.7. Theorem. *An abstract algebraic institution is strongly liberal iff every class of models definable in it is a strict quasi-variety.*

Proof. ($\Rightarrow$): Directly from our definitions by Fact 6.4 and arguments similar to those in the proof of Corollary 6.2, it easily follows that if an abstract algebraic institution is strongly liberal, then it strongly admits initial semantics and guarantees satisfiability. Hence, every class definable in it is a quasi-variety (by Theorem 4.4) and contains a terminal model (by Lemma 6.3), i.e., is a strict quasi-variety.

($\Leftarrow$): For any abstract algebraic institution, if every definable class of models is a strict quasi-variety, then this institution obviously guarantees satisfiability (since every definable class contains a terminal model, hence, is nonempty) and strongly admits initial semantics (by Theorem 4.4). Now, to use Theorem 5.1 here, we need the following lemma (notation from Theorem 5.1 and its proof).

6.8. Lemma. *If C, the initial model of $T3$, is reachable, then $\boldsymbol{F}_\sigma(A) = C|_{\iota'}$ is σ-reachable.*

Proof. Let $m: B \to C|_{\iota'}$, $m \in \boldsymbol{M}_{\Sigma 2}$, with $m|_\sigma$ an isomorphism. Then, let $h1: A \to (C|_{\iota'})|_\sigma$ be such that $C = E^\sigma_{h1}(C|_{\iota'})$ ($h1$ is well-defined by Lemma 5.3) and let $h2 = h1; (m|_\sigma)^{-1}$. By definition, there is $E(m|_\sigma): E_{h2}(B|_\sigma) \to E_{h1}((C|_{\iota'})|_\sigma)$ such that $E(m|_\sigma)|_\iota = m|_\sigma$. Now, by Lemma 3.3, there exists a unique $\Sigma 2(\sigma(A))$-morphism $m': E^\sigma_{h2}(B) \to C$ such that $m'|_{\sigma'} = E(m|_\sigma)$ and $m'|_{\iota'} = m$.

By Lemma 5.2, $E^{\sigma}_{h2}(B) \in \text{Mod}(T3)$. Hence, by the initiality of C, there is $f: C \to E^{\sigma}_{h2}(B)$. Moreover, $f; m'$ is an identity and so $(f; m')|_{\iota'} = f|_{\iota'}; m$ is an identity as well. By Fact 2.2, this proves that m is an isomorphism which completes the proof of Lemma 6.8. □

Proof of Theorem 6.7 (*continued*). This proof is simple now: if the institution strongly admits initial semantics and guarantees satisfiability, then the theory $T3$ has a model, hence, it has a reachable initial model, and so the σ-free model over A exists (by Theorem 5.1) and is σ-reachable (by the above lemma). □

7. Data constraints

In the early approaches to algebraic specification (e.g., [1]) a specification of an abstract data type was given by defining an algebraic signature and a list of axioms (equations) imposed on the operations of the data type, which describes a class of algebras (over the algebraic signature) which satisfy the axioms. The initial algebra in this class was considered to be the meaning (semantics) of the specification. Thus, initiality was treated as a standard requirement imposed at a meta-level on any specification. Of course, this works smoothly for simple examples and may easily be extended to handle parameterised specifications by allowing one to require freeness rather than initiality. However, when building bigger specifications it is necessary to combine specifications and, consequently, to require some parts of the specification to be interpreted in the standard ('initial') way independently from the interpretation of other parts, or even to allow some parts of the specification to be interpreted in a nonstandard ('loose') way. Then, it is convenient (and natural) to be able to avoid the necessity of imposing the requirement of initiality (or freeness) at the meta-level, once for the whole specification. As advocated, for example, in [13, 21, 34], this may be achieved by including the requirement of initiality in the specification itself, which guarantees that the specification is given the standard interpretation disregarding the context (bigger specification) in which it is used. To express such requirements we need a new kind of sentence (called *data constraints* in [13, 21]) as a part of the logical system we are working in. The following definition is adapted directly from [21].

Let us fix a liberal institution.

If $\sigma: T1 \to T2$ is a theory morphism and $B \in \text{Mod}(T2)$, then we say that B is *naturally σ-free* if it is ('naturally' isomorphic to) a σ-free model over its $T1$-part, i.e., more formally, if the counit morphism $\varepsilon_B = (\text{id}_{B|_\sigma})^{\#}: F_\sigma(B|_\sigma) \to B$ is an isomorphism.

By *Σ-data constraint*, for any signature Σ, we mean a pair $\langle \sigma: T1 \to T2, \theta: \Sigma 2 \to \Sigma \rangle$, Where $T1$ and $T2$ are theories, σ is a theory morphism, $\Sigma 2$ is the signature of $T2$, and θ is a signature morphism.

We say that Σ-model A satisfies the above constraint if $A|_\theta \in \text{Mod}(T2)$ and $A|_\theta$ is naturally σ-free.

Goguen and Burstall [21] have proved that if we accept Σ-data constraints as additional Σ-sentences in a liberal institution, then the resulting system is again an institution. It may also be proved that if the underlying institution is abstract algebraic, then the resultant institution is abstract algebraic as well. It is worth noting that it need not be liberal.

Note that in the above definition of the satisfaction relation for data constraints we do not use σ-free models over all models of $T1$; we only refer to σ-free models over $T1$-models that are, roughly, $T1$-parts of some models of $T2$.

Let $\sigma: T1 \to T2$ be a theory morphism. We say that $A \in \text{Mod}(T1)$ is *σ-consistent* with $T2$ if $A = B|_\sigma$ for some $B \in \text{Mod}(T2)$, i.e., if $A \in \text{Mod}(T2)|_\sigma$, where $\text{Mod}(T2)|_\sigma$ is the range of the σ-reduct functor $\_|_\sigma: \text{Mod}(T2) \to \text{Mod}(T1)$.

We call an institution *quasi-liberal* if for any theory morphism $\sigma: T1 \to T2$ there is a σ-free model over any model of $T1$ that is σ-consistent with $T2$, i.e., if for any theory morphism $\sigma: T1 \to T2$ the σ-reduct functor $\_|_\sigma: \text{Mod}(T2) \to \text{Mod}(T2)|_\sigma$ has a left adjoint $\boldsymbol{F}_\sigma: \text{Mod}(T2)|_\sigma \to \text{Mod}(T2)$.

7.1. Theorem. *An abstract algebraic institution is quasi-liberal iff it admits initial semantics.*

Proof. ($\Rightarrow$): Recall that for any theory T the initial model in $\text{Mod}(T)$ is just a ι-free model over the unique $\Sigma_\emptyset$-model ($\Sigma_\emptyset$ is the initial signature), where ι is the inclusion of the empty theory over $\Sigma_\emptyset$ into T. This initial model exists provided that T is satisfiable, since then the only $\Sigma_\emptyset$-model is ι-consistent with T.

($\Leftarrow$): If $\sigma: T1 \to T2$ is a theory morphism and $A \in \text{Mod}(T2)|_\sigma \subseteq \text{Mod}(T1)$, then $A = B|_\sigma$ for some $B \in \text{Mod}(T2)$. Hence (notation from Theorem 5.1 and its proof), $E^\sigma_{\text{id}_A}(B) \in \text{Mod}(T3)$ (by Lemma 5.2). Thus, theory $T3$ is satisfiable and, by the assumption, has an initial model—which, by Theorem 5.1, completes the proof of quasi-liberality. □

By Theorem 7.1, we can safely use data constraints in any algebraic institution that admits initial semantics, even if it is not liberal.

We call an abstract algebraic institution strongly quasi-liberal if, for any theory morphism $\sigma: T1 \to T2$, there is a σ-reachable σ-free model over any model of $T1$ that is σ-consistent with $T2$, i.e., if for any theory morphism $\sigma: T1 \to T2$ the σ-reduct functor $\_|_\sigma: \text{Mod}(T2) \to \text{Mod}(T2)|_\sigma$ has a left adjoint $\boldsymbol{F}_\sigma: \text{Mod}(T2)|_\sigma \to \text{Mod}(T2)$ such that, for any $A \in \text{Mod}(T2)|_\sigma$, $\boldsymbol{F}_\sigma(A)$ is σ-reachable.

Lemma 6.8 easily implies the following theorem. (We omit its proof which is similar to that of Theorem 7.1.)

7.2. Theorem. *An abstract algebraic institution is strongly quasi-liberal iff it strongly admits initial semantics.*

Remark. It is worth noting that the semantics of data constraints may be given without referring to left adjoints of the reduct functors. Namely, for any theory morphism $\sigma: T1 \to T2$ and $B \in \mathrm{Mod}(T2)$, B is naturally σ-free if and only if B is σ-free over $B|_\sigma$ with the unit morphism $\mathrm{id}_{B|_\sigma}$. This means that we can use data constraints in arbitrary institutions, without requiring the existence of free functors. However, in such an institution a data constraint may have no model even if the theories it involves are satisfiable.

8. Injectivity

In a series of very interesting papers, Andreka, Nemeti and Sain (cf. [3, 4, 5, 33]; see also, e.g., [7]) explored classes of morphisms (or, more generally, cones and trees) and the notion of injectivity w.r.t. these classes as categorical generalisation of the notions of, respectively, formulae and their satisfaction in a model. Along this line they obtained several Birkhoff-type characterisation theorems which hold in an arbitrary category satisfying rather mild assumptions. In this section we briefly review those of their results which we can directly apply in our framework.

Throughout this section let $\boldsymbol{K}$ be an arbitrary category with a factorisation system $\langle \boldsymbol{E}, \boldsymbol{M} \rangle$. We assume that $\boldsymbol{K}$ has an initial object Λ and all products (of sets). Moreover, we assume that $\boldsymbol{K}$ is $\boldsymbol{E}$-co-well-powered.

For any morphism $f: A \to B$ and an object $M \in |\boldsymbol{K}|$ we say that M is *injective* w.r.t. f if any morphism $g: A \to M$ factors through f, i.e., $g = f; h$ for some $h: B \to M$.

By a *cone* in $\boldsymbol{K}$ we mean any object $A \in |\boldsymbol{K}|$ together with a family of morphisms with domain A (although in this paper we only use cones where the family of morphisms is either empty or contains exactly one element, it is technically convenient to state this definition in its full generality).

Let $\gamma = \langle A, \{f_\beta : A \to B_\beta\}_{\beta<\alpha} \rangle$ be a cone in $\boldsymbol{K}$. We say that an object $M \in |\boldsymbol{K}|$ is injective w.r.t. γ if any morphism $g: A \to M$ factors through at least one morphism of γ, i.e., $g = f_\beta; h$ for some $\beta < \alpha$ and $h: B_\beta \to M$.

If Γ is a family of cones in $\boldsymbol{K}$, then we say that an object $M \in |\boldsymbol{K}|$ is injective w.r.t. Γ if it is injective w.r.t. all elements of Γ. $\mathrm{Inj}(\Gamma) \subseteq |\boldsymbol{K}|$ denotes the class of all objects which are injective w.r.t. Γ. We say that Γ *defines* $\mathrm{Inj}(\Gamma)$.

8.1. Definition. We call a class $K \subseteq |\boldsymbol{K}|$ of objects
- *implicational* if it is definable by a family of cones of the form $\langle A, \{e\} \rangle$ or $\langle A, \emptyset \rangle$, where $e \in \boldsymbol{E}$,
- *strictly implicational* if it is definable by a family of cones of the form $\langle A, \{e\} \rangle$, where $e \in \boldsymbol{E}$,
- *ground equational* if it is definable by a family of cones of the form $\langle \Lambda, \{e\} \rangle$, where $e \in \boldsymbol{E}$ (and Λ is initial in $\boldsymbol{K}$).

To justify the above definitions, let us consider (in the standard algebraic framework) a very simple example. Let Σ be an algebraic signature with exactly one sort and three constants a, b, and c; let A and B be the following Σ-algebras:

$$A: \underset{a=b}{\circ} \quad \underset{c}{\circ} \qquad\qquad B: \underset{a=b=c}{\circ}$$

Finally, let h^A and h be (the unique, by Fact 2.3(2)) Σ-homomorphisms from T_Σ (the initial Σ-algebra) to A and, respectively, from A to B.

Now, for any Σ-algebra C:

- C is injective w.r.t. $\langle T_\Sigma, \{h^A\}\rangle$ if and only if C satisfies the equation $a = b$ (as, by the definition of the initial algebra, the injectivity of C w.r.t. this cone is equivalent to the existence of a Σ-homomorphism from A to C).
- C is injective w.r.t. $\langle A, \emptyset\rangle$ if and only if C does not satisfy the equation $a = b$, or, equivalently, C satisfies $a = b \Rightarrow \mathbf{false}$ (as, by the definition, the injectivity of C w.r.t. this cone is equivalent to the fact that there is no Σ-homomorphism from A to C). We call formulae of this form, possibly with more than just one premise, conditional inequations.
- C is injective w.r.t. $\langle A, \{h\}\rangle$ if and only if either C does not satisfy the equation $a = b$ or C satisfies the equations $a = b$ and $b = c$, or, equivalently, if C satisfies the conditional equation $a = b \Rightarrow b = c$ (as, by Fact 2.3(2), the injectivity of C w.r.t. this cone is equivalent to the fact that either there is no Σ-homomorphism from A to C or there is a Σ-homomorphism from B to C).

In fact, in the standard algebraic framework it is always possible to characterise injectivity w.r.t. cones of the special form we consider using ground equations, conditional equations and conditional inequations respectively, as suggested by the above example. It may be necessary, however, to consider conditional equations and inequations with (universally quantified) variables and also with infinite sets of premises. On the other hand, it is also possible to characterise finitary conditional equations and inequations using injectivity w.r.t. so-called 'small' cones; this may be pushed even further to obtain a characterisation of arbitrary first-order logic formulae using injectivity w.r.t. 'small trees'. (As this does not directly relate to the results presented in this paper, we are not going to make this remark more precise; an interested reader may find more details in, e.g., [5, 33].)

8.2. Theorem. *A class of objects of **K** is*

(1) *implicational iff it is a quasi-variety,*

(2) *strictly implicational iff it is a strict quasi-variety,*

(3) *ground equational iff it is a ground variety.*

Proof. This is an easy adaptation to our framework of the proof given, e.g., in [3, 7, 33]. Because of its importance and because of slightly different assumptions (and even results—(3) has not been stated explicitly in the papers we refer to), we decided to repeat it here with the necessary modifications (and some simplifications which are possible in our framework).

We need the following lemma.

8.3. Lemma. *For any $e \in \boldsymbol{E}$, the class of objects injective w.r.t. e is a strict quasi-variety, i.e., is closed under products and submodels.*

Proof. Let $e: A \to B$, $e \in \boldsymbol{E}$.

Let P be a product of a family $\langle A_\beta \rangle_{\beta<\alpha}$ with projections $\pi_\beta : P \to A_\beta$. Let $f: A \to P$. If, for $\beta < \alpha$, A_β is injective w.r.t. e, then, for $\beta < \alpha$, there is $f_\beta : B \to A_\beta$ such that $e ; f_\beta = f ; \pi_\beta$. Hence, since P is a product of $\langle A_\beta \rangle_{\beta<\alpha}$, there is (a unique) $g: B \to P$ such that, for $\beta < \alpha$, $f_\beta = g ; \pi_\beta$. Moreover, since, for $\beta < \alpha$, $(e ; g) ; \pi_\beta = e ; f_\beta = f ; \pi_\beta$, we have $e ; g = f$, which proves that P is injective w.r.t. e.

Now, let $m: M \to C$, $m \in \boldsymbol{M}$, and $f: A \to M$. If C is injective w.r.t. e, then there is a morphism $g: B \to C$ such that $e ; g = f ; m$. Hence, by Lemma 2.1 there is a morphism $h: B \to M$ such that $e ; h = f$ (and $h ; m = g$), which proves that M is injective w.r.t. e. □

Proof of Theorem 8.2 (*continued*). Now, the proofs of the 'only if' parts of this theorem are straightforward.

(1) ($\Rightarrow$): Since intersection of quasi-varieties is a quasi-variety, by Lemma 8.3 it is sufficient to show that the class of objects injective w.r.t. $\langle A, \emptyset \rangle$ is closed under nonempty products and submodels. This, however, easily follows from the fact that an object B is injective w.r.t. $\langle A, \emptyset \rangle$ if and only if there is no morphism from A to B.

(2) ($\Rightarrow$): Obvious by Lemma 8.3 (since intersection of strict quasi-varieties is a strict quasi-variety).

(3) ($\Rightarrow$): Let, for $\beta < \alpha$, $e_\beta : \Lambda \to A_\beta$, $e_\beta \in \boldsymbol{E}$ and let $K = \mathrm{Inj}(\{e_\beta\}_{\beta<\alpha})$ (we identify morphisms with one-element cones). First note that, for $\beta < \alpha$, an object B is injective w.r.t. e_β iff there is a morphism from A_β to B. Hence, $K = \bigcap_{\beta<\alpha} \boldsymbol{Ext}(A_\beta)$ and so K is closed under products, subobjects, and extensions (since, by Facts 2.3(1) and 2.3(6) each $\boldsymbol{Ext}(A_\beta)$ is). Thus, by Lemma 2.4, K has a reachable initial object, say A. Moreover, since K is closed under extensions, $K = \boldsymbol{Ext}(A)$, i.e., K is a ground variety.

(2) ($\Leftarrow$): Let K be a strict quasi-variety. Then, let Γ be the family of all cones γ of the form $\langle A, \{e\} \rangle$, where $e \in \boldsymbol{E}$ such that $K \subseteq \mathrm{Inj}(\gamma)$. Obviously, $K \subseteq \mathrm{Inj}(\Gamma)$. To prove that $\mathrm{Inj}(\Gamma) \subseteq K$, consider arbitrary $B \in \mathrm{Inj}(\Gamma)$. Let E be the class of all factorisation epimorphisms with domain B and codomain in K. Then, let $\{e_\beta : B \to A_\beta\}_{\beta<\alpha}$ be a sub*set* of E such that any element of E is a composition of some e_β with an isomorphism (recall that K is $\boldsymbol{E}$-co-well-powered). Consider a product P of $\langle A_\beta \rangle_{\beta<\alpha}$ with projections $\pi_\beta : P \to A_\beta$. Let $f: B \to P$ be such that, for $\beta < \alpha$, $f ; \pi_\beta = e_\beta$. Then, let $f = e ; m$, with $e \in \boldsymbol{E}$ and $m \in \boldsymbol{M}$. Consider the cone $\langle B, \{e\} \rangle$.

Any element of K is injective w.r.t. e. For, let $g: B \to A$, where $A \in K$. Let $g = e_g ; m_g$, with $e_g \in \boldsymbol{E}$ and $m_g \in \boldsymbol{M}$. By our assumptions, $e_g = e_\beta ; i$, for some $\beta < \alpha$ and an isomorphism i. Then, $g = e_\beta ; i ; m_g = (f ; \pi_\beta) ; i ; m_g = e ; (m ; \pi_\beta ; i ; m_g)$.

Hence, B is also injective w.r.t. e, and so $\mathrm{id} = e\,;h$, for some morphism h. By Fact 2.2, e is an isomorphism and, so, B is a subobject of a product of elements of K. Thus, $B \in K$, which also proves that $K = \mathrm{Inj}(\Gamma)$.

(1) ($\Leftarrow$): Let K be a quasi-variety. Then, let Γ be a family of all cones γ of the form $\langle A, \{e\}\rangle$, where $e \in \boldsymbol{E}$, or $\langle A, \emptyset\rangle$ such that $K \subseteq \mathrm{Inj}(\gamma)$. Obviously, $K \subseteq \mathrm{Inj}(\Gamma)$. To prove that $\mathrm{Inj}(\Gamma) \subseteq K$, consider arbitrary $B \in \mathrm{Inj}(\Gamma)$. Let E be the class of all factorisation epimorphisms with domain B and codomain in K. If E is not empty, the proof is exactly the same as in case (2). If E is empty, then any object in K is injective w.r.t. $\langle B, \emptyset\rangle$ and, so, $\langle B, \emptyset\rangle \in \Gamma$, which yields a contradiction since B is not injective w.r.t. $\langle B, \emptyset\rangle$.

(3) ($\Leftarrow$): Let A be reachable and $h_A : \Lambda \to A$ be the unique morphism from the initial object Λ to A. By a remark in the proof of the 'only if' part, $\boldsymbol{Ext}(A) = \mathrm{Inj}(\{\langle \Lambda, h_A\rangle\})$, which completes the proof since, by Fact 2.3(3), $h_A \in \boldsymbol{E}$. □

Note that by the above theorem the requirement of definability of ground varieties (Definition 3.2(5)) in abstract algebraic institutions may be (re)formulated as follows: for any signature Σ and $e \in \boldsymbol{E}_\Sigma$ such that the domain of e is an initial Σ-model, there is a set of Σ-sentences $\Phi \subseteq |\boldsymbol{Sen}(\Sigma)|$ such that, for any Σ-model A, A is injective w.r.t. e iff $A \models \Phi$.

We can use Theorem 8.2 to reformulate the characterisation results from Sections 4, 6, and 7.

8.4. Corollary. (1) *An abstract algebraic institution strongly admits initial semantics (or equivalently, is strongly quasi-liberal) if and only if every class definable in it is implicational or, equivalently, for every signature Σ and Σ-sentence $\varphi \in |\boldsymbol{Sen}(\Sigma)|$ there is a set Γ of cones of the form $\langle A, \{e\}\rangle$ or $\langle A, \emptyset\rangle$, where $A \in |\boldsymbol{Mod}(\Sigma)|$ and $e \in \boldsymbol{E}_\Sigma$ such that, for every Σ-model M, $M \models \varphi$ iff M is injective w.r.t. Γ.*

(2) *An abstract algebraic institution is strongly liberal if and only if every class of models definable in it is strictly implicational, or, equivalently, for every signature Σ and Σ-sentence $\varphi \in |\boldsymbol{Sen}(\Sigma)|$ there is a set Γ of cones of the form $\langle A, \{e\}\rangle$, where $A \in |\boldsymbol{Mod}(\Sigma)|$ and $e \in \boldsymbol{E}_\Sigma$ such that, for every Σ-model M, $M \models \varphi$ iff M is injective w.r.t. Γ.*

9. A special case—partial algebras

Following the approach of Andreka and Nemeti [3] (cf. also [7, 33]) we give the definition of implicational, strictly implicational, and ground equational classes in the rather nonstandard terms of definability by a set of cones of a certain form (Section 8). To make our characterisation results directly useful, a more standard 'syntactic' form of this definition is required. This may be done for any particular basis of an abstract algebraic institution, i.e., only when notions of signature, model (together with their morphisms) and submodel are fixed.

The most obvious and perhaps most important special case of abstract algebraic institutions are standard algebraic institutions where signatures are standard algebraic signatures, models are just algebras, and factorisation epimorphisms and monomorphisms are just surjective and, respectively, injective homomorphisms. In this standard algebraic framework, ground varieties are exactly classes of algebras definable by ground equations, and quasi-varieties (respectively, strict quasi-varieties) are exactly classes of algebras definable by infinitary conditional equations and inequations (respectively, infinitary conditional equations). This leads to the characterisations of algebraic institutions which strongly admit initial semantics and those which are strongly liberal, which were originally formulated and proved in [29] (the former) and [37] (the latter). Here, we are not going to state them again more precisely. (Let us only mention that the appropriate specialisations of the results presented in this paper are slightly more general, since we require expressibility of only ground, rather than arbitrary, equations.) Instead, we formulate in this section the consequences of our results for abstract algebraic institutions of *partial algebras* (see [11, 12, 35]). Proofs of some basic facts, omitted here, may be extracted from results presented, for example, in [35].

Let $\Sigma = \langle S, \Omega\rangle$ be an algebraic signature.

A *partial Σ-algebra A* consists of an S-indexed family of carrier sets $|A| = \{|A|_s\}_{s\in S}$ and for each $f: s1,\dots, sn \to s$ a *partial* function $f_A: |A|_{s1}\times\cdots\times|A|_{sn}\to|A|_s$. Note that a *total* Σ-algebra is a partial Σ-algebra in which all these functions are total. A (*weak*) *Σ-homomorphism* from a partial Σ-algebra A to a partial Σ-algebra B, $h: A\to B$, is a family of (total) functions $\{h_s: |A|_s\to|B|_s\}_{s\in S}$ such that, for any $f: s1,\dots, sn\to s$ and $a_1\in|A|_{s1},\dots, a_n\in|A|_{sn}$,

$$f_A(a_1,\dots,a_n)\text{ defined}\Rightarrow f_B(h_{s1}(a_1),\dots,h_{sn}(a_n))\text{ defined and}$$
$$h_s(f_A(a_1,\dots,a_n)) = f_B(h_{s1}(a_1),\dots,h_{sn}(a_n))$$

(in [11] this would be called a *total* Σ-homomorphism). If, moreover, h satisfies the condition

$$f_B(h_{s1}(a_1),\dots,h_{sn}(a_n))\text{ defined}\Rightarrow f_A(a_1,\dots,a_n)\text{ defined},$$

then h is called a *closed Σ-homomorphism.*

The category of partial Σ-algebras $\boldsymbol{PAlg}(\Sigma)$ has partial Σ-algebras as objects and (weak) Σ-homomorphisms as morphisms; the composition of homomorphisms is the composition of their corresponding components as functions. (This obviously forms a category.) Note that this category has an initial object, which is the partial Σ-algebra with all carriers empty and, hence, all operations totally undefined, and all products of sets of partial Σ-algebras defined in the standard way. The terminal partial Σ-algebra is a total Σ-algebra with all carriers containing exactly one element.

We define a factorisation system for $\boldsymbol{PAlg}(\Sigma)$ as follows:

- The class of factorisation epimorphisms, $\boldsymbol{PE}_\Sigma$, is just the class of all epimorphisms in $\boldsymbol{PAlg}(\Sigma)$.

- The class of factorisation monomorphisms, $\boldsymbol{PM}_\Sigma$, is the class of all injective (i.e., 1–1) closed Σ-homomorphisms.

9.1. Fact. *$\langle \boldsymbol{PE}_\Sigma, \boldsymbol{PM}_\Sigma\rangle$ is a factorisation system for $\boldsymbol{PAlg}(\Sigma)$.*

Note that under this factorisation system a submodel corresponds to a partial subalgebra in the sense of [24, p. 80]: if A and B are partial Σ-algebras, then A is a subalgebra of B if $|A| \subseteq |B|$ and $|A|$ is closed under all operations (as defined in B).

Also note that epimorphisms in $\boldsymbol{PAlg}(\Sigma)$ need not be surjective. A Σ-homomorphism $h: A \to B$ is an epimorphism if and only if B has no proper subalgebra containing the (set-theoretic) image of $|A|$ under h.

For any S-sorted set $X = \{X_s\}_{s\in S}$, the (total) algebra of *Σ-terms with variables* X, denoted by $T_\Sigma(X)$, is defined as usual as 'the' initial total $\Sigma(X)$-algebra, where $\Sigma(X)$ is the enrichment of Σ by elements of X as constants of the appropriate sorts (see, e.g., [1, 14]). For any partial Σ-algebra A and any S-sorted function $v: X \to |A|$ (called a *valuation* of variables X), the *value* of a term $t \in |T_\Sigma(X)|_s$, $s \in S$, in A under v is denoted by $t_A(v)$. (Note that $t_A(v)$ may be undefined—see [12, 35] for a precise definition of this notion.) As before, we write T_Σ for $T_\Sigma(\emptyset)$ and refer to terms with no variables as *ground* terms. For a ground term t we write t_A rather than $t_A(\emptyset)$. Of course, when A is a total algebra, this coincides with the notation introduced in Section 2.

Note that a Σ-homomorphism $h: A \to B$ is an epimorphism (in $\boldsymbol{PAlg}(\Sigma)$) if and only if any element of $|B|$ is a value of a Σ-term with some variables X under a valuation which maps X into the image of $|A|$ under h. In particular, a partial Σ-algebra B is reachable in $\boldsymbol{PAlg}(\Sigma)$ if and only if every element of $|B|$ is the value in B of a ground Σ-term.

By a *Σ-equation* with variables X we mean any pair $\langle t, t'\rangle$ (written in the form $X: t = t'$), where t and t' are Σ-terms of the same sort with variables X. For any Σ-equation $X: t = t'$ and partial Σ-algebra A, we say that a valuation $v: X \to |A|$ is a *solution* of $X: t = t'$ if $t_A(v)$ and $t'_A(v)$ are both defined and equal. By $A_{X:t=t'}$ we denote the set of all solutions of $X: t = t'$ in A. We generalise this notion and notation to sets of equations (with the same set of variables) in the obvious way.

We say that a partial Σ-algebra A *satisfies* a set of Γ of Σ-equations with variables X (or that Γ holds in A), written $A \models \Gamma$, if any valuation $v: X \to |A|$ belongs to $A_{X:\Gamma}$ (is a solution of Γ). For any set Γ of Σ-equations, by $\mathrm{Mod}(\Gamma)$ we denote the class of all partial Σ-algebras which satisfy Γ. We say that Γ *defines* $\mathrm{Mod}(\Gamma)$.

This corresponds to the notion of existence-equations as developed in [3, 12, 35]. Note that we can easily specify the requirement of definedness of operations: a partial algebra A satisfies equation $X: t = t$ if and only if the value of t in A is defined under every valuation of variables X into A.

By a *ground Σ-equation* we mean any Σ-equation with the empty set of variables. By the *diagram* of a reachable partial Σ-algebra A we mean the set $\Delta^+(A)$ of all ground Σ-equations which hold in A. The following fact shows that this is consistent with the notation used in Section 5.

9.2. Fact. *If A is a reachable partial Σ-algebra, then for any partial Σ-algebra B there is a Σ-homomorphism from A to B if and only if* $B \models \Delta^+(A)$, *i.e.*, $\boldsymbol{Ext}(A) = \mathrm{Mod}(\Delta^+(A))$.

This shows that any ground variety of partial algebras is definable by ground equations. On the other hand, it is easy to see that any class definable by ground equations is closed under submodels and products, and so, by Lemma 2.4, contains a reachable initial partial algebra. This together with Fact 9.2 proves the following fact.

9.3. Fact. *A class of partial algebras is a ground variety* (*or, equivalently, is ground equational*) *if and only if it is definable by a set of ground equations.*

For any algebraic signature morphism $\sigma: \Sigma \to \Sigma'$, the σ-reduct functor $\_|_\sigma : \boldsymbol{PAlg}(\Sigma') \to \boldsymbol{PAlg}(\Sigma)$ is defined exactly as in the case of total algebras (see Section 2). Obviously, also the definition of translation of ground Σ-terms to ground Σ'-terms induced by σ given in Section 2 does not require any modification.

All the above notions combine to form the *institution of ground equations in partial algebras* $\boldsymbol{P}$GEQ:

- $\boldsymbol{Sign}_{\mathrm{PGEQ}}$ is the category of algebraic signatures $\boldsymbol{AlgSig}$.
- For any algebraic signature Σ, $\boldsymbol{Sen}_{\mathrm{PGEQ}}(\Sigma)$ is the set of all the ground Σ-equations; for any algebraic signature morphism $\sigma: \Sigma \to \Sigma'$, $\boldsymbol{Sen}_{\mathrm{PGEQ}}(\sigma)$ maps any ground Σ-equation $t = t'$ to the ground Σ'-equation $\sigma(t) = \sigma(t')$.
- For any algebraic signature Σ, $\boldsymbol{Mod}_{\mathrm{PGEQ}}(\Sigma)$ is $\boldsymbol{PAlg}(\Sigma)$; for any algebraic signature morphism $\sigma: \Sigma \to \Sigma'$, $\boldsymbol{Mod}_{\mathrm{PGEQ}}(\sigma)$ is the functor $\_|_\sigma : \boldsymbol{PAlg}(\Sigma') \to \boldsymbol{PAlg}(\Sigma)$.
- For any algebraic signature Σ, $\models_{\Sigma,\boldsymbol{P}\mathrm{GEQ}}$ is the satisfaction relation as defined above.

It is easy to check that $\boldsymbol{P}$GEQ (with the factorisation systems described above) is an abstract algebraic institution. Note that for any partial Σ-algebra A, a diagram signature for A, $\Sigma(A)$, may be given as the enrichment of Σ by a constant of the appropriate sort for each element of $|A|$.

By an *abstract algebraic institution of partial algebras* we mean an arbitrary abstract institution with the basis $\langle \boldsymbol{AlgSig}, \boldsymbol{Mod}_{\mathrm{PGEQ}}, \{\langle \boldsymbol{PE}_\Sigma, \boldsymbol{PM}_\Sigma\rangle\}_{\Sigma \in |\boldsymbol{AlgSig}|}\rangle$. Fact 9.3 proves that $\boldsymbol{P}$GEQ is the minimal abstract algebraic institution of partial algebras w.r.t. the following relation of reducibility of institutions (cf. [29]).

9.4. Definition. For any two abstract algebraic institutions INS1 and INS2 with the same basis, we say that INS1 is *reducible* to INS2 if any class of models definable in INS1 is also definable in INS2, or, equivalently, if, for any signature Σ and Σ-sentence φ in INS1, $\varphi \in |\boldsymbol{Sen}_{\mathrm{INS1}}(\Sigma)|$, φ is expressible in INS2, i.e., there is a set Φ of Σ-sentences in INS2, $\Phi \subseteq |\boldsymbol{Sen}_{\mathrm{INS2}}(\Sigma)|$, such that, for any $A \in |\boldsymbol{Mod}(\Sigma)|$, $A \models_{\mathrm{INS1}} \varphi$ iff $A \models_{\mathrm{INS2}} \Phi$.

Let $\Sigma = \langle S, \Omega\rangle$ be an algebraic signature.

Let Γ be an arbitrary set of Σ-equations with variables X. By a free $\langle \Sigma, \Gamma\rangle$-algebra we mean the partial Σ-algebra $F[X:\Gamma]$ constructed as follows.

Let $\Sigma(X)$ be the enrichment of Σ by elements of X as constants of the appropriate sorts, and let $\iota: \Sigma \to \Sigma(X)$ be the signature inclusion. Obviously, we can identify ground $\Sigma(X)$-terms with Σ-terms having variables X. By Fact 9.3, the set of ground $\Sigma(X)$-equations $\Gamma \cup \{x = x \mid x \in X_s, s \in S\}$ has a reachable initial model, which is a reachable partial $\Sigma(X)$-algebra, say $E(F)$. Now, define $F[X:\Gamma]$ to be the ι-reduct of $E(F)$. For $s \in S$, $x \in X_s$, by x^s_Γ we denote the value of the constant corresponding to x in $E(F)$.

9.5. Fact. *For any partial Σ-algebra A and $v \in A_{X:\Gamma}$ there is a unique Σ-homomorphism $v^\#: F[X:\Gamma] \to A$ such that $(v^\#)_s(x^s_\Gamma) = v_s(x)$ for $s \in S$, $x \in X_s$.*

9.6. Fact. *For any partial Σ-algebra A and Σ-homomorphism $h: F[X:\Gamma] \to A$, the valuation $v: X \to A$ defined by $v_s(x) = h_s(x^s_\Gamma)$ for $s \in S$, $x \in X_s$ is a solution of Γ in A.*

By an *infinitary conditional Σ-equation* with variables X we mean any pair $\langle \Gamma 1, \Gamma 2 \rangle$ (written as $X: \Gamma 1 \Rightarrow \Gamma 2$) of sets of equations with variables X. We say that a partial Σ-algebra A satisfies $X: \Gamma 1 \Rightarrow \Gamma 2$ if any solution of $\Gamma 1$ in A is a solution of $\Gamma 2$, i.e., $A_{X:\Gamma 1} \subseteq A_{X:\Gamma 2}$. If Ψ is a set of infinitary conditional Σ-equations, by $\mathrm{Mod}(\Psi)$ we denote the class of all partial Σ-algebras which satisfy every element of Ψ and we say that Ψ *defines* $\mathrm{Mod}(\Psi)$.

Consider an arbitrary infinitary conditional Σ-equation $X: \Gamma 1 \Rightarrow \Gamma 2$. Let $h^\#: F[X:\Gamma 1] \to F[X: \Gamma 1 \cup \Gamma 2]$ be the Σ-homomorphism defined by $(h^\#)_s(x^s_{\Gamma 1}) = x^s_{\Gamma 1 \cup \Gamma 2}$ ($h^\#$ is well-defined by Fact 9.5).

First observe that, by the above construction, $h^\#$ is an epimorphism. Then, note that a partial Σ-algebra A satisfies $X: \Gamma 1 \Rightarrow \Gamma 2$ if and only if it is injective w.r.t. $h^\#$. For, assume that A is injective w.r.t. $h^\#$ and let $v: X \to |A|$ be a solution of $\Gamma 1$. By injectivity of A, $v^\#$ factors through $h^\#$, say $v^\# = h^\#; h2$, where $h2: F[X: \Gamma 1 \cup \Gamma 2] \to A$. By the definitions of $v^\#$ and $h^\#$, $v_s(x) = h2_s(x^s_{\Gamma 1 \cup \Gamma 2})$. Hence, by Fact 9.6, v is a solution in A of $\Gamma 1 \cup \Gamma 2$ and, so, of $\Gamma 2$ as well, which proves that A satisfies $X: \Gamma 1 \Rightarrow \Gamma 2$.

Now, assume that A satisfies $X: \Gamma 1 \Rightarrow \Gamma 2$ and let $h1: F[X:\Gamma 1] \to A$. By Facts 9.5 and 9.6, $h1 = v^\#$ for some valuation $v: X \to |A|$ which is a solution of $\Gamma 1$ in A. By the hypothesis, v is a solution in A of $\Gamma 2$, and, hence, of $\Gamma 1 \cup \Gamma 2$ as well. Consider $h2: F[X: \Gamma 1 \cup \Gamma 2] \to A$ defined (unambiguously, by Fact 9.5) by $h2_s(x^s_{\Gamma 1 \cup \Gamma 2}) = v_s(x)$ for $s \in S$, $x \in X_s$. Now, $(h^\#; h2)_s(x^s_{\Gamma 1}) = v_s(x)$ for $s \in S$ and $x \in X_s$, which by Fact 9.5 proves that $h1 = h^\#; h2$ and, so, that A is injective w.r.t. $h^\#$.

The above arguments prove the following fact.

9.7. Fact. *Any class of partial Σ-algebras definable by infinitary conditional Σ-equations is strictly implicational.*

Now, let $e: A \to B$ be an epimorphism in $\boldsymbol{PAlg}(\Sigma)$. We show that the class of all partial Σ-algebras injective w.r.t. e is definable by an infinitary conditional Σ-equation.

Let $\Sigma(|A|)$ be the enrichment of Σ by a constant a^s of sort s for any $s \in S$, $a \in |A|_s$ and let $\iota: \Sigma \to \Sigma(|A|)$ be the signature inclusion. Then, let $E(A)$ and $E(B)$ be partial $\Sigma(|A|)$-algebras which are the ι-extensions of A and B, respectively, defined by $(a^s)_{E(A)} = a$ and $(a^s)_{E(B)} = e_s(a)$ for $s \in S$ and $a \in |A|_s$. Let $\Gamma_A = \Delta^+(E(A))$ and $\Gamma_B = \Delta^+(E(B))$ be the diagrams of $E(A)$ and $E(B)$, respectively. Note that $\Gamma_A \subseteq \Gamma_B$. $E(A)$ and $E(B)$ are reachable and, so, by Facts 9.2 and 2.3(5), they are initial $\Sigma(|A|)$-algebras which satisfy Γ_A and Γ_B, respectively. Thus, by the same arguments as those preceding Fact 9.7, a partial Σ-algebra is injective w.r.t. e if and only if it satisfies the infinitary conditional Σ-equation $|A|: \Gamma_A \Rightarrow \Gamma_B$ (with variables $|A|$). This proves the following fact.

9.8. Fact. *A class of partial algebras is definable by a set of infinitary conditional equations if and only if it is strictly implicational.*

The above fact and Corollary 8.4 directly imply the following theorem.

9.9. Theorem. *An abstract algebraic institution of partial algebras is strongly liberal if and only if every class of partial algebras definable in it is also definable by infinitary conditional equations, or, equivalently, for every algebraic signature Σ and Σ-sentence $\varphi \in |\mathbf{Sen}(\Sigma)|$ there is a set Ψ of infinitary conditional Σ-equations such that, for every partial Σ-algebra A, $A \models \varphi$ iff A satisfies Ψ.*

By an *infinitary conditional Σ-inequation* with variables X we mean a set Γ of Σ-equations with variables X (written in the form $X: \Gamma \Rightarrow \mathbf{false}$).

We say that a partial Σ-algebra A satisfies $X: \Gamma \Rightarrow \mathbf{false}$ if there is no solution of Γ in A. If Ψ is a set of infinitary conditional Σ-equations and inequations, by $\mathrm{Mod}(\Psi)$ we denote the class of all partial Σ-algebras which satisfy every element of Ψ and we say that Ψ *defines* $\mathrm{Mod}(\Psi)$.

It is easy to see that a partial Σ-algebra A satisfies $X: \Gamma \Rightarrow \mathbf{false}$ if and only if there is no Σ-homomorphism from $F[X:\Gamma]$ to A, i.e., A is injective w.r.t. the cone $\langle F[X:\Gamma], \emptyset\rangle$, where $F[X:\Gamma]$ is a free partial $\langle \Sigma, \Gamma\rangle$-algebra.

Moreover, for any partial Σ-algebra A, the infinitary conditional Σ-inequation $|A|: \Gamma_A \Rightarrow \mathbf{false}$ with variables $|A|$ exactly defines the class of all partial Σ-algebras which are injective w.r.t. the cone $\langle A, \emptyset\rangle$ (Γ_A is the diagram of $E(A)$ as defined in arguments preceding Fact 9.8). Thus, we can state the following fact.

9.10. Fact. *A class of partial Σ-algebras is definable by a set of infinitary conditional Σ-equations and inequations if and only if it is implicational.*

The above fact and Corollary 8.4 directly imply the following theorem.

9.11. Theorem. *An abstract algebraic institution of partial algebras strongly admits initial semantics if and only if every class of partial algebras definable in it is also* *definable by infinitary conditional equations and inequations, or, equivalently, for every algebraic signature Σ and Σ-sentence $\varphi \in |\mathbf{Sen}(\Sigma)|$ there is a set Ψ of infinitary conditional Σ-equations and inequations such that, for every partial Σ-algebra A, $A \models \varphi$ iff A satisfies Ψ.*

Theorems 9.9 and 9.11 state that the most general (w.r.t. reducibility) abstract algebraic institution of partial algebras which is strongly liberal is the (naturally defined) institution of infinitary conditional equations in partial algebras; the most general abstract algebraic institution of partial algebras which strongly admits initial semantics is the (naturally defined) institution of infinitary conditional equations and inequations in partial algebras.

Facts 9.8 and 9.10, which we used to specialise our characterisation of abstract algebraic institutions which strongly admit initial semantics to the case of partial algebras, may be deduced from similar results stated, e.g., in [3]. We decided, however, to give their proof in such detail here in order to convince the reader that at least part of these arguments may be generalised. Namely, any abstract algebraic institution determines a semantic notion of 'ground equation' (positive elementary sentence)—'ground equations' are exactly the sentences which define ground varieties. Of course, we cannot expect that a 'syntactic' characterisation of these 'ground equations' may be given without referring to a particular institution. What is possible, however, is that, when this syntactic notion of a 'ground equation' is given, quasi-varieties may be characterised in a uniform way, independent from any particular institution, as classes of models definable by universally quantified infinitary conditional 'equations' and 'inequations', exactly as we did here for partial algebras. We have developed this idea in much more detail in [39] using the notion of open formulae and quantification in an arbitrary institution outlined in [36].

10. Summary of results and final remarks

We recalled the notion of an institution introduced by Goguen and Burstall [21] to formalise the concept of a logical system for writing specifications. We specialised their extremely general definition and dealt with *abstract algebraic institutions*, i.e., institutions equipped with factorisation systems for the categories of models which satisfy a number of additional requirements. Namely, besides some purely technical conditions, we required that abstract algebraic institutions identify all isomorphic models, allow to define any ground variety of models and guarantee the existence of a diagram expansion for any model (Section 3).

In this framework, we generalised the characterisation of algebraic specification languages which strongly admit initial semantics due to Mahr and Makowsky [29] (cf. [37]). We proved (Section 4) that an abstract algebraic institution strongly admits initial semantics if and only if every class of models definable in it is implicational (Theorem 4.4 and Corollary 8.4).

Considering the problem of the existence of free functors (left adjoints to the reduct functors along an arbitrary theory morphism) we gave (Section 5) a basic construction of a free model of a theory over a model of a subtheory (w.r.t. an arbitrary theory morphism) which requires only the existence of initial models (Theorem 5.1). This led to the result that an abstract algebraic institution is liberal (i.e., guarantees the existence of a free functor for any theory morphism) if and only if *any* theory has an initial model (Corollary 6.2).

Another consequence of our basic construction is the characterisation of abstract algebraic institutions which are strongly liberal, i.e., which guarantee the existence of a model of a theory that is both free over and generated by a model of a subtheory w.r.t. any theory morphism. Namely (Theorem 6.7 and Corollary 8.4), an abstract algebraic institution is strongly liberal if and only if every class of models definable in it is strictly implicational.

Then, we pointed out that perhaps the requirement of liberality is too restrictive; we showed that data constraints are easy to deal with in institutions which are only quasi-liberal, i.e., which guarantee the existence of a free model of a theory over a model of a subtheory provided that this model is consistent with the whole theory. Quasi-liberality is equivalent to the existence of an initial model of any *satisfiable* theory (Theorem 7.1).

Finally, we specialised our results and 'syntactically' characterised the requirement of the existence of initial reachable models in abstract algebraic institutions of partial algebras (Section 9). Namely, we showed (Theorems 9.9 and 9.11) that the most general abstract algebraic institution of partial algebras which is strongly liberal is the institution of infinitary conditional (existence) equations; the most general abstract algebraic institution of partial algebras which strongly admits initial semantics is the institution of infinitary conditional (existence) equations and inequations.

Throughout this paper we deal with *reachable* initial models only and, hence, characterise abstract algebraic institutions which *strongly* admit initial semantics. Although this seems to be quite acceptable (many approaches to abstract data types are based on this restriction anyway), it would be very interesting to try to give this characterisation without requiring initial models to be reachable. In the standard algebraic framework this was recently done by Makowsky [30].

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