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NOTE

Factoring the Poincaré Polynomials for the Bruhat Order on S_n

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We consider the symmetric group S_n, whose elements are permutations written as words $w_1 w_2 \cdots w_n$. It is a graded poset with the Bruhat order $\preccurlyeq$. The Bruhat order was described combinatorially by Proctor [7]. We say that a set of numbers $\{a_1, ..., a_k\}$ is less than a set $\{b_1, ..., b_k\}$ if when the elements in the two sets are written in increasing order we have $a_i \leqslant b_i$ for $1 \leqslant i \leqslant k$. The following criterion is proved in [7]: Let $\mathbf{v}, \mathbf{w} \in S_n$, then $\mathbf{v} \preccurlyeq \mathbf{w}$ if and only if for each $1 \leqslant i \leqslant n$ we have $\{v_i, ..., v_n\} > \{w_i, ..., w_n\}$. The rank of an element $\mathbf{w} \in S_n$ is called the *length* of $\mathbf{w}$ and is denoted by $l(\mathbf{w})$. For $\mathbf{w} \in S_n$ consider the Poincaré polynomial $p_{\mathbf{w}}(t) = \sum_{\mathbf{v} \preccurlyeq \mathbf{w}} t^{l(\mathbf{v})}$. We prove the following theorem; the proof of the only if direction is combinatorial:

THEOREM 1.1. *Let* $\mathbf{w} \in S_n$. *The Poincaré polynomial* $p_{\mathbf{w}}(t)$ *factors into polynomials of the form* $1 + t + t^2 + \cdots + t^r$ *if and only if* $\mathbf{w}$ *does not contain a subsequence* $w_{i_1} w_{i_2} w_{i_3} w_{i_4}$ *of* 4 *elements with the same relative order as* 4231 *or* 3412.

The motivation for this result comes from Schubert varieties. Let B be the Borel subgroup of $SL_n(\mathbb{C})$ consisting of the upper triangular matrices. The Weyl group of type A is the symmetric group S_n. For $\mathbf{w} \in S_n$ let $X_{\mathbf{w}} = \overline{B\mathbf{w}B/B}$ be the Schubert variety of type A indexed by $\mathbf{w}$. Let $\mathrm{P}_{\mathbf{w}}(t)$ be the Poincaré polynomial of the cohomology ring of $X_{\mathbf{w}}$. Then $\mathrm{P}_{\mathbf{w}}(t) = p_{\mathbf{w}}(t^2)$. Lakshmibai and Sandhya [6] showed that $X_{\mathbf{w}}$ is smooth if and only if $\mathbf{w}$ does not contain a subsequence $w_{i_1} w_{i_2} w_{i_3} w_{i_4}$ of 4 elements with the same relative order as 4231 or 3412.

Thus, Theorem 1.1 is equivalent to:

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THEOREM 1.2. *A Schubert variety of type A is smooth if and only if the Poincaré polynomial of its cohomology ring factors into polynomials of the form* $\sum_{i=0}^{r} t^{2i}$.

As a referee pointed out, the "only if" assertion of Theorem 1.2 follows from Theorems 1 and 3 in [1]; the proofs of these theorems in [1] require Algebraic Geometry methods.

2. PROOF OF THEOREM 1.1

Let $\mathbf{w}$ be a permutation in S_n. Write $\mathbf{w} = \mathbf{w}' n \mathbf{w}''$. Let $\mathbf{u} = u_1 \cdots u_k$ be the subword of maximal length of $\mathbf{w}$ such that $u_1 = n$, u_2 is the largest number to the right of u_1 in $\mathbf{w}$, u_3 is the largest number to the right of u_2 in $\mathbf{w}$, ..., u_k is the largest number to the right of u_{k-1} in $\mathbf{w}$. (Then $u_k = w_n$ is the rightmost element of $\mathbf{w}''$ and $u_1 > u_2 > \cdots > u_k$.) Fix $\mathbf{w}$, $\mathbf{w}'$, $\mathbf{w}''$, and $\mathbf{u}$ as above.

LEMMA 2.1. *If* $n\mathbf{w}''$ *does not contain a subsequence of* 4 *elements with the same relative order as* 4231, *then for any* $w_i \in n\mathbf{w}'' \backslash \mathbf{u}$ *we have* $w_i < u_k$.

Proof. Assume the contrary, i.e., there exists a $w_i \in n\mathbf{w}'' \backslash \mathbf{u}$ such that $w_i > u_k$. Suppose w_i is between u_j and u_{j+1} in $n\mathbf{w}''$. By the choice of $\mathbf{u}$ it follows that $u_j > w_i < u_{j+1}$, so the subsequence $u_j w_i u_{j+1} u_k$ of $n\mathbf{w}''$ is order-equivalent to 4231, which is a contradiction. ▮

LEMMA 2.2. *If* $\mathbf{w}$ *does not contain a subsequence of* 4 *elements which is order-equivalent to* 4231 *or* 3412 *and there exists an element* $w_i \in \mathbf{w}'$ *such that* $w_i > u_k$, *then* $\mathbf{u} = n\mathbf{w}''$, *i.e.,* $\mathbf{w}''$ *is a decreasing sequence.*

Proof. Assume the contrary, i.e., there exists an element $w_j \in \mathbf{w}'' \backslash \mathbf{u}$. By Lemma 2.1 it follows that $w_j < u_k$, hence the subsequence $w_i n w_j u_k$ is order-equivalent to 3412, a contradiction. ▮

DEFINITION 2.3. Denote by Sm_n the set of permutations in S_n which do not contain a subsequence of 4 elements which is order-equivalent to 4231 or 3412. Define a map $\phi_n : \mathrm{Sm}_n \to S_{n-1}$ such that $\phi_n(\mathbf{w})$ is obtained from $\mathbf{w}$ by deleting u_1 as an element of $\mathbf{w}$, replacing u_1 as an element of $\mathbf{w}$ with u_2, u_2 with u_3, ..., u_{k-1} with u_k and leaving $\mathbf{w} \backslash \mathbf{u}$ unchanged. (Lemma 2.4 below shows that in fact ϕ_n is a map from Sm_n to Sm_{n-1}.)

LEMMA 2.4. *If* $\mathbf{w} \in \mathrm{Sm}_n$, *then* $\phi_n(\mathbf{w}) \in \mathrm{Sm}_{n-1}$.

Proof. Assume the contrary, i.e., $\mathbf{v}=\phi_n(\mathbf{w})$ contains a subsequence $v_{i_1}v_{i_2}v_{i_3}v_{i_4}$ which is order-equivalent to 4231 or 3412. If $\mathbf{w}''$ is a decreasing sequence, then $\mathbf{v}$ is obtained from $\mathbf{w}$ by simply removing n, so $v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}$ appear in $\mathbf{w}$ in the same order as in $\mathbf{v}$, hence $\mathbf{w}$ contains a sequence which is order-equivalent to 4231 or 3412, a contradiction. Therefore we can assume that $\mathbf{w}''$ is not a decreasing sequence. Then Lemmas 2.1 and 2.2 imply that for every $w_i \in \mathbf{w}\backslash\mathbf{u}$ we have $w_i < u_k$, hence $u_j = n-j+1$ for $1 \leqslant j \leqslant k$. Write $\mathbf{v}=\mathbf{v}'(n-1)\,\mathbf{v}''$ and note that $(n-1)\,\mathbf{v}''$ is order-equivalent to $n\mathbf{w}''\backslash w_n$. This implies that if $v_{i_1} \in (n-1)\,\mathbf{v}''$, then the sequence $w_{i_1}w_{i_2}w_{i_3}w_{i_4}$ in $\mathbf{w}$ is order-equivalent to $v_{i_1}v_{i_2}v_{i_3}v_{i_4}$, which is a contradiction. Hence we can assume that $v_{i_1} \in \mathbf{v}'$. We consider 2 cases:

Case 1. $v_{i_1}v_{i_2}v_{i_3}v_{i_4}$ is order-equivalent to 4231.

In this case $v_{i_1} > v_{i_2}, v_{i_3}, v_{i_4}$. Since $v_{i_1} \in \mathbf{v}'$, we have $v_{i_1} < u_k$, hence $v_{i_2}, v_{i_3}, v_{i_4} < u_k$. This implies that none of $v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}$ is in $\mathbf{u}$, hence $v_{i_1}=w_{i_1}, v_{i_2}=w_{i_2}, v_{i_3}=w_{i_3}, v_{i_4}=w_{i_4}$, so $w_{i_1}w_{i_2}w_{i_3}w_{i_4}$ is order-equivalent to 4231, a contradiction.

Case 2. $v_{i_1}v_{i_2}v_{i_3}v_{i_4}$ is order-equivalent to 3412.

Again $v_{i_1} < u_k$, hence $v_{i_3}, v_{i_4} < v_{i_1} < u_k$, so $v_{i_3}, v_{i_4} \notin \mathbf{u}$. Therefore $w_{i_1} = v_{i_1}$, $w_{i_3} = v_{i_3}$, and $w_{i_4} = v_{i_4}$. Also, $w_{i_2} = v_{i_2}$ if $v_{i_2} \notin \mathbf{u}$ and $w_{i_2} > v_{i_2}$ if $v_{i_2} \in \mathbf{u}$. This shows that $w_{i_2} > w_{i_1}, w_{i_3}, w_{i_4}$. Therefore the sequence $w_{i_1}w_{i_2}w_{i_3}w_{i_4}$ is order-equivalent to 3412, a contradiction. ∎

LEMMA 2.5. *Let* $\mathbf{w} \in S_n$ *and* $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ *be such that* $w_{i_1} = n$, $w_{i_2} = n-1, ..., w_{i_k} = n-k+1$. *Let* $\bar{\mathbf{w}}$ *be the word obtained from* $\mathbf{w}$ *by replacing each of* $w_{i_1}, ..., w_{i_k}$ *with* $n-k+1$. *Then*

$$p_{\mathbf{w}}(t) = p_{\bar{\mathbf{w}}}(t)\; p_{kk-1\ldots1}(t) = p_{\bar{\mathbf{w}}}(t) \prod_{i=1}^{k-1} (1+t+t^2+\ldots+t^i).$$

Proof. Let $\mathbf{v} \in S_n$ be such that $\mathbf{v} \preccurlyeq \mathbf{w}$. Let $v_{j_1} \cdots v_{j_k}$ be the subsequence of $\mathbf{v}$ corresponding to $w_{i_1} \cdots w_{i_k}$, i.e., $v_{j_1} \cdots v_{j_k}$ is a permutation of $\{n, n-1, ..., n-k+1\}$. Denote by $\bar{\mathbf{v}}$ be the word obtained from $\mathbf{v}$ by replacing each of $v_{j_1}, ..., v_{j_k}$ with $n-k+1$, so $l(\mathbf{v}) = l(\bar{\mathbf{v}}) + l(v_{j_1} \cdots v_{j_k})$. Since $\mathbf{v} \preccurlyeq \mathbf{w}$, we conclude that $\bar{\mathbf{v}} \preccurlyeq \bar{\mathbf{w}}$. Note also that $\bar{\mathbf{v}}$ and $v_{j_1}, ..., v_{j_k}$ are uniquely determined by $\mathbf{v}$ and vice-versa. Let T be the set of pairs $(\boldsymbol{\sigma}, \boldsymbol{\tau})$, where $\boldsymbol{\sigma}$ is a permutation of the multiset $\{1, ..., n-k, n-k+1, ..., n-k+1\}$ in which $n-k+1$ appears k times, $\boldsymbol{\sigma} \preccurlyeq \bar{\mathbf{w}}$, and $\boldsymbol{\tau}$ is a permutation of $\{n, n-1, ..., n-k+1\}$. The above discussion shows that the map

$$\psi : \{\mathbf{v} \in S_n \mid \mathbf{v} \preccurlyeq \mathbf{w}\} \to T$$

$$\mathbf{v} \mapsto (\bar{\mathbf{v}}, v_{j_1} \cdots v_{j_k})$$

is a bijection such that if $\psi(\mathbf{v})=(\boldsymbol{\sigma}, \boldsymbol{\tau})$, then $l(\mathbf{v})=l(\boldsymbol{\sigma})+l(\boldsymbol{\tau})$. This shows that

$$\begin{aligned} p_{\mathbf{w}}(t) &= p_{\bar{\mathbf{w}}}(t)\, p_{nn-1, ..., n-k+1}(t) = p_{\bar{\mathbf{w}}}(t)\, p_{kk-1, ..., 1}(t) \\ &= p_{\bar{\mathbf{w}}}(t) \prod_{i=1}^{k-1} (1+t+t^2+\cdots+t^i), \end{aligned}$$

which concludes the proof. ∎

THEOREM 2.6. *If* $\mathbf{w}\in \mathrm{Sm}_n$, *then* $p_{\mathbf{w}}(t)=(1+t+t^2+\cdots+t^{k-1})\, p_{\phi_n(\mathbf{w})}(t)$.

Proof.

Case 1. $\mathbf{u}=n\mathbf{w}''$, i.e., $\mathbf{w}''$ is a decreasing sequence.

Let $\mathbf{v}\in S_n$ be such that $\mathbf{v}\preccurlyeq\mathbf{w}$. Then

$$\{v_{n-k+1}, ..., v_n\} > \{w_{n-k+1}, ..., w_n\}. \tag{1}$$

Since $w_{n-k+1}=n$ it follows that $v_{n-k+l}=n$ for some l with $1\leqslant l\leqslant k$. Let $\boldsymbol{\sigma}$ be the permutation in S_n obtained from $\mathbf{v}$ by arranging the last k elements of $\mathbf{v}$ in decreasing order. Then $\mathbf{v}\preccurlyeq\boldsymbol{\sigma}$ and $\sigma_{n-k+1}=n$. By (1) and the fact that $\mathbf{w}''$ is decreasing it follows that $\sigma_{n-k+i}\geqslant w_{n-k+i}$ for $1\leqslant i\leqslant k$, hence $\{\sigma_{n-k+i}, ..., \sigma_n\} > \{w_{n-k+i}, ..., w_n\}$ for $1\leqslant i\leqslant k$. Since $\{\sigma_j, ..., \sigma_n\} = \{v_j, ..., v_n\}$ for $1\leqslant j\leqslant n-k$ and $\mathbf{v}\preccurlyeq\mathbf{w}$ it follows that $\{\sigma_j, ..., \sigma_n\} \geqslant \{w_j, ..., w_n\}$ for $1\leqslant j\leqslant n-k$. Therefore $\boldsymbol{\sigma}\preccurlyeq\mathbf{w}$. Since $\boldsymbol{\sigma}\backslash n$ is obtained from $\mathbf{v}\backslash n$ by arranging the last $k-1$ elements of $\mathbf{v}\backslash n$ in decreasing order it follows that $\mathbf{v}\backslash n\preccurlyeq\boldsymbol{\sigma}\backslash n$. Since n is in the same position in $\boldsymbol{\sigma}$ as in $\mathbf{w}$ and $\boldsymbol{\sigma}\preccurlyeq\mathbf{w}$ it follows that $\boldsymbol{\sigma}\backslash n\preccurlyeq\mathbf{w}\backslash n$. This implies that $\mathbf{v}\backslash n\preccurlyeq\mathbf{w}\backslash n=\phi_n(\mathbf{w})$. Note also that $\mathbf{v}$ is uniquely determined by $\mathbf{v}\backslash n$ and l. This shows that the map

$$\begin{aligned} \psi: \{\mathbf{v}\in S_n \mid \mathbf{v}\preccurlyeq\mathbf{w}\} &\to \{(\boldsymbol{\tau}, l) \mid \boldsymbol{\tau}\in S_{n-1}, \boldsymbol{\tau}\preccurlyeq\phi_n(\mathbf{w}), 1\leqslant l\leqslant k\} \\ \mathbf{v} &\mapsto (\mathbf{v}\backslash n, l) \end{aligned}$$

is a bijection such that if $\psi(\mathbf{v})=(\boldsymbol{\tau}, l)$, then $l(\mathbf{v})=l(\boldsymbol{\tau})+k-l$. Therefore $p_{\mathbf{w}}(t)=(1+t+t^2+\cdots+t^{k-1})\, p_{\phi_n(\mathbf{w})}(t)$.

Case 2. $\mathbf{u}\neq n\mathbf{w}''$, i.e., $\mathbf{w}''$ is not a decreasing sequence.

In this case Lemmas 2.1 and 2.2 imply that for any $w_i\in\mathbf{w}\backslash\mathbf{u}$ we have $w_i<u_k$, i.e., $u_j=n-j+1$ for $1\leqslant j\leqslant k$. As in Lemma 2.5, let $\bar{\mathbf{w}}$ be the word obtained from $\mathbf{w}$ by replacing each of $u_1, ..., u_k$ with $n-k+1$. By Lemma 2.5 we have that $p_{\mathbf{w}}(t)=p_{kk-1\cdots 1}(t)\, p_{\bar{\mathbf{w}}}(t)$. Note that $\bar{\mathbf{w}}\backslash\bar{w}_n=\bar{w}_1\cdots\bar{w}_{n-1}$ is the word $\overline{\phi_n(\mathbf{w})}$ which is obtained from $\phi_n(\mathbf{w})$ by replacing

each of the occurrences of $n-1, n-2, ..., n-k+1$ by $n-k+1$, so applying Lemma 2.5 again we get $p_{\phi_n(\mathbf{w})}(t) = p_{k-1k-2\cdots 1}(t)\, p_{\bar{\mathbf{w}}\backslash \bar{w}_n}(t)$. Since $\bar{w}_i \leqslant \bar{w}_n = n-k+1$ for $1 \leqslant i \leqslant n$, it follows that $p_{\bar{\mathbf{w}}\backslash \bar{w}_n}(t) = p_{\bar{\mathbf{w}}}(t)$, hence

$$p_{\phi_n(\mathbf{w})}(t) = p_{k-1k-2\cdots 1}(t)\, p_{\bar{\mathbf{w}}}(t).$$

Therefore we obtain that

$$p_{\mathbf{w}}(t) = \frac{p_{kk-1\cdots 1}(t)}{p_{k-1k-2\cdots 1}(t)}\, p_{\phi_n(\mathbf{w})}(t) = (1+t+t^2+\cdots+t^{k-1})\, p_{\phi_n(\mathbf{w})}(t),$$

which completes the proof. ∎

Proof of Theorem 1.1. Suppose that $p_{\mathbf{w}}(t)$ factors into polynomials of the form $1+t+t^2+\cdots+t^r$. In particular, $p_{\mathbf{w}}(t)$ is symmetric. Applying [3] we conclude that $X_{\mathbf{w}}$ is rationally smooth. Deodhar [4] showed that rational smoothness is equivalent to smoothness for Schubert varieties of type A. By [6] it follows that $\mathbf{w}$ avoids the patterns 4231 and 3412.

Now suppose that $\mathbf{w}$ avoids the patterns 4231 and 3412. By induction on the number of elements of $\mathbf{w}$ it follows immediately from Theorem 2.6 and Lemma 2.4 that $p_{\mathbf{w}}(t)$ factors into polynomials of the form $1+t+t^2+\cdots+t^r$. ∎

Remark 2.7. For S_n a factorization theorem of Chevalley, cf. [5, §3.15], states that for the maximal element $\boldsymbol{\tau} = n\, n-1 ... 1 \in S_n$ the polynomial $p_{\boldsymbol{\tau}}(t)$ factors as

$$p_{\boldsymbol{\tau}}(t) = \prod_{i=1}^{n-1} (1+t+t^2+\cdots+t^i).$$

This factorization follows immediately by induction from Theorem 2.6.

Remark 2.8. Theorem 2.6 gives an algorithm for computing the Poincaré polynomial $p_{\mathbf{w}}(t)$ of any $\mathbf{w} \in \mathrm{Sm}_n$ as a product

$$p_{\mathbf{w}}(t) = \prod_{i=1}^{n-1} (1+t+t^2+\cdots+t^{a_i}). \tag{2}$$

Moreover, Reiner [8] observed that by induction on n one immediately obtains from Theorem 2.6 expressions for the powers $a_1, ..., a_{n-1}$ appearing in (2) in terms of the inversions of $\mathbf{w}$. Namely, for $1 \leqslant k \leqslant n-1$ let λ_k be the number of pairs (i, j) such that $1 \leqslant i < j \leqslant n$ and k is the largest integer with the property that there exists a sequence $i = i_0 < i_1 < \cdots < i_k = j$ with $w_{i_0} > w_{i_1} > \cdots > w_{i_k}$. Then $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{n-1}$ and, assuming $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_{n-1}$, we have that $(a_1, ..., a_{n-1})$ is the conjugate partition of $(\lambda_1, ..., \lambda_{n-1})$.

Remark 2.9. Recently the factorization ideas in this paper were extended to Schubert varieties of types B and C by Billey in [2], where she showed that rational smoothness for such varieties is characterized by pattern avoidance

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