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Algebraic Inversion of the Laplace Transform

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Abstract—A new algebraic scheme for inverting Laplace transforms of smooth functions is presented. Expansion of the Laplace transform $F(s)$ in descending powers of s is used to construct the Taylor series of the corresponding time function $f(t)$ This is done through entirely algebraic evaluations of $F(s)$ at symmetric points around circles in the complex plane Test functions are used to examine the method and the results show good convergence over a broad region near $t = 0$. The method is especially well-suited to computer-based inversion of Laplace transform. © 2005 Elsevier Ltd All rights reserved.

1. INTRODUCTION

There are many problems whose solution may be found in terms of a Laplace transform. Simple transforms can often be inverted using readily available tables. More complex functions can be analytically inverted through the complex inversion formula,

$$f(t) = \frac{1}{i2\pi} \int_{c-i\infty}^{c+i\infty} e^{st} F(s)\, ds, \tag{1.1}$$

where c is a positive real number, such that all the poles of the function $F(s)$ lie at the left of the line $s = c$. In many cases, the resulting functions are not easy to invert analytically and there is need for numerical schemes.

A variety of different methods exist for numerically inverting the Laplace transform. There exists no universal method, but different types of methods work well for different classes of functions. Some methods numerically evaluate the inverse through expanding the complex inversion formula (1 1) into a cosine series [1] or a more general Fourier type series [2]. Such methods work well for discontinuous functions. Other methods involve implementation of the requisite quadrature methods for integrating the definition of the Laplace transform; combined with evaluations of $F(s)$ at selected values of s result in a system of linear equations to be solved [3]. Another class of methods hinge on the very important result given by Widder [4,5] and its subsequent improvement, the post-Widder inversion formula,

$$f(t) = \lim_{n\to\infty} \frac{(-1)^n}{n!} \left[F^{(n)}\left(\frac{n+1}{t}\right)\right] \left(\frac{n+1}{t}\right)^{n+1}, \tag{1.2}$$

where $F^{(n)}(s)$ denotes the n^{th} derivative of the function $F(s)$.

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This formula can be used both analytically and numerically, but in numerical implementation the need to take derivatives of the transformed function is a disadvantage. Another method related to the above result is given in [6]. Other major families of methods include those in [7–9].

An important theorem is developed in [5] and is related to the method presented in the following.

If $F(s)$ can be written in a series of descending powers of s,

$$F(s) = \sum_{n=0}^{\infty} \frac{A_n}{s^{n+1}}, \qquad \text{for } s > r, \tag{1.3}$$

then the time function $f(t) = L^{-1}[F(s)]$, can be expressed by the series,

$$f(t) = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!}, \qquad \text{for } 0 \le t < \infty, \tag{1.4}$$

where r is some positive number for which (1.3) is convergent. Use of this theorem requires explicit knowledge of the expansion (1.3), i.e., the coefficients A_n need to be evaluated. In this work, we construct $f(t)$ in a series, in terms of expressions that include only $F(s)$ and $f(0)$. The scheme is especially well-suited for numerical use.

2. THE ALGORITHM

This algorithm is based on the expansion of the definition of the Laplace transform,

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt, \tag{2.1}$$

into an asymptotic series of descending powers of s using the Laplace method. This expansion is valid if $f(t)$ is differentiable at $t = 0$ and for $f(t) \le e^{Bt}$ as $t \to \infty$ for some positive real B. Successively integrating (2.1) by parts, we have

$$\begin{aligned} F(s) &= -\frac{1}{s} e^{-st} f(t) \Big|_0^{\infty} - \frac{1}{s} \int_0^{\infty} e^{-st} f'(t)\, dt \\ &\vdots \\ &= \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \frac{f''(0)}{s^3} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{s^{n+1}} \end{aligned} \tag{2.2}$$

or

$$sF(s) = f(0) + \frac{f'(0)}{s} + \frac{f''(0)}{s^2} + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{s^n}. \tag{2.3}$$

By substituting $t = 1/s$ we have,

$$\frac{1}{t} F\left(\frac{1}{t}\right) = f(0) + t f'(0) + t^2 f''(0) + \cdots = \sum_{n=0}^{\infty} t^n f^{(n)}(0). \tag{2.4}$$

The basic idea of this method is to express the terms $t^n f^{(n)}(0)$ in series that involve algebraic expressions of just $F(s)$, $f(0)$, and higher powers of t. These expressions will then be used to construct the Taylor series of the function $f(t)$, around $t = 0$.

Consider the N roots of a positive real number M in the complex plane,

$$\omega_n(M,N) = \sqrt[N]{M} = \left|\sqrt[N]{M}\right| e^{i(2n\pi/N)}, \qquad n = 0,1,2,\dots,N-1. \tag{2.5}$$

The roots have the important property,

$$\sum_{n=1}^{N} (\omega_n(M,N))^m = \begin{cases} NM^L, & \text{for } m = LN, \quad L = 1,2,3,\dots, \\ 0, & \text{for } m = \text{other integer}. \end{cases} \tag{2.6}$$

Next, we define

$$S_N(M,t) = \sum_{n=1}^{N} \left[\frac{1}{(\omega_n(M,N))\,t} F\left(\frac{1}{(\omega_n(M,N))\,t}\right) - f(0)\right] \tag{2.7}$$

Using (2.4), as well as property (2.6), we arrive at the expression,

$$S_N(M,t) = NMt^N f^{(N)}(0) + NM^2 t^{2N} f^{(2N)}(0) + NM^3 t^{3N} f^{(3N)}(0) + \cdots. \tag{2.8}$$

Thus, an expression for $t^N f^{(N)}(0)$ is obtained,

$$\begin{aligned} t^N f^{(N)}(0) &= \frac{S_N(M,t)}{NM} - Mt^{2N} f^{(2N)}(0) - M^2 t^{3N} f^{(3N)}(0) - \cdots \\ &= \frac{S_N(M,t)}{NM} - \sum_{n=1}^{\infty} M^n t^{(n+1)N} f^{((n+1)N)}(0). \end{aligned} \tag{2.9}$$

We generalize (2.9) for different values of M as a function of N,

$$t^N f^{(N)}(0) = \frac{S_N(M_N,t)}{NM_N} - \sum_{n=1}^{\infty} M_N^n t^{(n+1)N} f^{((n+1)N)}(0). \tag{2.10}$$

The Taylor series of the function $f(t)$ around $t = 0$ is given by

$$f(t) = f(0) + tf'(0) + \frac{t^2}{2!} f''(0) + \cdots = f(0) + \sum_{n=1}^{\infty} \frac{t^n}{n!} f^{(n)}(0). \tag{2.11}$$

By substituting (2.10), term by term in the expression (2.11), we arrive at the new series that involves the terms S_n,

$$\begin{aligned} f(t) &= f(0) + \frac{C_1}{M_1} S_1(M_1,t) + \frac{C_2}{M_2} S_2(M_2,t) + \cdots + \frac{C_N}{M_N} S_3(M_3,t) + \cdots \\ &= f(0) + \sum_{n=1}^{\infty} \frac{C_n}{M_n} S_n(M_n,t). \end{aligned} \tag{2.12}$$

The coefficients C_n can be evaluated through the iterative algorithm,

$$C_n = \frac{1}{n!} - \sum_{i}^{p} M_{\varphi_i}^{(n/\varphi_i - 1)} C_{\varphi_i}, \qquad C_1 = 1. \tag{2.13}$$

By φ_i, we denote the p integers that exactly divide the integer n including $\varphi_1 = 1$, but excluding n itself.

Expression (2.12) is the new series for the inverse Laplace function. The next section describes its implementation.

3. EXAMPLES OF IMPLEMENTATION

On implementing this scheme numerically, series (2.12) is constructed, and thus, the selection of the series of numbers M_n should be addressed. The series M_n has to be chosen, such that $(|\sqrt[N]{M_N}|t)^{-1}$ is inside the region of convergence of (2.3) for the interval of t, for which we require good results. It is therefore sufficient to select M_n that satisfy the inequality,

$$\left(\left|\sqrt[N]{M_N}\right| t\right)^{-1} \gg r_s, \tag{3.1}$$

where r_s is the modulus of the singularity of $sF(s)$ farthest from the origin in the complex plane.

The evaluations of the function $F(s)$, that are used for the computation of the terms $S_n(M_n, t)$, are made around circles in the complex plane that enclose all the singularities of $F(s)$ and consequently of $sF(s)$. It is thus clear that the magnitudes of the numbers M_n chosen, should be decreasing with increasing n. Consideration of machine accuracy should also be made, so a bound exists for the decrease in magnitude. An optimal series of M_n was not pursued in this work, but for the test functions discussed in the next section, the following series was used,

$$M_1 = 0.01, \qquad M_{n+1} = 0.05 M_n. \tag{3.2}$$

This selection works very well, but we should take into account that in the test functions, the modulus of the largest root of the function $F(s)$, is at most, of the order of unity. The coefficients C_n are calculated directly from this choice for M_n, and are the same for all of the functions evaluated in this section. The first ten C_n are listed in Table 1.

Table 1. First ten coefficients C_n corresponding to M_n of form (3.2)

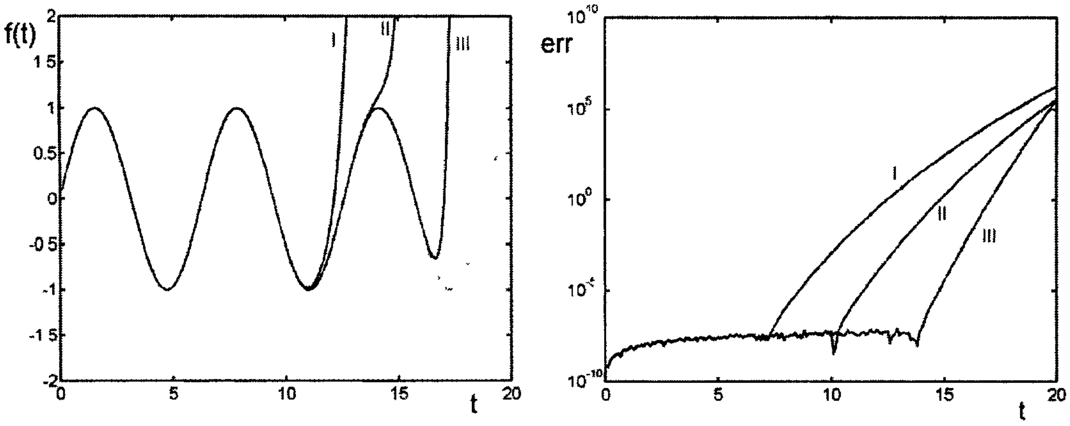
n	M_n	C_n	n	M_n	C_n
1	1 0000e − 02	1 0000e − 00	6	3.1250e − 09	1.3846e − 03
2	5.0000e − 04	4.9000e − 01	7	1 5625e − 10	1.9841e − 04
3	2.5000e − 05	1.6657e − 01	8	7.8125e − 12	2.4750e − 05
4	1 2500e − 06	4.1421e − 02	9	3 9063e − 13	2.7556e − 06
5	6.2500e − 08	8.3333e − 03	10	1.9531e − 14	2.7505e − 07

Table 2. Test functions $F(s)$ examined and their corresponding Laplace inverse $f(t)$

Test Functions	$F(s)$	$f(t) = L^{-1}[F(s)]$
1	$\frac{1}{s^2+1}$	$\sin t$
2	$\frac{1}{(s+1)^2}$	te^{-t}
3	$\frac{1}{\sqrt{s^2+1}}$	$J_0(t)$
4	$\frac{e^{-1/s}}{s^2}$	$\sqrt{t} J_1(2\sqrt{t})$

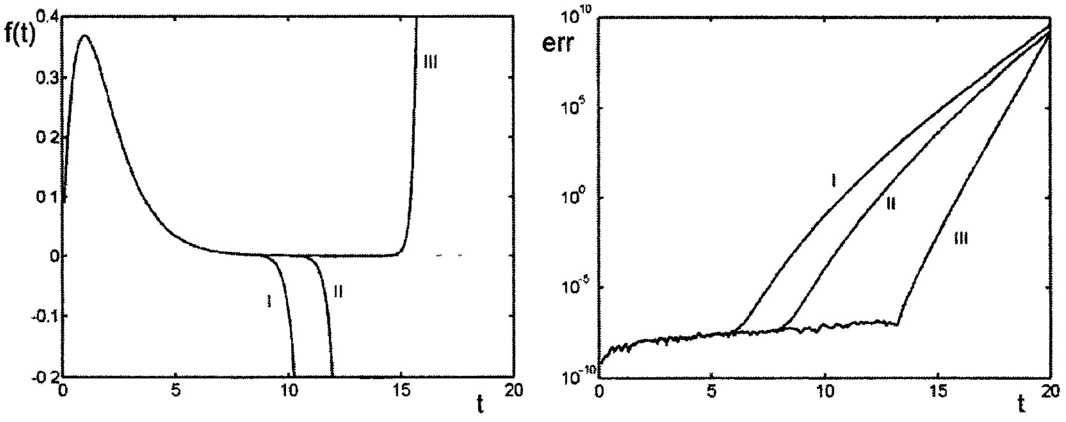
The four test functions given in Table 2 were chosen to test the method and in this section the results are presented. Graphs that plot the approximation through truncating series (2.12) are given and compared with the exact function (shown in the Figures 1a–4a with dotted lines). In all cases $f(0)$ was prescribed and the series (2.12) was truncated for 30, 40, and 80 terms (denoted in the graphs by I, II, and III correspondingly).

Logarithmic scale graphs for the absolute error err $= |f(t) - f_{\text{app}}(t)|$ are presented in Figures 1b–4b. All numerical computations were performed using MATLAB, imaginary parts were ignored.



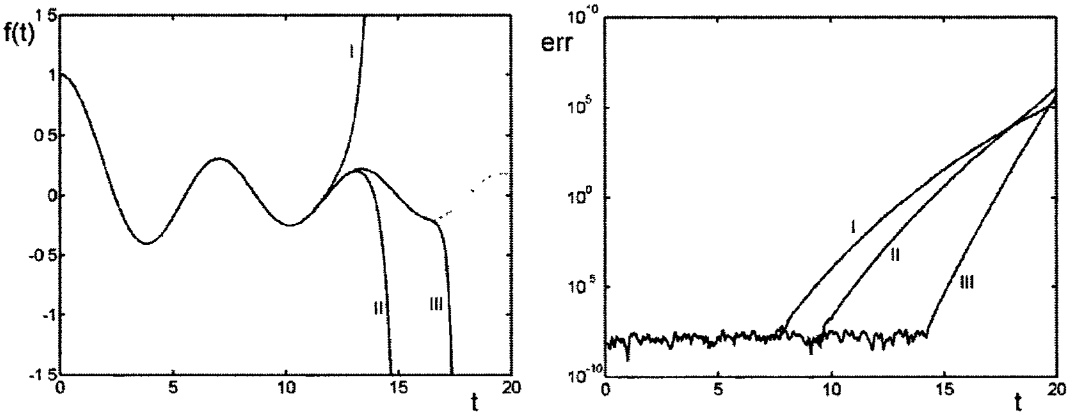
(a) (b)

Figure 1. (a) Approximation for $f(t) = \sin(t)$ with I: 30 terms, II: 40 terms, III 80 terms. (b) Absolute error in log scale



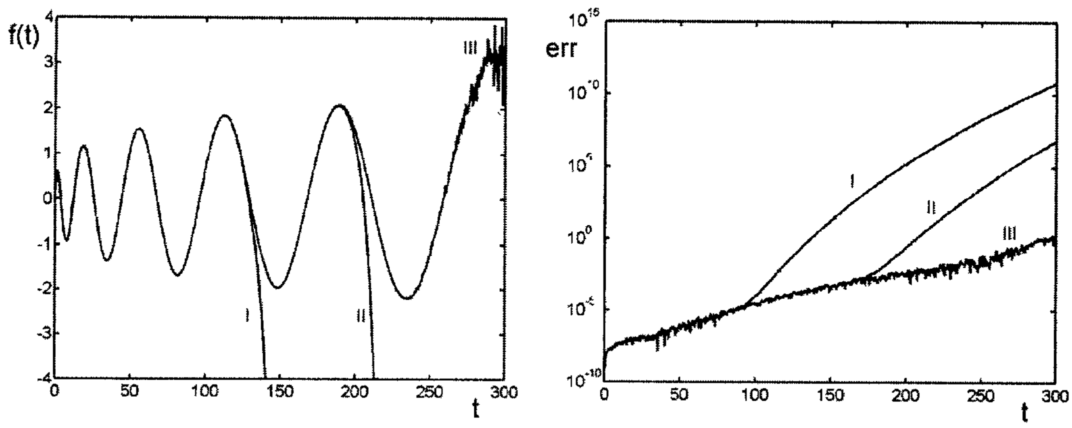
(a) (b)

Figure 2. (a) Approximation for $f(t) = te^{-t}$ with I. 30 terms, II: 40 terms, III. 80 terms (b) Absolute error in log scale.



(a) (b)

Figure 3 (a) Approximation for $f(t) = J_o(t)$ with I· 30 terms, II 40 terms, III. 80 terms (b) Absolute error in log scale



(a) (b)

Figure 4 (a) Approximation for $f(t) = \sqrt{t}J_1(2\sqrt{t})$ with I: 30 terms, II: 40 terms, III: 80 terms. (b) Absolute error in log scale

We have to note, that in the course of approximating the inverse Laplace transform of the function $F(s) = 1/\sqrt{s^2+1}$, care has to be taken in order to evaluate the correct root in the term $\sqrt{s^2+1}$. This is equivalent to selecting the correct analytic continuation of the function $F(s)$, in the region where the Laplace transform integral (2.1) does not converge, i.e., for the values of s with a negative real part. In this particular case, when evaluating $F(s)$ in the right half of the complex plane (for the values of s with a positive real part), the root selected was the one in the right half plane, when evaluating $F(s)$ in the left half plane (for values of s with a negative real part), the root selected was the one in the left half plane. So, for evaluation in the left half plane the expression $F(s) = -1/\sqrt{s^2+1}$ was used.

From the results shown, one can note that the approximation performs very well for a region in time near $t = 0$. Good convergence near $t = 0$ is expected, since the method approximates the Taylor series. At some threshold the error increases rapidly and the approximation diverges. The extent of the region for which the behavior of the approximation is good depends on the function $F(s)$, on the number of terms in the series (2.11) taken, and on the selection of the series of numbers M_n. For the same selection of the series of numbers M_n, the method will work in a more extended region for functions $F(s)$, with singularities of smaller magnitude. The results from the test functions show that the approximation for the function $F(s) = e^{-1/s}/s^2$, works well for a greater range than those approximations for the other test functions, since it has singularities only at $t = 0$, while for the other functions, the positions of the singularities have unit magnitude.

In each plot of error, a distinction between numerical error (noisy lines) and error from having insufficient terms in the Taylor series (smooth lines) is clear. The exception is the function in Figure 4, where for the 80 term approximation, the numerical noise dominates before the limit of the Taylor approximation expires.

4. LIMITATIONS AND DISCUSSION

Since the method is based on constructing the Taylor series around $t = 0$, this approximation method focuses on continuous and differentiable functions $f(t)$. The function to be approximated and all its derivatives have to be finite at $t = 0$ In order for the expansion (2.2) to be valid, $f(t)$ must be bounded by an exponential e^{Bt} for some positive B as $t \to \infty$. The function must also have a finite number of singularities, so that the condition (3.1) can be met.

For the implementation of this method, the function in Laplace space $F(s)$ has to be explicitly given as well as the value $f(0)$. In practice, $f(0)$ is often known from existing initial conditions that are defined in the specific problem to be solved. In the more general cases, that $f(0)$ is not known beforehand, it can be analytically evaluated from the following well-known expression which originates from (2.3),

$$f(0) = \lim_{s \to \infty} [sF(s)]. \tag{4.1}$$

The advantage of the method presented in this paper lies in the fact that it only uses algebraic expressions to construct and, in many cases, accelerate the Taylor series of $f(t)$. The coefficients C_n are easy to compute and one can compute as many C_n as needed without much effort through the iterative algorithm (2.12). The only limitation on the numerical aspect is that the decrease in magnitude of the numbers M_n is bounded by machine accuracy. So, even though the fact that the numbers M_n get very small with n, does not affect the calculation of S_n (since the values $|\sqrt[N]{M_n}|$ do not decrease rapidly and can even be constructed to remain constant), the existence of M_n in the calculations is in itself a problem, when the order of magnitude approaches the machine accuracy. In relation to (3.1), we would like to take M_n very small, but at the same time we have to make sure that these M_n are large enough that they are within machine accuracy until the series is truncated.

For the functions to which the method can be applied, the results are very good in a region of t near $t = 0$. From the error graphs given we can see that the error is extremely small over a broad initial region.

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