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Multifractal Measures and a Weak Separation Condition

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We define a new separation property on the family of contractive similitudes that allows certain overlappings. This property is weaker than the open set condition of Hutchinson. It includes the well-known class of infinite Bernoulli convolutions associated with the P.V. numbers and the solutions of the two-scale dilation equations. Our main purpose in this paper is to prove the multifractal formalism under such condition. © 1999 Academic Press

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1. INTRODUCTION

Let μ be a bounded regular Borel measure on $\mathbb{R}^d$ that has compact support. For each $x \in \mathbb{R}^d$ let

$$\alpha(x) = \lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta}$$

be the local dimension of μ at x. Let $K(\alpha) = \{x: \alpha(x) = \alpha\}$ and let $f(\alpha)$ be the Hausdorff dimension of $K(\alpha)$. We call $f(\alpha)$ the *dimension spectrum* (or *singularity spectrum*) of μ and loosely refer to μ as a *multifractal* measure if $f(\alpha) \neq 0$ for a continuum of α. Multifractal measures and dimension spectra were first proposed by physicists to study various multifractal models arising from natural phenomena (e.g., Mandelbrot [M], Frisch and Parisi [FP], Halsey *et al.* [H]): In fully developed turbulence they are

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used to investigate the intermittent behavior in the regions of high vorticity. In diffusion-limited aggregation, they are used to describe the probability of a random walk landing to the neighborhood of a given site on the aggregate. In dynamical system theory they are used to measure how often a given region of the attractor is visited.

In order to determine the function $f(\alpha)$, Hentschel and Procaccia [HP], Halsey *et al.* [Ha], and Frisch and Parisi [FP] introduced the following calculation. Let $\{Q_i(\delta)\}_i$ denote the family of δ-mesh cubes $\prod_{i=1}^d [n_i\delta, (n_i+1)\delta)$, $n_i \in \mathbb{Z}$, and let

$$\tau(q) = \lim_{\delta \to 0^+} \frac{\ln \sum_i \mu(Q_i(\delta))^q}{\ln \delta}$$

be the *L^q-spectrum* (moment scaling exponent). Note that τ is concave, we let τ^* denote the *concave conjugate* of τ (also known as the *Legendre transformation* of τ), i.e.,

$$\tau^*(\alpha) = \inf\{q\alpha - \tau(q): q \in \mathbb{R}\}. \tag{1.1}$$

They showed heuristically the relationship of τ and f:

If the measure μ is constructed from the cascade algorithm and if τ and f are smooth and concave, then $\tau^*(\alpha) = f(\alpha)$ and dually, $f^*(q) = \tau(q)$.

We call such relationship the *multifractal formalism*; it is also called the *thermodynamic formalism* because of the analogue of the Gibbs state, the pressure and the variational principle in thermodynamics (see Bowen [B], Landford [L] and Bohr and Rand [BR]). There are a number of cases that the above principle can actually be verified rigorously. Among those are the hyperbolic cookie-cutter maps and the critical maps on the circle with golden rotation number (Rand [R], Collet *et al.* [CLP]), the maximal measures associated with rational maps on the complex plane (Lopes [Lo]), the Moran construction and the digraph construction of self-similar measures with totally disconnected supports (Cawley and Mauldin [CM], Edgar and Mauldin [EM]), the random constructions of the above measures (Olsen [O2], Falconer [F3], Arbeiter and Patzschke [AP]), and the extension of the principle to various types of dimensions [O1]. In all the above cases some type of separation condition has to be imposed so that the attractor can be separated into disjoint components through the generating transformations. The simplest and most elegant case is the one due to Cawley and Mauldin [CM] (see also [Ri]), in which $\tau(q)$ can be given explicitly by

$$\sum_{i=1}^N p_i^q \rho_i^{-\tau(q)} = 1, \tag{1.2}$$

where the p_i's are the probability weights and the ρ_i's are the contraction ratios of the similitudes defining μ. By using such an identity together with the Birkhoff's individual ergodic theorem they verified the multifractal formalism by constructing a measure ν satisfying

$$\lim_{\delta \to 0} \frac{\nu(B_\delta(x))}{\delta^{\tau^*(\alpha)}} \leqslant C \qquad \forall x \in K(\alpha). \tag{1.3}$$

The mass distribution principle (i.e., Frostman's Lemma, see [F2]) then implies that $f(\alpha) := \dim_{\mathscr{H}} K(\alpha) \geqslant \tau^*(\alpha)$. The reverse inequality can be obtained much more easily by a standard argument from the definitions.

In another direction Daubechies and Lagarias [DL2] considered the multifractal structure of the solutions of certain two-scale dilation equations (where the local dimension of the measure is replaced by the local Lipschitz exponent). They showed that the singularity spectrum can have jumps, but the multifractal formalism still holds with respect to the convex hull. The basic differences of this dilation equation case and the previous measure case are that the functions are not necessarily monotone (they correspond to signed measures), and in the cascade construction of the functions, the maps do not satisfy the separation condition as before.

Despite all the investigations mentioned above, the exact range of validity for the multifractal formalism is still not known. The purpose of this paper is to study this principle for a larger class of self-similar measures defined by similitudes that allow certain overlapping. For $i = 1, ..., N$, let $S_i\colon \mathbb{R}^d \to \mathbb{R}^d$ be the contractive similitudes such that

$$S_i(x) = \rho_i R_i x + b_i,$$

where $0 < \rho_i < 1$, $b_i \in \mathbb{R}^d$ and R_i are orthogonal transformations. For a set of probability weights $\{p_1, ..., p_N\}$, let μ be the self-similar measure satisfying

$$\mu = \sum_{i=1}^{N} p_i \mu \circ S_i^{-1}.$$

Let $\Omega = \{1, ..., N\}^{\mathbb{N}}$. For $k \in \mathbb{N}$, we define a stopping time $t_k\colon \Omega \to \mathbb{N}$ by assigning each $\omega = (j_1, ..., j_i, ...) \in \Omega$ the value

$$t_k(\omega) = \min\{i\colon \rho_{j_1} \cdots \rho_{j_i} \leqslant \rho^k\}$$

and let

$$\Lambda_k = \{(j_1, ..., j_{t_k(\omega)})\colon \omega = (j_1, ..., j_i, ...) \in \Omega\}.$$

For any index $I=(i_1, ..., i_n)$ we let $S_I(x_0)$ denote $S_{i_1} \circ \cdots \circ S_{i_n}(x_0)$. We are interested in the following property of $\{S_i\}_{i=1}^N$:

There exist $x_0 \in \mathbb{R}^d$ *and* $b>0$ *such that for any* $z=S_I(x_0)$ *and for any* J, $J' \in \Lambda_k$, *either*

$$S_J(z)=S_{J'}(z) \qquad \text{or} \quad |S_J(z)-S_{J'}(z)| \geqslant b\rho^k.$$

We call a slight variant of this the *weak separation property* (WSP) (Definition 6.2). As is well known, self-similar measures can be obtained by iterating $\{S_j\}$ initiated on any compact set or at any point [Hut]. The main idea of the WSP is that instead of considering "set" separation in the recursive application of the S_i's, we consider "point" separation for the iterated points that are distinct. This allows us to include more interesting examples.

It is easy to show that the well known open set condition will imply the WSP (Section 6, Example 1). Our basic nontrivial example is the self-similar measure μ defined by the similitudes

$$S_1(x)=\rho x, \qquad S_2(x)=\rho x+(1-\rho), \tag{1.4}$$

where $1/2<\rho<1$ and the weights on each map is 1/2. In contrast to the case where $0<\rho<1/2$ which gives the Cantor measures, the case for $1/2<\rho<1$ is rather complicated due to the overlapping of the attractor [0, 1] induced by the two similitudes. Such measures are also known as infinitely convolved Bernoulli measures (ICBM) and have been studied for a long time. It was conjectured in the 30's that they should be absolutely continuous. Erdös [E] first disproved this by showing that if ρ^{-1} is a P.V. number, then the Fourier transformation of μ does not tend to zero (the converse was proved by Salem [S]) and hence μ is singular. (Recall that $\beta>1$ is called a P.V. number if β is an algebraic integer and all its conjugates have moduli less than 1.) Up to now there is still no clear classification of the absolute continuity and the singularity of such measures ([L1, L2]). In Section 6, Example 2, we use a result of Garsia [G] and show that if ρ^{-1} is a P.V. number, then the corresponding maps $\{S_1, S_2\}$ will have the WSP.

In Section 6, Example 3, we observe that for the functions defined by the two-scale dilation equations, the corresponding similitudes do not satisfy the open set condition but possess the WSP. This provides us with yet another important class of examples of WSP.

Let z be as in the definition of WSP and let $z_k=S_J(z)$, $J \in \Lambda_k$. For each n we can consider the n-step path of the z_k's which starts from z and ends at z_n. If $\{S_i\}_{i=1}^N$ satisfies the open set condition, then there is essentially a unique path from z to z_n. One of the key properties of the WSP is that the number of such (distinct) paths is bounded by ℓ^n for some fixed integer ℓ (Proposition 6.3).

In our consideration of the multifractal formalism, we make no *a priori* assumption on the smoothness of the functions. Instead of using the differentiability on τ, we will employ the dual property on τ^*, namely, the strict concavity. Recall that a function h on $\mathbb{R}$ is said to be *strictly concave* at x if there exists $c \in \mathbb{R}$ such that

$$h(y) < h(x) + c(y - x) \qquad \text{for all} \quad y \neq x. \tag{1.5}$$

It turns out that this property is easier to handle. Also, for technical reasons we will replace the sum $\sum_i \mu(Q_i(\delta))^q$ of the δ-cubes in the definition of $\tau(q)$ by $\sup \sum_i \mu(B_\delta(x_i))^q$ where $\{B_\delta(x_i)\}_i$ is a countable family of disjoint closed δ-balls centered at $x_i \in \operatorname{supp}(\mu)$, the support of μ. Let $N_\delta(\alpha - \varepsilon, \alpha + \varepsilon)$ be the supremum of the number of such disjoint balls satisfying

$$\delta^{\alpha+\varepsilon} \leqslant \mu(B_\delta(x_i)) < \delta^{\alpha-\varepsilon}.$$

Under this setting we have the following counting formulation (Corollary 4.3, Theorem 5.1):

THEOREM A. *Let* $\alpha \in (\operatorname{Dom} \tau^*)^\circ$. *Then*

$$\tau^*(\alpha) \geqslant \lim_{\varepsilon \to 0^+} \varlimsup_{\delta \to 0^+} \frac{\ln N_\delta(\alpha - \varepsilon, \alpha + \varepsilon)}{-\ln \delta}.$$

If in addition τ^* *is strictly concave at* α, *then*

$$\tau^*(\alpha) = \lim_{\varepsilon \to 0^+} \varlimsup_{\delta \to 0^+} \frac{\ln N_\delta(\alpha - \varepsilon, \alpha + \varepsilon)}{-\ln \delta}.$$

(E^o *denotes the interior of* E.)

Note that similar theorems have been proved in a number of papers (See [F2], [Ri] and the references there. Their proofs use a theorem of large deviation.) Our proof here depends only on some elementary $\varepsilon - \delta$ arguments. Moreover, the argument and the theorem are strengthened (Lemma 5.3) to prove Theorem B.

It is easy to see that the domain of τ^* is the union of two intervals: the interval of the points such that the infimum in (1.1) is attained for $q \geqslant 0$ and the other interval for $q < 0$ (Proposition 3.5). Let us denote the first interval by $\operatorname{Dom}^+ \tau^*$. Also, we denote the Hausdorff dimension of a set E by $\dim_{\mathscr{H}}(E)$. Our main theorem is the following (Theorem 6.6):

THEOREM B. *Suppose* $\{S_i\}_{i=1}^N$ *has the* WSP *and* τ^* *is strictly concave at* $\alpha \in (\mathrm{Dom}^+ \tau^*)^\circ$. *Then*

$$f(\alpha) := \dim_{\mathscr{H}} \left\{ x\colon \lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \alpha \right\} = \tau^*(\alpha).$$

The main idea of the proof of Theorem B is to use the counting expression of τ^* in Theorem A to construct a measure ν as in (1.3). Unlike the case with the open set condition, we do not have an explicit expression of $\tau(q)$ like (1.2) and it is not clear how the ergodic theorem can be applied. Instead we depend on the special property of the WSP mentioned above that the number of distinct n-step paths from one point to another is uniformly bounded by ℓ^n. The construction of ν is lengthy but elementary; the complication stems from the fact that each state z_k in the path can be visited by more than one point and hence extra effort is needed to estimate the probability. (See Section 6 and the outline at the end of that section.)

Note that the separation assumption in [CM] and [Ri] is stronger than the open set condition, we can use Theorem B to improve their theorem.

COROLLARY C. *Suppose* $\{S_i\}_{i=1}^N$ *satisfies the open set condition. Then the* L^q*-spectrum* τ *is given by* (1.2) *and the multifractal formalism* $f(\alpha) = \tau^*(\alpha)$ *holds for* $\alpha \in (\mathrm{Dom}^+ \tau^*)^\circ$.

Also, the following is the first known case that the multifractal formalism holds even though overlapping of the S_i's occurs.

COROLLARY D. *For* $\rho = (\sqrt{5}-1)/2$, *the* L^q*-spectrum* $\tau(q)$ *of the ICBM* μ *is differentiable for* $0 < q < \infty$, *and* μ *satisfies the multifractal formalism for* $\alpha = \tau'(q)$, $0 < q < \infty$.

The formula for τ in the above corollary is given in Section 9. That $\tau(q)$, $0 < q < \infty$ is differentiable and is actually the L^q-spectrum are proved in [LN1].

For the convenience of the reader we begin our investigation with a brief exposition of the needed properties of concave and conjugate concave functions (Section 2). In Section 3 we prove some preliminary properties of the L^q-spectrum $\tau(q)$ and its conjugate $\tau^*(\alpha)$ and introduce some basic dimension notations. In Sections 4 and 5 we prove Theorem A as well as the easy part of the multifractal formalism: $f(\alpha) \leqslant \tau^*(\alpha)$. Also, Theorem A is strengthened in Lemma 5.3 for later use. In these sections a few of the propositions may have already appeared in the literature heuristically or rigorously. We choose to include them here for the sake of continuation and completeness in our development. The remaining Sections 6, 7, 8 are devoted to the reverse inequality $f(\alpha) \geqslant \tau^*(\alpha)$. The definition of weak separation property together with some basic setup and estimations are given in Section 6. In Section 7, Theorem B is proved up to an arbitrarily small ε and the complete proof is done in Section 8.

In this paper we have not considered the other important class of examples that satisfy the WSP: the self-similar functions defined by the two-scale dilation equation as in [DL2]. Since it involves signed coefficient, we will deal with it in a separate paper. In regard to the explicit calculation of the L^q-spectrum $\tau(q)$, besides the elegant formula (1.2), $\tau(2)$ has been calculated for the ICBM defined in (1.4) for those $1/2 < \rho < 1$ with ρ^{-1} equal to a P.V. number ([L1], [L2]). In [LN2], we extended this calculation to $\tau(q)$ for all integers $q \geqslant 2$. If $\rho = (\sqrt{5} - 1)/2$, the reciprocal of the golden ratio, we can make use of the *second-order* self-similar identities of Strichartz *et al.* [STZ] and obtain a formula for $\tau(q)$ [LN1]. Combining with another sharp calculation by Hu [Hu] of the two end points of $\operatorname{Dom} \tau^*$, we have now much better understanding of the ICBM associated with the golden ratio. We will discuss this and some other remarks in Section 9.

2. CONJUGATE CONCAVE FUNCTIONS

We summarize some known facts about concave functions on $\mathbb{R}$ and their conjugates, which will be needed in the sequel. A complete treatment for such functions on $\mathbb{R}^d$ can be found in [Ro].

Let $\tau: \mathbb{R} \to [-\infty, \infty)$ be a concave function. (It is important to include the value $-\infty$.) We define its *effective domain* as

$$\operatorname{Dom} \tau = \{x: -\infty < \tau(x) < \infty\}.$$

To avoid triviality we assume $\operatorname{Dom} \tau$ is nonempty and τ is upper semicontinuous. Note that τ is the infimum of all affine functions $h \geqslant \tau$, i.e.,

$$\tau(x) = \inf\{h(x): h(y) = \alpha y - r, h \geqslant \tau\}.$$

Then $h \geqslant \tau$ if and only if $r \leqslant \alpha x - \tau(x)$ for all $x \in \mathbb{R}$. We define $\tau^*: \mathbb{R} \to [-\infty, \infty)$ by

$$\tau^*(\alpha) = \inf\{\alpha x - \tau(x): x \in \mathbb{R}\}.$$

Then τ^* is upper semicontinuous and concave. We call τ^* the *concave conjugate* of τ. (See Fig. 1.)

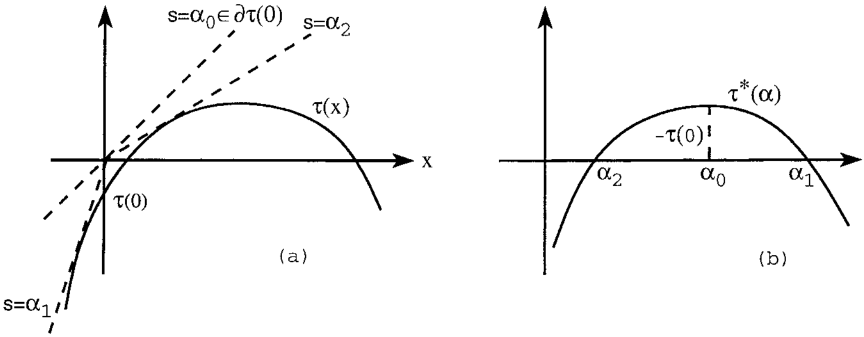

FIG. 1. A concave function τ and its concave conjugate τ^* (s means slope).

PROPOSITION 2.1. *Let τ and τ^* be as above. Then*

(i) $\tau^{**} = \tau$;

(ii) $\tau(x) + \tau^*(\alpha) \leqslant \alpha x$, *for all* $x, \alpha \in \mathbb{R}$.

For $x \in \operatorname{Dom} \tau$, we let $\partial\tau(x) \subseteq \mathbb{R}$ be the subdifferential (following the terminology of Rockafeller) of τ at x, i.e.,

$$\partial\tau(x) = \{\alpha\colon \tau(y) \leqslant \tau(x) + \alpha(y - x) \text{ for all } y \in \mathbb{R}\}.$$

If $x \in (\operatorname{Dom} \tau)^\circ$, then it is easy to show that $\partial\tau(x) = [\tau'_+(x), \tau'_-(x)]$ is a nonempty bounded interval. If x is on the boundary of $\operatorname{Dom} \tau$, then $\partial\tau(x)$ may be empty. (e.g., if $\tau(x) = \sqrt{1 - x^2}$ for $-1 \leqslant x \leqslant 1$ and $\tau(x) = -\infty$ otherwise, then $\partial\tau(-1) = \partial\tau(1) = \emptyset$). τ is said to be *smooth* at $x \in (\operatorname{Dom} \tau)^\circ$ if $\partial\tau(x)$ is a singleton, i.e., τ has a unique tangent line with slope $\alpha \in \partial\tau(x)$ and α is the derivative of τ at x. It is clear that $\partial\tau(x)$ is a decreasing function with jumps $[\tau'_+(x), \tau'_-(x)]$ at possibly countably many points where τ is not differentiable. We will use the following assertions frequently.

PROPOSITION 2.2. *The following are equivalent*:

(i) $\alpha \in \partial\tau(x)$;

(ii) $\alpha y - \tau(y)$ *achieves its minimum at* $y = x$;

(iii) $\tau^*(\alpha) + \tau(x) = \alpha x$.

It follows from Proposition 2.1(i) and Proposition 2.2(iii) that $\partial\tau^*$ is the inverse of $\partial\tau$ in the sense that

$$x \in \partial\tau^*(\alpha) \Leftrightarrow \alpha \in \partial\tau(x). \tag{2.1}$$

PROPOSITION 2.3. *Let*

$$\begin{aligned}\alpha_{\min} &= \inf\{\alpha : \alpha \in \partial\tau(x),\ x \in \operatorname{Dom} \tau\}\\ \alpha_{\max} &= \sup\{\alpha : \alpha \in \partial\tau(x),\ x \in \operatorname{Dom} \tau\}.\end{aligned} \tag{2.2}$$

Then $(\operatorname{Dom} \tau^*)^\circ = (\alpha_{\min}, \alpha_{\max})$.

τ is said to be *strictly concave* at x if there exists $\alpha \in \mathbb{R}$ such that

$$\tau(y) < \tau(x) + \alpha(y - x) \qquad \text{for all} \quad y \neq x. \tag{2.3}$$

Note that such an α must belong to $\partial\tau(x)$ by definition, and it is easy to show that $\alpha \in (\operatorname{Dom} \tau^*)^\circ$. For such α, if we let $\tau_\alpha(y) = \tau(y) - \alpha y$, then (2.3) is equivalent to

$$\tau_\alpha(y) < \tau_\alpha(x) \qquad \text{for all} \quad y \neq x.$$

Furthermore, if we assume $x \in (\operatorname{Dom} \tau)^\circ$ and let

$$\eta(\varepsilon; \alpha) = \inf\{\tau_\alpha(x) - \tau_\alpha(y) : |x - y| \geqslant \varepsilon\},$$

then η is a strictly increasing function of ε, and $\eta \searrow 0$ if and only if $\varepsilon \searrow 0$. For concave functions, smoothness and strict concavity have a very nice duality relationship. We include a simple proof here for completeness.

PROPOSITION 2.4. *Suppose τ is a concave function on $\mathbb{R}$.*

(i) *If τ is smooth at $x \in (\operatorname{Dom} \tau)^\circ$ with $\partial\tau(x) = \{\alpha\}$, then τ^* is strictly concave at α.*

(ii) *If τ^* is strictly concave at α and the corresponding strict inequality (2.3) holds for some $x \in \partial\tau^*(\alpha)$, then τ is smooth at x.*

Proof. (i) If τ^* were not strictly concave at α, then there would exist $\alpha_1 \neq \alpha$ such that

$$\tau^*(\alpha_1) = \tau^*(\alpha) + x(\alpha_1 - \alpha),$$

so by Proposition 2.2(iii), $\tau^*(\alpha_1) + \tau(x) = \alpha_1 x$. This implies, by the same proposition, that $\alpha_1 \in \partial\tau(x)$, contradicting the assumption that $\partial\tau(x)$ is a singleton.

(ii) Condition (2.3) adjusted to the present setting is

$$\tau^*(\xi) < \tau^*(\alpha) + x(\xi - \alpha) \qquad \text{for all} \quad \xi \neq \alpha. \tag{2.4}$$

We first note that this implies that $x \in (\mathrm{Dom}\, \tau)^{\circ}$. Now suppose τ were not smooth at x. Then there would exist $\alpha_1 \in \partial\tau(x) \backslash \{\alpha\}$ and hence $x \in \partial\tau^*(\alpha_1)$ such that

$$\tau^*(\xi) \leqslant \tau^*(\alpha_1) + x(\xi - \alpha_1) \qquad \text{for all} \quad \xi \neq \alpha_1.$$

In particular, $\tau^*(\alpha_1) \geqslant \tau^*(\alpha) + x(\alpha_1 - \alpha)$, which contradicts (2.3).

Remark. Suppose τ^* is strictly concave at α. By (ii), τ is smooth at x if $\partial\tau^*(\alpha)$ is a singleton $\{x\}$; if $\partial\tau^*(\alpha)$ is not a singleton, then τ is smooth at all $x \in (\partial\tau^*(\alpha))^{\circ}$. ($\tau$ may not be smooth at those x on the boundary of $\partial\tau^*(\alpha)$.)

3. L^q-SPECTRUM

Throughout this paper we assume that μ is a positive bounded regular Borel measure on $\mathbb{R}^d$ with bounded support. For $\delta > 0$ and $q \in \mathbb{R}$, we define the *L^q-spectrum* of μ by

$$\tau(q) = \varliminf_{\delta \to 0^+} \frac{\ln S_\delta(q)}{\ln \delta} \tag{3.1}$$

with

$$S_\delta(q) = \sup \sum_i \mu(B_\delta(x_i))^q,$$

where $\{B_\delta(x_i)\}_i$ is a countable family of disjoint closed δ-balls centered at $x_i \in \mathrm{supp}(\mu)$ and the supremum is taken over all such families. Note that $0 < S_\delta(q) < \infty$ holds for all $q \in \mathbb{R}$. Let

$$[n_1\delta, (n_1+1)\delta) \times \cdots \times [n_d\delta, (n_d+1)\delta), \qquad (n_1, ..., n_d) \in \mathbb{Z}^d$$

be a family of δ-mesh cubes in $\mathbb{R}^d$ and let $\{Q_i(\delta)\}_i$ denote the collection of all such cubes that intersect $\mathrm{supp}(\mu)$. For $q \geqslant 0$, $\tau(q)$ has an equivalent and more familiar expression obtained by taking cubes from the δ-mesh instead of packing with the disjoint δ-balls:

PROPOSITION 3.1. *For $q \geqslant 0$, $\tau(q) = \varliminf_{\delta \to 0^+} (\ln \sum_i \mu(Q_i(\delta))^q / \ln \delta)$.*

Proof. Let $\{B_\delta(x_i)\}_i$ be a collection of disjoint closed balls centered at $x_i \in \mathrm{supp}(\mu)$. Let $\mathfrak{I}_i$ be the subfamily of cubes in $\{Q_i(\delta)\}_i$ that intersect $B_\delta(x_i)$. $\mathfrak{I}_i$ has at most $C_1 = 3^d$ elements. Hence

$$\begin{aligned}\sum_i \mu(B_\delta(x_i))^q &\leqslant \sum_i \Big(\sum_{j\in\mathfrak{J}_i} \mu(Q_j(\delta))\Big)^q \\ &\leqslant (C_1^{q-1}+1)\sum_i \sum_{j\in\mathfrak{J}_i} \mu(Q_j(\delta))^q \\ &\leqslant (C_1^{q-1}+1)\, C_2 \sum_j \mu(Q_j(\delta))^q,\end{aligned}$$

where C_2 is the maximum number of disjoint balls $B_\delta(x_i)$ that intersect a fixed $Q_j(\delta)$. This implies that

$$\tau(q) = \underline{\lim}_{\delta\to 0^+} \frac{\ln S_\delta(q)}{\ln \delta} \geqslant \underline{\lim}_{\delta\to 0^+} \frac{\ln \sum_i \mu(Q_i(\delta))^q}{\ln \delta}.$$

To prove the reverse inequality we write $\bar\delta = \sqrt{d}\,\delta$. For each $Q_i(\delta)$ with positive μ measure, we choose an $x_i \in \text{supp}(\mu) \cap Q_i(\delta)$ and let $\mathscr{C}$ denote the collection of balls $B_{\bar\delta}(x_i)$. Note that $B_{\bar\delta}(x_i) \supseteq Q_i(\delta)$. Choose $B_{\bar\delta}(x_{i_1}) \in \mathscr{C}$ such that $\mu(B_{\bar\delta}(x_{i_1})) = \max\{\mu(B_i)\colon B_i \in \mathscr{C}\}$. Suppose $B_{\bar\delta}(x_{i_1})$, ..., $B_{\bar\delta}(x_{i_k})$ are chosen. We choose $B_{\bar\delta}(x_{i_{k+1}})$ among those balls in $\mathscr{C}$ that are disjoint from the previous k balls and has maximum μ measure. This way we obtain a sequence of disjoint balls $\{B_{\bar\delta}(x_{i_k})\}_k$ whose union has sufficiently large measure. Note that the number of balls in $\mathscr{C}$ that intersect a fixed $B_{\bar\delta}(x_{i_k})$ is bounded by a fixed number C' independent of δ. (C' can be taken to be $(1+2\sqrt{d})^d$.) Hence

$$S_{\bar\delta}(q) \geqslant \sum_k \mu(B_{\bar\delta}(x_{i_k}))^q \geqslant \frac{1}{C'}\sum_i \mu(B_{\bar\delta}(x_i))^q \geqslant \frac{1}{C'}\sum_i \mu(Q_i(\delta))^q.$$

It follows that $\tau(q) \leqslant \underline{\lim}_{\delta\to 0^+}(\ln \sum_i \mu(Q_i(\delta))^q/\ln\delta)$.

Remark. For $q<0$, the equality in the proposition fails. In fact, if μ is the Lebesgue measure restricted on $[0, 1]$, then according to (3.1), $\tau(q) = q-1$ for all $q\in\mathbb{R}$. However if we use the above δ-mesh with $\delta_n = (1-2^{-n})/n$ then the interval covering the right end point of $[0, 1]$ has measure 2^{-n} and therefore $\sum_i \mu(Q_i(\delta))^q > 2^{-qn}$. Consequently for $q<0$,

$$\underline{\lim}_{\delta\to 0^+} \frac{\ln \sum_i \mu(Q_i(\delta))^q}{\ln\delta} = \lim_{n\to\infty} \frac{\ln \sum_i \mu(Q_i(\delta_n))^q}{\ln \delta_n} = -\infty.$$

Note also that in [Ri], the case $q<0$ is treated by enlarging the cubes $Q_i(\delta)$ in the definition of $\tau(q)$ by three times. It can be shown that this is equivalent to our disjoint ball approach.

According to the notation in [St], we define the lower L^q-dimension of μ by

$$\underline{\dim}_q(\mu) = \tau(q)/(q-1), \qquad q > 1,$$

and for $q = 1$,

$$\underline{\dim}_1(\mu) = \varliminf_{\delta \to 0^+} \frac{\inf \sum_i -\mu(B_\delta(x_i)) \ln \mu(B_\delta(x_i))}{-\ln \delta},$$

where $\{B_\delta(x_i)\}_i$ is a disjoint family of balls as before and the infimum is taken over all such families. Note that $\dim_1(\mu)$ is known as the *entropy* dimension, and the L^q-dimension, $q > 1$, is called the *generalized dimension* by Hentschel and Procaccia [HP], which is actually Rényi's extension of the entropy dimension. For $q = \infty$ and $-\infty$, we let

$$\begin{aligned} \underline{\dim}_\infty(\mu) &= \varliminf_{\delta \to 0^+} \frac{\ln(\sup_x \mu(B_\delta(x)))}{\ln \delta}, \\ \underline{\dim}_{-\infty}(\mu) &= \varliminf_{\delta \to 0^+} \frac{\ln(\inf_x \mu(B_\delta(x)))}{\ln \delta}, \end{aligned} \tag{3.2}$$

where the supremum and infimum are taken over all $x \in \operatorname{supp}(\mu)$. Similarly we can define the corresponding upper dimensions. Note that $\underline{\dim}_\infty(\mu)$ can also be defined by taking cubes from the δ-mesh as in the case $0 < q < \infty$. However, the analogous expression for $\underline{\dim}_{-\infty}(\mu)$ might give a different value. This can be seen by using the example in the above remark. We denote the lower and upper *box* dimensions of a set E by $\underline{\dim}_{\mathscr{B}}(E)$ and $\overline{\dim}_{\mathscr{B}}(E)$ respectively.

PROPOSITION 3.2. *Let μ and τ be as above. Then τ is an increasing concave function with $\tau(1) = 0$ and $\tau(0) = -\overline{\dim}_{\mathscr{B}}(\operatorname{supp}(\mu))$.*

Proof. We can assume that μ is a probability measure. Then $q_1 < q_2$ implies that $\mu(B_\delta(x_i))^{q_1} \geqslant \mu(B_\delta(x_i))^{q_2}$ and the increasing property of τ follows. The concavity of τ is a consequence of the Hölder's inequality applied to $1/\lambda$ and $1/(1-\lambda)$, $0 < \lambda < 1$:

$$\sum_i \mu(B_\delta(x_i))^{\lambda q_1 + (1-\lambda) q_2} \leqslant \left(\sum_i \mu(B_\delta(x_i))^{q_1}\right)^\lambda \left(\sum_i \mu(B_\delta(x_i))^{q_2}\right)^{1-\lambda}$$

for $q_1, q_2 \in \operatorname{Dom} \tau$. The rest of the proposition follows from a direct check of the definitions (see [F2]).

We have the following dichotomy concerning $\operatorname{Dom} \tau$.

PROPOSITION 3.3. *Dom* $\tau = \mathbb{R}$ *if and only if* $\overline{\dim}_{-\infty}(\mu) < \infty$; *Dom* $\tau = [0, \infty)$ *if and only if* $\overline{\dim}_{-\infty}(\mu) = \infty$.

Proof. Without loss of generality, assume that supp(μ) is contained in a cube of size l. Since $-\infty < \tau(0) < \tau(q)$ for all $q > 0$ (Proposition 3.2), it follows that $[0, \infty) \subseteq \text{Dom}\,\tau$. We will prove the following two claims which will imply the proposition.

(i) $\overline{\dim}_{-\infty}(\mu) < \infty \Rightarrow \text{Dom}\,\tau = \mathbb{R}$;

(ii) $\overline{\dim}_{-\infty}(\mu) = \infty \Rightarrow \text{Dom}\,\tau = [0, \infty)$.

Write $b = \overline{\dim}_{-\infty}(\mu)$ and let $\varepsilon > 0$. The definition of $\overline{\dim}_{-\infty}(\mu)$ implies that there exists $\delta_\varepsilon > 0$ such that for all $0 < \delta < \delta_\varepsilon$,

$$\mu(B_\delta(x)) > \delta^{b+\varepsilon} \qquad \forall x \in \text{supp}(\mu).$$

Let $q < 0$ and let $\{B_\delta(x_i)\}_i$ be a collection of disjoint closed balls centered at $x_i \in \text{supp}(\mu)$. This family contains at most $C(l/\delta)^d$ elements, where C is the volume of the unit ball in $\mathbb{R}^d$. Hence

$$\sum_i \mu(B_\delta(x_i))^q \leqslant C\left(\frac{l}{\delta}\right)^d \delta^{q(b+\varepsilon)},$$

which yields, by taking supremum over all collections $\{B_\delta(x_i)\}_i$,

$$S_\delta(q) \leqslant Cl^d\, \delta^{q(b+\varepsilon)-d}.$$

It follows that

$$\tau(q) \geqslant q(b+\varepsilon) - d > -\infty, \qquad \forall q < 0$$

and this proves claim (i). To prove (ii) we let $q < 0$. Then for any $\delta > 0$,

$$(\inf\{\mu(B_\delta(x)) \colon x \in \text{supp}(\mu)\})^q \leqslant S_\delta(q).$$

Taking logarithm on both sides of the inequality, dividing by $\ln \delta$ and then letting $\delta \searrow 0$, we have $-\infty = q\,\overline{\dim}_{-\infty}(\mu) \geqslant \tau(q)$. This implies that $\tau(q) = -\infty$ and hence (ii) follows.

PROPOSITION 3.4. (i) $\alpha_{\min} = \lim_{q\to\infty} \underline{\dim}_q(\mu) = \underline{\dim}_\infty(\mu) \leqslant \overline{\dim}_\infty(\mu) \leqslant d$; (ii) $\alpha_{\max} = \overline{\dim}_{-\infty}(\mu)$.

Proof. (i) Assume that supp(μ) is contained in a cube Q of size l. We first show that $\overline{\dim}_\infty(\mu) \leqslant d$. For any $\delta > 0$, Q can be covered by

$c_\delta = ([l/\delta]+1)^d$ cubes Q_i of side length δ. At least one of the Q_i satisfies $\mu(Q_i) \geqslant \mu(Q)/c_\delta$. It follows that

$$\overline{\dim}_\infty(\mu) \leqslant \overline{\lim_{\delta \to 0^+}} \frac{\ln(\sup_i \mu(Q_i))}{\ln \delta} \leqslant \overline{\lim_{\delta \to 0^+}} \frac{-\ln c_\delta + \ln \mu(Q)}{\ln \delta} = d.$$

To show that $\alpha_{\min} = \underline{\dim}_\infty(\mu)$, we let $a = \underline{\dim}_\infty(\mu)$ and let $\varphi(q) = aq - \tau(q)$. That $\sup_x \mu(B_\delta(x))^q \leqslant S_\delta(q)$ implies that $aq \geqslant \tau(q)$, i.e., $\varphi(q) \geqslant 0$. We claim that $\varphi(q)$ is decreasing. Hence φ is bounded and $\lim_{q\to\infty} \varphi(q)/q = 0$. By the concavity of τ, $\alpha_{\min} = \lim_{q \to \infty} \tau(q)/q$ and the identities in (i) will follow. To prove the claim, we let $\varepsilon > 0$. Then there exists $\delta_\varepsilon > 0$ such that

$$\sup_x \mu(B_\delta(x)) < \delta^{a-\varepsilon}, \qquad \text{for all} \quad 0 < \delta < \delta_\varepsilon.$$

Note that for $0 < \delta < \delta_\varepsilon$, $1 < q_1 < q_2$ and any disjoint collection of closed balls $\{B_\delta(x_i)\}_i$ with center $x_i \in \operatorname{supp}(\mu)$, we have

$$\sum_i (\mu(B_\delta(x_i))/\delta^{a-\varepsilon})^{q_1} \geqslant \sum_i (\mu(B_\delta(x_i))/\delta^{a-\varepsilon})^{q_2}.$$

This implies that

$$\frac{\ln S_\delta(q_1)}{\ln \delta} - (a-\varepsilon)\, q_1 \leqslant \frac{\ln S_\delta(q_2)}{\ln \delta} - (a-\varepsilon)\, q_2.$$

By letting $\delta \to 0$ and then $\varepsilon \to 0$, we have $\varphi(q_2) \leqslant \varphi(q_1)$ as claimed.

(ii) If $\operatorname{Dom} \tau = [0, \infty)$, then it follows from the definition of $\alpha_{\max}$ and Proposition 3.3 that $\alpha_{\max} = \overline{\dim}_{-\infty}(\mu) = \infty$. Assume $\operatorname{Dom} \tau = \mathbb{R}$ and let $b = \overline{\dim}_{-\infty}(\mu)$. We can prove as in (i) above that $\varphi(q) = bq - \tau(q)$ is a non-negative increasing function on $(-\infty, 0)$ and $\alpha_{\max} = \lim_{q\to -\infty} \tau(q)/q = b$.

PROPOSITION 3.5. *Let τ^* be the concave conjugate of τ. Then*

(i) $(\operatorname{Dom} \tau^*)^\circ = (\alpha_{\min}, \alpha_{\max}) \subseteq (0, \infty)$ *and* $\tau^* \geqslant 0$ *on* $\operatorname{Dom} \tau^*$.

(ii) *Let* $\alpha_0 \in \partial\tau(0)$. *Then* τ^* *has a maximum at* α_0 *with* $\tau^*(\alpha_0) = -\tau(0)$. *Consequently* τ^* *is increasing on* $[\alpha_{\min}, \alpha_0]$ *and is decreasing on* $[\alpha_0, \alpha_{\max})$.

Proof. (i) Since $\alpha_{\min} \geqslant 0$ (Proposition 3.4(i)) and $(\operatorname{Dom} \tau^*)^\circ = (\alpha_{\min}, \alpha_{\max})$ (Proposition 2.3), we have $(\operatorname{Dom} \tau^*)^\circ \subseteq (0, \infty)$. To show that $\tau^* \geqslant 0$ we observe that if $\alpha \in (\operatorname{Dom} \tau^*)^\circ$, then $\alpha \in \partial\tau(q)$ for some $q \in \operatorname{Dom} \tau$. If $q \geqslant 0$, then

$$\tau^*(\alpha) = q\alpha - \tau(q) \geqslant \inf\{q'\alpha_{\min} - \tau(q') \colon q' \in \operatorname{Dom} \tau\} \geqslant 0.$$

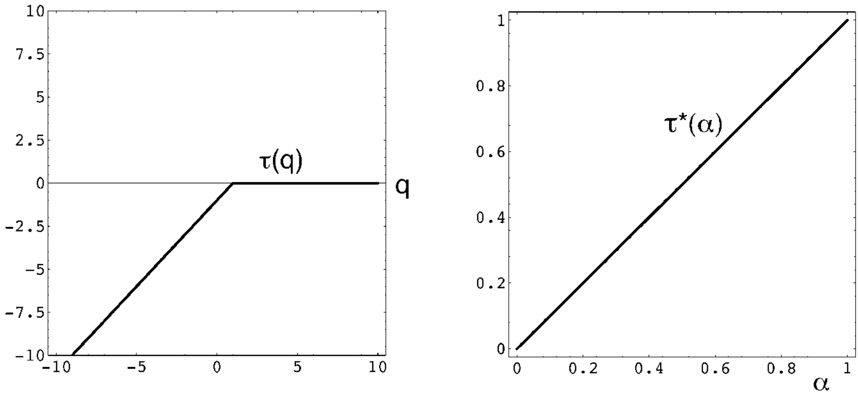

FIG. 2. $\tau(q)$ and its concave conjugate $\tau^*(\alpha)$ for the sum of Lebesgue and Dirac measures in Example 1.

(The last inequality is contained in the proof of Proposition 3.4(i) that $\varphi(q') \geqslant 0$.) Similarly, if $q < 0$, we can prove that

$$\tau^*(\alpha) \geqslant \inf\{q\alpha_{\max} - \tau(q) : q \in \operatorname{Dom} \tau\} \geqslant 0.$$

The upper semicontinuity of τ^* implies that it is nonnegative on $\operatorname{Dom} \tau^*$.

(ii) $\alpha_0 \in \partial\tau(0)$ implies that $0 \in \partial\tau^*(\alpha_0)$, i.e., $\tau^*(\alpha) \leqslant \tau^*(\alpha_0) + 0(\alpha - \alpha_0) = \tau^*(\alpha_0)$ for all $\alpha \in \mathbb{R}$. Therefore τ^* has a maximum at α_0, and the claim follows.

EXAMPLES. 1. Let $\mu = m + \delta_0$, where m denotes the Lebesgue measure on $[0, 1]$ and δ_0 denotes the Dirac measure at 0. Then it follows directly from definition that

$$\tau(q) = \begin{cases} 0 & \text{if } q \geqslant 1 \\ q - 1 & \text{if } q < 1 \end{cases}$$

and $\tau^*(\alpha) = \alpha$ on $\operatorname{Dom} \tau^* = [0, 1]$. (See Fig. 2.)

2. The following example is the simplest multifractal measure ([H], [F2, Example 17.1]). Fix $p \in (0, 1)$ and let μ be the self-similar measure satisfying

$$\mu = p\mu \circ S_1^{-1} + (1 - p)\,\mu \circ S_2^{-1},$$

where $S_1(x) = \frac{1}{3}x$ and $S_2(x) = \frac{1}{3}x + \frac{2}{3}$. Then by (1.2),

$$\tau(q) = \frac{\ln(p^q + (1 - p)^q)}{-\ln 3}.$$

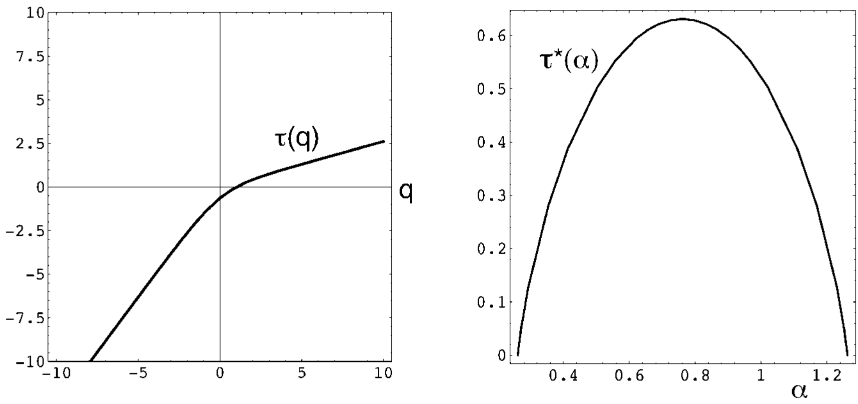

FIG. 3. $\tau(q)$ and its concave conjugate $\tau^*(\alpha)$ for the weighted Cantor measure in Example 2 with $p=1/4$.

Since τ is differentiable, the subdifferential of τ is the derivative. It follows that $\tau^*(\alpha)=\alpha q-\tau(q)$, with

$$\alpha=\tau'(q)=\frac{p^q\ln p+(1-p)^q\ln(1-p)}{-(p^q+(1-p)^q)\ln 3}.$$

(See Fig. 3.)

3. The following example shows the possibilities for $\tau^*(\alpha_{\min})$ or $\tau^*(\alpha_{\max})$ to be strictly positive ([H], [T]). Let $S_1(x)=\frac{1}{3}x$, $S_2(x)=\frac{1}{3}x+\frac{1}{3}$ and $S_3(x)=\frac{1}{3}x+\frac{2}{3}$. Let $0<p_1<\frac{1}{2}$, $p_2=1-2p_1$ and let μ be the self-similar measure on $[0,1]$ defined by

$$\mu=p_1\mu\circ S_1^{-1}+p_2\mu\circ S_2^{-1}+p_1\mu\circ S_3^{-1}.$$

Then $\tau^*(\alpha)=\alpha q-\tau(q)$ with

$$\tau(q)=\frac{\ln(2p_1^q+p_2^q)}{-\ln 3}\qquad\text{and}\qquad\alpha=\tau'(q)=\frac{2p_1^q\ln p_1+p_2^q\ln p_2}{-(2p_1^q+p_2^q)\ln 3}.$$

To calculate $\alpha_{\min}$ and $\alpha_{\max}$, we consider the following two cases.

Case 1. $p_1<p_2$. A direct calculation yields

$$\alpha_{\min}=\lim_{q\to\infty}\tau'(q)=-\frac{\ln p_2}{\ln 3},\qquad\alpha_{\max}=\lim_{q\to-\infty}\tau'(q)=-\frac{\ln p_1}{\ln 3},$$

$$\tau^*(\alpha_{\min})=\lim_{q\to\infty}\tau^*(\alpha(q))=0,\qquad\tau^*(\alpha_{\max})=\lim_{q\to-\infty}\tau^*(\alpha(q))=\frac{\ln 2}{\ln 3}.$$

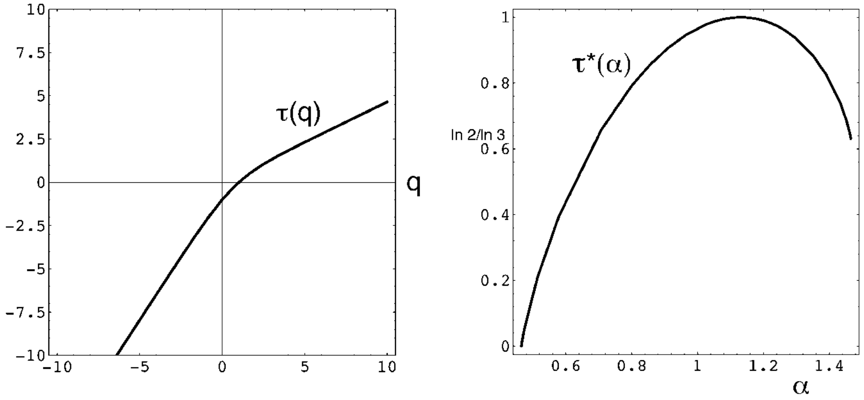

FIG. 4. $\tau(q)$ and its concave conjugate $\tau^*(\alpha)$ for the measure in Example 3 with $P1 = 0.2$.

Case 2. $p_1 > p_2$. Similarly,

$$\alpha_{\min} = -\frac{\ln p_1}{\ln 3}, \qquad \alpha_{\max} = -\frac{\ln p_2}{\ln 3}, \qquad \tau^*(\alpha_{\min}) = \frac{\ln 2}{\ln 3}, \qquad \tau^*(\alpha_{\max}) = 0.$$

(See Figs. 4 and 5.)

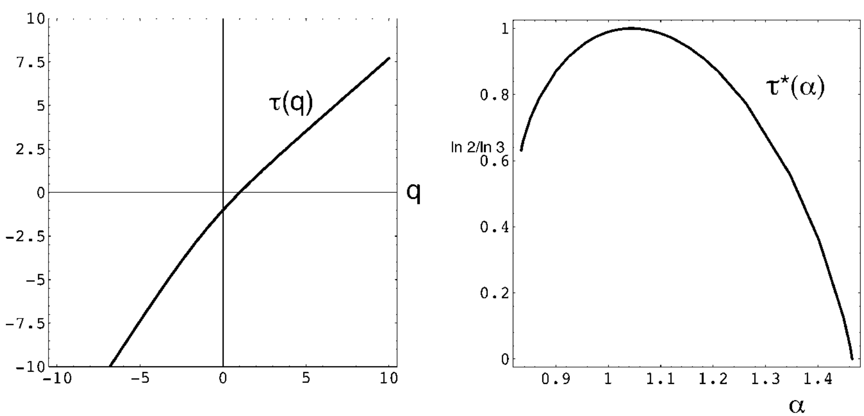

FIG. 5. $\tau(q)$ and its concave conjugate $\tau^*(\alpha)$ for the measure in Example 3 with $P1 = 0.4$.

4. THE DIMENSION SPECTRA

Throughout the rest of the paper we will explore the relationship between $\tau^*(\alpha)$ and the Hausdorff dimension of the set of x such that $\mu(B_\delta(x)) \approx \delta^\alpha$ as $\delta \to 0$. More precisely, we define

$$\bar{K}(\alpha) = \left\{ x: \overline{\lim_{\delta \to 0^+}} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \alpha \right\},$$

$$\underline{K}(\alpha) = \left\{ x: \underline{\lim}_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \alpha \right\},$$

$$\bar{U}(\alpha) = \left\{ x: \overline{\lim_{\delta \to 0^+}} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leqslant \alpha \right\},$$

$$\underline{L}(\alpha) = \left\{ x: \underline{\lim}_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \geqslant \alpha \right\}.$$

We also denote the closed interval $\partial\tau(0)$ by $[\alpha_0^-, \alpha_0^+]$. Our first theorem in this direction is

THEOREM 4.1. *Let* $\alpha \in (\mathrm{Dom}\, \tau^*)^\circ = (\alpha_{\min}, \alpha_{\max})$. *Then*

(i) *If* $\alpha_{\min} < \alpha < \alpha_0^+$, *then* $\dim_{\mathscr{H}} \bar{K}(\alpha) \leqslant \dim_{\mathscr{H}} \bar{U}(\alpha) \leqslant \tau^*(\alpha)$.

(ii) *If* $\alpha_0^+ \leqslant \alpha < \alpha_{\max}$, *then* $\dim_{\mathscr{H}} \underline{K}(\alpha) \leqslant \dim_{\mathscr{H}} \underline{L}(\alpha) \leqslant \tau^*(\alpha)$.

We remark that in [O1, Corollary 2.23], a similar result is proved for a class of measures satisfying certain scaling property. In order to prove the theorem we will develop some counting devices. Let $\mathscr{B}_\delta$ denote a disjoint family of closed balls of radii δ centered at points in $\mathrm{supp}(\mu)$. For $\alpha, \alpha_1, \alpha_2 \in (\mathrm{Dom}\, \tau^*)^\circ$, $\alpha_1 < \alpha_2$, we define the counting functions

$$N_\delta(\alpha) = \sup_{\mathscr{B}_\delta} \#\{B: B \in \mathscr{B}_\delta, \mu(B) \geqslant \delta^\alpha\},$$

$$\tilde{N}_\delta(\alpha) = \sup_{\mathscr{B}_\delta} \#\{B: B \in \mathscr{B}_\delta, \mu(B) < \delta^\alpha\},$$

$$N_\delta(\alpha_1, \alpha_2) = \sup_{\mathscr{B}_\delta} \#\{B: B \in \mathscr{B}_\delta, \delta^{\alpha_2} \leqslant \mu(B) < \delta^{\alpha_1}\}.$$

LEMMA 4.2. *Let* $\alpha_{\min} < \alpha < \alpha_0^+$, $q \in \partial\tau^*(\alpha)$ *and* $\xi > 0$. *Then for any* $\varepsilon > 0$, *there exists* $\delta_\varepsilon > 0$ *such that for all* $0 < \delta < \delta_\varepsilon$,

$$N_\delta(\alpha \pm \varepsilon) \leqslant \delta^{-\tau^*(\alpha) - (\xi \pm q)\varepsilon}.$$

For $\alpha_0^+ \leqslant \alpha < \alpha_{\max}$, *the above holds with* $\tilde{N}_\delta$ *replacing* N_δ.

Proof. Note that if $\alpha_{\min} < \alpha < \alpha_0^+$ and if $q \in \partial\tau^*(\alpha)$, then $q \geqslant 0$. Let $\mathscr{B}_\delta$ be a family of disjoint closed balls with centers in $\operatorname{supp}(\mu)$ such that for all $B \in \mathscr{B}_\delta$,

$$\mu(B) \geqslant \delta^{\alpha+\varepsilon}.$$

Then

$$S_\delta(q) \geqslant \sum_{\mathscr{B}_\delta} \mu(B)^q \geqslant \delta^{q(\alpha+\varepsilon)} \# \mathscr{B}_\delta.$$

By taking supremum over all such families of balls, we have

$$S_\delta(q) \geqslant \delta^{q(\alpha+\varepsilon)} N_\delta(\alpha+\varepsilon). \tag{4.1}$$

In view of $\tau(q) = \underline{\lim}_{\delta\to 0^+}(\ln S_\delta(q)/\ln\delta)$, there exists $\delta_\varepsilon > 0$ such that for $0 < \delta < \delta_\varepsilon$,

$$S_\delta(q) \leqslant \delta^{\tau(q)-\xi\varepsilon}. \tag{4.2}$$

Combining (4.1) and (4.2) we have, for $0 < \delta < \delta_\varepsilon$,

$$N_\delta(\alpha+\varepsilon) \leqslant \delta^{-q(\alpha+\varepsilon)} \cdot \delta^{\tau(q)-\xi\varepsilon} = \delta^{-\tau^*(\alpha)-(\xi+q)\varepsilon}.$$

The proof for $N_\delta(\alpha-\varepsilon)$ is the same. For $\alpha_0^+ \leqslant \alpha < \alpha_{\max}$, the corresponding $q \in \partial\tau^*(\alpha)$ is zero or negative. We let $\mathscr{B}_\delta$ be a disjoint family of balls with centers in $\operatorname{supp}(\mu)$ such that for all $B \in \mathscr{B}_\delta$,

$$\mu(B) < \delta^{\alpha+\varepsilon} \qquad \text{or} \quad \mu(B) < \delta^{\alpha-\varepsilon},$$

and proceed as above.

As a direct consequence we have

COROLLARY 4.3. *If $\alpha_{\min} < \alpha < \alpha_0^+$, then*

$$\lim_{\varepsilon\to 0^+} \overline{\lim_{\delta\to 0^+}} \frac{\ln N_\delta(\alpha+\varepsilon)}{-\ln\delta} \leqslant \tau^*(\alpha).$$

If $\alpha_0^+ \leqslant \alpha < \alpha_{\max}$, then

$$\lim_{\varepsilon\to 0^+} \overline{\lim_{\delta\to 0^+}} \frac{\ln \tilde{N}_\delta(\alpha-\varepsilon)}{-\ln\delta} \leqslant \tau^*(\alpha).$$

Consequently for any $\alpha \in (\text{Dom } \tau^*)^\circ$,

$$\lim_{\varepsilon \to 0^+} \overline{\lim_{\delta \to 0^+}} \frac{\ln N_\delta(\alpha - \varepsilon, \alpha + \varepsilon)}{-\ln \delta} \leqslant \tau^*(\alpha).$$

Remark. It is clear that for $\alpha \in (\text{Dom } \tau^*)^\circ = (\alpha_{\min}, \alpha_{\max})$, the two families corresponding to $N_\delta(\alpha)$ and $\tilde{N}_\delta(\alpha)$ are nonempty. Hence both $\overline{\lim}_{\delta \to 0^+} \ln N_\delta(\alpha)/(-\ln \delta)$ and $\overline{\lim}_{\delta \to 0^+} \ln \tilde{N}_\delta(\alpha)/(-\ln \delta)$ are nonnegative, so is τ^* on $(\text{Dom } \tau^*)^\circ$.

Proof of Theorem 4.1. (i) Let $q \in \partial \tau^*(\alpha)$ be nonnegative and let $\varepsilon > 0$. Applying Lemma 4.2 with $\xi = 1$, we can find n_ε such that for all $n \geqslant n_\varepsilon$,

$$N_{2^{-n}}(\alpha + \varepsilon) \leqslant 2^{n(\tau^*(\alpha) + \eta)},$$

where $\eta = (1 + q)\varepsilon$.

For each $x \in \bar{U}(\alpha)$, we choose $n_x \in \mathbb{N}$ sufficiently large so that for all $n \geqslant n_x$,

$$\mu(B_{2^{-n}}(x)) \geqslant 2^{-n(\alpha + \varepsilon)}.$$

For each $0 < \delta < 2^{-n_\varepsilon}$, let $\mathcal{G}_\delta$ denote the family of balls with centers in $\bar{U}(\alpha)$, diameter less than δ and satisfy the above inequality. Then $\mathcal{G}_\delta$ is a Vitali cover of $\bar{U}(\alpha)$. Let $s = \tau^*(\alpha) + 2\eta$. By the Vitali covering theorem ([F1, Theorem 1.10]), there exists a disjoint subcollection $\{B_i\}$ of $\mathcal{G}_\delta$ such that either

$$\sum_i |B_i|^s = \infty \qquad \text{or} \qquad \mathcal{H}^s \left(U(\alpha) \Big\backslash \bigcup_i B_i \right) = 0.$$

The first case is impossible since

$$\begin{aligned}
\sum_i |B_i|^s &= \sum_{2^{-n+1} < \delta} \sum_{|B_i| = 2^{-n+1}} |B_i|^s \\
&\leqslant \sum_{2^{-n+1} < \delta} 2^{(-n+1)s} 2^{n(\tau^*(\alpha) + \eta)} \\
&\leqslant \frac{2^s}{2^\eta - 1} = C < \infty.
\end{aligned}$$

Consequently the second identity must hold and therefore

$$\begin{aligned}\mathscr{H}^s_\delta(\bar{U}(\alpha)) &\leqslant \mathscr{H}^s_\delta\Big(\bar{U}(\alpha)\cap\bigcup_i B_i\Big)+\mathscr{H}^s_\delta\Big(\bar{U}(\alpha)\Big\backslash\bigcup_i B_i\Big)\\ &\leqslant \mathscr{H}^s_\delta\Big(\bigcup_i B_i\Big)+\mathscr{H}^s\Big(\bar{U}(\alpha)\Big\backslash\bigcup_i B_i\Big)\\ &\leqslant \sum_i |B_i|^s+0.\end{aligned}$$

It follows that $\mathscr{H}^s_\delta(\bar{U}(\alpha))\leqslant C$ and hence $\dim_{\mathscr{H}}\bar{U}(\alpha)\leqslant s=\tau^*(\alpha)+2\eta$. The theorem follows by letting $\varepsilon\searrow 0$.

(ii) In this case we take a zero or negative $q\in\partial\tau^*(\alpha)$ and let $\varepsilon>0$. Applying Lemma 4.2 with $\xi=1$, we can find n_ε such that for all $n\geqslant n_\varepsilon$,

$$\tilde{N}_{2^{-n}}(\alpha-\varepsilon)\leqslant 2^{n(\tau^*(\alpha)+\eta)},$$

where $\eta=(1-q)\varepsilon$. For each $x\in\underline{L}(\alpha)$, we choose n_x sufficiently large so that for all $n\geqslant n_x$,

$$\mu(B_{2^{-n}}(x))<2^{-n(\alpha-\varepsilon)}.$$

For each $0<\delta<2^{-n_\varepsilon}$, we define $\tilde{\mathscr{G}}_\delta$ to be the family of balls with centers in $\underline{L}(\alpha)$, diameter less than δ and satisfy the above inequality. Then the same argument as above yields (ii).

5. THE COUNTING FUNCTIONS

Our next step toward the equality $\dim_{\mathscr{H}}\bar{K}(\alpha)=\tau^*(\alpha)$ in Theorem 4.1 is to obtain a reverse inequality in Lemma 4.2 and Corollary 4.3. The additional assumption is that τ^* is strictly concave at α. Recall that the dual relationship is that τ is smooth at q where $q\in\partial\tau^*(\alpha)$ (Proposition 2.4).

THEOREM 5.1. *Let $\alpha\in(\mathrm{Dom}\,\tau^*)^\circ$ and suppose τ^* is strictly concave at α. Then*

$$\tau^*(\alpha)=\lim_{\varepsilon\to 0^+}\ \overline{\lim_{\delta\to 0^+}}\ \frac{\ln N_\delta(\alpha-\varepsilon,\alpha+\varepsilon)}{-\ln\delta}.$$

Proof. We first consider the case $\alpha_{\min}<\alpha<\alpha_0^+$. There exists a nonnegative $q\in\partial\tau^*(\alpha)$ satisfying the corresponding inequality for strict concavity

(2.4). Let $\varepsilon > 0$. According to the remark on strictly concave functions, there exists $0 < \eta < \varepsilon$ such that for all $|u - \alpha| \geqslant \varepsilon$,

$$\tau^*(\alpha) - q\alpha \geqslant \tau^*(u) - qu + (q+1)\eta. \tag{5.1}$$

Let $\mathscr{P}$ be an η-partition of $[\alpha_{\min}, \alpha_0^+) \backslash (\alpha - \varepsilon, \alpha + \varepsilon]$ such that all subintervals I_j satisfy $|I_j| = \eta$ except the two containing $\alpha_{\min}$ and α_0^+; the two exceptional subintervals satisfy $|I_j| \leqslant \eta$. Let α_j and α_{j+1} be the endpoints of I_j and let l be the number of such subintervals in $\mathscr{P}$. For $\delta > 0$ sufficiently small we have

(i) $l\delta^{\eta/4} \leqslant 1$ and

(ii) $N_\delta(\alpha_{j+1}) \leqslant \delta^{-\tau^*(\alpha_j) - \eta/4 - q\eta}$ (by putting $\xi = 1/4$ in Lemma 4.2).

We also recall that (Proposition 3.2),

$$\tau^*(\alpha_0^+) = -\tau(0) = \overline{\dim}_{\mathscr{B}}(\operatorname{supp}(\mu)).$$

Hence for all $\delta > 0$ small enough (depending on η) and for any family $\mathscr{B}_\delta$ of disjoint closed balls with centers in $\operatorname{supp}(\mu)$, we have

(iii) $\# \mathscr{B}_\delta \leqslant \delta^{-\tau^*(\alpha_0^+) - \eta/2}$.

Now assume that $\delta > 0$ is small enough so that (i), (ii) and (iii) hold. We write

$$\sum \mu(B)^q = \Big(\sum_I + \sum_{II} + \sum_{III} \Big) \mu(B)^q.$$

Here $\sum_I$ is the sum over all $B \in \mathscr{B}_\delta$ such that $\delta^{\alpha+\varepsilon} \leqslant \mu(B) < \delta^{\alpha - \varepsilon}$, $\sum_{II}$ is the sum over all B's satisfying

$$\delta^{\alpha - \varepsilon} \leqslant \mu(B) \qquad \text{or} \qquad \delta^{\alpha_0^+} \leqslant \mu(B) < \delta^{\alpha+\varepsilon},$$

and $\sum_{III}$ is the sum over the rest of the B's, i.e., $\mu(B) < \delta^{\alpha_0^+}$. (See Fig. 6.)

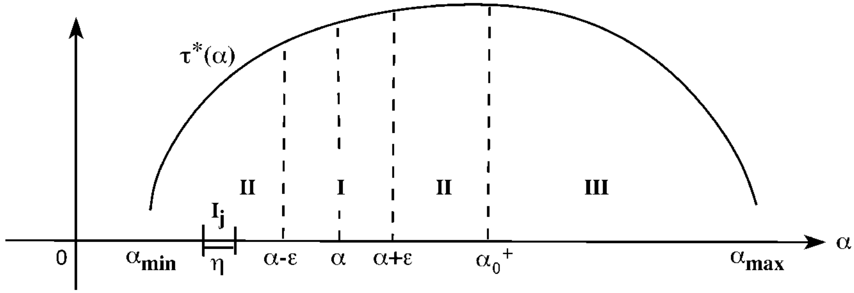

FIG. 6. Partitioning the domain of $\tau^*(\alpha)$ into three regions I, II, and III.

We first estimate $\sum_{II} \mu(B)^q$:

$$\begin{aligned}\sum_{II} \mu(B)^q &= \sum_j \sum \{\mu(B)^q : \delta^{\alpha_{j+1}} \leqslant \mu(B) < \delta^{\alpha_j}, B \in \mathscr{B}_\delta\} \\ &\leqslant \sum_j \delta^{\alpha_j q} N_\delta(\alpha_{j+1}) \\ &\leqslant \sum_j \delta^{\alpha_j q} \delta^{-\tau^*(\alpha_j) - \eta/4 - q\eta} \qquad \text{(by (ii))} \\ &\leqslant \sum_j \delta^{\tau(q) + 3\eta/4} \qquad \text{(by (5.1))} \\ &\leqslant \delta^{\tau(q) + \eta/2} \qquad \text{(by } (i)\text{)}.\end{aligned}$$

To estimate $\sum_{III} \mu(B)^q$, we use (iii) above:

$$\begin{aligned}\sum_{III} \mu(B)^q &\leqslant \delta^{q\alpha_0^+ - \tau^*(\alpha_0^+) - \eta/2} \\ &\leqslant \delta^{\tau(q) + (q+1)\eta - \eta/2} \\ &< \delta^{\tau(q) + \eta/2}.\end{aligned}$$

We hence have

$$\sum_{\mathscr{B}_\delta} \mu(B)^q \leqslant \delta^{q(\alpha - \varepsilon)} N_\delta(\alpha - \varepsilon, \alpha + \varepsilon) + 2\delta^{\tau(q) + \eta/2},$$

and taking supremum over all such families $\mathscr{B}_\delta$, we have

$$S_\delta(q) \leqslant \delta^{q(\alpha - \varepsilon)} N_\delta(\alpha - \varepsilon, \alpha + \varepsilon) + 2\delta^{\tau(q) + \eta/2}.$$

Since $\underline{\lim}_{\delta \to 0^+} \ln S_\delta(q)/\ln \delta = \tau(q)$, it follows that

(iv) there exists a decreasing sequence $\delta_k \searrow 0$ such that for every $\delta = \delta_k$,

$$3\delta^{\tau(q) + \eta/2} \leqslant S_\delta(q).$$

Let $\delta = \delta_k$ be small enough so that (i), (ii), (iii) and (iv) hold. Then

$$3\delta^{\tau(q) + \eta/2} \leqslant \delta^{(\alpha - \varepsilon) q} N_\delta(\alpha - \varepsilon, \alpha + \varepsilon) + 2\delta^{\eta/2 + \tau(q)}.$$

Therefore

$$\delta^{\eta/2 + \varepsilon q - \tau^*(\alpha)} \leqslant N_\delta(\alpha - \varepsilon, \alpha + \varepsilon). \tag{5.2}$$

Hence

$$\tau^*(\alpha) - \eta/2 - \varepsilon q \leqslant \overline{\lim_{\delta \to 0^+}} \frac{\ln N_\delta(\alpha - \varepsilon, \alpha + \varepsilon)}{-\ln \delta}. \tag{5.3}$$

Since $\eta \searrow 0$ as $\varepsilon \searrow 0$, we have

$$\tau^*(\alpha) \leqslant \lim_{\varepsilon \to 0^+} \overline{\lim_{\delta \to 0^+}} \frac{\ln N_\delta(\alpha - \varepsilon, \alpha + \varepsilon)}{-\ln \delta}.$$

This combined with Corollary 4.3 proves the theorem for the case $\alpha_{\min} < \alpha < \alpha_0^+$.

For $\alpha_0^+ \leqslant \alpha < \alpha_{\max}$, we replace (5.1) by

$$\tau^*(\alpha) - q\alpha \geqslant \tau^*(u) - qu + (1 - q)\eta$$

where $q \in \partial\tau^*(\alpha)$ is zero or negative. Let $\mathscr{B}_\delta$ be a family of disjoint balls with centers in $\operatorname{supp}(\mu)$ and write

$$\sum_{\mathscr{B}_\delta} \mu(B)^q = \left(\sum_I + \sum_{II} + \sum_{III}\right) \mu(B)^q.$$

Here $\sum_I$ is the sum over all B's such that $\delta^{\alpha+\varepsilon} \leqslant \mu(B) < \delta^{\alpha-\varepsilon}$, $\sum_{II}$ is the sum over all B's satisfying

$$\mu(B) < \delta^{\alpha+\varepsilon} \qquad \text{or} \qquad \delta^{\alpha-\varepsilon} \leqslant \mu(B) < \delta^{\alpha_0^+},$$

and $\sum_{III}$ is the sum over the rest of the B's, i.e., $\delta^{\alpha_0^+} \leqslant \mu(B)$. We can then continue the proof as above.

Remark 1. In Theorem 5.1, if we assume further that $\lim_{\delta \to 0^+} \ln S_\delta(q)/\ln \delta$ exists, then (iv) holds for all $\delta > 0$ sufficiently small. Consequently (5.2) holds for all $\delta > 0$ sufficiently small and $\lim_{\varepsilon \to 0^+} \underline{\lim}_{\delta \to 0^+} \ln N_\delta(\alpha - \varepsilon, \alpha + \varepsilon)/(-\ln \delta) = \tau^*(\alpha)$. Together with Corollary 4.3 we have

$$\lim_{\varepsilon \to 0^+} \underline{\lim_{\delta \to 0^+}} \frac{\ln N_\delta(\alpha - \varepsilon, \alpha + \varepsilon)}{-\ln \delta} = \lim_{\varepsilon \to 0^+} \overline{\lim_{\delta \to 0^+}} \frac{\ln N_\delta(\alpha - \varepsilon, \alpha + \varepsilon)}{-\ln \delta} = \tau^*(\alpha).$$

Remark 2. In Theorem 5.1, if $\alpha_{\min} < \alpha < \alpha_0^+$, then we can choose $\delta > 0$ arbitrarily small so that $N_\delta(\alpha - \varepsilon)$ is much smaller than $N_\delta(\alpha - \varepsilon, \alpha + \varepsilon)$. Indeed, let $q \in \partial\tau^*(\alpha)$ be positive and satisfy the inequality for strict concavity and let $\eta = (q - \xi)\,\varepsilon > 0$, where $\xi > 0$ is as in Lemma 4.2. Then there exists $\delta_\varepsilon > 0$ such that for all $0 < \delta < \delta_\varepsilon$,

$$N_\delta(\alpha - \varepsilon) \leqslant \delta^{-\tau^*(\alpha) + \eta}.$$

Also from (5.2) we have, for $0<\varepsilon'<\varepsilon$ there exists a sequence δ_k such that for all $\delta=\delta_k$,

$$\delta^{-\tau^*(\alpha)+\eta'} \leqslant N_\delta(\alpha-\varepsilon', \alpha+\varepsilon')(\leqslant N_\delta(\alpha-\varepsilon, \alpha+\varepsilon))$$

and $\eta'(<\eta) \searrow 0$ as $\varepsilon' \searrow 0$. It follows that for $\varepsilon'(<\varepsilon)$ small enough and $\delta=\delta_k$,

$$N_\delta(\alpha-\varepsilon) \leqslant \delta^{-\tau^*(\alpha)+\eta} \leqslant \delta^{-\tau^*(\alpha)+\eta'} \leqslant N_\delta(\alpha-\varepsilon, \alpha+\varepsilon). \tag{5.4}$$

Remark 3. If we consider

$$\sup_{\mathscr{B}_\delta} \#\{B\colon B\in\mathscr{B}_\delta,\ C_1\delta^{\alpha+\varepsilon}<\mu(B)<C_2\delta^{\alpha-\varepsilon}\}$$

and

$$\sup_{\mathscr{B}_\delta} \#\{B\colon B\in\mathscr{B}_\delta,\ \mu(B)\geqslant C_1\delta^{\alpha-\varepsilon}\}$$

for some fixed C_1, C_2 instead of $N_\delta(\alpha-\varepsilon, \alpha+\varepsilon)$, $N_\delta(\alpha-\varepsilon)$, then Theorem 5.1 and (5.4) still hold with the same sequence δ_k. The only adjustment is to let δ_k start from a larger k.

Remark 4. For each $0<\delta<1$, let $\{Q_i(\delta)\}_i$ denote the collection of all δ-mesh cubes that intersect $\operatorname{supp}(\mu)$ and let

$$N_\delta^Q(\alpha)=\#\{i\colon \mu(Q_i(\delta))\geqslant\delta^\alpha\};$$
$$N_\delta^Q(\alpha_1, \alpha_2)=\#\{i\colon \delta^{\alpha_2}\leqslant\mu(Q_i(\delta))<\delta^{\alpha_1}\}.$$

If we assume that $\alpha_{\min}<\alpha<\alpha_0^+$, then Lemma 4.2 and Corollary 4.3 hold with N_δ^Q replacing N_δ. Moreover, if τ^* is strictly concave at α, then we have the following analogue of Theorem 5.1.

$$\tau^*(\alpha)=\lim_{\varepsilon\to 0^+}\ \overline{\lim_{\delta\to 0^+}}\ \frac{\ln N_\delta^Q(\alpha-\varepsilon, \alpha+\varepsilon)}{-\ln\delta}.$$

To simplify notation we write

$$s_\delta \approx \delta^{a\pm\eta} \qquad \text{to mean} \quad \delta^{a+\eta}\leqslant s_\delta<\delta^{a-\eta}. \tag{5.5}$$

We also let $\mathscr{B}_\delta(\mu; \alpha\pm\varepsilon)$ denote a collection of disjoint δ-balls with $\mu(B)\approx\delta^{\alpha\pm\varepsilon}$.

Note that in Theorem 5.1, the choice of δ_k depends only on $\underline{\lim}_{\delta\to 0^+}(\ln S_\delta(q)/\ln\delta)=\tau(q)$. By modifying the proofs of Lemma 4.2 and Theorem 5.1 we have:

Corollary 5.2. *Let* $\alpha \in (\mathrm{Dom}\,\tau^*)^\circ$ *and let* $q \in \partial\tau^*(\alpha)$ *satisfy the corresponding inequality* (2.4) *for strict concavity. Then for* $\varepsilon > 0$ *there exist a sequence* $\delta_k \searrow 0$ *and* $\mathscr{B}_{\delta_k}(\mu; \alpha \pm \varepsilon)$ *such that*

$$\# \mathscr{B}_{\delta_k}(\mu; \alpha \pm \varepsilon) \approx \delta_k^{-\tau^*(\alpha) \pm \eta}$$

and

$$\sum \{\mu(B)^q : B \in \mathscr{B}_{\delta_k}(\mu; \alpha \pm \varepsilon)\} \approx \delta_k^{\tau(q) \pm \eta}$$

for sufficiently large k (*depending only on* ε *and* η) *and* $\eta = \eta(\varepsilon)$ *with* $\lim_{\varepsilon \to 0} \eta(\varepsilon) = 0$.

Remark. Observe that $S_\delta(q) \approx \delta_k{}^{\tau(q) \pm \eta'}$ from the definition of $\tau(q)$. The corollary states that the sum is concentrated on the family $\mathscr{B}_{\delta_k}(\mu; \alpha \pm \varepsilon)$ which has cardinality $\approx \delta_k{}^{-\tau^*(\alpha) \pm \eta}$. This is the original heuristic idea of Halsey *et al.* [H].

In the following we will strengthen the corollary to a family of measures for approximation purposes. The result will be needed in the next section.

Lemma 5.3. *Let* $r > 1$ *and let* $\{\mu_\delta\}_{\delta > 0}$ *be a family of positive regular Borel measures on* $\mathbb{R}^d$ *such that for all* $x \in \mathrm{supp}(\mu)$ *and* $0 < \delta < 1$,

$$\mu(B_{\delta/r}(x)) \leqslant \mu_\delta(B_\delta(x)) \leqslant \mu(B_{r\delta}(x)). \tag{5.6}$$

Suppose $\alpha \in (\alpha_{\min}, \alpha_0^+)$ *and* τ^* *is strictly concave at* α *with* $q \in \partial\tau^*(\alpha)$ *satisfying the corresponding inequality* (2.4) *for strict concavity. Then for* $\varepsilon > 0$ *there exists a sequence* $\delta_k \searrow 0$ *and* $\mathscr{B}_{\delta_k}(\mu_{\delta_k}; \alpha \pm \varepsilon)$ *such that*

$$\# \mathscr{B}_{\delta_k}(\mu_{\delta_k}; \alpha \pm \varepsilon) \approx \delta_k^{-\tau^*(\alpha) \pm \eta}$$

and

$$\sum \{\mu_{\delta_k}(B_i)^q : B_i \in \mathscr{B}_{\delta_k}(\mu_{\delta_k}; \alpha \pm \varepsilon)\} \approx \delta_k^{\tau(q) \pm \eta}$$

for k *sufficiently large* (*depending only on* ε *and* η) *and* $\eta = \eta(\varepsilon) > 0$ *with* $\lim_{\varepsilon \to 0} \eta(\varepsilon) = 0$.

Furthermore we can choose the above $\mathscr{B}_{\delta_k}(\mu_{\delta_k}; \alpha \pm \varepsilon)$ *so that for any disjoint collection* $\mathscr{D}$ *of* δ_k*-balls* B *satisfying* $\mu(B) \geqslant \delta_k^{\alpha - \varepsilon}$,

$$\# \mathscr{D} \leqslant \delta_k^s \, \# \mathscr{B}_{\delta_k}(\mu)$$

for some $s > 0$ (*depending only on* μ, r *and* ε).

Proof. Since we are fixing α and ε here, we write $\mathscr{B}_\delta(\mu)$ for $\mathscr{B}_\delta(\mu; \alpha \pm \varepsilon)$. We can find $s, \eta > 0$ and a decreasing sequence $\delta_k \searrow 0$ such that for each δ_k, there exists a family $\mathscr{B}_{\delta_k}(\mu)$ satisfying

$$\# \mathscr{B}_{\delta_k}(\mu) \approx \delta_k^{-\tau^*(\alpha) \pm \eta/2} \tag{5.7}$$

(by Corollary 5.2) and by (5.4) for any disjoint family $\mathscr{D}$ of δ_k-balls such that each member B satisfies $\mu(B) \geqslant \delta_k^{\alpha - \varepsilon}$,

$$\# \mathscr{D} \leqslant \delta_k^s \; \# \mathscr{B}_{\delta_k}(\mu). \tag{5.8}$$

The main task is to replace the μ in (5.7) by μ_{δ_k}. Let $\tilde{\mathscr{B}}$ be the family of balls $\tilde{B}$ of radius $r^2\delta_k$ concentric with $B \in \mathscr{B}_{\delta_k}(\mu)$. Since the δ_k-balls in $\mathscr{B}_{\delta_k}(\mu)$ are disjoint, we can select a maximal subcollection $\tilde{\mathscr{B}}^1$ of $\tilde{\mathscr{B}}$ and a constant C (equal to the number of disjoint δ_k-balls that can be packed inside a $(2r^2+2)\,\delta_k$-ball) so that the distance between any two balls in $\tilde{\mathscr{B}}^1$ is greater than $2\delta_k$ and

$$\# \tilde{\mathscr{B}}^1 \geqslant C^{-1} \# \tilde{\mathscr{B}} = C^{-1} \; \# \mathscr{B}_{\delta_k}(\mu). \tag{5.9}$$

Let $\tilde{\mathscr{B}}^2$ be the subfamily of balls in $\tilde{\mathscr{B}}^1$ that do not intersect any δ_k-ball B with $\mu(B) \geqslant \delta_k^{\alpha-\varepsilon}$. By (5.8) and (5.9) we have, for k satisfying $(1 - C\delta_k^s) \geqslant 1/2$,

$$\begin{aligned} \# \mathscr{B}_{\delta_k}(\mu) &\geqslant \# \tilde{\mathscr{B}}^2 \geqslant \# \tilde{\mathscr{B}}^1 - \delta_k^s \; \# \mathscr{B}_{\delta_k}(\mu) \geqslant \# \tilde{\mathscr{B}}^1 - C\delta_k^s \; \# \tilde{\mathscr{B}}^1 \geqslant \tfrac{1}{2} \# \tilde{\mathscr{B}}^1 \\ &\geqslant (2C)^{-1} \; \# \mathscr{B}_{\delta_k}(\mu). \end{aligned}$$

Also if we let $C_1 (\leqslant (\sqrt{2}\, r^2)^d)$ be the number of δ_k-balls required to cover an $r^2\delta_k$-ball, then

$$\mu(\tilde{B}) < C_1 \delta_k^{\alpha-\varepsilon} \qquad \text{for any} \quad \tilde{B} \in \tilde{\mathscr{B}}^2.$$

Let $\mathscr{B}^*$ be the family of all balls B^* with radius $r\delta_k$ concentric with $\tilde{B} \in \tilde{\mathscr{B}}^2$. Then by (5.6)

$$\delta_k^{\alpha+\varepsilon} \leqslant \mu(B) \leqslant \mu_{r\delta_k}(B^*) \leqslant \mu(\tilde{B}) < C_1 \delta_k^{\alpha-\varepsilon}$$

and

$$(2C)^{-1} \, \delta_k^{-\tau^*(\alpha)+\eta/2} \leqslant \# \mathscr{B}^* \leqslant \delta_k^{-\tau^*(\alpha)-\eta/2}.$$

Now for k large, by absorbing the constants $C_1, (2C)^{-1}$ into the exponents, we can let $\mathscr{B}_{r\delta_k}(\mu_{r\delta_k}; \alpha \pm 2\varepsilon) = \mathscr{B}^*$. Then

$$\# \mathscr{B}_{r\delta_k}(\mu_{r\delta_k}; \alpha \pm 2\varepsilon) \approx \delta_k^{-\tau^*(\alpha) \pm \eta}$$

and

$$\sum \{\mu_{r\delta_k}(B^*)^q \colon B^* \in \mathscr{B}_{r\delta_k}(\mu_{r\delta_k}; \alpha \pm 2\varepsilon)\}$$
$$\approx \delta_k^{-\tau^*(\alpha) \pm \eta}(r\delta_k)^{q(\alpha \pm 3\varepsilon)} \approx \delta_k^{\tau(q) \pm (\eta + 2q\varepsilon)}.$$

The first assertion of the lemma follows by obvious adjustments of η and $r\delta_k$. The second part is a trivial consequence of this and (5.8).

6. THE WEAK SEPARATION PROPERTY

For $i = 1, ..., N$, let $S_i \colon \mathbb{R}^d \to \mathbb{R}^d$ be the contractive similitudes such that

$$S_i(x) = \rho_i R_i x + b_i,$$

where $0 < \rho_i < 1$, $b_i \in \mathbb{R}^d$ and R_i are orthogonal transformations. Let $\{p_1, ..., p_N\}$ be probability weights and μ be the *self-similar* measure satisfying

$$\mu = \sum_i p_i \mu \circ S_i^{-1}. \tag{6.1}$$

Let $\Omega = \{1, ..., N\}^{\mathbb{N}}$, P be the product measure on Ω defined by $\{p_i\}_{i=1}^N$ and $X_n \colon \Omega \to \{1, ..., N\}$ be the projection of Ω onto the n-th coordinate. For any Borel probability measure ν, suppose we let $T(\nu)$ be the probability measure defined by

$$T(\nu) = \sum_{i=1}^N p_i \nu \circ S_i^{-1}.$$

Then for *any* $x_0 \in \mathbb{R}^d$, the iterates $T^n(\delta_{x_0})$ of the point mass measure δ_{x_0} converge weakly to μ ([Hut]). Without loss of generality, we fix $x_0 = 0$. Let $Y_n = S_{X_n} \circ \cdots \circ S_{X_1}(0)$, then

$$Y_1 = b_{X_1}, \qquad Y_2 = \rho_{X_2} R_{X_2}(b_{X_1}) + b_{X_2}$$

and inductively,

$$\begin{aligned} Y_n &= \rho_{X_n} \cdots \rho_{X_2} R_{X_n} \circ \cdots \circ R_{X_2}(b_{X_1}) + \cdots + \rho_{X_n} R_{X_n}(b_{X_{n-1}}) + b_{X_n} \\ &= \sum_{j=1}^n \rho_{X_n} \cdots \rho_{X_{j+1}} R_{X_n} \circ \cdots \circ R_{X_{j+1}}(b_{X_j}). \end{aligned}$$

On the other hand if we let $Z_n = S_{X_1} \circ \cdots \circ S_{X_n}(0)$, then Z_n has the same distribution as Y_n since $\{X_n\}$ are i.i.d. random variables. In this case

$$\begin{aligned} Z_n &= \rho_{X_1} \cdots \rho_{X_{n-1}} R_{X_1} \circ \cdots \circ R_{X_{n-1}}(b_{X_n}) + \cdots + \rho_{X_1} R_{X_1}(b_{X_2}) + b_{X_1} \\ &= \sum_{j=1}^{n} \rho_{X_1} \cdots \rho_{X_{j-1}} R_{X_1} \circ \cdots \circ R_{X_{j-1}}(b_{X_j}) \end{aligned}$$

instead. The basic difference between the sequences $\{Y_n\}$ and $\{Z_n\}$ is that $\{Y_n\}$ does not converge pointwise (actually it is chaotic since each iteration adds in a term b_{X_n}), but $\{Z_n\}$ does since

$$|Z_{n+1} - Z_n| \leqslant |\rho_{X_1} \cdots \rho_{X_n} R_{X_1} \circ \cdots \circ R_{X_n}(b_{X_{n+1}})| \leqslant (\max_i \rho_i)^n (\max_i |b_i|).$$

The trade off is that $\{Y_n\}$ is a Markov process but $\{Z_n\}$ is not unless $\rho_i = \rho$ and $R_i = I$.

Throughout the rest of this paper we will use

$$\rho = \min\{\rho_i : i = 1, ..., N\}, \qquad r_0 = (\max_i |b_i|) \sum_{j=1}^{\infty} (\max_i \rho_i)^j.$$

For $J = (j_1, ..., j_i) \in \{1, ..., N\}^i$, we let $|J| = i$ and

$$X_J = (X_{j_1}, ..., X_{j_i}), \qquad S_J = S_{j_1} \circ \cdots \circ S_{j_i}, \qquad \rho_J = \rho_{j_1} \cdots \rho_{j_i}.$$

Also we use $\mathbf{J}_n = (J_1, ..., J_n)$ to denote the multi-index of the J_i's. We define, for $k \in \mathbb{N}$, the stopping time $t_k : \Omega \to \mathbb{N}$ by assigning each $\omega = (j_1, ..., j_i, ...) \in \Omega$ the value

$$t_k(\omega) = \min\{i : \rho_{(j_1, ..., j_i)} \leqslant \rho^k\}$$

and let

$$\Lambda_k = \{J = (j_1, ..., j_{t_k(\omega)}) : \omega = (j_1, ..., j_i, ...) \in \Omega\}.$$

Note that for each k, $\Lambda_k \neq \emptyset$ and if $J \in \Lambda_k$ then $\rho^{k+1} < \rho_J \leqslant \rho^k$. Let Z denote the random variable defined as the pointwise limit of Z_n. Also, we let

$$\Xi = Z(\Omega), \qquad \Xi_k = Z_{t_k}(\Omega)$$

and let μ, μ_{t_k} be the distribution measures on Ξ and Ξ_k respectively. Then $\{Z_{t_k}\}$ converges to Z everywhere which implies that $\{\mu_{t_k}\}$ converges to μ in distribution and μ satisfies (6.1).

PROPOSITION 6.1. *If $z \in \mathbb{R}^d$ and $c \geqslant 1 + r_0$, then for any $k \in \mathbb{N}$,*

$$\mu(B_{\rho^k}(z)) \leqslant \mu_{t_k}(B_{c\rho^k}(z)) \leqslant \mu(B_{2c\rho^k}(z)). \tag{6.2}$$

Consequently the family $\{\mu_{t_k}\}_{k=1}^{\infty}$ satisfies the condition in Lemma 5.3.

Proof. For $z \in \mathbb{R}^d$, let $E = \{|Z_{t_k} - z| \leqslant c\rho^k\}$. Then $\mu_{t_k}(B_{c\rho^k}(z)) = P(E)$ and (6.2) is a direct consequence of the following claim:

$$Z^{-1}(B_{\rho^k}(z)) \subseteq E \subseteq Z^{-1}(B_{2c\rho^k}(z)).$$

To prove the first inclusion we let $Z(\omega') \in B_{\rho^k}(z)$. Then

$$\begin{aligned} |Z_{t_k(\omega')}(\omega') - z| &\leqslant |Z_{t_k(\omega')}(\omega') - Z(\omega')| + |Z(\omega') - z| \\ &\leqslant \rho^k r_0 + \rho^k \leqslant c\rho^k. \end{aligned}$$

For the second inclusion, observe that if $\omega' \in E$ and $Z(\omega') = z'$, then

$$\begin{aligned} |z' - z| &\leqslant |Z(\omega') - Z_{t_k(\omega')}(\omega')| + |Z_{t_k(\omega')}(\omega') - z| \\ &\leqslant \rho^k r_0 + c\rho^k \leqslant 2c\rho^k. \end{aligned}$$

In regard to the last assertion, we can adjust (6.2) to the form in (5.6) as follows: Let $r' = 1 + r_0$, $r = 2r'/\rho$. For $\delta > 0$ let $k > 0$ be such that $r'\rho^k < \delta \leqslant r'\rho^{k-1}$ and let $\mu_\delta = \mu_{t_k}$. Then for all $z \in \mathbb{R}^d$,

$$\begin{aligned} \mu(B_{r^{-1}\delta}(z)) &\leqslant \mu_{t_k}(B_{r'\rho^k}(z)) \leqslant \mu_\delta(B_\delta(z)) \\ &\leqslant \mu_{t_k}(B_{r'\rho^{k-1}}(z)) \leqslant \mu(B_{2r'\rho^{k-1}}(z)) \leqslant \mu(B_{r\delta}(z)), \end{aligned}$$

and the family $\{\mu_{t_k}\}$ satisfies the condition in Lemma 5.3.

DEFINITION 6.2. A family of similitudes $\{S_i\}_{i=1}^N$ is said to have the *weak separation property* (WSP) if there exist $z_0 \in \mathbb{R}^d$ and $\ell \in \mathbb{N}$ such that for any $z = S_I(z_0)$, every closed ρ^k-ball contains at most ℓ distinct $S_J(z)$, $J \in \Lambda_k$. ($S_J(z)$ can be repeated, i.e., we allow $S_J(z) = S_{J'}(z)$ for $J, J' \in \Lambda_k$, $J \neq J'$.)

Remark 1. For any starting point z_0, the resulting invariant measure is the same. For convenience we will take $z_0 = 0$ all the time. It is easy to see that $\{S_i\}_{i=1}^N$ will have the WSP if there exists $b > 0$ such that for any $J_1, J_2 \in \Lambda_k$, $k \in \mathbb{N}$, and for any $z = S_I(0)$, either

$$S_{J_1}(z) = S_{J_2}(z) \qquad \text{or} \quad |S_{J_1}(z) - S_{J_2}(z)| \geqslant b\rho^k.$$

Our examples in the following will satisfy this slightly stronger condition, which is essentially the "separation property" we mean. The term "weak" is because it is weaker than the separation by the well known *open set condition* (See Example 1).

Remark 2. Note that

$$S_{J_i}(z) = \rho_{J_i} R_{J_i} z + S_{J_i}(0), \qquad i = 1, ..., N.$$

Hence if $\{\ln \rho_i\}_{i=1}^N$ generates a lattice and $\{R_i\}_{i=1}^N$ generates a finite group, then we have the following equivalent definition: there exists ℓ' such that every ρ^k-ball contains at most ℓ' distinct $S_J(0)$, $J \in \Lambda_k$.

Remark 3. For any $r > 1$ we can find $[2r]^d$ 1-balls to cover an r-ball. Hence under the weak separation assumption, every $(r\rho^k)$-ball with $r > 1$ contains at most $\ell[2r]^d$ distinct $S_J(z)$, $J \in \Lambda_k$. (Here $[\,\cdot\,]$ is the greatest integer function.)

Remark 4. Suppose $\{S_i\}_{i=1}^N$ has the WSP. Let $a \in \mathbb{N}$ be fixed and for $J = (j_1, ..., j_i) \in \Lambda_k$, let $\tilde{J} = (j_1, ..., j_{i \pm m})$ for some $0 \leqslant m \leqslant a$. Then there exists ℓ' such that for any $z = S_I(0)$, every ρ^k-ball contains at most ℓ' distinct $S_{\tilde{J}}(z)$. Indeed for $\tilde{J} = (j_1, ..., j_{i+m})$ with $m = 1$, consider $S_{(J,\, j)}(z) = S_J(S_j(z))$. There are N of the $S_j(z)'s$ and, combining with the definition of WSP, we can conclude that there are at most $N\ell$ distinct $S_{(J,\, j)}(z)$ in any ρ^k-ball. The general case $0 \leqslant m \leqslant a$ follows from the same argument. On the other hand for $\tilde{J} = (j_1, ..., j_{i-m})$ with $m \leqslant a$, we let k_1 be the largest index so that the initial segment J_1 of $\tilde{J}$ is in Λ_{k_1} and write $\tilde{J} = (J_1, J_2)$. Then $|J_2| \leqslant [\ln \rho / \ln \rho_{\max}] + 1 := n_0$. There are at most N^{n_0} distinct $S_{J_2}(z)$. By the WSP, there are at most ℓN^{n_0} distinct $S_{J_1}(S_{J_2}(z))$ inside any ρ^{k_1}-ball and the assertion follows.

There are three major classes of examples possessing the WSP.

EXAMPLE 1. Suppose $\{S_i\}_{i=1}^N$ satisfies the open set condition [Hut]. Then it has the WSP.

Proof. Let U be an open set guaranteed by the open set condition and choose any $z_0 \in U$. Then $B_r(z_0) \subseteq U$ for some $r > 0$. Let $J_1, J_2 \in \Lambda_k$ with $J_1 \neq J_2$ and let

$$J_1 = (i_1, ..., i_n), \qquad J_2 = (j_1, ..., j_m).$$

Let l be the first integer such that $i_l \neq j_l$ and let

$$J_1' = (i_l, ..., i_n), \qquad J_2' = (j_l, ..., j_m).$$

The open set condition implies that $S_{J_1'}(U) \cap S_{J_2'}(U) = \varnothing$ so that $S_{J_1}(U) \cap S_{J_2}(U) = \varnothing$. Since

$$\rho^{k+1} \operatorname{diam} U \leqslant \operatorname{diam} S_{J_i}(U) \leqslant \rho^k \operatorname{diam} U, \qquad i = 1, 2,$$

we have $|S_{J_1}(z_0) - S_{J_2}(z_0)| \geqslant (2\rho r)\rho^k$. The same holds for $z = S_I(z_0)$ and Remark 1 implies that $\{S_i\}_{i=1}^N$ has the WSP.

EXAMPLE 2. For $0 < \rho < 1$, we define

$$S_1(x) = \rho x, \qquad S_2(x) = \rho x + (1 - \rho).$$

It is easy to see that if $\frac{1}{2} < \rho < 1$, then $\{S_1, S_2\}$ does not satisfy the open set condition. However it has the WSP if ρ^{-1} is a P.V. number. Recall that an algebraic integer $\alpha > 1$ is a P.V. number if all its algebraic conjugates have moduli less than 1.

Let $\alpha > 1$ be an algebraic integer and let $\alpha_1, \alpha_2, ..., \alpha_s$ denote the conjugates of α and σ denote the number of i such that $|\alpha_i| = 1$. It is proved in ([G, Lemma 1.51]) that for $P(x)$ a polynomial of degree at most n, with integer coefficients and height $M := \max\{|a_i|: a_i \text{ is a coefficient of } P(x)\}$, if $P(\alpha) \neq 0$, then

$$|P(\alpha)| \geqslant \frac{\prod_{|\alpha_i| \neq 1} ||\alpha_i| - 1|}{(n+1)^\sigma \left(\prod_{|\alpha_i| > 1} |\alpha_i|\right)^{n+1} M^s}.$$

It follows that if ρ^{-1} is a P.V. number, then for $J, J' \in \Lambda_k$ with $S_J(0) \neq S_{J'}(0)$,

$$|S_J(0) - S_{J'}(0)| = (1 - \rho) \left| \sum_{i=0}^{k-1} \rho^i X_i - \sum_{i=0}^{k-1} \rho^i X'_i \right|$$

$$= (1 - \rho)\, \rho^k\, |P(\rho^{-1})| \geqslant (1 - \rho)\, \rho^k \prod_{|\alpha_i| \neq 1} (1 - |\alpha_i|) = b\rho^k,$$

where $P(x) = \sum_{i=1}^k (X_{k-i} - X'_{k-i}) x^i$ is a polynomial of height 1. This implies that $\{S_1, S_2\}$ has the WSP.

EXAMPLE 3. In wavelet theory, a fundamental equation is the two-scale dilation equation

$$\phi(x) = \sum_{i=0}^N c_i \phi(2x - i), \qquad x \in \mathbb{R},$$

where $\sum c_i = 2$, $c_i \in \mathbb{R}$. The non-zero L^1-solution has compact support in $[0, N]$ [DL1]. Let $S_i(x) = x/2 + i/2$, $i = 0, ..., N$. Note that if we let $\mu(-\infty, x] = \int_{-\infty}^x \phi(t)\, dt$, then μ satisfies $\mu = \sum_{i=0}^N (c_i/2)\, \mu \circ S_i^{-1}$ as in (6.1). (The coefficients need not be positive here.) Again, if $N > 2$, the family

$\{S_i\}_{i=0}^N$ does not satisfy the open set condition, but for any $J_1, J_2 \in \Lambda_k$, either

$$S_{J_1}(0) = S_{J_2}(0) \qquad \text{or} \quad |S_{J_1}(0) - S_{J_2}(0)| \geqslant \frac{1}{2^k}.$$

Remark 1 implies that $\{S_i\}_{i=0}^N$ has the WSP.

For $k < k'$, $z_k \in Z_k(\Omega)$, $z_{k'} \in Z_{k'}(\Omega)$, we say that $z_{k'}$ can be reached by z_k if there exists $\omega = (i_1, ..., i_k, ..., i_{k'}, ...) \in \Omega$ such that $z_k = Z_k(\omega)$ and $z_{k'} = Z_{k'}(\omega)$.

PROPOSITION 6.3. *Suppose $\{S_i\}_{i=1}^N$ has the WSP. Then for $k < k'$ and for $z_{k'} \in Z_{t_{k'}}(\Omega)$, there exist at most $\ell_1 = \ell([2r_0]^d + 1)$ distinct $z_k \in Z_{t_k}(\Omega)$ that can reach $z_{k'}$, where $r_0 = (\max_i |b_i|) \sum_{j=1}^{\infty} (\max_i \rho_i)^j$.*

Proof. Suppose z_k can reach $z_{k'}$. Then

$$|z_k - z_{k'}| \leqslant \left| \sum_{i=k+1}^{k'} \rho_{X_1} \cdots \rho_{X_{i-1}} R_{X_1} \circ \cdots \circ R_{X_{i-1}}(b_{X_i}) \right| \leqslant r_0 \rho^k,$$

i.e., $z_k \in B_{r_0 \rho^k}(z_{k'})$. By the WSP, there are at most ℓ distinct z_k in any $(r_0 \rho^k)$-ball if $r_0 \leqslant 1$ and, by Remark 3, at most $\ell[2r_0]^d$ of such z_k if $r_0 > 1$. This implies the proposition.

For the rest of this section we assume that $\{S_i\}_{i=1}^N$ has the WSP. For each fixed k, the random variable $Z_{t_k}: \Lambda_k \to \Xi_k$ is given by $Z_{t_k}(J) = S_J(0)$. We will consider the products Ξ_k^n and $\Xi_k^{\mathbb{N}}$. Let $\iota_n: \Xi_k^n \to \Lambda_k^n$ be a selection from the inverse $(Z_{t_k}^n)^{-1}$, i.e., for $\xi = (\xi_1, \xi_2, ..., \xi_n) \in \Xi_k^n$, $\iota_n(\xi) \in (Z_{t_k}^{-1}(\xi_1), Z_{t_k}^{-1}(\xi_2), ..., Z_{t_k}^{-1}(\xi_n))$. Let Λ_{kn} be the index space for the stopping time t_{kn}. For n a positive integer (or ∞), we use $\mathbf{J}_n = (J_1, ..., J_n)$ to abbreviate the notation and define

$$\kappa: \Lambda_k^n \to \Lambda_{kn}, \qquad \kappa(\mathbf{J}_n) = \tilde{\mathbf{J}}_n$$

where $J_i = (\tilde{J}_i, J'_i)$ and $\tilde{\mathbf{J}}_i = (\tilde{J}_1, ..., \tilde{J}_i) \in \Lambda_{ki}$ for $i \leqslant n$. (i.e., we are dropping the indices J'_i at each stage i of stopping.) For $m \geqslant n$ ($m \in \mathbb{N}$ or $m = \infty$) and $\xi = (\xi_1, ..., \xi_m)$, we use the notation $(\xi \mid n) = (\xi_1, ..., \xi_n)$ to denote the restriction of $\xi \in \Xi_k^m$ to Ξ_k^n. Suppose ι_n is chosen. We impose the condition on the inductive choice of ι_{n+1} so that for $\xi = (\xi_1, ..., \xi_n, \xi_{n+1})$ and $\iota_{n+1}(\xi) = (J_1, ..., J_n, J_{n+1})$, we have $Z_{t_{km}}(\kappa(J_1, ..., J_m)) = Z_{t_{km}}(\kappa(\iota_m(\xi \mid m)))$ for all $m \leqslant n$. We also define

$$g: \Xi_k^n \to \Xi_{kn}, \qquad g(\xi) = Z_{t_{kn}}(\kappa(\iota_n(\xi))) = S_{\tilde{\mathbf{J}}_n}(0)$$

and $g: \Xi_k^{\mathbb{N}} \to \Xi$ by $g(\xi) = \lim_{n\to\infty} Z_{t_{kn}}(\kappa(\iota_n(\xi\,|\,n))) = \lim_{n\to\infty} S_{\tilde{\mathbf{J}}_n}(0)$. It follows from the definitions that the following diagram commutes:

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{\ \kappa\ } & \Lambda_{kn} \\ \uparrow{\scriptstyle \iota_n} & & \downarrow{\scriptstyle Z_{t_{kn}}} \\ \Xi_k^n & \xrightarrow{\ g\ } & \Xi_{kn} \end{array}$$

In the following we will develop some properties of g. Since the product space Ξ_k^n has simpler structure, we will treat Ξ_k^n as a "coding" space for Ξ_{kn} through g. Note that the map g is not necessarily surjective. For each ξ let

$$z_i = g(\xi\,|\,i) = S_{\tilde{\mathbf{J}}_i}(0), \qquad \tilde{\xi}_i = S_{\tilde{J}_i}(0) = S^{-1}_{\tilde{\mathbf{J}}_{i-1}}(z_i).$$

Then $\{z_i\}_{i=1}^n$ can be considered as a (non-Markov) path with "increment" $\tilde{\xi}_i$ as in Fig. 7.

Remark 1. Using the above notations we observe that for $J_i = (\tilde{J}_i, J'_i)$, then $|J'_i| \leqslant a = 2[\ln \rho/\ln \rho_{\max}] + 1$, $i = 1, ..., n$. Indeed it is trivial for $i = 1$ since $J_1 = \tilde{J}_1$. For $2 \leqslant i \leqslant n$,

$$\rho^{(i-1)k+1} \leqslant \rho_{\tilde{\mathbf{J}}_{i-1}} \leqslant \rho^{(i-1)k}$$

implies that $\rho^{ik+2} \leqslant \rho_{(\tilde{\mathbf{J}}_{i-1}, J_i)} \leqslant \rho^{ik}$ and hence

$$\rho^{ik+2}(\rho_{J'_i})^{-1} \leqslant \rho_{(\tilde{\mathbf{J}}_{i-1}, \tilde{J}_i)} \leqslant \rho^{ik}.$$

It follows that $|J'_i|$ is at most $2[\ln \rho/\ln \rho_{\max}] + 1$ in order that $\rho^{ik+1} \leqslant \rho_{\tilde{\mathbf{J}}_i} \leqslant \rho^{ik}$.

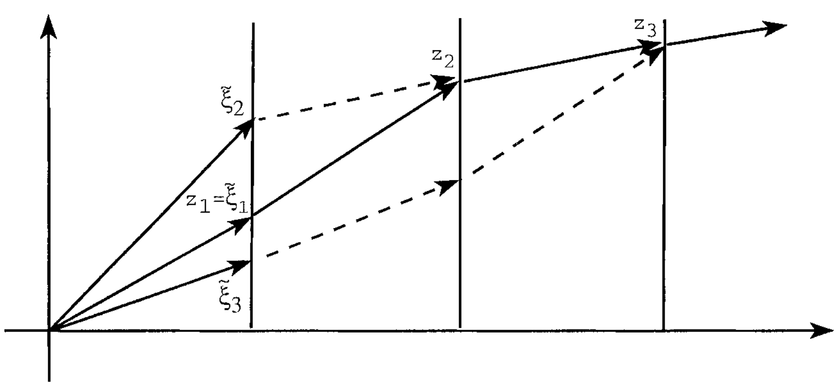

FIG. 7. $z_1 = S_{\tilde{J}_1}(0)$, $z_2 = S_{\tilde{\mathbf{J}}_2}(0) = S_{\tilde{J}_1}(\tilde{\xi}_2)$, $z_3 = S_{\tilde{\mathbf{J}}_3}(0) = S_{\tilde{\mathbf{J}}_2}(\tilde{\xi}_3)$.

Remark 2. For $J \in Z_{t_k}^{-1}(\xi)$ such that $S_J(0) = \xi$, let $J = (\tilde{J}, J')$ with $|J'| \leqslant a$ where a is as in Remark 1. Then

$$|\xi - S_{\tilde{J}}(0)| = |S_J(0) - S_{\tilde{J}}(0)| = |S_{\tilde{J}}(S_{J'}(0)) - S_{\tilde{J}}(0)| \leqslant r_0\, \rho_{\tilde{J}} \leqslant r_0\, \rho^{-a} \rho^k. \tag{6.3}$$

It follows from Remark 4 of Definition 6.2 that there are at most $\ell_2 = \ell'[2r_0\, \rho^{-a}]^d$ distinct $\tilde{\xi} = S_{\tilde{J}}(0)$; they are in $B_{r_0 \rho^{-a} \rho^k}(\xi)$. (Similarly there are at most ℓ_2 distinct $\xi = S_J(0)$ in $B_{r_0 \rho^{-a} \rho^k}(\tilde{\xi})$.)

PROPOSITION 6.4. *Suppose there exists a constant c such that for any path $(z_1, ..., z_{i-1})$, $2 \leqslant i \leqslant n$, there are no more than c non-identical $S_{\tilde{\mathbf{J}}_{i-1}}$ satisfying $\tilde{\mathbf{J}}_{i-1} \in \kappa(Z_{t_k}^{-1}(\xi_1), ..., Z_{t_k}^{-1}(\xi_{i-1}))$ and $S_{\tilde{\mathbf{J}}_{i-1}}(0) = z_{i-1}$. Then the map $g\colon \Xi_k^n \to \Xi_{kn}$ is at most ℓ_3^{n-1} to 1 where $\ell_3 = c\ell_1 \ell_2$ and ℓ_1, ℓ_2 are defined in Proposition 6.3 and Remark 2 respectively.*

Proof. Let $z = z_n \in g(\Xi_k^n) \subseteq \Xi_{kn}$, Proposition 6.3 implies that there are at most ℓ_1 of the $z_{n-1} \in \Xi_{k(n-1)}$ that can reach z. Apply this again, there are at most ℓ_1^2 of the $z_{n-2} \in \Xi_{k(n-2)}$ that can reach z. Inductively we can conclude that there are at most ℓ_1^{n-1} different paths $\{0, z_1, ..., z_{n-1}\}$ to reach z.

For each such path $\{0, z_1, ..., z_{n-1}, z\}$, let $\boldsymbol{\xi} = (\xi_1, ..., \xi_n) \in \Xi_k^n$ such that $g(\boldsymbol{\xi} \,|\, i) = z_i$ for all $1 \leqslant i \leqslant n$. Let $\iota(\boldsymbol{\xi}) = \mathbf{I}_n$ and let $\kappa(\iota(\boldsymbol{\xi})) = \tilde{\mathbf{I}}_n$ as defined above. Then

$$z_1 = S_{\tilde{\mathbf{I}}_1}(0), \qquad z_2 = S_{\tilde{\mathbf{I}}_2}(0), ..., z = S_{\tilde{\mathbf{I}}_n}(0).$$

Let $S_{\tilde{\mathbf{I}}_1}(0) = \tilde{\xi}_1$ and for $2 \leqslant i \leqslant n$ let

$$S_{\tilde{\mathbf{I}}_i}(0) = S_{\tilde{\mathbf{I}}_{i-1}}^{-1}(z_i) = \tilde{\xi}_i.$$

The hypothesis and an induction argument imply that the number of possible $(\tilde{\xi}_1, ..., \tilde{\xi}_n)$ is no more than c^{n-1}.

Remark 2 above implies that for each $\tilde{\xi}_i$, there are at most ℓ_2 different ξ_i, $2 \leqslant i \leqslant n$, so that there are $(c\ell_2)^{n-1}$ different $(\xi_1, ..., \xi_n)$ for each path $\{0, z_1, ..., z_{n-1}\}$ to reach z. The proposition now follows by combining the two conclusions in the above paragraphs.

The hypothesis of Proposition 6.4 is clearly satisfied if $\{\ln \rho_i\}_{i=1}^N$ generates a lattice and $\{R_i\}_{i=1}^N$ generates a finite group. The general case is discussed below.

In order to establish a relationship between the probability of $\boldsymbol{\xi} \in \Xi_k^n$ and $z = g(\boldsymbol{\xi}) \in \Xi_{kn}$, we consider the diagram

$$\begin{array}{ccc}
(\Lambda_k^n, P) & \xrightarrow{\ \kappa\ } & (\Lambda_{kn}, P) \\
{\scriptstyle (Z_{t_k})^n} \downarrow & & \downarrow {\scriptstyle Z_{t_{kn}}} \\
(\Xi_k^n, (\mu_{t_k})^n) & \xrightarrow{\ g\ } & (\Xi_{kn}, \mu_{t_{kn}})
\end{array}$$

where P is the measure induced from (Ω, P). For the special case $\rho_i = \rho$ and $R_i = I$, κ is the identity map and it can be shown that for $z_n \in g(\Xi_k^n)$,

$$\mu_{t_{kn}}(z_n) = \sum \left\{ \prod \mu_{t_k}(\xi_i) \colon g(\xi_1, ..., \xi_n) = z_n \right\}.$$

In this case g is at most ℓ_1^{n-1} to 1 and hence the sum is over at most ℓ_1^{n-1} elements. In order to extend this identity to the general case, we need to choose a specific $\imath$ on Ξ_k^n so that for each $\boldsymbol{\xi}$, $\{g(\boldsymbol{\xi} \mid m)\}_{m=1}^n$ corresponds to a path of sufficiently large probability.

For $n = 1$, we define $\imath_1 \colon \Xi_k \to \Lambda_k$ by $\imath_1(\xi_1) = J_1 \in Z_{t_k}^{-1}(\xi_1)$. Then $g_1 = Z_{t_k} \circ \kappa \circ \imath_1 \colon \Xi_k \to \Xi_k$ is the identity. Suppose we have defined $\imath_m, g_m$ for $1 \leqslant m \leqslant n-1$ such that $g_m = Z_{t_{km}} \circ \kappa \circ \imath_m$ and for $\boldsymbol{\xi} = (\xi_1, ..., \xi_{n-1})$,

$$\imath_{n-1}(\boldsymbol{\xi}) = \mathbf{I}_{n-1} \qquad \text{and} \qquad g_m(\boldsymbol{\xi} \mid m) = S_{\tilde{\mathbf{I}}_m}(0) = z_m, \qquad 1 \leqslant m \leqslant n-1.$$

For $\boldsymbol{\xi} = (\xi_1, ..., \xi_n)$, let

$$\mathbf{E}_{n-1} = \{\mathbf{J}_{n-1} \in \Lambda_k^{n-1} \colon S_{J_m}(0) = \xi_m,\, S_{\tilde{\mathbf{J}}_m}(0) = z_m,\, 1 \leqslant m \leqslant n-1\}$$

and

$$\mathbf{F}_n = \{\mathbf{J}_n = (\mathbf{J}_{n-1}, J_n) \in \Lambda_k^n \colon \mathbf{J}_{n-1} \in \mathbf{E}_{n-1},\, S_{J_n}(0) = \xi_n\}.$$

Also let

$$\tilde{\mathbf{E}}_{n-1} = \kappa(\mathbf{E}_{n-1}) \qquad \text{and} \qquad \tilde{\mathbf{F}}_n = \kappa(\mathbf{F}_n).$$

We decompose $\tilde{\mathbf{F}}_n$ as

$$\tilde{\mathbf{F}}_n(u, v) = \{\tilde{\mathbf{J}}_n \in \tilde{\mathbf{F}}_n \colon S_{\tilde{J}_n}(0) = u,\, S_{\tilde{\mathbf{J}}_{n-1}}(u) = v\}.$$

By Remark 2, the number of such u is at most ℓ_2. Also, for each u and each $\tilde{\mathbf{J}}_{n-1} \in \tilde{\mathbf{E}}_{n-1}$,

$$|S_{\tilde{\mathbf{J}}_{n-1}}(u) - z_{n-1}| = |S_{\tilde{\mathbf{J}}_{n-1}}(u) - S_{\tilde{\mathbf{J}}_{n-1}}(0)| \leqslant r_0 \rho^{k(n-1)}.$$

By the WSP, there are at most ℓ_1 distinct v associated with each u. Altogether there are at most $\ell_1 \ell_2$ choices of (u, v). We choose the (u, v) so that v has the largest $\mu_{t_{kn}}$ probability and denote it by $(\tilde{\xi}_n, z_n)$. We define $\imath_n(\boldsymbol{\xi}) = \mathbf{I}_n$ to be any choice of $\{\mathbf{J}_n \in \mathbf{F}_n \colon \tilde{\mathbf{J}}_n \in \tilde{\mathbf{F}}_n(\tilde{\xi}_n, z_n)\}$ and define

$$g_n(\boldsymbol{\xi}) = z_n = S_{\tilde{\mathbf{I}}_n}(0).$$

This completes the inductive choice of z_n and g_n. For convenience, we use g to denote all the g_n. Note that $\xi \in \Xi_k^n$, $\{g(\xi|m)\}_{m=1}^{n-1} = \{z_m\}_{m=1}^{n-1}$ is a path to z_n and for $\xi \in \Xi_k^{\mathbb{N}}$, $\{g(\xi|m)\}_{m=1}^{n-1}$ converges. Let z be the limit and let $g(\xi) = z$.

We will use this g throughout the rest of the paper. Suppose there exist d linearly independent vectors $u_1, ..., u_d$ of the form $S_I(0)$. Fix $(\xi_1, ..., \xi_n)$ and use the above notation. The WSP implies that for each u_j, there are no more than ℓ_1 distinct $z = S_{\tilde{\mathbf{J}}_{i-1}}(u_j)$, $\tilde{\mathbf{J}}_{i-1} \in \tilde{\mathbf{E}}_{i-1}$. Altogether there are no more than $d\ell_1$ distinct $z = S_{\tilde{\mathbf{J}}_{i-1}}(u_j)$, $1 \leqslant j \leqslant d$. Since $S_{\tilde{\mathbf{J}}_{i-1}}$ is uniquely determined by $S_{\tilde{\mathbf{J}}_{i-1}}(u_j)$, $1 \leqslant j \leqslant d$, we conclude that there are at most $(d\ell_1)^d$ non-identical $S_{\tilde{\mathbf{J}}_{i-1}}$. The conclusion of Proposition 6.4 holds with $c = (d\ell_1)^d$. (If such d vectors do not exist, we can project the S_i onto the smallest subspace containing the self-similar set to begin with.) We have in addition the following proposition which is required in the proof of Lemma 7.2(ii).

PROPOSITION 6.5. *For* $z \in g(\Xi_k^n)$,

$$\sum \left\{ \prod \mu_{t_k}(\xi_i) : \xi = (\xi_1, ..., \xi_n), g(\xi) = z \right\} \leqslant \ell_4^n \mu_{t_{kn}}(z), \tag{6.4}$$

where $\ell_4 = \ell_1 \ell_2^2$.

Proof. In the definitions above, the sets $\mathbf{F}_n$, $\tilde{\mathbf{F}}_n(\tilde{\xi}_n, z_n)$ depend on ξ_n (also on $(z_1, ..., z_{n-1})$). We will denote them by $\mathbf{F}_n(\xi_n)$ and $\tilde{\mathbf{F}}_n(\xi_n; \tilde{\xi}_n, z_n)$ respectively here. For $z = z_n \in g(\Xi_k^n)$ we use $\mathscr{P}_{n-1}$ to denote the paths $\{z_1, ..., z_{n-1}\}$, $z_i \in g(\Xi_k^i)$ that can reach z_n. For each z_{n-1}, we consider $\tilde{\xi}_n$ (more precisely $\rho_{\tilde{\mathbf{J}}_{n-1}} R_{\tilde{\mathbf{J}}_{n-1}}(\tilde{\xi}_n)$) as an increment to reach z_n. Let

$$\mathbf{G}_n(\tilde{\xi}_n) = \{\xi_n : \tilde{\mathbf{F}}_n(\xi_n; \tilde{\xi}_n, z_n) \neq \varnothing\}.$$

Then $\xi_n \in \mathbf{G}_n(\tilde{\xi}_n)$ means there exist $J_n \in \Lambda_k$ and $\tilde{\mathbf{J}}_{n-1} \in \tilde{\mathbf{E}}_{n-1}$ such that

$$S_{J_n}(0) = \xi_n, \qquad S_{\tilde{J}_n}(0) = \tilde{\xi}_n, \qquad S_{\tilde{\mathbf{J}}_n}(0) = S_{\tilde{\mathbf{J}}_{n-1}}(\tilde{\xi}_n) = z_n.$$

Hence

$$\begin{aligned}
\mu_{t_{kn}}(z_n) &= \sum \{P(\tilde{\mathbf{J}}_n) : \tilde{\mathbf{J}}_n \in \Lambda_{t_{kn}} - S_{\tilde{\mathbf{J}}_n}(0) = z_n\} \\
&\geqslant \sum_{\mathscr{P}_{n-1}, \tilde{\xi}_n} \sum \{P(\tilde{\mathbf{J}}_{n-1}) \, P(\tilde{J}_n) : \tilde{\mathbf{J}}_n \in \tilde{\mathbf{F}}_n(\xi_n; \tilde{\xi}_n, z_n), \xi_n \in \mathbf{G}_n(\tilde{\xi}_n)\} \\
&\geqslant (\ell_1 \ell_2)^{-1} \sum_{\mathscr{P}_{n-1}, \tilde{\xi}_n} \sum_{\xi_n \in \mathbf{G}_n(\tilde{\xi}_n)} \sum \{P(\tilde{\mathbf{J}}_{n-1}) \, P(J_n) : \\
&\qquad \tilde{\mathbf{J}}_{n-1} \in \tilde{\mathbf{E}}_{n-1}, S_{J_n}(0) = \xi_n\} \\
&= (\ell_1 \ell_2)^{-1} \sum_{\mathscr{P}_{n-1}, \tilde{\xi}_n} \left(\sum \{P(\tilde{\mathbf{J}}_{n-1}) : \tilde{\mathbf{J}}_{n-1} \in \tilde{\mathbf{E}}_{n-1}\} \right) \\
&\qquad \times \left(\sum \{\mu_{t_k}(\xi_n) : \xi_n \in \mathbf{G}_n(\tilde{\xi}_n)\} \right)
\end{aligned}$$

$$\geqslant (\ell_1 \ell_2)^{-n} \sum_{(\tilde{\xi}_1, ..., \tilde{\xi}_n)} \Big\{ \prod \mu_{t_k}(\xi_i): $$

$$\boldsymbol{\xi} = (\xi_1, ..., \xi_n),\, S_{I_i}(0) = \xi_i,\, S_{\tilde{I}_i}(0) = \tilde{\xi}_i,\, 1 \leqslant i \leqslant n \Big\}$$

$$\geqslant (\ell_1 \ell_2)^{-n} \ell_2^{-n} \sum_{\boldsymbol{\xi}} \Big\{ \prod \mu_{t_k}(\xi_i): \boldsymbol{\xi} = (\xi_1, ..., \xi_n),\, g(\boldsymbol{\xi}) = z_n \Big\}.$$

(The first inequality is because the sum is replaced by one which sums over a smaller family; the second inequality is by the choice of $(\tilde{\xi}_n, z_n)$ in the construction; the third inequality is by induction; the extra factor ℓ_2^n arises in the last inequality because each $(\tilde{\xi}_1, ..., \tilde{\xi}_n)$ can come from ℓ_2^n different $(\xi_1, ..., \xi_n)$.)

Our main theorem is the following (the reverse inequality in Theorem 4.1(i).)

THEOREM 6.6. *Suppose $\{S_i\}_{i=1}^N$ has the WSP. Let μ be the self-similar measure corresponding to the weights $\{p_1, ..., p_N\}$. Suppose τ^* is strictly concave at $\alpha \in (\alpha_{\min}, \alpha_0^+)$. Then*

$$\dim_{\mathscr{H}} \left\{ z\colon \lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(z))}{\ln \delta} = \alpha \right\} = \tau^*(\alpha).$$

We will make use of the mass distribution principle ([F2], p. 61):

Let $K \subseteq \mathbb{R}^d$ be a Borel subset and let ν be a positive measure on $\mathbb{R}^d$. Suppose there exists $0 < c < \infty$ such that $\overline{\lim}_{\delta \to 0^+} \nu(B_\delta(x))/\delta^s < c$ for all $x \in K$ and $\nu(K) > 0$. Then $\dim_{\mathscr{H}}(K) \geqslant s$.

Since the proof of Theorem 6.6 is rather long and technical, we will explain the main idea here and leave the details in the next two sections. We use Proposition 6.1 to select a large k and a subset $E \subseteq \Xi_k$ such that

$$\#E \approx \rho^{k(-\tau^*(\alpha) \pm \eta)} \qquad \text{and} \qquad \mu_{t_k}(\xi) \approx \rho^{k(\alpha \pm \varepsilon)}, \qquad \xi \in E.$$

Note that all the ξ's in E have "almost" equal probabilities. We define the uniform probability measure on E by assigning the probability $(\#E)^{-1}$ to each $\xi \in E$ and let Q be the product measure on $\Xi_k^{\mathbb{N}}$. Then Q is concentrated on $E^{\mathbb{N}} \subseteq \Xi_k^{\mathbb{N}}$.

We will use $\Xi_k^{\mathbb{N}}$ as the "coding" space through the map $g\colon \Xi_k^n \to \Xi_{kn}$ and $\Xi_k^{\mathbb{N}} \to \Xi$. Let ν_n and ν be the induced measures of Q on Ξ_{kn} and Ξ respectively. We can obtain a good control of ν and ν_n from Lemma 7.2 and

Lemma 7.4, namely, there exists a subset $H \in E^{\mathbb{N}}$ such that for $K = g(H)$, $K_n = g(H \mid n)$ and for $z_n \in K_n$,

$$\nu(K) \geqslant \tfrac{1}{2} \qquad \text{and} \qquad \nu_n(z_n) \approx \rho^{kn(\tau^*(\alpha) \pm \eta')}.$$

Hence the scaling exponent of ν at $z \in K_n$ is of order $\tau^*(\alpha) \pm \eta'$. Furthermore, if we let

$$K_\varepsilon(\alpha) = \left\{ z \in \Xi \colon \alpha - \varepsilon < \varliminf_{\delta \to 0^+} \frac{\ln \mu(B_\delta(z))}{\ln \delta} \right.$$
$$\left. \leqslant \varlimsup_{\delta \to 0^+} \frac{\ln \mu(B_\delta(z))}{\ln \delta} < \alpha + \varepsilon \right\}, \tag{6.5}$$

we can readjust the k so that $K_\varepsilon(\alpha) \supseteq K$. The mass distribution principle will imply that the Hausdorff dimension of the set in (6.5) is greater than or equal to $\tau^*(\alpha) - \eta'$ so that $\lim_{\varepsilon \to 0^+} \dim_{\mathscr{H}} K_\varepsilon(\alpha) = \tau^*(\alpha)$. (Theorem 7.5).

There are two major technicalities in the proof of the theorem.

(i) In order to construct the set K with the above property, we need to choose E to be those $z \in \Xi$ such that the $\mu_{t_{kn}}$-probability of each z_n is mainly determined by the probability of the paths $\{0, z_1, ..., z_{n-1}, z_n\}$ with $z_i \in g(E_k^i)$. This construction depends on Proposition 6.5 and is done in Section 7.

(ii) Note that $\bigcap_{\varepsilon > 0} K_\varepsilon(\alpha) = \{z \colon \lim_{\delta \to 0^+} \ln \mu(B_\delta(z)) / \ln \delta = \alpha\}$. However, this does not imply that $\lim_{\varepsilon \to 0} \dim_{\mathscr{H}}(\bigcap_{\varepsilon > 0} K_\varepsilon(\alpha)) = \dim_{\mathscr{H}}(K(\alpha))$. In order to replace the set (6.5) by $\{z \in g(E^{\mathbb{N}}) \colon \lim_{\delta \to 0^+} \ln \mu(B_\delta(z)) / \ln \delta = \alpha\}$, we have to replace ε by a sequence $\{\varepsilon_k\} \searrow 0$ and the fixed k, Ξ_k^n etc. have to be adjusted accordingly. This is proved in Section 8.

7. SOME AUXILIARY RESULTS

In the rest of the paper we assume that $\{S_i\}_{i=1}^N$ has the WSP and that τ^* is strictly concave at $\alpha \in (\alpha_{\min}, \alpha_0^+)$ without explicitly mentioning.

For fixed ε and α we let

$$E_k = E_k(\varepsilon) := \{\xi \in \Xi_k \colon \rho^{k(\alpha + \varepsilon)} \leqslant \mu_{t_k}(\xi) < \rho^{k(\alpha - \varepsilon)}\},$$

$$F_k = F_k(\varepsilon) := \{\xi \in \Xi_k \colon \rho^{k(\alpha - \varepsilon)} \leqslant \mu_{t_k}(\xi)\}.$$

It follows from Lemma 5.3 and Proposition 6.1 that for $\varepsilon > 0$, we can choose $k \in \mathbb{N}$ arbitrarily large and $\eta' > \eta > 0$ such that

$$\# E_k \approx \rho^{k(-\tau^*(\alpha) \pm \eta)} \qquad \text{and} \qquad \# F_m \leqslant \rho^{m(-\tau^*(\alpha) + \eta')} \qquad \text{for all} \quad m \geqslant k. \tag{7.1}$$

Intuitively the set F_k contains those points with large probability; they are relatively few in view of (7.1). The complement of $E_k \cup F_k$ contains all $\xi \in \Xi_k$ that have small probability; i.e., $\mu_{t_k}(\xi) < \rho^{k(\alpha+\varepsilon)}$, they do not contribute much in our estimation. Hence the major part in Ξ_k comes from E_k and the following lemmas are aiming at exploiting this basic observation.

Lemma 7.1. *Let $c > 0$. Then for $\varepsilon > 0$, we can choose $\eta, \eta' > 0$, k, E_k and F_m, $m \geqslant k$ to satisfy* (7.1), *and furthermore for any n, the number of elements in the set*

$$D = \{\xi \in E_k^n : \operatorname{dist}(g(\xi \mid j), F_{kj}) < c\rho^{kj} \text{ for some } 1 \leqslant j \leqslant n\}$$

is less than $\frac{1}{2}(\# E_k^n)$.

Proof. For any finite subset F of $\mathbb{R}^d$ and for any $1 \leqslant j \leqslant n$, let

$$D_j = \{\xi \in E_k^n : \operatorname{dist}(g(\xi \mid j), F) < c\rho^{kj}\}.$$

We claim that

$$\# D_j \leqslant C\ell_3{}^j (\# F)(\# E_k^{n-j}), \tag{7.2}$$

where $C = \ell([2c]^d + 1)$ and ℓ_3 is defined as in Proposition 6.4. Indeed for each $y \in F$, the WSP implies that there exist at most C of the $g(\xi \mid j)$ in the $c\rho^{kj}$-neighborhood of y. Altogether there are at most $C(\# F)$ of such $g(\xi \mid j)$. By Proposition 6.4, there are at most $C\ell_3^j(\# F)$ of $(\xi_1, ..., \xi_j) \in E_k^j$ giving rise to these $g(\xi \mid j)$. Furthermore, for each such $(\xi_1, ..., \xi_j)$, there are at most $\# E_k^{n-j}$ of $(\xi_{j+1}, ..., \xi_n)$ such that $(\xi_1, ..., \xi_j, \xi_{j+1}, ..., \xi_n) \in E_k^n$. The estimation of $\# D_j$ in (7.2) is hence as described.

Now let $\eta, \eta' > 0$, k, E_k, F_m be as in (7.1). We assume that k is large enough so that

$$C \sum_{j=1}^{\infty} (\ell_3 \rho^{k(\eta' - \eta)})^j < \tfrac{1}{2}. \tag{7.3}$$

Applying (7.2) with $F=F_{kj}$ and (7.3), we have

$$\# D \leqslant \sum_{j=1}^{n} \# D_j \leqslant C \sum_{j=1}^{n} \ell_3^j(\# F_{kj})(\# E_k^{n-j})$$
$$\leqslant C\left(\sum_{j=1}^{n} (\ell_3 \rho^{k(\eta'-\eta)j}\right)(\# E_k^n) < \tfrac{1}{2}(\# E_k^n).$$

LEMMA 7.2. *Let $c>0$ and let*

$$H_n = H_n(\varepsilon) := \{\xi \in E_k^n \colon \operatorname{dist}(g(\xi\,|\,j), F_{kj}) \geqslant c\rho^{kj} \text{ for all } 1 \leqslant j \leqslant n\}.$$

Then for any $\varepsilon>0$, there exist $\eta>0$ and k (depending on ε, η) such that for all $n \in \mathbb{N}$,

(i) $\# H_n \geqslant \frac{1}{2} \# E_k^n$;

(ii) $\mu_{t_{kn}}(z_n) \approx \rho^{kn(\alpha \pm 2\varepsilon)}$, $\mu_{t_{kn}}(B_{c\rho^{kn}}(z_n)) \approx \rho^{kn(\alpha \pm 2\varepsilon)}$ *for all $z_n \in g(H_n)$. In particular,* $g(H_n) \subseteq E_{kn}(2\varepsilon)$.

Proof. (i) follows directly from Lemma 7.1. To prove (ii) we let $z_n = g(\xi)$ with $\xi \in H_n$. That $\operatorname{dist}(g(\xi\,|\,j), F_{kj}) \geqslant c\rho^{kj}$ and that $B_{c\rho^{kn}}(z_n)$ contains at most $C=\ell([2c]^d+1)$ elements in $\Xi_{t_{kn}}$ imply

$$\mu_{t_{kn}}(B_{c\rho^{kn}}(z_n)) \leqslant C\rho^{kn(\alpha-\varepsilon)}.$$

Also Proposition 6.5 implies that

$$\ell_4^{-n} \rho^{kn(\alpha+\varepsilon)} \leqslant \mu_{t_{kn}}(z_n).$$

The statements in (ii) follow directly from these observations provided that we choose $k > \ln(C\ell_4)/(-\varepsilon \ln \rho)$ to begin with.

By using Lemma 7.2 we can strengthen the first counting in (7.1) which will be needed in the next section.

THEOREM 7.3. *For $\varepsilon>0$, there exist $\eta>0$ with $\lim_{\varepsilon \to 0^+} \eta = 0$ and k (depending on ε and η) such that for any $m \geqslant k$,*

$$\# E_m(\varepsilon) \approx \rho^{m(-\tau^*(\alpha) \pm \eta)}. \tag{7.4}$$

Proof. Lemma 4.2 implies that there exists $\eta>0$ such that $\# E_m(\varepsilon) \leqslant \rho^{m(-\tau^*(\alpha)-\eta)}$ for m large enough. We need only prove the opposite inequality. We first claim that there exist $\eta > 0$ and k_1 such that for all $j \in \mathbb{N}$,

$$\rho^{k_1 j(-\tau^*(\alpha)+\eta/2)} \leqslant \# E_{k_1 j}(\varepsilon/2). \tag{7.5}$$

Indeed by applying Lemma 7.2 to $\varepsilon/4$, we can find $\eta > 0$ and k_1 so that for all $j \in \mathbb{N}$, $g(H_j(\varepsilon/4)) \subseteq E_{k_1 j}(\varepsilon/2)$ and using Proposition 6.4,

$$\rho^{k_1 j(-\tau^*(\alpha)+\eta/2)} \leqslant \tfrac{1}{2}\ell_3^{-j} \rho^{k_1 j(-\tau^*(\alpha)+\eta/4)} \leqslant \ell_3^{-j} \# H_j(\varepsilon/4) \leqslant \# E_{k_1 j}(\varepsilon/2).$$

Now let $m \in \mathbb{N}$. Then there exists j such that $k_1 j \leqslant m < k_1(j+1)$. For each $z \in E_{k_1 j}(\varepsilon/2)$, the WSP implies that there exists at least one $z_m \in \Xi_m$ that can be reached by z such that

$$\mu_{t_m}(z_m) \geqslant \ell_1^{-1} \mu_{t_{k_1 j}}(z)(\min_i p_i)^{k_1(\ln \rho/\ln \rho_{\max})} \geqslant \rho^{k_1 j(\alpha+\varepsilon)},$$

provided that j is large enough, and hence $z_m \in E_m(\varepsilon) \cup F_m(\varepsilon)$. Proposition 6.3 implies that each such z_m can be reached by no more than ℓ_1 elements from $E_{k_1 j}(\varepsilon/2)$. It follows that

$$\#(E_m(\varepsilon) \cup F_m(\varepsilon)) \geqslant \ell_1^{-1} \# E_{k_1 j}(\varepsilon/2) \geqslant \ell_1^{-1} \rho^{k_1 j(-\tau^*(\alpha)+\eta/2)}. \tag{7.6}$$

In view of the estimation of $\# F_m(\varepsilon)$ in (7.1) we have

$$\begin{aligned} \# E_m(\varepsilon) &\geqslant \ell_1^{-1} \rho^{k_1 j(-\tau^*(\alpha)+\eta/2)} - \rho^{m(-\tau^*(\alpha)+\eta')} \\ &= \rho^{m(-\tau^*(\alpha)+\eta/2)}(\ell_1^{-1} \rho^{-(m-k_1 j)(-\tau^*(\alpha)+\eta/2)} - \rho^{m(\eta'-\eta/2)}). \end{aligned}$$

The first term in the parentheses is not less than $\ell_1^{-1} \rho^{-k_1(-\tau^*(\alpha)+\eta/2)}$ and the second term tends to 0 as $m \to \infty$. Hence if we let $k = k_1 j_0$ for some large j_0, then for $m \geqslant k$,

$$\# E_m(\varepsilon) \geqslant \rho^{m(-\tau^*(\alpha)+\eta)}.$$

For $\varepsilon > 0$ we choose k as in Lemma 7.2 and define a probability measure on Ξ_k by assigning the probability $1/\# E_k$ to each element in E_k. Let Q and Q_n be the product measures on $\Xi_k^{\mathbb{N}}$ and Ξ_k^n respectively. Then Q and Q_n are supported by $E_k^{\mathbb{N}}$ and E_k^n. Let v and v_n be the induced measures on Ξ and Ξ_{kn} respectively. Then v_n converges to v weakly.

LEMMA 7.4. *Let $c > 0$ be as in Lemma 7.2. Then for $\varepsilon > 0$, we can choose η, k, E_k such that for any $n \in \mathbb{N}$,*

(i) $\nu(B_{\rho^{kn}}(z)) \approx \rho^{kn(\tau^*(\alpha) \pm 2\eta)}$ *for all* $z \in g(E_k^{\mathbb{N}})$;

(ii) *there exists a subset* $K \subseteq g(E_k^{\mathbb{N}})$ *with* $\nu(K) \geqslant 1/2$ *such that for* $z = g(\xi) \in K$, $z_n = g(\xi \mid n)$, *we have*

$$\mu_{t_{kn}}(z_n), \qquad \mu_{t_{kn}}(B_{c\rho^{kn}}(z_n)) \approx \rho^{kn(\alpha \pm 2\varepsilon)}.$$

Proof. (i) Note that $\# E_k \approx \rho^{k(-\tau^*(\alpha) \pm \eta)}$. For $\xi \in E_k^{\mathbb{N}}$, let $z_n = g(\xi \mid n)$ and let $L(z_n)$ be the cylinder set in $E_k^{\mathbb{N}}$ with initial segment $(\xi_1, ..., \xi_n)$. Then $g(L(z_n)) \subseteq B_{r_0\rho^{kn}}(z_n)$. By the WSP, there are at most ℓ_1 of the z'_n in $B_{r_0\rho^{kn}}(z_n) \cap g(E_k^n)$. Each such z'_n is the image under g_n of at most ℓ_3^{n-1} of $(\xi'_1, ..., \xi'_n) \in \Xi_k^n$ (Proposition 6.4). It follows that

$$(\# E_k)^{-n} \leqslant \nu_n(B_{r_0\rho^{kn}}(z_n)) \leqslant (\ell_1 \ell_3)^n \, (\# E_k)^{-n}.$$

In view of the above estimation of $\# E_k$, we can choose k large enough so that

$$\nu_n(B_{r_0\rho^{kn}}(z_n)) \approx \rho^{kn(\tau^*(\alpha) \pm 3\eta/2)}.$$

We can now conclude (i) by passing the inequality to ν (adjust the k again), using

$$\nu(B_{\rho^{kn}}(z)) \leqslant \nu_n(B_{r'\rho^{kn}}(z)) \leqslant \nu(B_{2r'\rho^{kn}}(z)) \qquad \text{with} \quad r' = 1 + r_0,$$

which follows from the same argument as in Proposition 6.1. (ii) Let

$$H = \{\xi \in E_k^{\mathbb{N}} : \operatorname{dist}(g(\xi \mid j), F_{kj}) \geqslant c\rho^{kj} \text{ for all } j\}.$$

By applying Lemma 7.2(i) to the complement of H, we have $\nu_n(g(H_n)) \geqslant 1/2$ for all n. Let $K = g(H)$, then $\nu(K) \geqslant 1/2$. The rest of the assertion follows from Lemma 7.2(ii).

THEOREM 7.5. *Suppose* $\{S_i\}_{i=1}^N$ *has the WSP and* τ^* *is strictly concave at* $\alpha \in (\alpha_{\min}, \alpha_0^+)$. *Let*

$$K_\varepsilon(\alpha) = \left\{x \colon \alpha - \varepsilon < \varliminf_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} \leqslant \varlimsup_{\delta \to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} < \alpha + \varepsilon\right\}.$$

Then

$$\lim_{\varepsilon \to 0^+} \dim_{\mathscr{H}} K_\varepsilon(\alpha) = \tau^*(\alpha).$$

Proof. Let $\gamma > 0$ be arbitrary. Let $\eta > 0$, k and K be chosen as in Lemma 7.4 with $\varepsilon/3$ replacing ε and $\eta < \gamma/3$. By using Lemma 7.4(ii) and

Proposition 6.1 with a suitable c, say $c=2\gamma'$, we can show that $K\subseteq K_{\varepsilon}(\alpha)$. Since $\nu(K)\geqslant 1/2$ and for $z\in K$,

$$\lim_{\delta\to 0^+} \nu(B_{\delta}(z))/\delta^{\tau^*(\alpha)-\gamma}=0,$$

the mass distribution theorem quoted in Section 6 yields $\dim_{\mathscr{H}} K_{\varepsilon}(\alpha)\geqslant \tau^*(\alpha)-\eta$ and hence

$$\lim_{\varepsilon\to 0^+} \dim_{\mathscr{H}} K_{\varepsilon}(\alpha)\geqslant \tau^*(\alpha). \tag{7.7}$$

The theorem follows by combining this with Theorem 4.1.

8. PROOF OF THE MULTIFRACTAL FORMALISM

In order to improve (7.7) to

$$\dim_{\mathscr{H}} \left\{z\colon \lim_{\delta\to 0^+} \frac{\ln \mu(B_{\delta}(z))}{\ln \delta}=\alpha\right\}\geqslant \tau^*(\alpha),$$

we need to modify Theorem 7.5 and show that for z belonging to some set similar to the K in Lemma 7.4, then $\lim_{\delta\to 0^+}(\ln \mu(B_{\delta}(z))/\ln \delta)$ exists and equals α. One obvious approach is to replace the subset $E_k^{\mathbb{N}}(\varepsilon)\subseteq \Xi$ by

$$E_{k_0}(\varepsilon_0)\times E_{k_1}(\varepsilon_1)\times \cdots \times E_{k_n}(\varepsilon_n)\times \cdots$$

where $\varepsilon_n \searrow 0$ and k_n are chosen corresponding to ε_n as before. However, one setback of this consideration is that there may be "large gaps" between k_{n-1} and k_n in the sense that the ratio

$$\frac{\rho^{k_0+\cdots+k_{n-1}}}{\rho^{k_0+\cdots+k_n}}=\rho^{-k_n}$$

is not bounded. The corresponding estimation for $\mu(B_{\delta}(z))$ then holds only for δ "close" to the sequence of values $\{\rho^{k_0+\cdots+k_n}\}_{n=0}^{\infty}$ and we cannot conclude that $\lim_{\delta\to 0^+}(\ln \mu(B_{\delta}(z))/\ln \delta)$ exists. An adjustment to this is to use blocks as follows

$$(E_{k_0}(\varepsilon_0)\times E_{k_0+1}(\varepsilon_0)\times \cdots \times E_{k_1-1}(\varepsilon_0))\times (E_{k_1}(\varepsilon_1)\times \cdots \times E_{k_2-1}(\varepsilon_1))\times \cdots \tag{8.1}$$

For $n, m\in \mathbb{N}$ with $n\leqslant m$, we let $s(n,m)=\sum_{i=n}^{m} i$. We will consider the spaces $\prod_{i=k}^{n} \Xi_i$ and $\Lambda_{t_{s(k,n)}}$ instead of Ξ_k^n and $\Lambda_{t_{kn}}$. By applying the same construction as in Section 7, we can define ι and $g\colon \prod_{i=k}^{n}(\Xi_i, \mu_{t_i})\to \Xi_{t_{s(k,n)}}$.

Propositions 6.4 and 6.5 will hold the same way. We introduce a few more notations: Let

$$\mathbf{E}_{n,m}(\varepsilon) = E_n(\varepsilon) \times E_{n+1}(\varepsilon) \times \cdots \times E_m(\varepsilon).$$

For two given sequences $\{\varepsilon_n\}_{n=0}^{\infty}$ and $\{k_n\}_{n=0}^{\infty}$, let $\boldsymbol{\varepsilon}_n = (\varepsilon_0, ..., \varepsilon_n)$ and for $m \geqslant k_n$, let

$$\mathbf{E}_{k_0,m}(\boldsymbol{\varepsilon}_n) = \mathbf{E}_{k_0,k_1-1}(\varepsilon_0) \times \cdots \times \mathbf{E}_{k_n,m}(\varepsilon_n).$$

Furthermore we fix $c > 0$ and define, for $m \geqslant k_0$,

$$\mathbf{H}_m(\boldsymbol{\varepsilon}_0) = \bigcap_{j=k_0}^{m} \{\xi \in \mathbf{E}_{k_0,m}(\varepsilon_0) \colon \operatorname{dist}(g(\xi \mid j-k_0+1), F_{s(k_0,j)}(\varepsilon_0)) \geqslant c\rho^{s(k_0,j)}\}$$

and for $m \geqslant k_n$, $n \geqslant 1$,

$$\mathbf{H}_m(\boldsymbol{\varepsilon}_n) = \bigcap_{j=k_n}^{m} \{\xi \in \mathbf{H}_{k_n-1}(\boldsymbol{\varepsilon}_{n-1}) \times \mathbf{E}_{k_n,m}(\varepsilon_n) \colon$$
$$\operatorname{dist}(g(\xi \mid j-k_0+1), F_{s(k_0,j)}(\varepsilon_n)) \geqslant c\rho^{s(k_0,j)}\}. \qquad (8.2)$$

Lastly, we let

$$\mathbf{H} = \{\xi \colon (\xi_{k_0}, ..., \xi_m) \in \mathbf{H}_m(\boldsymbol{\varepsilon}_n)\ \forall n \text{ and } \forall m \geqslant k_n\}.$$

The construction of the sequence in (8.1) is contained in the following proposition. We will repeatedly use the fact that for $\varepsilon > 0$, there exist $\eta' > \eta > 0$ and $k \in \mathbb{N}$ such that for $n \geqslant k$,

$$\# E_n(\varepsilon) \approx \rho^{n(-\tau^*(\alpha) \pm \eta/2)}, \qquad \# F_n(\varepsilon) \leqslant \rho^{n(-\tau^*(\alpha)+\eta')}. \qquad (8.3)$$

(See (7.1) and Theorem 7.3).

Proposition 8.1. *Let $c > 0$ be fixed. Let $\{\varepsilon_n\}_{n=0}^{\infty}$ be a sequence such that $0 < \varepsilon_{n+1} < \varepsilon_n/2$ for all $n \geqslant 0$. Then there exist sequences $\{k_n\}$, $\{\eta_n\}$ with $0 < \eta_{n+1} < \eta_n/2$ such that for $m \geqslant k_n$,*

(i) $\# \mathbf{E}_{k_0,m}(\boldsymbol{\varepsilon}_n) \approx \rho^{s(k_0,m)(-\tau^*(\alpha) \pm \eta_n)}$;

(ii) $\# \mathbf{H}_m(\boldsymbol{\varepsilon}_n) \geqslant (1 - \sum_{i=2}^{n} 1/2^i)\ \# \mathbf{E}_{k_0,m}(\boldsymbol{\varepsilon}_n)$.

(iii) *For $z \in g(\mathbf{H}_m(\boldsymbol{\varepsilon}_n))$,*

$$\mu_{t_{s(k_0,m)}}(z) \approx \rho^{s(k_0,m)(\alpha \pm 2\varepsilon_{n-1})}, \qquad \mu_{t_{s(k_0,m)}}(B_{c\rho^{s(k_0,m)}}(z)) \approx \rho^{s(k_0,m)(\alpha \pm 2\varepsilon_{n-1})}. \qquad (8.4)$$

Proof. For $n = 0$, the above assertions follow from the same argument as in the proofs of Lemmas 7.1, 7.2 and Theorem 7.3. For $n = 1$, we first choose $0<\eta_1<\eta_1'<\eta_0/2$ and $k_1 \in \mathbb{N}$ with respect to ε_1 such that for $m \geqslant k_0$,

$$\#\mathbf{E}_{k_0, m}(\boldsymbol{\varepsilon}_1) \approx \rho^{s(k_0, m)(-\tau^*(\alpha) \pm \eta_1)} \quad \text{and} \quad \#F_{s(k_0, m)}(\varepsilon_1) \leqslant \rho^{s(k_0, m)(-\tau^*(\alpha) + \eta_1')}.$$

(i.e., (8.3)) and hence (i) is satisfied. We then apply the same arguments of Lemmas 7.1 and 7.2 (choose a larger k_1 if necessary) to show that condition (ii) is also satisfied. It follows from the definition of $\mathbf{H}_m(\boldsymbol{\varepsilon}_1)$ and an analog of Proposition 6.5 that there exists ℓ_4 such that for $m \geqslant k_1$ and $z \in g(\mathbf{H}_m(\boldsymbol{\varepsilon}_1))$,

$$\ell_4^{-(m-k_0+1)} \rho^{s(k_0, m)\alpha + (s(k_0, k_1-1)\varepsilon_0 + s(k_1, m)\varepsilon_1)} \leqslant \mu_{t_{s(k_0, m)}}(z) \leqslant \rho^{s(k_0, m)(\alpha - \varepsilon_1)}.$$

Note that $(s(k_0, k_1-1)\varepsilon_0 + s(k_1, m)\varepsilon_1)/s(k_0, m) < \varepsilon_0$. Hence, as in the proof of Lemma 7.2, assertion (iii) holds (again we can make k_1 even larger if necessary). Similarly for $n=2$, we can choose the corresponding terms so that (i) and (ii) hold and that for $m \geqslant k_2$,

$$\ell_4^{-(m-k_0+1)} \rho^{s(k_0, m)\alpha + (s(k_0, k_1-1)\varepsilon_0 + s(k_1, k_2-1)\varepsilon_1 + s(k_2, m)\varepsilon_2)}$$
$$\leqslant \mu_{t_{s(k_0, m)}}(z) \leqslant \rho^{s(k_0, m)(\alpha - \varepsilon_2)}$$

with

$$\frac{s(k_0, k_1-1)\varepsilon_0 + s(k_1, k_2-1)\varepsilon_1 + s(k_2, m)\varepsilon_2}{s(k_0, m)} < 2\varepsilon_1.$$

The previous argument implies (iii) holds also. The proposition follows by induction.

Let $\{k_n\}$, $\{\eta_n\}$ be chosen as in Proposition 8.1. For $m \geqslant k_0$, we let $\sigma(m)$ be the unique integer such that

$$k_{\sigma(m)} \leqslant m < k_{\sigma(m)+1}.$$

We also let $E_m(\varepsilon_{\sigma(m)})$ be given the uniform distribution. Let Q be the induced product measure on the product space $\mathbf{E} := \prod_{m=k_0}^{\infty} E_m(\varepsilon_{\sigma(m)})$ and let Q_m be the m-th product measure. The induced measures ν on Ξ and ν_m on $\Xi_{t_{s(k_0, m)}}$ are given by $\nu = Q \circ g^{-1}$ and $\nu_m = Q_m \circ g^{-1}$. It follows from Proposition 8.1 that (see Lemma 7.4)

LEMMA 8.2. (i) *For any* $\boldsymbol{\xi} = (\xi_1, \xi_2, ...) \in \mathbf{E}$, $z = g(\boldsymbol{\xi})$ *and for any* $m \geqslant k_0$,

$$\nu(B_{\rho^{s(k_0, m)}}(z)) \approx \rho^{s(k_0, m)(\tau^*(\alpha) \pm 3\eta_{\sigma(m)})}.$$

(ii) *Let* $K = g(\mathbf{H})$. *Then* $\nu(K) \geqslant 1/2$. *Moreover, for* $z = g(\xi) \in K$, $z_i = g(\xi|i)$ *and* $m = k_0 + i - 1$, *we have*

$$\mu_{t_{s(k_0,\, m)}}(z_i) \approx \rho^{s(k_0,\, m)(\alpha \pm 2\varepsilon_{\sigma(m)-1})}$$

and

$$\mu_{t_{s(k_0,\, m)}}(B_{c\rho^{s(k_0,\, m)}}(z_i)) \approx \rho^{s(k_0,\, m)(\alpha \pm 2\varepsilon_{\sigma(m)-1})}.$$

Proof of Theorem 6.6, *the multifractal formalism.* Let $\gamma > 0$ be arbitrary. By observing that for $z = g(\xi) \in g(\mathbf{H})$ and $z_i = g(\xi \mid i)$,

$$\mu_{t_{s(k_0,\, m)}}(z_i) \leqslant \mu(B_{r'\rho^{s(k_0,\, m)}}(z)) \leqslant \mu_{t_{s(k_0,\, m)}}(B_{2r'\rho^{s(k_0,\, m)}}(z_i))$$

where $r' = 1 + r_0$ and $m = k_0 + i - 1$, we can show by applying Lemma 8.2(ii) with $c = 2r'$ that

$$K \subseteq \left\{ z\colon \lim_{\delta \to 0^+} \frac{\ln \mu(B_\delta(z))}{\ln \delta} = \alpha \right\}.$$

Moreover, $\nu(K) \geqslant 1/2$, and for each $z \in K$, Lemma 8.2(i) implies that

$$\lim_{\delta \to 0^+} \frac{\nu(B_\delta(z))}{\delta^{\tau^*(\alpha) - \gamma}} = 0.$$

By applying the mass distribution theorem and then letting $\gamma \searrow 0$, we have $\dim_{\mathscr{H}} K \geqslant \tau^*(\alpha)$. The theorem follows by combining this and Theorem 4.1.

9. REMARKS

It is known that there are multifractal measures for which the L^q-spectra $\tau(q)$ have discontinuous first derivatives and the singularity spectra $f(\alpha) = \dim_{\mathscr{H}} K(\alpha)$ are not concave (see e.g., [BR], [CS], [DL2]). The existence of such spectra can be heuristically explained as the lack of uniformity by which the scaling is distributed. In all the known examples they are either non-generic [BR], or the measures are actually absolutely continuous [CS] or include signs [DL2]. We do not know whether this will occur for the self-similar measures associated with the WSP. Nevertheless Theorem 6.6 shows that if the similitudes $\{S_i\}_{i=1}^N$ have the WSP then $\tau^*(\alpha)$ is the convex hull of $f(\alpha)$. It is because $f(\alpha) \leqslant \tau^*(\alpha)$ and equality holds at every strictly concave point $(\alpha, \tau^*(\alpha))$ which determines τ^* and the convex hull of f.

The Gibbs measure, the pressure and the large deviation theorem in thermodynamics provide convenient tools to study the dynamics of multifractal measures [BR], [BMP], [CLP]. However such formulation depends on the symbolic representation of the system into a coding space and the dynamics as a shift. With only the WSP the representation of a point to the coding space is not necessarily unique and it is not clear how such approach can be set up.

Theorem 6.6 is applicable to the self-similar measures defined by similitudes satisfying the open set condition (Section 6, Example 1) since τ is differentiable and therefore τ^* is strictly concave. It extends the result of Cawley and Mauldin [CM] and Riedi [Ri] that the attractor K of $\{S_i\}_{i=1}^N$ has to be totally disconnected.

COROLLARY 9.1. *Suppose $\{S_i\}_{i=1}^N$ satisfies the open set condition. Then the L^q-spectrum τ is given by*

$$\sum_{i=1}^N p_i^q \rho_i^{-\tau(q)} = 1,$$

and the multifractal formalism $f(\alpha) = \tau^*(\alpha)$ holds for $\alpha \in (\mathrm{Dom}^+ \tau^*)^\circ$.

We will discuss the more interesting case of the ICBM μ defined by

$$S_1(x) = \rho x, \qquad S_2(x) = \rho x + (1 - \rho)$$

with $1/2 < \rho < 1$ and ρ^{-1} a P.V. number and with probability 1/2 on each map (Section 6, Example 2). In [L1] by using the equivalent integral form

$$I_\delta(q) = \delta^{-1} \int \mu(B_\delta(x))^q \, dx$$

instead of the sum

$$S_\delta(q) = \sup \sum_i \mu(B_\delta(x_i))^q$$

in (3.1), the L^2-spectrum $\tau(2)$ of μ was calculated. The expression is in terms of the maximal eigenvalue of a representing matrix. Moreover it was shown that $I_\delta(2)$ satisfies

$$\frac{1}{\delta^{(1+\tau(2))}} I_\delta(2) = p(\delta) + o(1) \qquad \text{as} \quad \delta \to 0,$$

where p is a continuous multiplicatively periodic function satisfying $p(\rho\delta) = p(\delta)$. In [LN2], this algorithm is extended to calculate $\tau(q)$ for all integers $2 \leqslant q < \infty$. For $\rho = (\sqrt{5} - 1)/2$, approximately, $\dim_2(\mu) \approx 0.9924$,

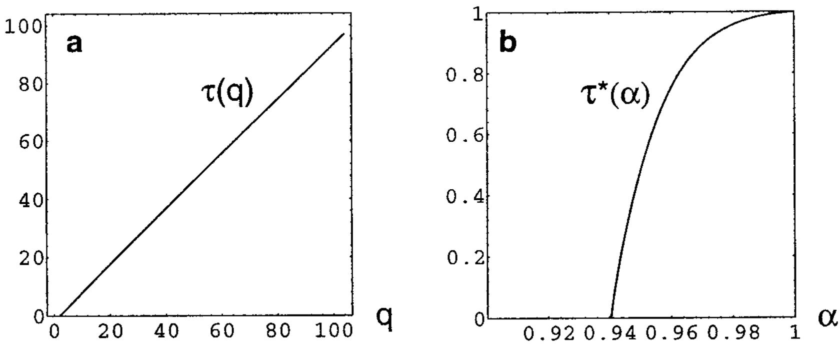

FIG. 8. (a) The approximate $\tau(q)$ curve for the ICBM with ρ^{-1} equal to the golden ratio. The graph is plotted for integer q from 1 to 100 (It is not a straight line). (b) The corresponding dimension spectrum.

$\dim_3(\mu) \approx 0.9897$, $\dim_4(\mu) \approx 0.9875$. Also in [Hu], Hu showed that $\dim_\infty(\mu) = -(1/2) - \log 2/\log \rho \approx 0.9404$. It is seen that for $q > 0$ the dimension spectrum lies in a very narrow region close to 1. (See Fig. 8.)

We have more information for the case $\rho = (\sqrt{5}-1)/2$. In [STZ], Strichartz *et al.* defined another family of three maps in terms of S_1 and S_2:

$$T_0 x = S_1 S_1 x = \rho^2 x,$$
$$T_1 x = S_2 S_1 S_1 x = \rho^3 x + \rho^2,$$
$$T_2 x = S_2 S_2 x = \rho^2 x + 1 - \rho^2.$$

Note that $(0, 1)$ is the disjoint union of $T_i(0, 1)$, $i = 1, 2, 3$, so that the T_i's satisfy the open set condition. Also, the measure μ satisfies a set of "second order" self-similar identities defined by the T_i's. By using these and Theorem 6.6, it is proved in [LN1] that

THEOREM 9.2. *Let $\rho = (\sqrt{5}-1)/2$. Then the L^q-spectrum $\tau(q)$ of the ICBM μ is differentiable for $0 < q < \infty$, and hence μ satisfies the multifractal formalism for $\alpha = \tau'(q)$, $0 < q < \infty$. Moreover $\tau(q)$ is defined by*

$$\sum_{k=0}^{\infty} \rho^{-(2k+3)\,\tau(q)} \left(\sum_{|J|=k} c_J^q \right) = 1, \qquad 0 < q < \infty$$

where $J = (j_1, ..., j_k)$, $j_i = 0$ or 1 and

$$c_J = \tfrac{1}{4}\,[0, 1, 0]\, M_J \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \textit{with} \quad M_0 = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \qquad M_1 = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

The entropy dimension of μ *is given by*

$$\tau'(1) = \frac{1}{9 \ln \rho} \sum_{k=0}^{\infty} \Big(\sum_{|J|=k} c_J \ln c_J \Big).$$

The number 9 in the second expression comes from $\sum_{k=0}^{\infty} (2k+3) \times \sum_{|J|=k} c_J$. We remark that the entropy dimension of the ICBM for $\rho = (\sqrt{5}-1)/2$ had also been considered in [LP], [AY], [AZ] and its value is approximately 0.9957 [AZ]; the calculation from the above formula is close to this number but needs more iterations. Finally the above technique to reduce an overlapping case to a nonoverlapping case seems to be quite restrictive. Besides the golden ratio, Ho found another P.V. number to have the same property (ρ^{-1} satisfies $x^3 - 2x^2 + x - 1 = 0$), but most of them fail. The question of obtaining formulas of $\tau(q)$ for the other P.V. numbers is hence still open.

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REFERENCES

[AP] M. Arbeiter and N. Patzschke, Random self-similar multifractals, *Math. Nachr.* **181** (1996), 5–42.

[AY] J. C. Alexander and J. A. Yorke, Fat baker's transformations, *Ergodic Theory Dynamical Syst.* **4** (1984), 1–23.

[AZ] J. C. Alexander and D. Zagier, The entropy of a certain infinitely convolved Bernoulli measure, *J. London Math. Soc.* **44** (1991), 121–134.

[B] R. Bowen, "Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms," Lecture Notes in Mathematics, No. 470, Springer-Verlag, Berlin, 1975.

[BMP] G. Brown, G. Michon, and J. Peyrière, On the multifractal analysis of measures, *J. Stat. Phys.* **66** (1992), 775–790.

[BR] T. Bohr and D. Rand, The entropy function for characteristic exponents, *Physica D* **25** (1987), 387–398.

[CLP] P. Collet, J. L. Lebowitz, and A. Porzio, The dimension spectrum of some dynamical systems, *J. Stat. Phys.* **47** (1987), 609–644.

[CM] R. Cawley and R. D. Mauldin, Multifractal decompositions of Moran fractals, *Adv. Math.* **92** (1992), 196–236.

[CS] A. Csordás and P. Szépfalusy, Singularities in Rényi information as phase transitions in chaotic states, *Phys. Rev. A* **39** (1989), 4767–4777.

[DL1] I. Daubechies and J. Lagarias, Two-scale difference equations I. Existence and global regularity of solutions, *SIAM J. Math. Anal.* **22** (1991), 1388–1410.

[DL2] I. Daubechies and J. Lagarias, On the thermodynamic formalism for multifractal functions, *Rev. Math. Phys.* **6** (1994), 1033–1070.

[E] P. Erdös, On a family of symmetric Bernoulli convolutions, *Amer. J. Math.* **61** (1939), 974–976.

[EM] G. A. Edgar and R. D. Mauldin, Multifractal decompositions of digraph recursive fractals, *Proc. London Math. Soc.* **65** (1992), 604–628.

[F1] K. J. Falconer, "The Geometry of Fractal Sets," Cambridge Tracts in Mathematics, No. 85, Cambridge Univ. Press, New York/London, 1985.

[F2] K. J. Falconer, "Fractal Geometry-Mathematical Foundations and Applications," Wiley, New York, 1990.

[F3] K. J. Falconer, The multifractal spectrum of statistically self-similar measures, *J. Theoret. Probab.* **7** (1994), 681–702.

[FP] U. Frisch and G. Parisi, On the singularity structure of fully developed turbulence, *in* "Proc. Int. Sch. Phys., "Enrico Fermi" Course LXXXVIII," pp. 84–88, North Holland, Amsterdam, 1985.

[G] A. M. Garsia, Arithmetic properties of Bernoulli's convolutions, *Amer. J. Math.* **102** (1962), 409–432.

[H] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Fractal measures and their singularities: The characterization of strange sets, *Phys. Rev. A* **33** (1986), 1141–1151.

[HP] H. Hentschel and I. Procaccia, The infinite number of generalized dimensions of fractals and strange attractors, *Physica D* **8** (1983), 435–444.

[Hu] T. Y. Hu, The local dimension of the Bernoulli convolutions associated with the golden number, *Trans. Amer. Math. Soc.* **349** (1997), 2917–2940.

[Hut] J. E. Hutchinson, Fractal, and self-similarity, *Indiana Univ. Math. J.* **30** (1981), 713–747.

[L] O. Landford, Entropy and equilibrium states in classical statistical mechanics, *in* "Statistical Mechanics and Mathematics Problems," Lecture Notes in Physics, No. 20, Springer-Verlag, Berlin.

[L1] K. S. Lau, Fractal measures and mean p-variations, *J. Funct. Anal.* **108** (1992), 427–457.

[L2] K. S. Lau, Dimension of a family of singular Bernoulli convolutions, *J. Funct. Anal.* **116** (1993), 335–358.

[LN1] K. S. Lau and S. M. Ngai, L^q-spectrum of the Bernoulli convolution associated with the golden ratio, *Studia Math.* **131** (1998), 225–251.

[LN2] K. S. Lau and S. M. Ngai, L^q-spectrum of Bernoulli convolutions associated with P.V. numbers, preprint, 1996.

[Lo] A. O. Lopes, The dimension spectrum of the maximal measure, *SIAM J. Math. Anal.* **20** (1989), 1243–1254.

[LP] F. Ledrappier and A. Porzio, A dimension formula for Bernoulli convolutions, *J. Statist. Phys.* **76** (1994), 1307–1327.

[M] B. B. Mandelbrot, Intermittent turbulence in self-similar cascades: Divergence of high moments and dimension of the carrier, *J. Fluid Mech.* **62** (1974), 331–358.

[O1] L. Olsen, A multifractal formalism, *Adv. Math.* **116** (1995), 82–196.

[O2] L. Olsen, "Random Geometrically Graph Directed Self-Similar Multifractals," Pitman Research Notes in Mathematics Series **307** (1994).

[R] D. A. Rand, The singularity spectrum $f(\alpha)$ for cookie-cutters, *Ergodic Theory Dynamical Syst*. **9** (1989), 527–541.

[Ri] R. Riedi, An improved multifractal formalism and self-similar measures, *J. Math. Anal. Appl*. **189** (1995), 462–490.

[Ro] R. T. Rockafellar, "Convex Analysis," Princeton Univ. Press, Princeton, NJ, 1970.

[S] R. Salem, "Algebraic Numbers and Fourier Transformations," Heath Math. Monographs, Boston, 1962.

[St] R. S. Strichartz, Self-similar measures and their Fourier transforms III, *Indiana U. Math. J*. **42** (1993), 367–411.

[STZ] R. S. Strichartz, A. Taylor, and T. Zhang, Densities of self-similar measures on the line, *Experiment. Math*. **4** (1995), 101–128.

[T] T. Tél, Fractals, multifractals, and thermodynamics, *Zschr. f. Naturforsch A* **43** (1988), 1154–1174.