Discrete Applied Mathematics 8 (1984) 279-287
North-Holland

COMPARISON OF MEAN DISTANCE IN SUPERPOSED NETWORKS

M. PAOLI
E.R.A. C.N.R.S. No. 452, Laboratoire de Recherche en Informatique, L.R.I. Bâtiment 490, Université de Paris-Sud, 91405 Orsay Cedex, France

Received 2 May 1983

In this paper, we consider the problem of the superposition of two networks in the structure of local telecommunication networks: the physical one, determined by the ways of the means of communication, and the interconnection one. We compare the mean-distance of the graphs associated to these two networks.

Dans cet article, nous considérons le problème de la superposition de deux réseaux dans la structure de réseaux locaux d'interconnexion: le premier, déterminé par le trajet des moyens de communication sera appelé réseau physique, le second est le réseau d'interconnexion. Nous comparons les diamètres moyens des graphes associés à ces réseaux.

Introduction

The following problem has been suggested by people working in the French National Telecommunication Center (C.N.E.T. Lannion).

In the structure of a telecommunication network, we can distinguish two networks. First, suppose we draw on a map the routes that the means of communications (cables for example) follow. Then we get what we call the physical network P, which can be modelled by a graph whose vertices are the intersections of the routes and whose links are the parts of the routes that connected two vertices without any intersection between them. On the other hand, there is the interconnection network I, modelled by a graph whose vertices, the switching centers, correspond to the nodes of the network P, but whose links (or edges) represent the connection between these centers. Such a link can exist only if there exists in P a path between the corresponding nodes of P. In practice, the link which connects two nodes of I travels through the path of P. The network I is said to be superposed on the network P.

The problem which we are interested in has been suggested in the study of future local networks (for example TV networks) in a town. In this case, the physical network is related to the configuration of the streets and the existence of underground networks.

0166-218X/84/$3.00 © 1984, Elsevier Science Publishers B.V. (North-Holland)

The network P is usually given. Its construction depends on cost considerations, on the topography and on the localization of the users. Thus, most often the average path length between two nodes is rather large. On the other hand, there is a freedom for the construction of the interconnection network I. The choice of the connections can improve the average path length between two nodes and therefore, reduce the message delay and the capacity of commutation centers needed for the transmission of the messages between the nodes.

However, the number of links in I is bounded by two constraints:

(1) the number of links issued from a switching center is bounded (for practical constraints),

(2) the capacity of a link of P, which is the number of links of I that can pass through it, is also bounded.

In this paper we will study relation between the mean-distance (or average path length) of the two networks.

Definitions and notation

Definitions and notation not given here can be found in [1].

The vertex-set and the edge-set of a graph G will be denoted, respectively, by $V(G)$ and $E(G)$.

The graphs associated with the physical and the interconnection networks will be denoted, respectively, by g and G, with $V(g)=V(G)$. We will always assume that g and G are simple connected graphs.

We can model the practical problem by giving a one-to-one correspondence ϕ between the set of edges of G and a set of paths of g that we will denote by C. As the graph must be simple, there cannot be several distinct paths in C between any given pair of vertices of G.

The distance between two vertices x and y of g (resp. of G) will be denoted by $d(x,y)$ (resp. $D(x,y)$). We call length of an edge E of the graph G, and denote it by $l(E)$, the number of edges of g in the path $\phi(E)$.

Let Δ be the maximum degree of G and α be $\max_{e\in E(g)} |\{c\in C \mid e\in E(c)\}|$. In practice conditions (1) and (2) mean that Δ and α are bounded.

Let the mean distance of respectively g and G by:

$$\bar{d}=\frac{1}{\upsilon(\upsilon-1)}\sum_{x,y\in V(g)} d(x,y),$$

$$\bar{D}=\frac{1}{\upsilon(\upsilon-1)}\sum_{x,y\in V(G)} D(x,y) \quad \text{where } \upsilon=|V(g)|.$$

Example. In Fig. 1, we give a schema of two superposed networks. In Figs. 2(a) and 2(b) we give the graphs g and G associated respectively to the networks P and I. In this example, the parameters are $\alpha=2$, $\delta=4$, where δ is the maximum degree of g,

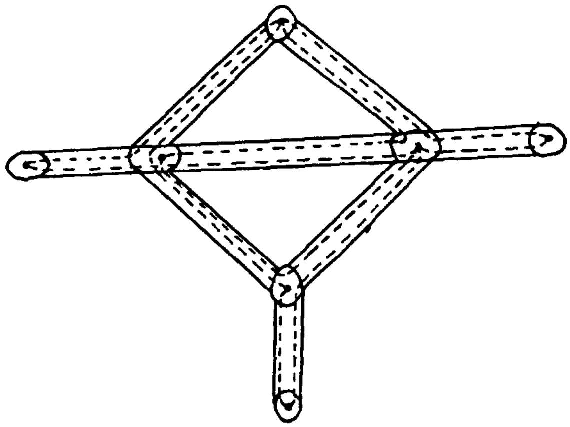

Fig. 1.

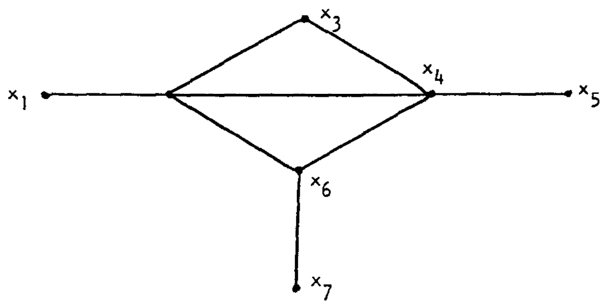

Fig. 2(a). Graph g.

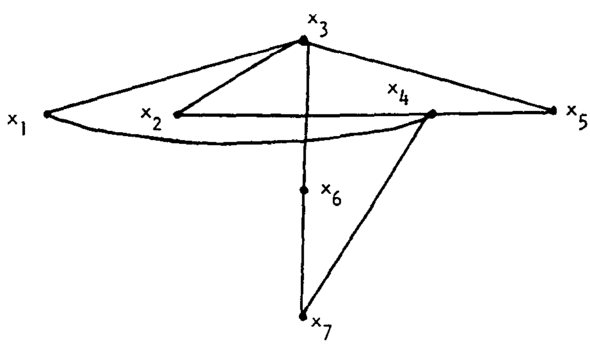

Fig. 2(b). Graph G.

$\Delta=4$, $\bar{d}=38/21\simeq 1.81$, $\bar{D}=34/21\simeq 1.62$.

We note that

$$d(x_1,x_2)=1 \quad \text{and} \quad D(x_1,x_2)=2,$$

$$d(x_1,x_4)=2 \quad \text{and} \quad D(x_1,x_4)=1.$$

Hence

$$d(x_1,x_2) < D(x_1,x_2) \quad \text{and} \quad d(x_1,x_4) > D(x_1,x_4).$$

In what follows, we will avoid cases where there exist vertices of g, x and y, such that $d(x,y) < D(x,y)$, to be sure that $\bar{D} \le \bar{d}$.

Supplementary conditions

Condition A. $E(g)$ is contained in C (which means that every edge of g is also an edge of G). Therefore, for any x, y of $V(g)$, $d(x,y) \ge D(x,y)$ and then $\bar{D} \le \bar{d}$.

Furthermore, to simplify the calculations we will suppose that

Condition B. g is regular of degree δ and G is regular of degree Δ. Therefore $\Delta \le \delta\alpha$.

Relationship between $\bar{d}$ and $\bar{D}$

G. Vautrin [3] asked if there was a simple relation between $\bar{d}$ and $\bar{D}$ and, in particular, if it was true that $\bar{d} \le (\delta\alpha/\Delta)\bar{D}$. We will give two relations and we will show, by exhibiting infinite families of counterexamples, that the relation $\bar{d} \le (\delta\alpha/\Delta)\bar{D}$ is not true.

Proposition 1. *If condition A is satisfied, then*

$$\bar{D} \le \bar{d} \le L\bar{D}, \quad \text{where } L = \sup_{E \in E(G)} l(E).$$

Furthermore, equality holds on the right of this inequality if and only if G is isomorphic to g (and then $\bar{d} = \bar{D}$*).*

Proof. First, condition A implies that $\bar{D} \le \bar{d}$.

Let us denote by $C_{x,y}$ the shortest path between x and y in G. The set $\{\phi(E) \mid E \in C_{x,y}\}$ defines a path (not necessarily elementary) between x and y; then

$$d(x,y) \le \sum_{E \in C_{x,y}} l(E).$$

Therefore

$$\sum_{x,y \in V(g)} d(x,y) \le \sum_{x,y \in V(G)} \sum_{E \in C_{x,y}} l(E)$$
$$\le \sum_{x,y \in V(G)} L\,|E(C_{x,y})|.$$

As $|E(C_{x,y})| = D(x,y)$, we get $\bar{d} \le L\bar{D}$.

Furthermore, if G is isomorphic to g, then $E(g)$ is in one-to-one correspondence with C; since $C \supseteq E(g)$ (condition A) $C = E(g)$ and $L = 1$. Therefore $\bar{d} = \bar{D} = L\bar{D}$. Conversely, if $\bar{d} = L\bar{D}$, then, for any pair (x, y) of vertices of g:

$$d(x, y) = \sum_{E \in C_{x,y}} l(E) = L\, D(x, y).$$

Therefore, for any pair of adjacent vertices x and y in the graph g, $d(x, y) = 1 = L\, D(x, y) = 1$ (condition A). The equality $L = 1$ means that $C \subsetneq E(g)$ and G is isomorphic to g. □

Let us give two definitions: An edge of g is said to be *saturated* if it is contained in α distinct paths of C.

The *average length* of the edges of G is

$$\bar{L} = \frac{1}{|E(G)|} \sum_{E \in E(G)} l(E).$$

Proposition 2. *If condition* B *is satisfied, then* $\bar{L} \le \delta\alpha/\Delta$. *Furthermore, equality holds if and only if all the edges of* g *are saturated.*

Proof. From the definition α, we deduce that

$$\sum_{c \in C} |E(c)| \le \alpha\, |E(g)|.$$

Since the graphs g and G are regular (condition B) their number of edges are $|E(g)| = \frac{1}{2}\delta\upsilon$, resp. $|E(G)| = \frac{1}{2}\Delta\upsilon$. Therefore

$$\frac{1}{|E(G)|} \sum_{E \in E(G)} l(E) \le \frac{2}{\Delta\upsilon}\, \alpha \frac{\delta\upsilon}{2} \Rightarrow \bar{L} \le \frac{\alpha\delta}{\Delta}.$$

Furthermore, equality holds if and only if $\sum_{c \in C} |E(c)| = \alpha\, |E(g)|$, which means that all edges of g are saturated. □

These two propositions seem to give support to Vautrin's conjecture. We first give a counterexample in the case $\delta = 2$, and then in the general case.

Counterexample in the case $\delta = 2$

Let g be the cycle C_υ, whose vertices are denoted by $x_0, x_1, \ldots, x_{\upsilon-1}$. Let $\upsilon = 2qL + 1$ with $L = 2b$, b and q being positive integers such that $b < q$. Let two vertices x_i and x_j be joined by an edge in G if and only if $|j - i| = 1$ or L. In other words, C consists of $E(g)$ and of all the paths of the form $(x_i, x_{i+1}, \ldots, x_{i+L})$, adding modulo υ. Conditions A and B are fulfilled and $\delta = 2$ and $\Delta = 4$. (See Fig. 3 for the case $\upsilon = 9$, $L = 2$.)

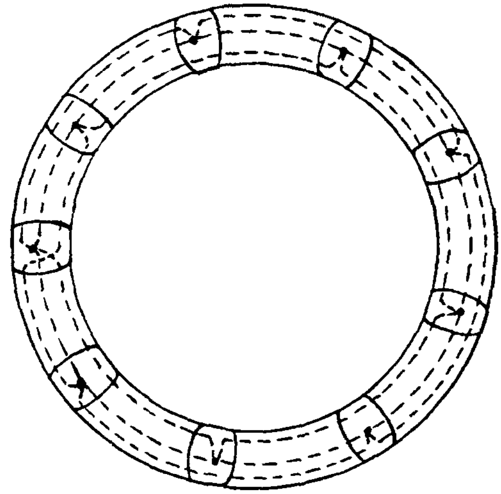

Fig. 3. $\bar{d}=2.5$, $\bar{D}=1.5$, $\bar{d}/\bar{D}=(2qb+1)/(q+b)=5/3$.

Proposition 3. *With the definitions of g and G given above,*

$$\frac{\bar{d}}{\bar{D}}=\frac{2qb+1}{q+b}.$$

Proof. The mean distance of C_υ is $\bar{d}=\frac{1}{2}(2qb+1)$. We have, obviously, for any x_i of $V(g)$:

$$\sum_{\substack{0\le j\le \upsilon-1\\ j\ne i}} D(x_i,x_j)=\sum_{1\le j\le \upsilon-1} D(x_0,x_j)=2\sum_{1\le j\le qL} D(x_0,x_j).$$

Therefore

$$\bar{D}=\frac{2}{\upsilon-1}\sum_{i=1}^{qL} D(x_0,x_i).$$

Let D be the diameter of G, that is $D=\max_{1\le i\le qL}(D(x_0,x_i))$. Let Γ_i be the set of vertices of $\{x_1,\dots,x_{qL}\}$ at distance i from the vertex x_0. $\bar{D}$ can be written as:

$$\bar{D}=\frac{2}{\upsilon-1}\sum_{i=1}^{D} i\,|\Gamma_i|.$$

The sets Γ_i for $i=1,2,\dots,D$ are easily determined:

$$\Gamma_i=\{x_{\beta L+\gamma}\mid \beta+\gamma=i, 0\le\gamma\le b, 0\le\beta L+\gamma\le qL\}$$
$$\cup\{x_{\beta L-\gamma}\mid \beta+\gamma=i, 1\le\gamma\le -1, 0\le\beta L-\gamma\le qL\}.$$

The cardinality of Γ_i is equal to the number of pairs (β,γ) of nonnegative integers satisfying

$$\text{(a)}\ \begin{array}{l}\beta+\gamma=i,\\ 0\le\gamma\le b,\\ 0\le\beta L+\gamma\le qL,\end{array}\qquad\text{or}\qquad\text{(b)}\ \begin{array}{l}\beta+\gamma=i,\\ 1\le\beta L-\gamma\le qL,\\ 0\le\gamma\le b-1.\end{array}$$

Therefore, $D=q+b$ and

$$\begin{array}{ll}\text{if } 1\le i\le b, & \text{then } |\Gamma_i|=2i,\\ \text{if } b\le i\le q, & \text{then } |\Gamma_i|=2b,\\ \text{if } 1\le i\le q+b, & \text{then } |\Gamma_i|=2(b+q-i).\end{array}$$

Therefore

$$\begin{aligned}\sum_{i=1}^{D} i\,|\Gamma_i| &= \sum_{i=1}^{b} i\,2i+\sum_{i=b+1}^{q} i\,2b+\sum_{i=q+1}^{q+b} i\,2(b+q-i)\\ &=2\left[\sum_{i=1}^{b} i^2+\sum_{i=b+1}^{q} bi+\sum_{i=1}^{b}(q+i)(b-i)\right]\\ &=2\left[b\sum_{i=1}^{q} i+\sum_{i=1}^{b} qb-\sum_{i=1}^{b} qi\right]\\ &=qb(q+b).\end{aligned}$$

So

$$\bar{D}=\frac{2}{4qb}qb[q+b]=\frac{q+b}{2}.$$

Therefore, we deduce that $\bar{d}/\bar{D}=(2qb+1)/(q+b)$. □

Then, as q approaches infinity the quotient $\bar{d}/\bar{D}$ goes to $2b$. This means that asymptotically $\bar{D}=\bar{d}/L$.

Furthermore, it is obvious that all the edges of g are contained in the same number of paths of C, which will be α. Therefore, every edge of g is saturated and $\bar{L}=\delta\alpha/\Delta$. $\bar{L}$ is easily calculated: $\bar{L}=\frac{1}{2}(1+L)<L$ if $L>1$. Since $\lim_{q\to\infty}\bar{d}/\bar{D}=L$, there exists q_0 such that, if $q\ge q_0$

$$\bar{d}/\bar{L}>\bar{D}\ge\bar{d}/L,$$

which contradicts Vautrin's conjecture.

This example also shows that, given the maximum length of L of an edge of the graph G, we cannot improve (at least asymptotically) the bound of Proposition 1).

Remark. We can extend this example by deleting the restriction on the parity of v and L, but the calculations are longer.

Counterexample in the general case

Let us consider for g a regular graph of degree δ and girth at least $2L+1$. Let C be the set of all the paths of length at most L. Then G is a simple regular graph isomorphic to g^L, the L-th power of g.

Proposition 4. *With the definitions of g and G given above,*

$$\bar{d}/L \le \bar{D} < \bar{d}/L + 1.$$

Proof. Let x and y be two vertices of G. From the definition of G, we deduce that $D(x, y) = \lceil d(x, y)/L \rceil$ ($\lceil \beta \rceil$ being the least integer greater than or equal to β). Therefore

$$\frac{d(x, y)}{L} \le D(x, y) < \frac{d(x, y)}{L} + 1.$$

Since this inequality is satisfied for each pair of vertices x and y of g,

$$\bar{d}/L \le \bar{D} < \bar{d}/L + 1. \qquad \square$$

The regularity of g and the definition of G of Proposition 4 imply that each edge of the graph g is contained in the same number α of paths of C. Hence each edge of g is saturated and the average length of the edges of G is $\bar{L} = \delta\alpha/\Delta$.

Furthermore, we can calculate α and Δ, for $\delta \ge 3$. Since the girth of g is at least $2L+1$, the number of vertices of g at distance at most L from any vertex x of g is

$$\delta + \delta(\delta-1) + \cdots + \delta(\delta-1)^{L-1} = \Delta.$$

Therefore

$$\Delta = \frac{\delta}{\delta-2}[(\delta-1)^L - 1].$$

Let (x, y) be an edge of g. For $i = 1, 2, \ldots, L$, there are $(\delta-1)^{i-1}$ paths $(x_0, x_1, \ldots, x_i)$, of length i, such that $(x, y) = (x_j, x_{j+1})$ for j given in $\{0, 1, \ldots, i-1\}$. Therefore

$$\alpha = \sum_{i=1}^{L} i(\delta-1)^{i-1} = \frac{L(\delta-1)^{L+1} - (L+1)(\delta-1)^L + 1}{(\delta-2)^2}.$$

We note that the value of α and Δ and therefore, the value of $\bar{L}$ depend only on δ and L (and not on the number of vertices of the graph g).

Let $(g_i)_{i\in\mathbb{N}}$ be a family of regular graphs of degree $\delta \ge 3$ such that $\lim_{i\to+\infty} v_i = +\infty$ where $v_i = |V(g_i)|$. Then, if $\bar{d}_i$ is the mean distance of g_i, $\lim_{i\to+\infty} \bar{d}_i = +\infty$. Indeed, let n_i be the largest integer n such that

$$u_i = 1 + \sum_{j=0}^{n-1} \delta(\delta-1)^j \le v_i.$$

For any vertex x of g_i, the sum $\sum_{y\in V(g)-\{x\}} d(x, y)$ is greater than or equal to the sum of the distances between the root r and all the other vertices in a tree T with u_i vertices all of degree δ unless the leaves which are all at distance n_i from r. Therefore

$$\frac{1}{v_i-1}\sum_{y\in V(g)-\{x\}} d(x,y) \geq \frac{\delta-2}{\delta(\delta-1)^{n_i}}\sum_{j=1}^{n_i} j\delta(\delta-1)^{j-1}$$
$$\geq n_i - \frac{1}{(\delta-2)(\delta-1)} + \frac{1}{(\delta-2)(\delta-1)^{n_{i+1}}}.$$

So

$$\lim_{i\to\infty}\frac{1}{v_i-1}\sum_{y\in V(g)-\{x\}} d(x,y) = \infty$$

for each vertex x of g, and $\lim_{i\to\infty}\bar{d}_i = \infty$.

Let now g_i be a $(\delta, 2i+1)$-cage. By a result of Erdös and Sachs [2] such a cage exists for every $i \geq 1$. Furthermore, because of the lower bound on the number of vertices of a cage [see 4], $\lim_{i\to\infty} v_i = \infty$.

Note that $\bar{d}_i/L + 1 \leq \bar{d}_i/\bar{L}$ if and only if $\bar{d}_i \geq L\bar{L}/(L-\bar{L})$. Since $\lim_{i\to\infty}\bar{D}_i = \infty$, there exists an i_0 such that for any $i \geq i_0$, $\bar{d}_i \geq L\bar{L}/(L-\bar{L})$ and then,

$$\bar{D}_i \leq \frac{\bar{d}_i}{L} + 1 \leq \frac{\bar{d}_i}{\bar{L}} = \frac{\Delta}{\delta\alpha}\bar{d}_i.$$

Thus, these graphs g_i $(i \geq i_0)$ give counterexamples to Vautrin's conjecture for any values of $\delta + 3$ and L.

While the construction of G gives a rather good value for $\bar{D}$, it imposes rather great values on Δ and α. In practice, however, δ, Δ and α are given, and it would be interesting to find a function $f(\alpha,\delta,\Delta)$ (in Vautrin's conjecture it was $f(\alpha,\delta,\Delta) = \delta\alpha/\Delta$) such that we always have $\bar{d}/\bar{D} \leq g(\alpha,\delta,\Delta)$. But this seems a difficult problem.

Acknowledgement

We thank G. Vautrin for suggesting the problem.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (American Elsevier, New York, 1976).

[2] P. Erdös and S. Sachs, Reguläre graphen gegebener Taillenweite mit minimaler Knotenzahl, Wiss, Z. Uni. Halle (Math. Nat.) 12 (1963) 251-257.

[3] G. Vautrin, Structure d'un réseau partiellement maillé, Unpublished manuscript (1981).

[4] P.K. Wong, Cages - a survey, J. Graph Theory 6 (1982) 1-22.