Solutions for quasilinear elliptic problems with critical Sobolev–Hardy exponents

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Abstract

Let $N \geq 3$, $2 < p < N$, $0 \leq s < p$ and $p^*(s) := \frac{p(N-s)}{N-p}$. Via the variational methods and analytic technique, we prove the existence of nontrivial solution to the singular quasilinear problem

$$-\Delta_p u + |u|^{p-2}u = |u|^{p^*(s)-2}u |x|^s u + f(u), \quad u \in W^{1,p}_p(\mathbb{R}^N) \quad \text{for} \quad N \geq p^2 \quad \text{and suitable functions} \quad f(u).$$

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1. Introduction and main results

Consider the following quasilinear elliptic problem:

$$\begin{cases}
-\Delta_p u + |u|^{p-2}u = \frac{|u|^{p^*(s)-2}u}{|x|^{s}} u + f(u), & x \in \mathbb{R}^N, \\
u \to 0 \quad \text{as} \quad |x| \to +\infty,
\end{cases}$$

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where $N \geq 3$, $2 < p < N$, $0 \leq s < p$ and $p^*(s) := \frac{p(N-s)}{N-p}$ is the critical Sobolev–Hardy exponent for the embedding $W^{1,p}_r(\mathbb{R}^N) \hookrightarrow L^{p^*(s)}(\mathbb{R}^N, |x|^{-s})$ with
\[ W^{1,p}_r(\mathbb{R}^N) := \{ u \mid u \in W^{1,p}(\mathbb{R}^N), \ u(x) = u(|x|) \} . \]

Note that $p^*(0) = p^* := \frac{Np}{N-p}$ is the critical Sobolev exponent. Throughout this paper we assume that
\[ (f_1) \quad f(t) \in C^2(\mathbb{R}_+), \quad \lim_{t \to 0^+} f(t) = 0 \quad \text{and} \quad \lim_{t \to +\infty} f(t) = 0. \]

\[ (f_2) \quad \text{There exists } \tau > 0 \text{ small enough such that} \quad t(\tau f(t)) \geq (p - 1 + \tau)f(t) \geq 0 \quad \text{for all } t \geq 0. \tag{1.2} \]

\[ (f_3) \quad \text{f(t) is odd.} \]

Note that it follows directly from $(f_2)$ that
\[ \int_0^u f(t) \, dt \leq \frac{uf(u)}{p + \tau}. \tag{1.3} \]

We define the variational functional corresponding to (1.1) on $W^{1,p}_r(\mathbb{R}^N)$ by
\[ I(u) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p) \, dx - \frac{1}{p^*(s)} \int_{\mathbb{R}^N} |u|^{p^*(s)} \frac{|x|^s}{|x|} \, dx - \int_{\mathbb{R}^N} F(u) \, dx, \]
where $F(u) = \int_0^u f(t) \, dt$.

It should be mentioned that Ghoussoub and Yuan in [4] studied the following singular quasilinear problem:
\[ \begin{cases} -\Delta_p u = |u|^{p^*(s)-2} u + \lambda |u|^q \nabla u, & x \in \Omega, \\ u = 0, & x \in \partial \Omega, \end{cases} \tag{1.4} \]
where $\Omega$ is a bounded domain in $\mathbb{R}^N$ containing 0, $p \leq q < p^*$. Due to the invariance of $\int_{\Omega} |\nabla u|^p \, dx$ and $\int_{\Omega} \frac{|u|^{p^*(s)}}{|x|^s} \, dx$ with respect to the rescaling $u \mapsto u_\varepsilon = \varepsilon^{N-p} u(\varepsilon \cdot)$ and the existence of nontrivial entire solution of the limiting problem (see [4])
\[ \begin{cases} -\Delta_p u = \frac{|u|^{p^*(s)-2}}{|x|^s} u, & x \in \mathbb{R}^N, \\ u \to 0 \quad \text{as} \ |x| \to \infty, \end{cases} \]
the functional corresponding to problem (1.4),
\[ J(u) := \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \frac{1}{p^*(s)} \int_{\Omega} |u|^{p^*(s)} \frac{|x|^s}{|x|^s} \, dx - \frac{\lambda}{q} \int_{\Omega} |u|^q \, dx, \]
fails to satisfy the classical Palais–Smale (PS in short) condition in $W^{1,p}_0(\Omega)$. However, a local PS condition can be established. Indeed, define the best constant

$$A_s := \inf_{u \in W^{1,p}_0(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p \, dx}{\left( \int_\Omega \frac{|u|^{p^*(s)}}{|x|^s} \, dx \right)^{\frac{p}{p^*-s}}}.$$

suppose $\{u_n\} \subset W^{1,p}_0(\Omega)$ is a sequence such that $J(u_n) \leq c < p^{-s} \frac{N-s}{p(N-s)} A_s^{\frac{N-s}{p-s}}$, $J'(u_n) \to 0$ in $W^{-1}(\Omega) = (W^{1,p}_0(\Omega))^*$, then $\{u_n\}$ contains a strongly convergent subsequence. Using this local PS condition, Ghoussoub and Yuan in [4] proved the existence of positive solutions and sign-changing solutions to (1.4) in $W^{1,p}_0(\Omega)$ for suitable positive parameters $\lambda$, $N$, $p$ and $r$. Moreover, they found that for $\varepsilon > 0$, the functions

$$U_\varepsilon(x) = \left( \varepsilon(N-s) \left( \frac{N-p}{p-1} \right)^{p-1} \right)^{\frac{N-p}{N-s}} \left( \varepsilon + |x|^{\frac{p-s}{p-1}} \right)^{\frac{N-p}{p-s}}$$

solve the equation

$$-\Delta_p u = \frac{|u|^{p^*(s)-2} u}{|x|^s} \text{ in } \mathbb{R}^N \setminus \{0\}$$

and satisfy

$$\int_{\mathbb{R}^N} |\nabla U_\varepsilon|^p \, dx = \int_{\mathbb{R}^N} \frac{|U_\varepsilon|^{p^*(s)}}{|x|^s} \, dx = (A_s) \frac{N-s}{p-s},$$

$A_s$ is independent of $\Omega$ and is achieved by $U_\varepsilon$ on $\mathbb{R}^N$.

In the case when $\Omega$ is an unbounded domain in $\mathbb{R}^N$, the corresponding problem becomes more complicated, since the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is not compact for all $q \in [p, p^*)$. However, thanks to the Strauss’ lemma (see [6]), the embedding $W^{1,p}_{r}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is compact for all $q \in [p, p^*)$. Thus we can discuss the nontrivial solutions of (1.1) in $W^{1,p}_{r}(\mathbb{R}^N)$ by standard variational methods. But there are also some difficulties for (1.1), because the embedding $W^{1,p}_{r}(\mathbb{R}^N) \hookrightarrow L^{p^*(s)}(\mathbb{R}^N, |x|^{-s})$ is still not compact.

Many authors have studied the existence of nontrivial solutions for quasilinear elliptic problems with critical Sobolev exponent $2^*$ (see, for instance, [2–4,7]). However, to our knowledge, there are no results on the existence of nontrivial solutions for (1.1) as $0 < s < p$. Recently, Deng et al. in [2] studied problem (1.1) in the case $s = 0$, i.e., the nonsingular case of (1.1), obtained the existence results of nontrivial solutions and nodal solutions.

Inspired by [2] and [4], we continue to study the existence of nontrivial solutions for (1.1) in this paper. The main results we obtained are presented in the following theorem.
Theorem 1.1. Assume that $N \geq p^2$, $f(t)$ satisfy $(f_1)$, $(f_2)$ and $(f_3)$. Then problem (1.1) has a nontrivial solution $u \in W^{1,p}_r(\mathbb{R}^N)$ such that

$$I(u) < \frac{p-s}{p(N-s)}(A_2)^{\frac{N-s}{N-p}}.$$ 

This paper is organized as follows. In Section 2, firstly we prove the weak continuity of the operator $-\Delta_p u$. Then by standard mountain pass arguments, we manage to give the proof of Theorem 1.1. In the following discussion, we denote various positive constants as $C$ and omit $dx$ in integration for convenience.

2. Proof of the main results

We first give some definitions and lemmas.

Definition 2.1. Let $\{u_n\}$ be a sequence in $W^{1,p}_r(\Omega)$, if there exists a constant $c \in \mathbb{R}$ such that

$$I(u_n) \to c, \quad I'(u_n) \to 0 \quad \text{in} \quad (W^{1,p}_r(\Omega))^*$$

as $n \to \infty$, then $\{u_n\}$ is called a $(PS)_c$ sequence in $W^{1,p}_r(\Omega)$.

We say $I(u)$ satisfies $(PS)_c$ condition if any sequence $\{u_n\} \subset W^{1,p}_r(\Omega)$ satisfying $I(u_n) \to c$ and $I'(u_n) \to 0$ in $(W^{1,p}_r(\Omega))^*$ as $n \to \infty$ has a convergent subsequence.

Definition 2.2. Let $\{u_n\}$ be a sequence in $W^{1,p}_r(\Omega)$. If for any $\eta \geq 0$, there exists $T \geq 0$ such that for all $n \in \mathbb{N},$

$$\int_0^\infty |u_n(r)|^{p^*_r} r^{N-1} dr \leq \eta,$$

then we say $\{u_n\}$ is a tight sequence in $W^{1,p}_r(\Omega)$.

By a standard argument, we can obtain the following lemma by the concentration compactness (cf. [7]).

Lemma 2.1. Assume that $(f_1)$, $(f_2)$ and $(f_3)$ hold, $\{u_n\}$ is a tight sequence. Then there exists a subsequence (without loss of generality still denoted by $\{u_n\}$) and $u \in W^{1,p}_r(\mathbb{R}^N)$ such that

$$u_n \to u \quad \text{weakly in} \quad W^{1,p}(\mathbb{R}^N),$$

$$\nabla u_n \to \nabla u \quad \text{a.e. in} \quad \mathbb{R}^N,$$

$$|\nabla u_n|^{p-2} \nabla u_n \to |\nabla u|^{p-2} \nabla u \quad \text{weakly in} \quad L^{\frac{p}{p-2}}(\Omega)^N.$$ 

Lemma 2.2. Assume that $(f_1)$, $(f_2)$ and $(f_3)$ hold, $\{u_n\}$ is a $(PS)_c$ sequence in $W^{1,p}_r(\mathbb{R}^N)$. Then there exists a subsequence (still denoted by $\{u_n\}$) and $u \in W^{1,p}_r(\mathbb{R}^N)$ such that
\( u_n \rightarrow u \) weakly in \( W^{1,p}(\mathbb{R}^N) \),
\( \nabla u_n \rightarrow \nabla u \) a.e. in \( \mathbb{R}^N \),
\( |\nabla u_n|^{p-2}\nabla u_n \rightarrow |\nabla u|^{p-2}\nabla u \) weakly in \( [L^\infty(\Omega)]^N \).

**Proof.** By Lemma 2.1, it suffices to prove that \( \{u_n\} \) is a tight sequence in \( W^{1,p}_r(\mathbb{R}^N) \). To this end, suppose that \( \{u_n\} \) is a \((PS)_c\) sequence in \( W^{1,p}_r(\mathbb{R}^N) \), then as \( n \rightarrow \infty \) we have
\[
I(u_n) \rightarrow c, \quad I'(u_n) \rightarrow 0 \quad \text{in} \quad (W^{1,p}_r(\mathbb{R}^N))^*.
\]
Hence
\[
\frac{1}{p} \int_{\mathbb{R}^N} \left( |\nabla u_n|^p + |u_n|^p \right) - \frac{1}{p^*(s)} \int_{\mathbb{R}^N} \frac{|u_n|^{p^*(s)}}{|x|^s} - \int_{\mathbb{R}^N} F(u_n) = c + o(1), \tag{2.1}
\]
\[
\int_{\mathbb{R}^N} \left( |\nabla u_n|^p + |u_n|^p \right) - \int_{\mathbb{R}^N} \frac{|u_n|^{p^*(s)}}{|x|^s} - \int_{\mathbb{R}^N} f(u_n)u_n = \langle \xi_n, u_n \rangle, \tag{2.2}
\]
where \( \xi_n = I'(u_n) \). By (2.1), (2.2) and (1.3) we get
\[
\left( \frac{1}{p} - \frac{1}{p + \tau} \right) \int_{\mathbb{R}^N} f(u_n)u_n \leq c + o(1) + \frac{1}{p} \|\xi_n\|\|u_n\|,
\]
\[
\frac{p - s}{p(N - s)} \|u_n\|_{W^{1,p}}^p \leq \left( \frac{1}{p + \tau} - \frac{1}{p^*(s)} \right) \int_{\mathbb{R}^N} f(u_n)u_n
\]
\[
\quad + c + o(1) + \frac{1}{p^*(s)} \|\xi_n\|\|u_n\|
\]
\[
\leq C_1 + C_2\|u_n\|, \tag{2.3}
\]
where we have used (2.3) and the fact that \( \|\xi_n\| \rightarrow 0 \), \( C_1 \) and \( C_2 \) are positive constant.
Hence we conclude that \( \{u_n\} \) is a bounded sequence in \( W^{1,p}_r(\mathbb{R}^N) \), i.e., \( \|u_n\| \leq \bar{C} < \infty \).
On the other hand, it follows from Lemma 2.1 in [5] that
\[
\|u_n(r)\|_{r^{N-2}} \leq C\|u_n\|_{W^{1,p}} \leq \bar{C} < \infty.
\]
Thus
\[
\|u_n(r)\|_{r^{N-1}} \leq (\bar{C})^\tau r^{-(N(p-2)-1)/N-p}.
\]
Since \( \frac{N(p-2)}{N-p} + 1 > 1 \), for any \( \eta > 0 \), there exists some \( T > 0 \), such that
\[
\int_{T}^{\infty} \|u_n(r)\|_{r^{N-1}} dr \leq (\bar{C})^\tau \int_{T}^{\infty} r^{-(N)} dr \leq \eta
\]
holds for all \( n \in \mathbb{N} \). By Definition 2.2, \( \{u_n\} \) is a tight sequence and we complete the proof. \( \square \)
In the following, we discuss the existence of nontrivial solutions of (1.1) by the mountain pass theorem. For convenience, we denote \( \int_{\mathbb{R}^N} \| \cdot \|_{W^{1,p}} \| \cdot \|_{L^p} \) and \( W^{1,p}_{r}(\mathbb{R}^N) \) by \( \int \| \cdot \|_r \) respectively.

**Lemma 2.3.** Assume that \((f_1), (f_2)\) and \((f_3)\) hold, \( c \in \left(0, \frac{p-s}{p|N-s|} A_{\frac{N-s}{p-s}} \right) \). Then \( I(u) \) satisfies \((PS)_c\) condition.

**Proof.** Let \( \{u_n\} \in W^{1,p}_{r} \) be a \((PS)_c\) sequence, by the proof of Lemma 2.2, \( \{u_n\} \) is bounded in \( W^{1,p}_{r} \). Passing to a subsequence (still denoted by \( \{u_n\} \)), as \( n \to \infty \), we get that

\[
\left\{ \begin{array}{l}
\nabla u_n \to \nabla u \quad \text{weakly in } W^{1,p}_{r}, \\
u_n \to u \quad \text{in } L^q(\mathbb{R}^N), \quad q \in [p, p^*), \\
u_n \to u \quad \text{a.e. in } \mathbb{R}^N.
\end{array} \right.
\]

From Lemma 2.2, by subtracting a subsequence, we have that

\[
|\nabla u_n|^{p-2}\nabla u_n \to |\nabla u|^{p-2}\nabla u \quad \text{weakly in } L^{\frac{p}{p-1}}(\Omega)^N.
\]

By (2.4)–(2.6), using Lemma 2.1 in [6], we deduce that

\[
F(u_n) \to F(u) \quad \text{in } L^1(\mathbb{R}^N),
\]

\[
f(u_n)u_n \to f(u)u \quad \text{in } L^1(\mathbb{R}^N).
\]

It follows from Sobolev–Hardy inequality (see [4]) that \( |u_n|^{p^*(s)-2}u_n \) is bounded in \( L^{\frac{p^*(s)}{p^*(s)-1}}(\mathbb{R}^N, |x|^{-s}) \), thus we have that

\[
|u_n|^{p^*(s)-2}u_n \to |u|^{p^*(s)-2}u \quad \text{weakly in } L^{\frac{p^*(s)}{p^*(s)-1}}(\mathbb{R}^N, |x|^{-s}).
\]

Since \( I'(u_n) \to 0 \), by (2.4) and (2.7)–(2.10) we obtain that

\[
\int \left(|\nabla u|^p + |u|^p\right) - \int \frac{|u|^{p^*(s)}}{|x|^{s}} - \int f(u)u = 0.
\]

From (1.3),

\[
I(u) = \left(\frac{1}{p} - \frac{1}{p^*(s)}\right) \int \frac{|u|^{p^*(s)}}{|x|^{s}} + \frac{1}{p} \int f(u)u - \int F(u) > 0.
\]

On the other hand, since \( I(u_n) \to c, I'(u_n) \to 0 \) and \( \{u_n\} \) is bounded, we have that

\[
\frac{1}{p} \|u_n\|^p - \frac{1}{p^*(s)} \int \frac{|u_n|^{p^*(s)}}{|x|^{s}} - \int F(u) = c + o(1)
\]

and

\[
\|u_n\|^p - \int \frac{|u_n|^{p^*(s)}}{|x|^{s}} - \int f(u)u = o(1).
\]
Set \( v_n = u_n - u \), by Brezis–Lieb lemma [1] and Lemma 2.2, we infer that
\[
\int \frac{|u_n|^{p'(s)}}{|x|^s} = \int \frac{|u|^{p'(s)}}{|x|^s} + \int \frac{|v_n|^{p'(s)}}{|x|^s} + o(1)
\]
(2.15)
and
\[
\int |u_n|^p = \int |u|^p + o(1).
\]
(2.16)

It follows directly from (2.13)–(2.16) that
\[
I(u) + \frac{1}{p} \int |\nabla v_n|^p - \frac{1}{p^*(s)} \int \frac{|v_n|^{p^*(s)}}{|x|^s} = c + o(1)
\]
(2.17)
and
\[
\int |\nabla v_n|^p - \int \frac{|v_n|^{p^*(s)}}{|x|^s} = o(1).
\]

Without loss of generality, we may assume that \( \lim_{n \to \infty} \int |\nabla v_n|^p = k \). Then we get that
\[
\lim_{n \to \infty} \int \frac{|v_n|^{p^*(s)}}{|x|^s} = k.
\]

By Sobolev–Hardy inequality,
\[
\int |\nabla v_n|^p \geq A_k \left( \int \frac{|v_n|^{p^*(s)}}{|x|^s} \right)^{\frac{p}{p^*(s)}} \text{ for all } n \in \mathbb{N}.
\]

Then by taking \( n \to \infty \), we obtain
\[
k \geq A_k k^{\frac{p}{p^*(s)}}.
\]

If \( k > 0 \), then we have that \( k \geq (A_k)^{\frac{N-s}{N-2}} \). By (2.17) we deduce that
\[
I(u) \leq c - \frac{p-s}{p(N-s)} (A_k)^{\frac{N-s}{p(N-s)}} \leq 0,
\]
which contradicts to (2.12). Thus \( k = 0 \). By the definition of \( v_n \) we conclude that \( I(u) \) satisfies (PS)\(_c\) condition.

By Lemma 2.3 and the mountain pass theorem given in [1], we can verify the following lemma.

**Lemma 2.4.** Assume that \((f_1), (f_2)\) and \((f_3)\) hold. If there exists \( u_0 \in W^{1,p}_r \), \( u_0 \not\equiv 0 \), such that
\[
\sup_{t \geq 0} I(tu_0) < \frac{p-s}{p(N-s)} (A_k)^{\frac{N-s}{p(N-s)}},
\]
(2.18)
then (1.1) has at least one nontrivial weak solution.
In the following discussion, we prove that (2.18) holds naturally for $N \geq p^2$. To this end, for $\varepsilon > 0$ set

$$u_\varepsilon(x) = Q_\varepsilon \left( \varepsilon \frac{x}{p \sqrt{1 - \varepsilon}} + |x| \frac{p - N}{p - s} \right).$$

we can choose $Q > 0$ such that $u_\varepsilon$ satisfies

$$\int |\nabla u_\varepsilon|^p = \int |u_\varepsilon|^{p^*(\varepsilon)} |x|^{p - s} = (A_\varepsilon) \frac{N - s}{p - s}.$$

Let $0 \leq \phi(x) \leq 1$ be a cutting-off function in $C_0^\infty(\mathbb{R}^N) \cap W^{1,p}_r$ such that $\phi(x) = 1$ for $|x| \leq R$ and $\phi(x) = 0$ for $|x| \geq 2R$. Set $v_\varepsilon(x) = \phi(x)u_\varepsilon(x)$, then we have the following estimating results (see [4]).

**Lemma 2.5.** $v_\varepsilon(x)$ satisfies the following estimates:

1. $\|\nabla v_\varepsilon\|_p^p = (A_\varepsilon) \frac{N - s}{p - s} + O(\varepsilon \frac{N - p}{p - s})$.
2. $\int \frac{|v_\varepsilon|^{p^*(\varepsilon)}}{|x|^p} = (A_\varepsilon) \frac{N - s}{p - s} + O(\varepsilon \frac{N - s}{p - s})$.
3. $\|v_\varepsilon\|_r^r = \begin{cases} O(\varepsilon^{\frac{N-p}{r-1}}), & r > p^*(1 - \frac{1}{p}), \\
O(\varepsilon^{\frac{N-p}{r-1}}|\ln \varepsilon|), & r = p^*(1 - \frac{1}{p}), \\
O(\varepsilon^{\frac{N-p}{r-1}}), & r < p^*(1 - \frac{1}{p}), \end{cases}$
4. $\|v_\varepsilon\|_p^p = \begin{cases} O(\varepsilon^p), & N > p^2, \\
O(\varepsilon^p|\ln \varepsilon|), & N = p^2, \\
O(\varepsilon^\frac{N-p}{p-1}), & N < p^2, \end{cases}$
5. $\|v_\varepsilon\|_{p-1}^p = O(\varepsilon \frac{N-p}{p-1})$.
6. $\|v_\varepsilon\|_{p^*}^{p^*} = D + O(\varepsilon \frac{N-p}{p-1})$.

Here $D > 0$ is a constant, by $O(\varepsilon^\gamma)$ we denote the quantity $\alpha$ satisfying $|\alpha| < K\varepsilon$ as $\varepsilon \to 0^+$, $\gamma \geq 0$, $K$ is some positive constant.

**Lemma 2.6.** Assume that $(f_1)$, $(f_2)$ and $(f_3)$ hold. Then there exists $u_0 \in W^{1,p}_r$, $u_0 \not\equiv 0$, such that (2.18) holds for $N \geq p^2$.

**Proof.** We divide the proof into four steps.

1. We claim that for $\varepsilon$ small enough, there exists some $t_\varepsilon > 0$ such that

$$I(t_\varepsilon v_\varepsilon) = \sup_{t \geq 0} I(t v_\varepsilon)$$
and
\[0 < C_1 < t_e < C_2 < \infty, \quad (2.19)\]
where \(C_1, C_2\) are positive constants independent of \(\varepsilon\).

In fact, by \((f_1)\), there exists some \(t_e > 0\) such that
\[I(t_e v_\varepsilon) = \sup_{t \geq 0} I(t v_\varepsilon) \quad \text{and} \quad \frac{dI(t v_\varepsilon)}{dt} \bigg|_{t = t_e} = 0.\]

Thus we have that
\[
\frac{1}{t_e^{p^*(s)-p}} \int \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s} - \frac{1}{t_e^{p^*(s)-p}} \int \frac{f(t v_\varepsilon)}{|x|^s} v_\varepsilon = 0. \quad (2.20)
\]

By Lemma 2.5 and \((f_1)\), for any \(\delta > 0\), there exists some constants \(k_1, k_2, k_3 > 0\) such that
\[
k_1 \int (\delta t_\varepsilon^{p^*-1} v_\varepsilon^{p^*-1} + k_2 t_\varepsilon^{p^*-1} v_\varepsilon^{p^*-1}) v_\varepsilon = k_1 \int (\delta t_\varepsilon^{p^*-1} v_\varepsilon^{p^*-1} + k_2 v_\varepsilon^{p^*-1} \varepsilon vp_0^{p^*-p} \quad \text{as} \quad \varepsilon \to 0^+.
\]

From (2.20), as \(\varepsilon \to 0^+\) we get that
\[1 - t_\varepsilon^{p^*(s)-p} - \delta k_3 t_\varepsilon^{p^*-p} \leq 0,
\]
thus \(t_0 > 0\) and there exists some \(C_1 > 0\) such that
\[t_e > C_1 \quad \text{as} \quad \varepsilon \text{ small enough.}
\]

On the other hand, from Lemma 2.5 and (2.20), as \(\varepsilon \to 0^+\) there exists some positive constant \(C_2 > 0\) such that
\[t_e \leq \left( \int (|\nabla v_\varepsilon|^p + |v_\varepsilon|^p) \right)^{\frac{1}{p^*(s)-p}} \left( \int \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s} \right)^{\frac{1}{p^*(s)-p}} \leq C_2. \]

**Step 2.** As \(N > p^2\), we claim that
\[
I(t v_\varepsilon) = \frac{\varepsilon}{p} \int |\nabla v_\varepsilon|^p - \frac{\varepsilon}{p} \frac{p^*(s)}{p^*(s) - p} \int \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s} + \frac{\varepsilon}{p} \int |v_\varepsilon|^p - \int F(t v_\varepsilon)
\]
\[\leq \frac{p - s}{p(N - s)} (A_s)^{\frac{N-s}{p(N-s)}} + O(\varepsilon^p) - \int F(t v_\varepsilon). \quad (2.21)
\]

Indeed, define the function
\[g(t) := \frac{\varepsilon}{p} \int |\nabla v|^p - \frac{\varepsilon}{p} \frac{p^*(s)}{p^*(s) - p} \int \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s}, \quad t \in (0, \infty).
\]

Then \(g(t)\) attains its maximum at
\[t_0 := \left( \int |\nabla v|^p \right)^{\frac{1}{p^*(s)-p}} \left( \int \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s} \right)^{\frac{1}{p^*(s)-p}} = 1 + O(\varepsilon^\frac{n-p}{s-p}) \]
Note that \( \frac{N-p}{p-N} > p \) as \( N > p^2 \), then (2.21) follows.

Similarly, as \( N = p^2 \) we can also get

\[
I(t_s v_s) = \frac{t_s^p}{p} \int |\nabla v_s|^p - \frac{t_s^p r_s(t_s)}{p^*(s)} \int |v_s|^p r_s(t_s) + \frac{t_s^p}{p} \int |v_s|^p - \int F(t_s v_s)
\leq \frac{p}{p(N-s)} (A_s) \frac{N-p}{p-N} + O(e^p |\ln \epsilon|) - \int F(t_s v_s). \tag{2.22}
\]

**Step 3.** We claim that if \((f_1), (f_2)\) and \((f_3)\) hold, then

\[
\lim_{\epsilon \to 0^+} \epsilon^{N-p} |\ln \epsilon|^{-1} \int_0^{1/\epsilon} F\left( \frac{\epsilon^{N-p} r}{(1 + r^{p-N})^{N-p}} \right) r^{N-1} dr = +\infty. \tag{2.23}
\]

In fact, from \((f_2)\), for \( t > 0 \) we have that

\[
\left( \frac{f(t)}{t^{p+\tau}} \right)' = \frac{tf'(t) - (p-1+\tau)f(t)}{t^{p+\tau}} \geq 0.
\]

Thus there exists some constant \( C > 0 \) such that \( F(t) \geq Ct^{p+\tau} \). Therefore

\[
F\left( \frac{\epsilon^{N-p} r}{(1 + r^{p-N})^{N-p}} \right) \geq C \left( \frac{\epsilon^{N-p} r}{(1 + r^{p-N})^{N-p}} \right)^{p+\tau}
\]

Hence

\[
\epsilon^{N-p} |\ln \epsilon|^{-1} \int_0^{1/\epsilon} F\left( \frac{\epsilon^{N-p} r}{(1 + r^{p-N})^{N-p}} \right) r^{N-1} dr
\geq C \epsilon^{N-p} |\ln \epsilon|^{-1} \int_0^{1/\epsilon} \left( \frac{\epsilon^{N-p} r}{(1 + r^{p-N})^{N-p}} \right)^{p+\tau} r^{N-1} dr
\]

\[
= C \epsilon^{\frac{(p-N)}{p}} |\ln \epsilon|^{-1} \int_0^{1/\epsilon} \left( \frac{r^{N-1}}{(1 + r^{p-N})^{N-p}} \right)^{(p-N)\frac{p}{p-N}} dr \to +\infty \quad \text{as} \quad \epsilon \to 0^+.
\]

**Step 4.** We claim that if \( N \geq p^2 \), then \( I(t_s v_s) < \frac{p-N}{p(N-s)} (A_s) \frac{N-p}{p-N} \) as \( \epsilon \) small enough.

Indeed, by (2.21), it suffices to verify that

\[
g(t_0) = \frac{t_0^p}{p} \int |\nabla v|_s|^p - \frac{t_0^p r_0(t_0)}{p^*(s)} \int |v|^p r_0(t_0) + \frac{t_0^p}{p} \int |v|^p - \int F(t_0 v) = \frac{p}{p(N-s)} (A_s) \frac{N-p}{p-N} + O(e^p |\ln \epsilon|)
\]

and

\[
\int_0^{1/\epsilon} F\left( \frac{\epsilon^{N-p} r}{(1 + r^{p-N})^{N-p}} \right) r^{N-1} dr
\geq C \epsilon^{N-p} |\ln \epsilon|^{-1} \int_0^{1/\epsilon} \left( \frac{\epsilon^{N-p} r}{(1 + r^{p-N})^{N-p}} \right)^{p+\tau} r^{N-1} dr
\]

\[
= C \epsilon^{\frac{(p-N)}{p}} |\ln \epsilon|^{-1} \int_0^{1/\epsilon} \left( \frac{r^{N-1}}{(1 + r^{p-N})^{N-p}} \right)^{(p-N)\frac{p}{p-N}} dr \to +\infty \quad \text{as} \quad \epsilon \to 0^+.
\]
\[ \lim_{\varepsilon \to 0^+} \varepsilon^{-p} |\ln \varepsilon|^{-1} \int_{B_2 R} F(t_\varepsilon v_\varepsilon) \, dx = \lim_{\varepsilon \to 0^+} \varepsilon^{-p} |\ln \varepsilon|^{-1} \int_{B_2 R} F(t_\varepsilon v_\varepsilon) \, dx = +\infty. \tag{2.24} \]

It follows from \((f_2)\) and (2.19) that

\[ \lim_{\varepsilon \to 0^+} \varepsilon^{-p} |\ln \varepsilon|^{-1} \int_{B_2 R} F(t_\varepsilon v_\varepsilon) \, dx \geq \lim_{\varepsilon \to 0^+} \varepsilon^{-p} |\ln \varepsilon|^{-1} 1 \int_0^{R/\varepsilon} F\left( \frac{C \varepsilon^{p-N}}{r} \right) r^{N-1} \omega_N \, dr \]

\[ = C \lim_{\varepsilon \to 0^+} \varepsilon^{-p} |\ln \varepsilon|^{-1} \int_0^{R'/\varepsilon} F\left( \frac{\varepsilon^{p-N}}{(1+r/p^{1-q})^N} \right) r^{N-1} \omega_N \, dr, \]

where \(\omega_N\) is the area of the unit sphere in \(\mathbb{R}^N\) and \(C\) is a positive constant. As \(R' \geq 1\), then (2.24) is a consequence of (2.23).

As \(R' < 1\), from \((f_1)\) we have that

\[ Z_\varepsilon := \varepsilon^{-p} |\ln \varepsilon|^{-1} \int_{R'/\varepsilon}^{1/\varepsilon} F\left( \frac{\varepsilon^{p-N}}{(1+r/p^{1-q})^N} \right) r^{N-1} \, dr \]

\[ = \varepsilon^{-p} |\ln \varepsilon|^{-1} F\left( \frac{\varepsilon^{p-N}}{(1+(\theta \varepsilon^{-1})^{N-p})^N} \right) (\theta \varepsilon^{-1})^{N-1} (1-R') \varepsilon^{-1} \]

\[ \leq C \varepsilon^{N-p} |\ln \varepsilon|^{-1} F(C_1 \varepsilon^{p-N}) \varepsilon^{-N} \]

\[ \leq C \varepsilon^{-p} |\ln \varepsilon|^{-1} o(\varepsilon^{N-p}) \leq M \quad \text{as } N \geq p^2, \]

where \(\theta \in (R', 1)\), \(C\) and \(M\) are positive constants, by \(o(\varepsilon^7)\) we denote the quantity \(\beta > 0\) satisfying \(\beta \varepsilon^{-t} \to 0 \) as \(\varepsilon \to 0^+\). Hence \(Z_\varepsilon\) is bounded as \(\varepsilon \to 0^+\). It follows naturally that (2.24) is again a consequence of (2.23). Let \(u_0 = v_\varepsilon, \varepsilon\) small enough, from Step 4 we deduce that

\[ \sup_{t \geq 0} I(tu_0) = I(t_\varepsilon v_\varepsilon) < \frac{p-s}{p(N-s)} (A_{\lambda})^{N-p}. \]

The proof of this lemma is completed. \(\square\)

**Proof of Theorem 1.1.** The results of Theorem 1.1 follows from Lemmas 2.4 and 2.6. \(\square\)

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References