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Discrete Allocation Mechanisms: Dimensional Requirements for Resource-Allocation Mechanisms When Desired Outcomes Are Unbounded*,†,‡

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Contents. A. *Introduction.* 1. Preliminary remarks. 2. Continuum mechanisms. B. *Dimensional requirements in discrete message spaces.* 1. The questions to be studied and the plan of the paper. 2. Informational size of discrete message spaces. 3. Some results for pure-exchange economies. C. *The ϵ-optimality of rescaled round-off mechanisms.* 1. A proposition about rescaled approximations to continuum mechanisms. 2. Proof of Proposition III. D. *No denumerable-message mechanism on the set of linear-quadratic exchange economies exactly attains the lower bound on error implied by integer outcomes.* E. *Approximating the Walrasian mechanism without rescaling.*§ F. *Approximating the continuum Walrasian mechanism on the set of*

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† The origin of this research was a session on Welfare Economics at the September 1976 Atlantic City meeting of the American Economic Association. One of the authors chaired the session and the other was discussant of a paper by Stanley Reiter, called "Information and Performance in the (New)^2 Welfare Economics" (see References at end of this paper), which surveyed recent developments in the theory of resource allocation mechanisms. In the discussion which followed the formal part of the session, Jacob Marschak raised the question whether the various allocation-mechanism results dealing with continua are relevant in a realistic setting, where messages and actions are discrete. This stimulated the response that discrete results parallel to the continuum results seem plausible. The present work is an attempt to explore the conjectured parallelism.

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linear-quadratic exchange economies when the parameters (β_1, β_2) *are "unbounded"; the problem of coverage; fixed versus variable tolerances.*[§] 1. Mechanisms with rescaling and constant round-off tolerances. 2. Mechanisms with variable round-off tolerances and with rescaling.

A. Introduction

1. *Preliminary Remarks*

In an earlier paper (Hurwicz and Marschak, 1984) we developed techniques for the construction of informationally efficient privacy-preserving mechanisms with discrete message spaces for the case of bounded discrete outcome spaces. We then sought to apply these techniques to obtain discrete-mechanism analogs to results which have been obtained for continuum mechanisms (mechanisms whose message spaces are continua). Those results establish the message-space minimality of the Walrasian mechanism among all mechanisms which achieve Pareto optimality on wide classes of exchange economies and which obey appropriate regularity conditions. Efficiency is defined relative to a performance function—often called, in the present study, a "desired-outcome" function—which assigns a set of desired outcomes (often a one-element set) to each environment. A given discrete mechanism is efficient if there is no other mechanism with smaller or equal error, as well as smaller or equal informational costs, with one of these inequalities strict. Error is the largest possible distance between the desired outcome and the outcome generated by the mechanism. Our (1984) paper focused entirely on the case where the set of desired outcomes is *bounded* as one passes over the set of possible environments. Much of (1984) confined attention to *finite* mechanisms, in which the number of possible messages is finite and so is the number of possible outcomes. In that case, the natural cost measures for the study of efficiency are simply the number of possible messages and the number of possible outcomes.

By contrast, the present paper considers desired-outcome (performance) functions which are *unbounded* on the set of possible environments, and mechanisms in which the message space, while not a continuum, is nevertheless infinite. The discrete mechanisms on which we concentrate have integer tuples (or a slight generalization of integer tuples) as messages. (A *tuple* is a finite ordered sequence. A k-tuple has k elements. A two-tuple is an ordered pair, etc.) We seek to order such message spaces according to a suitable measure of size. Our central aim is again to explore analogs to the message-space minimality results obtained for continuum Walrasian mechanisms, where minimality now has to do with the measure of size which we propose. At the same time, the techniques needed to do so shed light on the

[§] Certain lengthy details of proofs in Sections E and F are contained in three appendixes which are omitted from the present paper but are available in unpublished form from the authors.

general question of efficient discrete mechanisms with infinite message spaces, relative to unbounded desired-outcome functions, where efficiency is now defined (on the cost side) with regard to the message-space size measure we consider. In particular, we obtain some results about transforming a continuum mechanism realizing a given performance (desired-outcome) function into a discrete mechanism that approximates the continuum mechanism and is also (in a sense to be specified) informationally efficient relative to that performance function.

We begin by briefly surveying—in Section 2—the results on continuum mechanisms.

2. *Continuum Mechanisms*

Let E_i, with typical element e_i, be the space of possible *characteristics* or *local environments* of the ith agent, $i = 1, \ldots, n$; $E = E_1 \times \cdots \times E_n$; M the message space; A the space of possible actions. An element $e = (e_1, \ldots, e_n)$ of E is called an *environment*. E is called the class of a priori admissible environments.

One way to describe the static aspects of a privacy-preserving resource allocation mechanism is to specify an *outcome function* h: $M \rightarrow A$ and n *equilibrium-condition functions* g_i: $M \times E \rightarrow Z_i$, where Z_i are appropriately chosen spaces with a null element written as 0. The interpretation is as follows: for any given $m \in M$, the ith agent is required to verify whether

$$g_i(m, e_i) = 0, \tag{1_A}$$

where $e_i \in E_i$ is his local environment. Since the agent need only know his own local environment, the mechanism is *privacy preserving*. If $g_i(m, e_i) = 0$, $i = 1, \ldots, n$, then the message m qualifies as an *equilibrium message* for e. A number of trials will be required until an equilibrium message is found. But, as in much of the literature, we do not deal with the dynamics of a mechanism and are not concerned with the number of trials required. The outcome function h then prescribes that the *equilibrium action* or *equilibrium outcome* $a = h(m)$ be taken. Thus, *a mechanism* on E is defined by a triple $\pi = (M, g, h)$, where $g = (g_1, \ldots, g_n)$, such that there exists an equilibrium message for each e in E. It will sometimes be convenient to state that requirement as "π covers E." A mechanism π has the *performance correspondence* F: $E \twoheadrightarrow A$ given by

$$F(e) = \{a : g(m, e) = 0 \text{ for some } m \in M;\ a = h(m)\},$$

where $g(m, e) = 0$ abbreviates $g_i(m, e) = 0$, $i = 1, \ldots, n$. We shall often say that "π *realizes* F on E." If E describes a class of economies, then we are particularly interested in performance correspondences whose values are Pareto optimal, i.e., correspondences such that, for every $e \in E$, every element of $F(e)$ is Pareto optimal in the economy defined by e.

An alternative specification of a mechanism is obtained by noting that (1_A) is a binary relation between m and e_i, and that it defines a correspondence μ_i: $E_i \twoheadrightarrow M$ given by

$$\mu_i(e) = \{m : g_i(m, e_i) = 0\}.$$

More generally, then, one may define a mechanism by requiring that the ith agent verify whether, for a given $m \in M$, it is the case that

$$m \in \mu_i(e_i).$$

If $m \in \mu_i(e_i)$ for $i = 1, \ldots, n$, then m qualifies as an equilibrium message for e. Equivalently, we define the correspondence μ: $E \twoheadrightarrow M$ by

$$\mu(e) = \bigcap_{i=1}^{n} \mu_i(e_i),$$

and we accept m as an *equilibrium message* if and only if $m \in \mu(e)$. The action is again chosen to be $a = h(m)$, and the performance correspondence F: $E \twoheadrightarrow M$ of a mechanism $\pi = (M, \mu, h)$ is defined by

$$F(e) = \{a : m \in \mu(e) \text{ for some } m \in M;\ a = h(m)\}.$$

We shall use both the "(M, g, h)" specification of a mechanism and the alternative "(M, μ, h)" specification; sometimes one of them will be convenient and sometimes the other.[1]

In much of the economist's experience, message spaces are Euclidean. For instance, the Walrasian process can be fitted into the preceding framework as follows. Let A be the space of net trades in an n-person exchange economy; thus $a \in A$ is an n-tuple $a = (a_1, \ldots, a_n)$, $a \in \mathbb{R}^{(l+1)n}$, where there are l goods other than numeraire. Let P be the normalized price space. Then the Walrasian equilibrium conditions can be written as

$$\left.\begin{array}{c} \dfrac{u_k^i(w_x^i + x^i, w_y^i - p \cdot x^i)}{u_y^i(w_x^i + x^i, w_y^i - p \cdot x^i)} - p_k = 0, \\ i = 1, \ldots, n-1, \\ \dfrac{u_k^n\left(w_x^n - \sum_1^{n-1} x^i, w_y^n - p \cdot \sum_1^{n-1} x^i\right)}{u_y^n\left(w_x^n - \sum_1^{n-1} x^i, w_y^n - p \cdot \sum_1^{n-1} x^i\right)} - p_k = 0, \end{array}\right\} k = 1, \ldots, l,$$

where y denotes the numeraire; p is in P; x^i is an l-tuple of nonnumeraire goods; w_x^i, w_y^i denote i's endowment; and subscripts for agent i's utility

[1] For the two-agent case we shall rewrite the triple (M, g, h) as a quadruple, namely (M, g_1, g_2, h).

function u^i indicate partial differentiation. Let $m = (x^1, \ldots, x^{n-1}, p)$. Then each message is defined by n l-tuples of real numbers and so the dimension of the message space (the space of such m's) is $n \cdot l$.

Now in this version of the Walrasian mechanism the equilibrium-condition functions g_i are specified by the left-hand sides of the above equations and $h(m)$ is the projection function, i.e.,

$$h(x^1, \ldots, x^{n-1}, p) = (x^1, \ldots, x^{n-1})$$

(x^n equals $-\ \Sigma_1^{n-1} x^i$). A central question studied in the literature is whether there exists a mechanism, guaranteeing the existence and Pareto optimality of equilibrium allocations for the same class of environments as the Walrasian mechanism, but with a message space of lower dimension. If no restrictions are imposed on the functions g (or the correspondences μ_i), then the answer is in the affirmative due to the existence of the Peano mapping from (say) $\mathbb{R}^1$ onto $\mathbb{R}^k$ for any k finite. Let γ be such a Peano mapping. Then, for any mechanism with multidimensional M one can substitute a mechanism with a one-dimensional message space, say a subset of $\mathbb{R}^1$, through the replacement of the equilibrium equation

$$g_i(m, e_i) = 0$$

by

$$\tilde{g}_i(\tilde{m}, e_i) = 0,$$

where

$$\tilde{m} \in \tilde{M} \subseteq \mathbb{R}^1$$

and

$$\tilde{g}_i(\tilde{m}, e_i) = g_i(\gamma(\tilde{m}), e_i), \qquad \forall(\tilde{m}, e_i, i).$$

Then the outcome function h is replaced by $\tilde{h}$ such that

$$\tilde{h}(\tilde{m}) = h(\gamma(\tilde{m})), \qquad \forall \tilde{m}.$$

Let $g(m, e) = 0$; then, since γ is onto, there is $\tilde{m} \in M$ such that $\gamma(\tilde{m}) = m$. Hence, if a is an equilibrium outcome of the (M, g, h) mechanism, it is also an equilibrium outcome of the $(\tilde{M}, \tilde{g}, \tilde{h})$ mechanism. On the other hand, let $\tilde{g}(\tilde{m}, e) = 0$. Then $g(m, e) = 0$ for $m = \gamma(\tilde{m})$. Hence, every equilibrium outcome of the new mechanism is also an equilibrium outcome of the original mechanism. Thus (M, g, h) and $(\tilde{M}, \tilde{g}, \tilde{h})$ are both privacy preserving and have the same performance. Consequently we can construct a privacy-preserving one-dimensional mechanism with Walrasian (hence Pareto-optimal) performance unless some regularity restrictions are imposed so as to rule out equilibrium-condition functions such as $\tilde{g} = g \cdot \gamma$ with γ Peano.

Two types of restrictions have been used for this purpose. They can best be stated by writing both (M, g, h) and $(\tilde{M}, \tilde{g}, \tilde{h})$ in their alternative forms, namely (M, μ, h) and $(\tilde{M}, \tilde{\mu}, \tilde{h}')$, respectively. We have

$$\tilde{\mu}(e) = \{\tilde{m} \in \tilde{M} : \tilde{\gamma}(\tilde{m}) \in \mu(e)\},$$

i.e.,

$$\tilde{\mu}(e) = \gamma^{-1}[\mu(e)],$$

where γ^{-1} is the correspondence defined by $\gamma^{-1}(m) = \{\tilde{m} : m = \gamma(\tilde{m})\}$. It is seen that $\tilde{\mu}$ inherits the irregularity of γ^{-1}. In fact, it turns out that γ^{-1}, and hence $\tilde{\mu}$, violates a smoothness condition on correspondences, called *local threadedness*. A correspondence is locally threaded if at each point of its domain there is a neighborhood such that in that neighborhood there is a continuous selection.

Let (M, μ, h), with M Euclidean, be a privacy-preserving nonwasteful[2] mechanism on a class of n-person $(l + 1)$-commodity exchange economies which includes the classic ones (convex preferences, etc.). From the results of Mount and Reiter (1974), Walker (1977), and Osana (1978), it follows that, if the equilibrium correspondence μ is locally threaded, then the dimension of M cannot be lower than nl, the dimension used by the above Walrasian mechanism.[3]

But this result leaves open the possibility that there might be some *non*-Euclidean message space which, in some sense, would be "smaller" than the nl-dimensional Euclidean space. To deal with this possibility, Mount and Reiter defined a quasi ordering of topological spaces which, for Euclidean spaces E^k agrees with the Euclidean dimension, but is meaningful even in spaces for which dimension may not be defined. According to their definition, the (*informational*) size of a topological space A is at least as great as that of topological space B (written $A \geq^{\mathrm{MR}} B$) if any only if there exists a continuous surjective function f: $A \rightarrow B$ whose inverse (correspondence) f^{-1} is locally threaded.

Since it is desirable that a given space have size at least as great as its subspaces, a property lacking in $\geq^{MR}$, two alternatives have been proposed by Walker. The first (denoted $\geq^{\mathrm{W}}$) is defined by $A \geq^{\mathrm{W}} B$ if and only if there

[2] A mechanism is *nonwasteful* if all its equilibrium allocations are Pareto optimal.

[3] Alternatively, it has been shown that dim $M \geq nl$ if it is assumed that μ^{-1} has a Lipschitz-continuous selection (Hurwicz, 1971).

exists a subspace[4] A' of A such that $A' \geq^{MR} B$. The second is the Fréchet "dimension type" (denoted $\geq^{F}$) defined by $A \geq^{F} B$ if and only if there is a subspace of A which is homeomorphic to B. In subsequent work both alternatives have been used ($\geq^{W}$ by Osana, for Edgeworth Box economies, $\geq^{F}$ by Sato for economies with public goods).[5]

The typical proof of minimal size requirements for message spaces goes as follows. Suppose we wish to show that a privacy-preserving mechanism (M, μ, h) that is nonwasteful over a class of environments E must satisfy the requirement $M \geq M'$, where M' is some given message space and $\geq$ represents a quasi ordering such as $\geq^{MR}$, $\geq^{W}$, or $\geq^{F}$. We find a set $\tilde{E} \subseteq E$ of *test environments* such that $E \geq \tilde{E}$. We shall say[6] that a subset $\tilde{E}$ of E has the *uniqueness property* (with respect to the Pareto correspondence) if and only if: for all $\tilde{e}, \tilde{\tilde{e}} \in \tilde{E}$ [where $\tilde{e} = (\tilde{e}_1, \tilde{e}_2)$, $\tilde{\tilde{e}} = (\tilde{\tilde{e}}_1, \tilde{\tilde{e}}_2)$], if there exists some a which is Pareto optimal for the four environments $\tilde{e}$, $\tilde{\tilde{e}}$, $(\tilde{e}_1, \tilde{\tilde{e}}_2)$, $(\tilde{\tilde{e}}_1, \tilde{e}_2)$, then

$$\tilde{e} = \tilde{\tilde{e}}.$$ [7]

Now, let (M, μ, h) be a privacy-preserving mechanism on E, whose subset $\tilde{E}$ has the uniqueness property. Then it can be shown that the restriction of μ to $\tilde{E}$, written $\tilde{\mu}: \tilde{E} \twoheadrightarrow M$, is an *injective* correspondence, i.e.,

$$\tilde{\mu}(\tilde{e}) \cap \mu(\tilde{\tilde{e}}) \neq \emptyset \Rightarrow \tilde{e} = \tilde{\tilde{e}}.$$ [8]

That is to say, the inverse $\tilde{\mu}^{-1}$ is a single-valued function from M onto $\tilde{E}$. It then turns out that, under certain regularity conditions, the size of M must be at least as great as that of $\tilde{E}$. In particular, let $\tilde{E}$ be homeomorphic to a finite-dimensional Euclidean space and let M be a Hausdorff space. Suppose, furthermore, that $\tilde{\mu}$ is locally threaded. Then $M \geq^{F} \tilde{E}$; i.e., $\tilde{E}$ can be embedded homeomorphically in M.[9] We may also note that $M \geq^{F} \tilde{E}$ implies $M \geq^{W} \tilde{E}$.

When both spaces are Euclidean, $M \geq \tilde{E}$ implies $\dim M \geq \dim \tilde{E}$. Thus for such spaces the local threadedness of μ (which is inherited by $\tilde{\mu}$) is a

[4] Possibly A itself.

[5] Hurwicz (1977) and Calsamiglia (1977) only consider finite-dimensional Euclidean message spaces. (See Calsamiglia, 1977, p. 274.)

[6] This definition is only given here for $n = 2$ (since our applications do not go beyond this case).

[7] This concept can be defined more generally for any social choice correspondence O^*: $E \twoheadrightarrow A$. We say that $E^* \subseteq E$ has the uniqueness property with respect to O^* if and only if: for all $\tilde{e}, \tilde{\tilde{e}} \in E^*$, if $O^*(\tilde{e}) \cap O^*(\tilde{\tilde{e}}) \cap O^*(\tilde{e}_1, \tilde{\tilde{e}}_2) \cap O^*(\tilde{\tilde{e}}_1, \tilde{e}_2) \neq \emptyset$, then $\tilde{e} = \tilde{\tilde{e}}$.

[8] See Hurwicz, 1977 (Single-Valuedness Lemma); Calsamiglia, 1977 (Injectiveness Lemma).

[9] See Sato, 1981, p. 53, Lemma 1. Analogous theorems for other orderings and/or spaces are found in Mount and Reiter, 1974; Walker, 1977; Chander, 1983.

sufficient condition to force the message space to be at least of the dimension of $\tilde{E}$. But there is in the Euclidean case an alternative to the assumption of local threadedness of μ. (See Hurwicz, 1977.) For suppose that M and $\tilde{E}$ are Euclidean and in addition that $\tilde{\mu}^{-1}$: $M \underset{\text{onto}}{\rightarrow} \tilde{E}$ is Lipschitzian. Then (Apostol, 1957, p. 257, Theorem 10–8) again $\dim M \geq \dim \tilde{E}$.

We see, therefore, that the conclusion $\dim M \geq \dim \tilde{E}$, where $\tilde{E}$ is a class of test environments with the uniqueness property, can be obtained either by assuming the equilibrium correspondence itself (restricted to $\tilde{E}$) to have a continuity property (viz., local threadedness) or by assuming its inverse to have a stronger property, viz., having a Lipschitzian selection (the latter is sufficient since under our other assumptions $\tilde{\mu}^{-1}$ is single valued). [This need for a stronger property on the inverse is not surprising because, in effect, we are imposing conditions designed to rule out a space-filling (e.g., Peano) curve from M onto $\tilde{E}$. Now a space-filling curve such as Peano's *is* continuous but its inverse does not have (and cannot have) local threads. Hence to rule out such a mapping, say δ, we can either require local threadedness of its inverse δ^{-1} or some property of δ stronger than continuity. It only remains to note that in our context δ corresponds to $\tilde{\mu}^{-1}$ and hence δ^{-1} to $\tilde{\mu}$. Hence we require either local threadedness of $\tilde{\mu}$ or the Lipschitz continuity (which, of course, is stronger than ordinary continuity) of $\tilde{\mu}^{-1}$.]

Using such a framework it follows from the work of Mount and Reiter (1974), Osana (1978), and Sato (1981), that if μ is locally threaded, privacy preserving, interior valued, and nonwasteful, and if, for each i, E_i contains a class $\tilde{E}_i$ of Cobb–Douglas environments, then its message space will have the Mount–Reiter, Walker, or Fréchet size at least as great as that of the Euclidean space of dimension nl, provided that either the message space is Hausdorff or the correspondence $(\mu \mid \tilde{E})^{-1}$ is upper or lower hemicontinuous. Thus the Walrasian mechanism attains the lower bound on size in a class of topological spaces much broader than Euclidean. An analogous result for public-goods economies has been established by Sato with respect to a variant of the Lindahl mechanism.

B. Dimensional Requirements in Discrete Message Spaces

1. *The Questions to Be Studied and the Plan of the Paper*

How does the situation change when the message space and the space of permissible outcomes become discrete while the class E of a priori admissible environments remains a continuum? We shall divide the inquiry into three main questions. They will be formulated specifically for the case in which E is the class of (parametrized) linear-quadratic two-person two-commodity exchange economies (a class considered also in our (1984) paper), but the concepts required will be general ones, and some of the results established in answering the three questions will be general as well.

It will be helpful, in the present preliminary sketch of the paper, to present the three questions not in the order of their subsequent detailed treatment, but rather in the *reverse* order.

(i) *Can the lower bound on error implied by a given space of permissible outcomes be attained for a discrete message space?*

In the continuum literature just summarized, the possible Pareto optima for a typical class E of exchange economies—e.g., the class of Cobb–Douglas economies—is a continuum. Consequently the outcome space $h\ [\mu(E)]$ for a continuum mechanism (M, μ, h) which realizes a Pareto-optimal performance function on E is also a continuum. For discrete mechanisms, neither the set of equilibrium messages nor the set of equilibrium outcomes is a continuum. We shall be considering discrete mechanisms (on a set E) whose messages and outcomes are tuples of integers; i.e., the *permissible outcome space* is no longer a continuum, even though the *correct* (e.g., Pareto-optimal) outcome space remains a continuum. That means that for some environments in E the correct (Pareto-optimal) outcome cannot be an equilibrium outcome of such a discrete mechanism. For some sets E—including the linear-quadratic set which we shall mainly study—that means in turn that there is a positive lower bound to the mechanism's *error* on E. Here error means—for the case in which each environment has a unique Pareto-optimal outcome—the maximum distance, over all economies e in E, between the Pareto-optimal outcome for e and the equilibrium outcome which the discrete mechanism yields for e.[10]

For the class of (parametrized) two-person, two-commodity linear-quadratic exchange economies—a class which we henceforth denote E^*—the correct (Pareto-optimal) outcomes comprise a continuum in the real line. For any integer n, moreover, there is an economy e in E^* for which the correct outcome, to be denoted $\phi(e)$, equals $n + \frac{1}{2}$. In the discrete mechanisms which we shall study—approximations to the continuum Walrasian mechanism on E^*—the permissible outcome space is that of the integers; i.e., every equilibrium outcome is contained in

$$N \equiv \{\ldots, -1, 0, +1, \ldots\}.$$

Clearly, for any mechanism on E^* with integers as the permissible outcomes, a lower bound on error is $\frac{1}{2}$. For let such a mechanism, say $\pi = (M, \mu, h)$, dictate some $x \in N$ as the equilibrium outcome when the environment e satisfies $\phi(e) = n + \frac{1}{2}$, $n \in N$. The closest permissible outcomes being n

[10]When there is a *set* of Pareto optima for every e, then "error at e" may be defined as the maximum distance, over all the equilibrium outcomes for e, between the equilibrium outcome and the nearest Pareto optimum. "Error" is then the supremum of "error at e" over all e in E.

and $n + 1$, the distance $|x - \phi(e)|$ cannot be below $\frac{1}{2}$. Hence $\epsilon_\phi(\pi)$, the *error of the mechanism* π, defined by

$$\epsilon_\phi(\pi) \equiv \sup_{e\in E} \sup_{m\in\mu(e)} |h(m) - \phi(e)|,$$

must satisfy

$$\epsilon_\phi(\pi) \geq \tfrac{1}{2}.$$

Our first question, then, concerns mechanisms on E^* using integer tuples as messages and integers as outcomes[11]—i.e., mechanisms $\pi = (M^{(k)}, \mu, h)$ satisfying

$$M^{(k)} = M_1 \times \cdots \times M_k;\ k \geq 1 \text{ an integer; for } r = 1, \ldots, k$$

$$\text{and for some real } b_r,\ M_r \subseteq \{\ldots, -2b_r, -b_r, 0, +2b_r \ldots\} \quad (1_B)$$

$$h[\mu(E)] \subseteq N. \quad (2_B)$$

The question is whether among such mechanisms there exists one which is capable of *exactly attaining* the lower bound of $\frac{1}{2}$ as its error. We show in Section D (Proposition IV) that the answer is in the negative and remains so when $M^{(k)}$ is replaced by *any* denumerable message space. The proposition generalizes to a class of environments broader than E^* and a class of desired-outcome functions broader than the function ϕ which is associated with E^*.

(ii) *Can one get arbitrarily close to the lower bound on error when messages are integer k-tuples?*

Given the negative answer to the previous question, it is natural to ask next whether one can find a mechanism on E^* (our set of linear-quadratic exchange economies) which satisfies (1_B) for some k, satisfies (2_B), and has an error arbitrarily close to the lower bound of $\frac{1}{2}$. Here the answer turns out to be in the affirmative, at least for large subsets of E^*.

One gets arbitrarily close to an error of $\frac{1}{2}$ by using a discrete mechanism which (i) has a message space in $M^{(2)}$; (ii) approximates the continuum Walrasian mechanism in a "round-off" manner; and (iii) rescales every message, i.e., each agent divides any message by a suitable positive constant in forming his response. Thus given his environment e_i, agent i accepts a message m as an equilibrium message if and only if $|g_i(m^*, e_i)| \leq \delta_i$, where $\delta_i > 0$ is a "round-off tolerance," m^* is the rescaled version of m, and g_i is the equilibrium-condition function of the continuum Walrasian mechanism. In fact, the merits of such "rescaled round-off" approximations hold in settings far more general than our linear-quadratic exchange-economy set E^*.

[11] The analysis we shall give in Sections C and D extends to the case where the permissible outcomes lie not in N but in $\{\ldots, -2b, -b, 0, b, 2b, \ldots\}$, b real, $b \neq 1$.

We show in Proposition III of Section C that for a wide class of environment sets E, of real-valued desired-outcome functions ϕ defined on E, and of continuum mechanisms realizing ϕ, such a rescaled approximation to the continuum mechanism can always be made—by suitably choosing the rescaling constant—arbitrarily close to the error permitted by an integer outcome space. The conclusion of Proposition III holds when E, ϕ, and the continuum mechanism realizing ϕ on E meet a certain collection of regularity requirements called "Condition A." Condition A is met, in particular, by a subset $E^{*\prime}$ of our set E^* (namely a subset in which each of two specific parameters of the four that define a parametrized economy lies in an arbitrary compact subset of positive reals), by the restriction of the Pareto-optimality function ϕ to $E^{*\prime}$, and by the continuum Walrasian mechanism on $E^{*\prime}$. For the subset $E^{*\prime}$, moreover, the *ordinary* round-off approximation to the continuum mechanism—the discrete approximation in which *no* rescaling occurs—has a lower bound on error which is *greater* than $\frac{1}{2}$, no matter what values we choose for the round-off tolerances δ_i among all those values for which the discrete mechanism covers[12] $E^{*\prime}$. That is shown in Section E, which is based on Appendix 1 (see footnote §). Thus if we did not permit rescaling in our approximation to the Walrasian mechanism, then we could *not* bring error arbitrarily close to $\frac{1}{2}$.

(iii) *Can one get arbitrarily close to the lower bound on error with a message space "smaller" than the space of integer pairs?*

The final question is whether the same arbitrary closeness to the lower bound of $\frac{1}{2}$ can be achieved by a mechanism on E^* whose message space is, in some suitable sense, "smaller" than the space of integer pairs which we use in the rescaled discrete Walrasian mechanism. In particular, can it be achieved if the message space is $M^{(1)}$? In the absence of further restrictions the answer is in the affirmative because of the existence of well-known one-to-one correspondences between $M^{(k)}$ and $M^{(1)}$ for any finite k (both $M^{(k)}$ and $M^{(1)}$ are denumerable). But it is possible to define a concept of informational size for discrete sets in a manner quite analogous to the Mount–Reiter approach for topological spaces. This leads to the formulation of regularity of "smoothness" conditions: one may then indeed have a lower bound on the "dimension" k when one considers all smooth mechanisms which use some message space in the collection $\{M^{(1)}, M^{(2)}, \ldots\}$ and which realize a given performance function on a given environment set. A mechanism which, in effect, codes each message in $M^{(k)}$ into a message in $M^{(k')}$, with $k' < k$, violates the smoothness condition. Our general definition of informational size and the associated definition of smooth—or "pseudo-Lipschitzian" as we call it—are given in the next section (B2). In Section B3, we again consider

[12] Recall that a mechanism is said to cover an environment set if it has an equilibrium message for every environment in the set.

the class E^* of linear-quadratic exchange economies, and we take as the performance function to be realized the *set-valued* function which assigns to any economy e in E^* all outcomes within a given distance ϵ of the Pareto-optimal outcome $\phi(e)$. We study—analogously to the procedure used in the continuum case—a suitable "test" class of economies in E^*. We thereby show (Proposition I) that no mechanism on E^* which is pseudo-Lipschitzian on an appropriate subset of E^*, using integer tuples as messages and having a message space smaller than $M^{(2)}$—no matter what its outcome space—can realize the given (set-valued) performance function on E^*. In Proposition II, we demonstrate that Proposition I is not vacuous—that there exists an integer-message mechanism, pseudo-Lipschitzian on a subset with the uniqueness property, and realizing the set-valued performance function on nontrivial subsets of E^*. In particular, we verify in Proposition II that on a subset of $E^{*\prime}$ our rescaled discrete Walrasian mechanism is indeed pseudo-Lipschitzian and realizes the given set-valued performance function. (Recall that $E^{*\prime}$ is the subset of E^* on which the rescaled mechanism's error was shown to be arbitrarily close to $\frac{1}{2}$.)

By virtue of Propositions I, II, III, we will have shown that for a certain class E of (two-person two-commodity) exchange economies, the rescaled discrete Walrasian mechanism has minimal message-space size among all smooth (pseudo-Lipschitzian) integer-outcome mechanisms on E which have errors arbitrarily close to the lower bound implied by integer outcomes. That class E has the following properties: (1) It includes the test economies used in proving Proposition I and it includes the subset of $E^{*\prime}$ on which the rescaled discrete Walrasian mechanism was shown to be pseudo-Lipschitzian; (2) the continuum Walrasian mechanism π on E realizes a desired-outcome function ϕ with respect to which error is defined, and the triple (E, ϕ, π) obeys Condition A; (3) the discrete rescaled round-off approximation to π is pseudo-Lipschitzian on E. So we indeed have a first analog (a somewhat limited one) to the dimensional-minimality results of the continuum literature.

In Section F we deal with the fact that Condition A—under which the rescaled approximation to a continuum mechanism was shown to have the merits described in Proposition III—appears to be a rather stringent one when applied to the *full* set E^* of linear-quadratic exchange economies and the continuum Walrasian mechanism π on E^*. The triple (E^*, π, ϕ), where ϕ is the Pareto-optimal desired-outcome function, fails to meet Condition A, because for the full set E^* a certain uniform-continuity requirement on π (part of Condition A) fails to hold; the requirement *does* hold for the smaller set $E^{*\prime}$ considered above. One therefore asks: Even though (E^*, π, ϕ) fails that uniform-continuity requirement—which is, after all, only a *sufficient* condition for the conclusion of Proposition III—is it nevertheless the case that the rescaled integer-outcome approximation to π on the full set E^* has an error arbitrarily close to the error implied by integer outcomes? The answer is in the negative. In Section F1 (based on Appendix 2) we show that the rescaled mechanism cannot even be made to *cover* the full set E^*.

In Section F2, (based on Appendix 3), we resolve the coverage difficulty and restore the error-optimality property of the rescaled approximation by liberalizing the round-off rules and settling for a set somewhat smaller than E^* (but still unbounded). The constant round-off tolerance δ_i is replaced by a function $\hat{\delta}_i$ on person i's local-environment set. (Thus the privacy-preserving property of the mechanism is not lost.) For such a *variable round-off* approximation, both coverage and error optimality are restored, not for the full set E^*, but rather for a set $\hat{E}$ which can be made arbitrarily close to E^*. Thus we have further support for the hypothesis of parallelism between continuum situations and their discrete-approximation counterparts.

Furthermore, it can be shown that the variable round-off mechanism is pseudo-Lipschitzian on a subset of $\hat{E}$ which has the uniqueness property. This rules out the existence of a mechanism of equal accuracy in approximating the continuum Walrasian mechanism on all of $\hat{E}$), whose (discrete) message space is "one-dimensional."

2. *Informational Size of Discrete Message Spaces*

We seek a definition of informational size which will, in particular, distinguish between the spaces $M^{(n')}$ and $M^{(n'')}$ defined in the previous section, where $n' \neq n''$. If one wants to adapt the MR definition to such spaces, then one natural suggestion is to associate with each space $M^{(n)}$ the "discrete" topology, whose open sets are all the subsets of $M^{(n)}$. But since, for that topology, every subset of $M^{(n)}$ is both closed and open, one sees that the MR definition assigns the *same* size to $M^{(n')}$ as to $M^{(n'')}$. It is conceivable that there is some other topology with respect to which the MR size (or the Walker or Fréchet size) is different for $M^{(n')}$ than for $M^{(n'')}$. We shall, however, use an alternative approach, based on norms and on the Lipschitz properties.[13]

We start with some definitions.

DEFINITION 1. A function[14] $f: X \to Y$ is said to be *Lipschitz continuous* (or *Lipschitzian*) if and only if there exists $K > 0$ such that

$$\|f(x') - f(x'')\| \leq K \cdot \|x' - x''\| \qquad \text{for all } x', x'' \in X.$$

DEFINITION 2. A function $g: A \to B$ is said to be *Lipschitz sectioned* if and only if the inverse correspondence has a Lipschitzian selection, i.e., if

[13] The fact that our approach using norms serves our purposes, together with the apparent need for *uniform* continuity in certain proofs, suggests that in a more general treatment of our problem it may be natural to use not only topology but also uniform structures and uniform spaces. The choice of norm may matter (Dugundji, 1966, p. 201, Ex. 1, last sentence).

[14] The term "function" means single valued, unless the contrary is specified.

and only if there exists a Lipschitzian function s: $g(A) \rightarrow A$ such that $s(b) \in g^{-1}(b)$ for all $b \in g(A)$.

EXAMPLE. The standard[15] one-to-one mapping ψ: $\overline{N} \underset{\text{onto}}{\rightarrow} \overline{N}^2$, where $\overline{N}$ is the set of nonnegative integers and $\overline{N}^2$ is the set of nonnegative integer pairs, is Lipschitzian (with $K = 1$) but, as we shall see from the proof of Lemma B1 below, it is not Lipschitz sectioned.

DEFINITION 3. We shall say that the (informational) *Lipschitz size* of space A is *at least as great as* that of space B, written $A \geq^{\mathrm{L}} B$, if and only if there exists a surjective function g: $A' \underset{\text{onto}}{\rightarrow} B$ which is Lipschitz sectioned, where A' is a subset of A. As usual, $A >^{\mathrm{L}} B$ means "$A \geq^{\mathrm{L}} B$ but not $B \geq^{\mathrm{L}} A$." $A \overset{\mathrm{L}}{\sim} B$ means "$A \geq^{\mathrm{L}} B$ and $B \geq^{\mathrm{L}} A$."

We now prove a lemma concerning integer r-tuples. We do so both to illustrate the size ordering and because the lemma is needed in the proof of Proposition I.

LEMMA B1. *Let r', r'' be positive integers with $r'' > r'$. The informational Lipschitz size of $N^{r'}$, the set of all integer r'-tuples, is strictly lower than the Lipschitz size of $N^{r''}$, the set of all integer r''-tuples; i.e., in symbols, $N^{r''} >^{\mathrm{L}} N^{r'}$.*

Proof. We show here that $N^2 >^{\mathrm{L}} N$ ($= N^1$). An analogous argument shows that $N^{r''} >^{\mathrm{L}} N^{r'}$ *for any* r', r'' with $r'' > r'$.

Step 1. First we show that $N^2 \geq^{\mathrm{L}} N$. Consider the projection function π: $N^2 \rightarrow N$, defined by $\pi(x, y) = x$. It is surjective (onto). It is also Lipschitz sectioned, since its inverse π^{-1} has the selection s defined by $s(\theta) = (\theta, 0)$ which is Lipschitzian, with $K = 1$. [The function s is Lipschitzian because, for $\theta' \neq \theta''$,

$$\frac{\|s(\theta') - s(\theta'')\|}{\|\theta' - \theta''\|} = \frac{\|(\theta', 0) - (\theta'', 0)\|}{|\theta' - \theta''|} = \frac{\|\theta' - \theta'', 0\|}{|\theta' - \theta''|} = \frac{|\theta' - \theta''|}{|\theta' - \theta''|} = 1.]$$

Step 2. It remains to be shown that it is not the case that $N \geq^{\mathrm{L}} N^2$. Suppose that $N \geq^{\mathrm{L}} N^2$. Then there exists a surjection g: $N' \underset{\text{onto}}{\rightarrow} N^2$, where $N' \subseteq N$ and g is Lipschitz sectioned. This means that g^{-1} has a Lipschitzian selection s. Hence there is a Lipschitzian one-to-one function from N^2 to N' and hence also from N^2 to N. [*Any* selection d from g^{-1} is one-to-one: Suppose $d = s(y^*)$ and $d = s(y^{**})$. Then $d \in g^{-1}(y^*)$ and $d \in g^{-1}(y^{**})$, i.e., $g(d) = y^*$ and $g(d) = y^{**}$. Hence $y^* = y^{**}$, since g is a function.] Consider, then, such a one-to-one function s from N^2 to N, where s is Lipschitzian with some given $K > 0$. Without loss of generality, we may set $s(0, 0) = 0$. For some integer $m > 0$, consider a subset of N^2, namely $J(m) =$

[15] See, e.g., Kamke, 1950, p. 2.

$\{(x, y) \in \epsilon N^2 : 0 \leq x \leq m, 0 \leq y \leq m\}$. The set $J(m)$ is an $(m + 1) \times (m + 1)$ square of points in the two-dimensional lattice N^2, and so $\#J(m) = (m + 1)^2$.

Now, for all z in $J(m)$, we have $\|z - (0, 0)\| = \|z\| \leq m$. Hence, since s is Lipschitzian with K, we must have $\|s(z) - s(0, 0)\| \leq K \cdot \|z - (0, 0)\|$, and, therefore, for all z in $J(m)$,

$$|s(z)| \leq K \cdot \|z\| \leq K \cdot m,$$

where $s(z)$ is an integer. Now the number of integers r satisfying the inequality $|r| \leq K \cdot m$ does not exceed $2Km + 1$. Because s is one-to-one, a contradiction is obtained if $\#J(m) > 2Km + 1$, i.e., if $(m + 1)^2 > 2Km + 1$. For a positive integer m, this is equivalent to $m > 2K - 2$. Hence, for any given $K > 0$, the choice $m = \max(2K - 1, 1)$ yields a contradiction. Q.E.D.

Remark B1. The argument in Step 2 implies also that there is no Lipschitzian one-to-one function from $\bar{N}^2$ to N, where $\bar{N}^2$ denotes the set of all pairs of *nonnegative* integers.

3. *Some Results for Pure-Exchange Economies*

We now proceed to apply these concepts to a set of Edgeworth-box economies. We show that if a discrete mechanism on this set obeys certain regularity conditions, if its message space is one of $N, N^2, N^3, \ldots$, and if its equilibrium outcome is always within a specified distance of the Pareto-optimal outcome, then the mechanism's message space must have Lipschitz size at least as great as N^2. The pattern of proof will be seen to be quite similar to that for continua.

We shall study a pure-exchange economy for two individuals and two goods, X and Y. The utility functions are quadratic in X and linear in Y, say

$$u^i = \alpha_i(w_i^x + x_i) - \tfrac{1}{2}\beta_i(w_i^x + x_i)^2 + w_i^y + y_i,$$

where w_i^x, w_i^y denote endowments (initial holdings prior to trade); x_i, y_i are net trades; $\alpha_i > 0$, $\beta_i > 0$, $w_i^x \geq 0$, $w_i^y \geq 0$; and we are assumed to be in the region where u^i increases with x_i. The marginal utilities with respect to X are, for a net increment x_i in i's holding of X,

$$\theta_i - \beta_i x_i, \qquad i = 1, 2,$$

where

$$\theta_i \equiv \alpha_i - \beta_i w_i^x.$$

The Walrasian equilibrium conditions are, accordingly,

$$\theta_i - \beta_i x_i = p, \qquad i = 1, 2;\ x_i + x_2 = 0,$$

where p is the normalized price of X (Y is the numeraire). Writing $x_1 = x$, this becomes

$$p + \beta_1 x - \theta_1 = 0 \tag{3_B}$$

$$p - \beta_2 x - \theta_2 = 0. \tag{4_B}$$

Solving, we obtain

$$x = \frac{\theta_1 - \theta_2}{\beta_1 + \beta_2} \equiv \phi(\theta, \beta) \tag{5_B}$$

$$p = \frac{\theta_1\beta_2 + \theta_2\beta_1}{\beta_1 + \beta_2}. \tag{6_B}$$

To every quadruple $(\theta_1, \beta_1, \theta_2, \beta_2)$, there correspond many (in fact a continuum) of Edgeworth-box economies $(w_x^1, \alpha_1, \beta_1, w_x^2, \alpha_2, \beta_2)$. We shall assume that an Edgeworth-box economy can occur if and only if (i) marginal utilities are nonnegative at every point in the box, and (ii) there is an interior Pareto optimum (with equality of marginal rates of substitution). The point $[(w_x^1 + x, w_y^1 + y), (w_x^2 - x, w_y^2 - y)]$ is an interior Pareto optimum if and only if $x \in [-w_x^1, w_x^2]$, $y \in [-w_y^1, w_y^2]$, and $x = \phi(\theta, \beta)$. Consequently, the set of parametrized economies $[(\theta_1, \beta_1), (\theta_2, \beta_2)]$ which correspond to the Edgeworth-box economies that can occur is the set

$\{[(\theta_1, \beta_1), (\theta_2, \beta_2) : \beta_1 > 0, \beta_2 > 0$; for some positive α_1, α_2 and some nonnegative w_x^1, w_x^2 we have (i) $\theta_1 = \alpha_1 - \beta_1 w_x^1$, $\theta_2 = \alpha_2 - \beta_2 w_x^2$, (ii) $\theta_1 - \beta_1 x \geq 0$ and $\theta_2 + \beta_2 x \geq 0$ for all x in $[-w_x^1, w_x^2]$, (iii) $\phi(\theta, \beta) \in [-w_x^1, w_x^2]\}$.

It is readily verified, however,[16] that this set is identical to the set

$$\begin{aligned} E^* &= \{[(\theta_1, \beta_1), (\theta_2, \beta_2)] : \theta_1 \geq 0, \theta_2 \geq 0, \beta_1 > 0, \beta_2 > 0\} \\ &= E_1^* \times E_2^*, \end{aligned}$$

[16] Suppose $\theta_1 < 0$, $\theta_2 < 0$. Requirements (ii) and (iii) imply that $\theta_1 - \beta_1[(\theta_1 - \theta_2)/(\beta_1 + \beta_2)] \geq 0$, i.e. $\theta_1[1 - \beta_1/(\beta_1 + \beta_2)] \geq -\beta_1\theta_2/(\beta_1 + \beta_2)$, which is impossible, since the term on the left is negative and the term on the right is positive. Suppose $\theta_1 > 0$, $\theta_2 < 0$. Then requirement (ii) implies that $\theta_2 + \beta_2, x \geq 0$ for $x = -w_x^1$, i.e., $\theta_2 \geq \beta_2 w_x^1$, which is impossible, since $w_x^1 \geq 0$. Similarly, $\theta_1 < 0$, $\theta_2 > 0$ is ruled out. *Only* $\theta_1 \geq 0$, $\theta_2 \geq 0$ is consistent with requirements (i), (ii), (iii) and with the inequalities $\beta_i > 0$, $\alpha_1 > 0$, $w_x^i \geq 0$, $i = 1, 2$; and *any* nonnegative (θ_1, θ_2) is consistent with those requirements and inequalities.

where

$$E_i^* = \{(\theta_i, \beta_i) : \theta_1 \geq 0, \beta_i > 0\}, \qquad i = 1, 2.$$

We shall write i's local environment as $e_i = (\theta_i, \beta_i)$; we also write $e = (e_1, e_2)$.

The *continuum Walrasian mechanism on* E^* is a quadruple $\pi = (M, g_1, g_2, h)$, where $M = \{(x, p) : (x, p) \in \mathbb{R}^2\}$; $h(m) = h((x, p)) = x$; and g_i, $i = 1, 2$, is a function from $M \times E_i^*$ to $\mathbb{R}$ given by the left-hand sides of (3_B), (4_B). The mechanism π is indeed a mechanism on E^* (a mechanism which covers E^*), since for each $e \in E^*$, there exists a message (x, p) satisfying $(3_B, 4_B)$. The mechanism π, moreover, *realizes the (point-valued) desired-outcome function* ϕ on E^*, since, for each e in E^*, we have $h(m) = \phi(e)$, where $m = (x, p)$ is the unique message satisfying (5_B), (6_B). The mechanism leaves unspecified the Y-trade which takes place. Hence, since *any* feasible Y-trade, when combined with the X-trade $x = \phi(e)$, comprises a Pareto optimum, we impose no bias in favor of any particular member of the set of Pareto optima for e when we take $\phi(e)$ to be our desired outcome for e.

If we now consider, instead of the continuum mechanism on E^*, any mechanism on E^* whose outcome function is integer valued (whether that mechanism has a discrete message space or not), then we can no longer hope to realize ϕ exactly. Since the range of the desired-outcome function ϕ on E^* includes the entire closed interval between some pair of successive integers, the most we can ask for is that for some specified $\epsilon > 0$

$$|x - \phi(e)| \leq \tfrac{1}{2} + \epsilon.$$

That desideratum gives us a *correspondence* O_ϵ^*, from E^* to the integers, defined by

$$x \in O_\epsilon^*(e) \Leftrightarrow |x - \phi(e)| \leq \tfrac{1}{2} + \epsilon.$$

We shall study a class of discrete mechanisms on E^* which have integers as outcomes and have, for some integer $k > 0$, the set N^k of integer k-tuples as the message space. We are interested in mechanisms which belong to that class and which *also* realize the correspondence O_ϵ^*; i.e., for every e in E^*, every equilibrium outcome lies in $O_\epsilon^*(e)$. An *example* of a mechanism in the class is $\pi^* = (N^2, g_1^*, g_2^*, h^*)$, a *round-off approximation* to the continuum Walrasian mechanism π on E^*. Since only integer pairs are now available as messages, rather than the points of the continuum $\mathbb{R}^2$ (as in the continuum Walrasian mechanism π), it is no longer possible to insist that for every e in E^*, there exist a message m for which $g_i(m, e_i) = 0$, $i = 1, 2$, where the g_i are the equilibrium-condition functions of the continuum Walrasian mechanism $\pi = (M, g_1, g_2, h)$. The most one can ask is that for some preassigned $\delta_i > 0$, $i = 1, 2$, the function g_i have a value within δ_i of zero. Thus we have, for the round-off approximation π^*, for every $e = (e_1, e_2)$ in E^*, and for all integer pairs[17] (x, p) in N^2

$$g_i^*[(x, p), e_i] = 0 \text{ if and only if } |g_i[(x, p), e_i]| \leq \delta_i,\ i = 1, 2, \quad (7_B)$$

$$\begin{aligned} h^*((x, p)) &= \text{the integer closest to } h((x, p)) \\ &= \text{the integer closest to } x, \end{aligned} \quad (8_B)$$

with ties broken downward, where the *round-off tolerances* δ_1, δ_2 are positive numbers such that for every $e = (e_1, e_2)$ in E^*,

$$\text{there exists } (x, p) \in N^2 \text{ for which } g_i^*[(x, p), e_i] = 0,\ i = 1, 2. \quad (9_B)$$

If the pair (δ_1, δ_2) has the property (9_B) for all e in some set, then that pair is said to *achieve coverage of* that set. Note that, as always, we can write the mechanism π^* in the alternative form $\pi^* = (N^2, \mu^*, h^*)$, where, for $i = 1$, 2, μ^* is a correspondence from E to N^2 such that

$$\mu^*(e) = \bigcap_{i=1}^{2} \mu^*(e_i)$$

and μ_i^* is the individual correspondence given by

$$\mu_i^*(e_i) = \{m \in N^2 : |g_i(m, e_i)| \leq \delta_i\}.$$

The mechanism π^* realizes O_ϵ^* on E^* and only if

$$\sup_{e \in E^*} \sup_{(x,p) \in \mu^*(e)} |h^*[(x, p)] - \phi(e)| = \tfrac{1}{2} + \epsilon,$$

or, to use the notation introduced in Section B1,

$$\epsilon_\phi(\pi^*) = \tfrac{1}{2} + \epsilon.$$

We shall now leave the example π^* for the present, and shall consider instead any mechanism (N^k, μ, h) on E^* which realizes the correspondence O_ϵ^*. We ask: How "large"—in the sense of Definition 3—must the message space of such a mechanism be if the mechanism is to be "smooth," in a sense suggested by the following weakening of Definition 1?

[17] It would, of course, be natural—in approximating π—to limit the "price" component of each message to nonnegative integers. We shall not do so here.

DEFINITION 4. A function $f: A \to B$ is said to be *pseudo-Lipschitzian* if and only if, for every $c > 0$, there exists $K > 0$ such that for all $a', a'' \in A$ satisfying the inequality

$$\|a' - a''\| \geq c,$$

we have

$$\|f(a') - f(a'')\| \leq K \cdot \|a' - a''\|.$$

It is clear that a Lipschitzian function is pseudo-Lipschitzian. Also, if the domain A is N^k or the more general $M^{(k)}$ of (1_B), then a pseudo-Lipschitzian function is Lipschitzian. (The latter property will be used in what follows.) On the other hand, a pseudo-Lipschitzian function need not be continuous (e.g., $f(x) = [x]$, where $[x]$ is the largest integer not exceeding x), and a continuous function need not be pseudo-Lipschitzian (e.g., $f(x) = x^2, x \in R$).

We shall need to exhibit a set contained in E^* and having the uniqueness property (described in Section A2) with respect to O_ϵ^*. For any given $\eta > 2$ that purpose is served by the set

$$\begin{aligned} E_{\epsilon\eta} &= E_{\epsilon\eta}^1 \times E_{\epsilon\eta}^2, \\ E_{\epsilon\eta}^i &= \{e_i : e_i = (\theta_i, \beta_i),\ \beta_i = 1,\ \theta_i = n \cdot (4\epsilon + \eta) \\ &\qquad \text{for some } n \in \{0, 1, 2, \ldots\}\}. \end{aligned}$$

We note that $E_{\epsilon\eta}$ is isomorphic with $\overline{N}^2$, the set of all pairs of nonnegative integers. The set $E_{\epsilon\eta}$ will be our set of "test environments"; its role will be analogous to the role of the "test environment" set in the typical proof of dimensional minimality for continuum mechanisms (as described in Section A2). We shall establish

PROPOSITION I. *Let the restriction $\mu \mid E_{\epsilon\eta}$ to $E_{\epsilon\eta}$ of the equilibrium correspondence $\mu: E \twoheadrightarrow M$ have a pseudo-Lipschitzian selection, and let the mechanism (M, μ, h) realize O_ϵ^* on a subset E' of E^* containing $E_{\epsilon\eta}$. Assume that M, the message space, is one of the following: $\{0\}, N, N^2, \ldots$. Then*

$$M \geq^{\mathrm{L}} N^2.$$

Remark B2. By methods similar to those used here one can generalize Proposition I to the case of an arbitrary number of goods and agents.

Proposition I would not be of interest if a mechanism satisfying the conditions of the proposition with $M \overset{\mathrm{L}}{\sim} N^2$ were not known. But, in fact, the following holds.

PROPOSITION II. *Consider the discrete mechanism* $\pi^* = (N^2, g_1^*, g_2^*, h^*)$ *(a round-off approximation to the continuum Walrasian mechanism on* E^**) as a mechanism on the set* $E_{\epsilon\eta}$*; i.e., let* g_1^*, g_2^*, h^* *satisfy* (7_B), (8_B), (9_B) *for every* e *in* $E_{\epsilon\eta}$. *Then*

(i) *this mechanism is pseudo-Lipschitzian (i.e., the correspondence* $\mu^*: E_{\epsilon\eta} \twoheadrightarrow N^2$ *has a pseudo-Lipschitzian selection);*

(ii) *there exists* ϵ^* *such that for all* $\epsilon > \epsilon^*$, (δ_1, δ_2) *can be chosen so that* (7_B), (8_B), (9_B) *are satisfied for all* e *in* $E_{\epsilon\eta}$ *and* O_ϵ *is realized on* $E_{\epsilon\eta}$.

Remark B3. In Proposition III below, we consider a modified ("rescaled") version of the approximate price mechanism π^*. It can be shown that conclusion (i) of Proposition II also holds for such a modified mechanism, and, in addition, for ϵ arbitrarily small, the modified mechanism realizes O_ϵ on subsets of E^* larger than the set $E_{\epsilon\eta}$, namely on sets of the form $\{[(\theta_1, \beta_1), (\theta_2, \beta_2)] : \theta_1 \geq 0, \theta_2 \geq 0, \beta_i \in B_i, i = 1, 2\}$, where B_i $(i = 1, 2)$ is an arbitrary compact subset of the positive real line.

Remark B4. It can be shown (using an argument like that used in the proof of Proposition II) that the pseudo-Lipschitzian property holds for π^* on a subset of E^* larger than $E_{\epsilon\eta}$, namely the set $\{[(\theta_1, \beta_1), (\theta_2, \beta_2)] : \theta_1 \geq 0, \theta_2 \geq 0, \beta_1 = \gamma\beta_2; \beta_2 \geq \bar{\beta}\}$, where $\bar{\beta}$ and γ are arbitrary positive numbers.

Proof of Proposition I

Part 1. We first show that the set $E_{\epsilon\eta}$ has the uniqueness property with respect to the desired-outcome correspondence O_ϵ^*. Let $e' = [(\theta_1', 1), (\theta_2', 1)]$ and $e'' = [(\theta_1'', 1), (\theta_2'', 1)]$ belong to $E_{\epsilon\eta}$. Let x belong to the intersection of the four sets

$$O_\epsilon^*(e'), \qquad O_\epsilon^*(e''), \qquad O_\epsilon^*[(\theta_1', 1), (\theta_2'', 1)], \qquad O_\epsilon^*[(\theta_1'', 1), (\theta_2', 1)].$$

[Clearly, the points $[(\theta_1', 1), (\theta_2'', 1)]$ and $[(\theta_1'', 1), (\theta_2', 1)]$ also belong to $E_{\epsilon\eta}$.] That means that there exist four pairs (x^j, z^j) such that

$$x = x^j + z^j, \qquad j = 1, 2, 3, 4 \tag{10_B}$$

where $|z^j| \leq \epsilon + \frac{1}{2}$, $j = 1, 2, 3, 4$, and

$$x^1 = \phi(e'), \qquad x^2 = \phi(e''), \qquad x^3 = \phi[(\theta_1', 1), (\theta_2'', 1)],$$
$$x^4 = \phi[(\theta_1'', 1), (\theta_2', 1)].$$

In view of (10_B), we have

$$0 = x^2 - x^3 + z^2 - z^3. \tag{11_B}$$

Since $\phi(e) = \frac{1}{2}(\theta_1 - \theta_2)$ for e in $E_{\epsilon\eta}$, (11_B) can be written

$$(\theta_1'' - \theta_2'')/2 - (\theta_1' - \theta_2'')/2 + z^2 - z^3 = 0;$$

i.e.,

$$\theta_1'' - \theta_1' = 2(z^3 - z^2),$$

and, therefore,

$$|\theta_1'' - \theta_1'| \leq 2(\epsilon + \tfrac{1}{2} + \epsilon + \tfrac{1}{2}) = 4\epsilon + 2.$$

Similarly,

$$|\theta_2'' - \theta_2'| \leq 4\epsilon.$$

Since both e' and e'' are in $E_{\epsilon\eta}$ it follows (in view of the definition of $E_{\epsilon\eta}$ and the fact that $\eta > 2$) that $e' = e''$. Thus the uniqueness property has been established.

Part 2. Now suppose that the assertion $M \geq^{L} N^2$ is false. That means, in view of Lemma B1, that M cannot be in $\{N^3, N^4, \ldots\}$. Clearly, $M = \{0\}$ does not realize O_ϵ. So it must be that $M = N$. But then, since by hypothesis the correspondence $\mu \mid E_{\epsilon\eta}$ has a pseudo-Lipschitzian selection, since $E_{\epsilon\eta}$ is discrete (so that a pseudo-Lipschitzian function on $E_{\epsilon\eta}$ is also a Lipschitzian function), and since the uniqueness property just shown implies the injectiveness of μ on $E_{\epsilon\eta}$, there exists a one-to-one *Lipschitzian* function from $E_{\epsilon\eta}$ to N. Since, however, $E_{\epsilon\eta}$ is isomorphic to $\overline{N}^2$, the set of all pairs of non-negative integers, that means that there also exists a one-to-one Lipschitzian function from N_2 to N. But that is impossible (see Remark B1 following the proof of Lemma B1).

That completes the proof.

Proof of Proposition II

We only show here that conclusion (i) of the proposition holds, i.e., that the approximate Walrasian mechanism π^*, viewed as a mechanism on $E_{\epsilon\eta}$ and satisfying (7_B), (8_B), (9_B) for all e in $E_{\epsilon\eta}$, is pseudo-Lipschitzian. Conclusion (ii) will follow from results discussed in Section E (see Remark E1).

An integer-valued pair (x, p) lies in $\mu^*[(\theta_1, \beta_1), (\theta_2, \beta_1)]$, where $[(\theta_1, \beta_1), (\theta_2, \beta_2)] \in E_{\epsilon\eta}$, so that $\beta_1 = \beta_2 = 1$, if and only if

$$|p + x - \theta_1| \leq \delta_1,$$
$$|p - x - \theta_2| \leq \delta_2,$$
$$h(x, p) = x.$$

So $x = (\theta_1 - \theta_2)/2 + (y_1 - y_2)/2, \quad p = (\theta_1 + \theta_2)/2 + (y_1 + y_2)/2,$ where $|y_i| \le \delta_i$, $i = 1, 2$.

Now consider $e' = [(\theta_1', 1), (\theta_2', 1)]$, $e'' = [\theta_1'', 1), (\theta_2'', 1)]$ in $\boldsymbol{E}$(in particular they may lie in $\boldsymbol{E}_{\epsilon\eta}$). Then $\|e' - e''\| = \|\theta' - \theta''\|$, where $\theta' = (\theta_1', \theta_2')$, $\theta'' = (\theta_1'', \theta_2'')$.

Choose an arbitrary $c > 0$ (constant). We shall show that there exists $K > 0$ (depending on c) such that

$$\frac{\|(x', p') - (x'', p'')\|}{\|\theta' - \theta''\|} = K$$

for all (θ', θ'') satisfying

$$\|\theta' - \theta''\| \ge c,$$

where (x', p'), (x'', p'') denote the values taken by (x, p) for θ' and θ'', respectively. From the formula for x and the definition of the norm it follows—letting δ denote $\max(\delta_1, \delta_2)$—that

$$\begin{aligned}
|x' - x''| &= \tfrac{1}{2}\,|(\theta_1' - \theta_2' + y_1' - y_2') - (\theta_1'' - \theta_2'' + y_1'' - y_2'')| \\
&= \tfrac{1}{2}\,|(\theta_1' - \theta_1'') - (\theta_2' - \theta_2'') + (y_1' - y_1'') - (y_2' - y_2'')| \\
&\le \tfrac{1}{2}\,(|\theta_1' - \theta_1''| + |\theta_2' - \theta_2''|) \\
&\le + \tfrac{1}{2}\,(|y_1' - y_1''| + |y_2' - y_2''|) \\
&\le \tfrac{1}{2}\,(2)\max(|\theta_1' - \theta_1''|, |\theta_2' - \theta_2''|) \\
&\quad + \tfrac{1}{2}\,(2)\max(|y_1' - y_1''|, |y_2' - y_2''|) \\
&\le \|\theta' - \theta''\| + \delta \\
&\le \|\theta' - \theta''\| + (\delta)\left(\frac{\|\theta' - \theta''\|}{c}\right),
\end{aligned}$$

since $\|\theta' - \theta''\| \ge c$. Hence

$$\frac{|x' - x''|}{\|\theta' - \theta''\|} \le 1 + \delta/c.$$

A similar inequality holds for $|p' - p''|$. Therefore,

$$\begin{aligned}
\frac{\|(x', p') - (x'', p'')\|}{\|\theta' - \theta''\|} &= \max\left(\frac{|x' - x''|}{\|\theta' - \theta''\|}\, \frac{|p' - p''|}{\|\theta' - \theta''\|}\right), \\
&\le 1 + \delta/c.
\end{aligned}$$

So the equilibrium message correspondence μ^* is indeed pseudo-Lipschitzian for any $c > 0$; we only have to set $K = 1 + \delta/c$. Q.E.D.

C. The ϵ-Optimality of Rescaled Round-off Mechanisms

1. *A Proposition about Rescaled Approximations to Continuum Mechanisms*

In this section we show that, under certain regularity conditions, a continuum mechanism which exactly realizes a given performance function can be converted into a discrete round-off mechanism whose maximum error can be brought arbitrarily close to the lower bound implied by an integer-valued outcome set (to $\frac{1}{2}$ in the case just considered) by choosing a sufficiently high rescaling factor.

In the *continuum mechanism,* to be described formally below, both the message space and the outcome space are Euclidean, and the equilibrium equations must be satisfied exactly. By hypothesis, it realizes exactly the specified performance (desired-outcome) function. In the *discrete mechanism,* both the message space and the outcome space are discrete; specifically, the outcome space is the set of all integers, and the message space a finite Cartesian product of such sets. Because the outcome space consists of integers, the number $\frac{1}{2}$ is, as before, an obvious lower bound on maximum error, given that the correct values of the performance function include numbers of the form $\frac{1}{2} + n$, where n is an integer. In a discrete *round-off mechanism* the equilibrium equations of the continuum mechanism are only required to be satisfied approximately, with a specified accuracy; i.e., equations are replaced by inequalities. Furthermore, in a discrete round-off mechanism the *variables* are *rescaled* in the manner shown below. It then turns out that, by choosing a sufficiently high rescaling factor and a sufficiently high degree of approximation required in the equilibrium conditions, the maximum error in the discrete process can be made arbitrarily close to the lower bound $\frac{1}{2}$.

About the space of environments (economies) E we only specify that it is the Cartesian product of the n individual environments (spaces of individual characteristics) E_1, where n is the number of agents; i.e., $E = E_1 \times \cdots \times E_n$. We denote by ϕ the performance (desired-outcome) function which we wish to realize through our mechanism. Its domain is E, its range the set $\mathbb{R}$ of reals. (It appears that our results could be extended to the case of a multidimensional range, but this has not been carried out as yet.) Formally, $\phi: E \to \mathbb{R}$.

The continuum mechanism, denoted by π, is specified as usual by its message space M, its equilibrium functions g_i, $i \in I = \{1, \ldots, n\}$, and the outcome function h. It is assumed that $M = \mathbb{R}^k$, the k-dimensional Euclidean space. For each $i \in I$, we have $g_i: M \times E_i \to \mathbb{R}^{q_i}$. The outcome function is $h: M \to \mathbb{R}$. The equilibrium conditions are

$$g_i(m, e_i) = 0, \qquad i \in I.$$

These conditions are sometimes abbreviated as

$$g(m, e) = 0.$$

In what follows, the functions g_i, $i \in I$, and h will be assumed to satisfy some or all parts of the following regularity condition:

Condition A. (i) the outcome function h is uniformly continuous on M;

(ii) for each $i \in I$, the function g_i is Lipschitzian in m, uniformly on its domain $M \times E_i$;

(iii) there is a neighborhood Y of 0 in $\mathbb{R}^{q_1 + \cdots + q_n}$ such that $g(m, e) = y$ (i.e., $g_i(m, e_i) = y_i$, $y_i \in \mathbb{R}^{q_i}$, $i \in I$) is uniquely solvable for m on $E \times Y$, the solution being written as $m = f(y, e)$;

(iv) the solution $m = f(y, e)$ of $g(m, e) = y$ is continuous in y uniformly on $E \times Y$; i.e., the family $\{\tilde{f}_e\}_{e \in E}$ of functions $\tilde{f}_e: Y \to M$, where $\tilde{f}_e(y) \equiv f(y, e)$, is equicontinuous.

Remark C1. It is readily verified that Condition A is satisfied by the continuum Walrasian mechanism introduced in Section B2 when that mechanism is defined on a *subset* of the set E^* introduced in Remark B3, namely on a subset of the form

$$E^{*\prime} = \{[(\theta_1, \beta_1), (\theta_2, \beta_2)] : \theta_1 \geq 0,\ \theta_2 \geq 0,\ \beta_1 \in B_1,\ \beta_2 \in B_2\},$$

where B_1, B_2 are arbitrary compact subsets of the positive real line. The set E^{**}, with $\beta_1 = \beta_2 = 1$, is an example.

In conformity with our customary notation, we write $\mu(e) = f(0, e)$; i.e., $\mu(e)$ is the unique solution of $g(m, e) = 0$. We assume that the continuum mechanism $\pi = (M, g, h)$ *realizes* the performance function ϕ (on E), i.e., that

$$h(\mu(e)) = \phi(e) \qquad \text{for all } e \text{ in } E.$$

We shall now proceed to describe the corresponding *rescaled discrete round-off mechanism* $\pi_{s\delta}$. As before, denote by N the set of all integers. The new message space is N^k, the set of all integer k-tuples. (Thus $N = N^1$.)

The permissible range of the outcome function is N. Since the new outcome function is parametrized by the *scale factor* $s > 0$, it will be denoted by h_s. Thus $h_s: M' \to Z$, where $M' = N^k$ is the new message space. An element of the new message space will be written $z = (z_1, \ldots, z_k)$, where the z_j,

$j = 1, \ldots, k$, are integers. Also, we denote by $[a]$ the integer closest to the real number a, with ties resolved (say) downward (if ties are resolved upward instead, then the proposition to be proved remains true). Then the new outcome function is defined by

$$h_s(z) = [h(z/s)] \qquad (s > 0),$$

where h is the outcome function of the continuum mechanism π.

Let $\delta_1, \ldots, \delta_n$ be positive real numbers. The norm symbols will refer, as before, to the maximum norm. The equilibrium conditions of the discrete mechanism $\pi_{s\delta}$ (where δ abbreviates $(\delta_1, \ldots, \delta_n)$, are

$$\|g_i(z/s, e_i)\| \leq \delta_i, \qquad i \in I,\ s > 0,$$

where I denotes $\{1, \ldots, n\}$. These conditions are sometimes abbreviated as

$$\|g(z/s, e)\| \leq \delta.$$

(The symbol z/s denotes $(z_1/s, \ldots, z_k/s)$.) The g functions in the above inequalities are the equilibrium functions of the continuum mechanism π. (Clearly, by setting the scale factor $s = 0$ and also the round-off tolerances $\delta_i = 0$, $i \in I$, we would get back the continuum-mechanism equilibrium conditions, although applied to integer-valued messages. In general, because of the restriction of outcomes to integers, these more demanding equilibrium conditions—equalities—would be impossible to satisfy.)

We now state

PROPOSITION III. *Let the continuum mechanism $\pi = (M, g, h)$ on E, with $M = \mathbb{R}^k$, h: $M \to \mathbb{R}$, realize the performance function ϕ: $E \to \mathbb{R}$, and let it satisfy the above Condition* A. *Then, for every $\epsilon > 0$, there exist positive numbers s and δ, such that, for every e in E:*

there is a k-tuple z^* of integers z^* satisfying the inequalities
$$\|g_i(z^*/s, e_i)\| \leq \delta,\ i \in I; \tag{1c}$$

and

for every k-tuple z of integers, if $\|g_i(z/s,e_i)\| \leq \delta$, $i \in I$,
$$|[h(z/s)] - \phi(e)| = \tfrac{1}{2} + \epsilon. \tag{2c}$$

[Recall that $[a]$ denotes the integer closest to a, with ties resolved downward. Also that, for $z = (z_1, \ldots, z_k)$ and any nonzero number s, we write z/s for $(z_1/s, \ldots, z_k/s)$. Also note that here δ is a scalar.]

Remark C2. Conclusion (1_C) means that the discrete mechanism has an equilibrium message for every e in E. Conclusion (2_C) means that every equilibrium outcome of the discrete mechanism yields an error which does not exceed the lower bound of $\frac{1}{2}$ by more than the specified ϵ. The proposition, therefore, asserts that the outcome of the discrete mechanism can be brought arbitrarily close to the lower bound $\frac{1}{2}$ on error by a suitable choice of the scaling factor s and of the round-off tolerance δ.

2. *Proof of Proposition* III

Part 1. Given $\epsilon > 0$, the uniform continuity of h on M, postulated in Condition A (i), implies that there is an $r_\epsilon > 0$ such that

$$\|m - m'\| \leq r_\epsilon \text{ implies } |h(m) - h(m')| \leq \epsilon \text{ for all } m, m' \in M.$$

Since, by Condition A (iii)–(iv), $f(y, e)$ is well-defined and continuous in y, uniformly on $E \times Y$, it is the case that there is a number, say $\hat{\delta}(r_\epsilon)$, such that, for all y', y'' in the set Y,

$$\|y' - y''\| \leq \hat{\delta}(r_\epsilon) \text{ implies } \|f(y', e) - f(y'', e)\| \leq r_\epsilon.$$

Also, since Y is a neighborhood of the origin in $\mathbb{R}^q$, $q = \Sigma_{i\in I}\, q_i$, there exists for each y in $\mathbb{R}^q$ a positive number, say δ', such that

$$\|y\| \leq \delta' \text{ implies } y \in Y.$$

Now let

$$\delta = \min(\hat{\delta}(r_\epsilon), \delta')$$

and K the common Lipschitz constant for the functions g_i. (The existence of K is guaranteed by A.(ii).) Define the rescaling factor s by

$$s = K/\delta,$$

where δ is defined as above.

We shall now show that δ and s so defined satisfy the assertions of the proposition.

Part 2. Let m_e denote $f(0, e)$. Choose the k-tuple z^* of integers by setting its jth component

$$z_j^* = [sm_{ej}],$$

where $m_{e,j}$ is the jth component of m_e and, again, $[a]$ denotes the integer closest to a. Then

$$|z_j^*/s - m_{e,j}| = |([sm_{e,j}] - sm_{e,j})/s| \leq 1/(2s),$$

because $|[a] - a| \leq \frac{1}{2}$. Hence

$$z^*/s - m_e\| = \max_j |z_j^*/s - m_{e,j}| \leq 1/(2s).$$

But, by the Lipschitz property, we have

$$\|g_i(z^*/s, e_i) - g_i(m_e, e_i)\| \leq K \cdot \|z^*/s - m_e\|.$$

Therefore,

$$\|g_i(z^*/s, e_i) - g_i(m_e, e_i)\| \leq (K)(1/(2s)) = (K)(1/2(K/\delta)) = \delta/2 < \delta.$$

Since $g_i(m_e, e_i) = 0$ by the definition of m_e, we have shown that

$$\text{for any } \delta \text{ and } s \text{ constructed in Part 1, } \|g_i(z^*/s, e_i)\| < \delta,\ i \in I. \tag{3$_C$}$$

That is, we have established assertion (1_C).

Part 3. By A (iii), the equation system $g(m, e) = 0$ has for a given e a unique solution, say m_e. [In the alternative notation, we can write $\mu(e) = \{m_e\}$ (a one-element set).] By the uniform continuity of h on M, we have, for all $m \in M$, and with r_ϵ defined in Part 1 above,

$$\|m - m_e\| \leq r_\epsilon \text{ implies } |h(m) - h(m_e)| \leq e;$$

i.e., since $h(m_e) = \phi(e)$ (because the continuum mechanism π realizes ϕ).

$$\|m - m_e\| \leq r_\epsilon \text{ implies } |h(m) - \phi(e)| \leq \epsilon. \tag{4$_C$}$$

Now, again denoting by $[a]$ the integer closest to a, we have

$$\begin{aligned}
|[h(m)] - \phi(e)| &= |([h(m)] - h(m)) + (h(m) - \phi(e))| \\
&\leq |[h(m)] - h(m)| + |h(m) - \phi(e)| \\
&\leq \tfrac{1}{2} + |h(m) - \phi(e)|,
\end{aligned}$$

since $|[a] - a| \leq \frac{1}{2}$ always. Therefore, we have

$$\text{for all } (m, e),\ \|m - m_e\| \leq r_e \text{ implies } |[h(m)] - \phi(e)| \leq \tfrac{1}{2} + \epsilon. \tag{5$_C$}$$

Part 4. For $\delta > 0$ constructed in Part 1, for all y', y'' in the set Y, and for any y in $\mathbb{R}^{q_1 + \cdots + q_n}$,

$$\|y' - y''\| \leq \delta \text{ implies } \|f(y', e) - f(y'', e)\| \leq r_\epsilon, \tag{6$_C$}$$

$$\|y\| \leq \delta \text{ implies } y \in Y. \tag{7$_C$}$$

Now for a given $r \geq 0$, suppose that (m, e) satisfies

$$|g_i(m, e_i)| \leq \hat{\delta}(r),\ i \in I.$$

In view of (7_C), that is equivalent to the existence of some y which satisfies

$$g(m, e) = y, \qquad \|y\| \leq \delta,$$

so that $y \in Y$ by (7_C), and hence also

$$m = f(y, e).$$

Then, setting $y' = y$ and $y'' = 0$, the antecedent in (6_C) is satisfied (since $y \in Y$). Therefore,

$$\|f(y, e) - f(0, e)\| \leq r_\epsilon.$$

But $f(0, e) = \mu(e) = m_e$. So, we have $\|m - m_e\| \leq r_\epsilon$. In particular, if it happens that $m = z/s$ for some z in N^k and a positive number s, we have shown that

$$\|g_i(z/s, e_i)\| \leq \delta \text{ implies } \|z/s - m_e\| \leq r_\epsilon. \tag{8_C}$$

In view of (5_C) and (8_C), we have therefore established that

the assertion (2_C) holds for s and δ constructed in Part 1. (9_C)

Q.E.D.

Remark C3. Note that in Part 2 it is shown that we can achieve coverage of E—i.e., we can assure that an equilibrium message exists for every e in E—for *arbitrarily small* δ, by choosing the rescaling factor s sufficiently large. *All* we need for that result is the Lipschitzian property of the equilibrium-condition functions g_i of the continuum mechanism which we are approximating. That stands in sharp contrast to the difficulty of achieving coverage of the linear-quadratic exchange-economy set E^* when the approximating discrete mechanism (an approximation to the continuum Walrasian mechanism) does *not* use rescaling. That difficulty is discussed in Section F1 below.

D. No Denumerable-Message Mechanism on the Set of Linear-Quadratic Exchange Economies Exactly Attains the Lower Bound on Error Implied by Integer Outcomes

Consider again the set E^* of linear-quadratic exchange economies $e = [(\theta_1, \beta_1), (\theta_2, \beta_2)]$ and the subset

$$E^{**} = \{e = [(\theta_1, \beta_1), (\theta_2, \beta_2)] : \theta_1 \geq 0, \theta_2 \geq 0, \beta_1 = \beta_2 = 1\}.$$

Consider also the desired-outcome function $\phi = (\theta_1 - \theta_2)/(\beta_1 + \beta_2)$. On E^{**} that function takes the form $\frac{1}{2}(\theta_1 - \theta_2)$, and it will be convenient to let $\phi(\theta)$ denote $\frac{1}{2}(\theta_1 - \theta_2)$, where θ denotes (θ_1, θ_2). We now study *any* mechanism $\pi = [M,(\mu_1, \mu_2), h]$ on E^{**}, such that

—$h[\mu(E^{**})]$ (the set of equilibrium outcomes) lies in N (the set of all integers);

—M is a denumerable set.

We shall study the *error*

$$\epsilon_\phi(\pi) = \sup_{e \in E^*} \sup_{m \in \mu(\theta)} |\phi(\theta) - h(m)|,$$

where $\mu(\theta)$ is a convenient way of writing the set $\mu[(\theta_1, 1), (\theta_2, 1)]$ (the set of equilibrium messages for $e = [(\theta_1, 1), (\theta_2, 1)] \in E^{**}$). We shall establish

PROPOSITION IV. *Let $\pi = [M,(\mu_1, \mu_2), h]$ be a mechanism on the set E^{**}. Let M be denumerable, and let equilibrium outcomes be integers, i.e.,*

$$h[\mu(E^{**})] \in N.$$

Then there exists $\bar{\theta} = (\bar{\theta}_1, \bar{\theta}_2) \in E^{**}$ such that

$$|h(m) - \phi(\bar{\theta})| > \tfrac{1}{2} \qquad \textit{for some } m \textit{ in } \mu(\bar{\theta}). \tag{*}$$

Note that (*) implies that $\epsilon_\phi(\pi) > \frac{1}{2}$.

Remark D1. The proposition remains true for a class of sets much wider than E^{**}. It is sufficient that θ_i be real, $i = 1, 2$ (with β_1, β_2 again fixed), and that the intersection of the set of possible pairs $\theta_1, \theta_2)$ with the set $\{(\theta_1, \theta_2) \in \mathbb{R}^2 : \theta_2 = \theta_1 - 1\}$ have the power of the continuum.

Remark D2. The proposition remains true when $\phi = \frac{1}{2}(\theta_1 - \theta_2)$ is replaced by any function ϕ in a wide class. It is sufficient that for some real K, the set $\{\theta \in \mathbb{R}^2 : \phi = K\}$ be a set with the power of the continuum, while the sets $\{\theta \in \mathbb{R}^2 : \phi > K\}$ and $\{\theta \in \mathbb{R}^2 : \phi < K\}$ each contain more than one point.

Proof of Proposition IV

Part 1. Since the mechanism π covers the full set E^{**} we have

$$\mu(\theta) \equiv \mu_1(\theta_1) \cap \mu_2(\theta_2) \neq \emptyset \qquad \text{for all } \theta \equiv (\theta_1, \theta_2) \in \mathbb{R}^2_+ \tag{1_D}$$

($\mathbb{R}^2_+$ denotes the set of pairs of nonnegative real numbers and $\emptyset$ denotes the empty set.) Write

$$L = \{(r_1, r_2) \in \mathbb{R}^2_+ : r_2 = r_1 - 1\}. \tag{2_D}$$

Note first that there exist $m \in M$, $\theta' \in L$ and $\theta'' \in L$, $\theta' \neq \theta''$ such that

$$m \in \mu(\theta') \cap \mu(\theta''). \tag{3_D}$$

For suppose that (3_D) is false. Since M is denumerable, we may write it as an infinite ordered sequence

$$M = (m_1, m_2, m_3, \ldots).$$

Since (3_D) is supposed false, there exists *at most one* point $\theta \in L$ such that

$$m_1 \in \mu(\theta).$$

If such $\theta \in L$ does exist, denote it by $\theta^{(1)}$. Similarly, for any positive integer j there exists at most one point θ on the line L such that $m_j \in \mu(\theta)$. If such $\theta \in L$ does exist, denote it by $\theta^{(j)}$.

In this manner we obtain a sequence of points, say

$$(\theta^{(j_1)}, \theta^{(j_2)}, \ldots), \tag{4_D}$$

where $1 \leq j_1 < j_2 < \cdots$ and, for each j_k, the point $\theta^{(j_k)}$ is an element of L satisfying $m_{j_k} \in \mu(\theta^{(j_k)})$, and, by hypothesis

$$m_{j_k} \not\in \mu(\theta) \qquad \text{for all } \theta \in L \setminus \{\theta^{j_k}\}. \tag{5_D}$$

Now consider the set consisting of elements of L other than those in the sequence specified by (4_D), i.e., the set

$$L \setminus \{\theta^{(j_1)}, \theta^{(j_2)}, \ldots\}. \tag{6_D}$$

Note that this set has the power of the continuum (since L has that power and we are only subtracting a denumerable subset); a fortiori, it is nonempty.

Let θ be an element of the set in (6_D), and examine $\mu(\overline{\theta})$. Clearly, $m_1 \not\in \mu(\overline{\theta})$ because either there is no point θ in L such that $m_1 \in \mu(\theta)$ or $\theta^{(1)}$ is the only such point and, by construction, $\overline{\theta} \neq \theta^{(1)}$.

Similarly, $m_2 \not\in \mu(\overline{\theta})$, etc. Thus

$$m_k \not\in \mu(\overline{\theta}) \qquad \text{for all } k \in \{1, 2, \ldots\},$$

i.e.,

$$\mu(\bar{\theta}) = \emptyset,$$

which contradicts the covering assumption (1_D). Therefore, we have established the existence of $m \in M$, $\theta'\theta'' \in L$, $\theta' \neq \theta''$, satisfying (3_D).

Part 2. Now (3_D) implies that, for $\theta' = (\theta_1', \theta_2')$, $\theta'' = (\theta_1'', \theta_2'')$,

$$m \in \mu_1(\theta_1') \cap \mu_2(\theta_2') \cap \mu_1(\theta_1'') \cap \mu^2(\theta_2'').$$

But then we also have

$$m \in \mu_1(\theta_1') \cap \mu_2(\theta_2'') \qquad (7_D)$$

and

$$m \in \mu_1(\theta_1'') \cap \mu_2(\theta_2'). \qquad (8_D)$$

Since θ', $\theta'' \in L$, we may suppose without loss of generality that

$$\theta_i' < \theta_i'', \qquad i = 1, 2.$$

(See Fig. 1.) Write

$$\theta^* = (\theta_1', \theta_2'')$$
$$\theta^{**} = (\theta_1'', \theta_2').$$

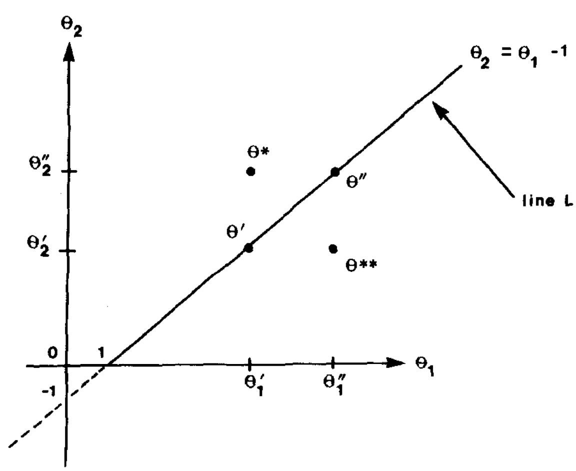

FIGURE 1

Note that θ^* is above L, while θ^{**} is below L. Thus, by (7_D), (8_D) we have

$$m \in \mu(\theta^*), \qquad \phi(\theta^*) < \tfrac{1}{2},$$

and

$$m \in \mu(\theta^{**}), \qquad \phi(\theta^{**}) > \tfrac{1}{2}.$$

Now, by hypothesis, $h(m)$ is an integer. So either $h(m) \leq 0$ or $h(m) \geq 1$. Suppose first that $h(m) \leq 0$. Then

$$|h(m) - \phi(\theta^{**})| > \tfrac{1}{2}.$$

On the other hand, suppose $h(m) \geq 1$. Then

$$|h(m) - \phi(\theta^*)| > \tfrac{1}{2}.$$

Thus, in either case, the conclusion of the proposition holds. Q.E.D.

E. Approximating the Walrasian Mechanism without Rescaling

To complete our assessment of discrete versions of the Walrasian mechanism on the set E^* of linear-quadratic exchange economies, we ask whether the rescaling which achieves the minimal-error result of Proposition III is essential. Can one achieve a similar result without it? The answer is in the negative: for our desired-outcome function $\phi[(\theta_1, \beta_1), (\theta_2, \beta_2)] = (\theta_1 - \theta_2)/(\beta_1 + \beta_2)$, any mechanism which approximates the continuum Walrasian mechanism in the round-off manner *without* rescaling and which covers a *subset* of E^*, namely the subset $E^{**} = \{[(\theta_1, \beta_1), (\theta_2, \beta_2)] : \theta_1 \geq 0, \theta_2 \geq 0, \beta_1 = \beta_2 = 1]$, has an infimum for the error *larger* than the infimum for the error implied by its outcome space. We shall prove this result for outcome spaces of the form $\{\ldots, -2\tau, -\tau, 0, \tau, 2\tau, \ldots\}$ with τ an arbitrary positive number. (Easy generalization of Proposition IV shows that, with rescaling, the greatest lower bound on error is then $\tau/2$.) A fortiori, the same is true for a mechanism covering the full set E^*.

For simplicity, we shall drop any reference to the (fixed) β_i's, and accordingly we replace E^{**} by the set

$$\overline{E} = \{\theta = (\theta_1, \theta_2) : \theta_1 \geq 0, \theta_2 \geq 0\}.$$

We henceforth regard θ_i as i's local environment. Stretching the use of the symbol ϕ slightly (as in Section D), we write $\phi(\theta) = \tfrac{1}{2}(\theta_1 - \theta_2)$. Now consider a discrete mechanism $\overline{\pi} = (\overline{M}, \overline{g}_1, \overline{g}_2, h)$ on $\overline{E}$—written $(\overline{M}, \overline{\mu}, \overline{h})$ in the alternative notation—where for some $\tau > 0, \rho > 0$

$$\overline{M} = \{(\tau k, \rho l) : l \geq 0 \text{ and } k, l \text{ are integers}\}$$

$$\overline{g}_1[(\tau k, \rho l), \theta_1] = 0 \qquad \text{if and only if } |\rho l + \tau k - \theta_1| \leq \delta_1 \tag{1_E}$$

$$\overline{g}_2[(\tau k, \rho l), \theta_2] = 0 \qquad \text{if and only if } |\rho l + \tau k - \theta_2| \leq \delta_2 \tag{2_E}$$

$$\overline{h}[(\tau k, \rho l)] = \tau k.$$

The mechanism $\overline{\pi}$ approximates the continuum Walrasian mechanism $\pi = (M, g_1, g_2, h)$ on E, where, to recall, $M = \{(x, p) : x \in \mathbb{R}, p \in \mathbb{R}_+\}$, $g_1 = p + x - \theta_1$, $g_2 = p - x - \theta_2$, and $h[(x, p)] = x$. The approximating mechanism does *not* use a rescaling factor. For generality, we do not insist that its messages be integer pairs; instead we now permit a "price grid" of mesh ρ and a "trade grid" of mesh τ. For additional generality, we may require (as we did not for the rescaled approximating mechanism) that the "price points" ρl be nonnegative; our main conclusion will be seen to hold whether or not we impose that requirement. The lower bound on error implied by the outcome space is now $\tau/2$; that generalizes the lower bound of $\frac{1}{2}$ considered in Section C.

For any $\theta = (\theta_1, \theta_2)$, the mechanism $\overline{\pi}$ has a set of equilibrium messages, namely all pairs $(\overline{x}, \overline{p}) = (\tau k, \rho l)$ satisfying (1_E), (2_E). To each such equilibrium message $\overline{x}, \overline{p})$ the outcome function $\overline{h}$ assigns the "trade projection," namely $\overline{x} = \tau k$. Note that this meets the requirement on the outcome function in Proposition III—generalized to the case of an outcome space $\{\ldots -2\tau, \tau, 0, \tau, 2\tau, \ldots\}$. That is true since for the outcome function h in the continuum mechanism being approximated, we have $h[(\tau k, \rho l)] = \rho l$ and hence

$$\rho l = \overline{h}[(\tau k, \rho l)] = \text{that element of the outcome space of } \overline{\pi} \text{ which is closest to } h[(\tau k, \rho l)].$$

If $(\tau k, \rho l)$ is an equilibrium message for $\theta = (\theta_1, \theta_2)$—i.e., $(\tau k, \rho l) \in \overline{\mu}(\theta)$—then

$$\rho l + \tau k = \theta_1 + y_1, \qquad |y_1| \leq \delta_1 \tag{3_E}$$

$$\rho l - \tau k = \theta_2 + y_2, \qquad |y_2| \leq \delta_2 \tag{4_E}$$

and hence

$$\begin{aligned} \overline{h}[(\tau k, \rho l)] &= \tau k = \tfrac{1}{2}(\theta_1 - \theta_2) + \tfrac{1}{2}(y_1 - y_2) \\ &= \phi(\theta) + \tfrac{1}{2}(y_1 - y_2), \end{aligned} \tag{5_E}$$

so that

$$|\phi(\theta) - \bar{h}(\tau k, \rho l)| \leq \frac{\delta_1 + \delta_2}{2}, \qquad \text{all } (\tau k, \rho l) \in \bar{\mu}(\theta). \tag{6$_E$}$$

But the inequality in (6_E) in fact becomes an equality for some $(\theta, \tau k, \rho l)$. To see that, let $\bar{l}$, $\bar{k}$ be integers such that

$$\bar{l} > 0, \qquad \rho\bar{l} + \tau\bar{k} > \delta_1, \qquad \rho\bar{l} > \tau\bar{k}, \qquad 2\tau\bar{k} - \delta_1 - \delta_2 > 0.$$

Let $\bar{\theta} = (\bar{\theta}_1, \bar{\theta}_2)$, $\bar{\theta}_1 = -\delta_1 + \rho\bar{l} + \tau\bar{k}$, $\bar{\theta}_2 = \delta_2 + \rho\bar{l} - \tau\bar{k}$. Then we have $(\bar{\theta}_1, \bar{\theta}_2) \in \bar{E}$; $\phi(\bar{\theta}) > 0$; $(\tau\bar{k}, \rho\bar{l}) \in \bar{\mu}(\bar{\theta})$, since ($3_E$), ($4_E$) hold for $l = \bar{l}$, $k = \bar{k}$, $\theta_1 = \bar{\theta}_1$, $\theta_2 = \bar{\theta}_2$, $y_1 = \delta_1$, $y_2 = -\delta_2$; and, in view of (5_E),

$$\bar{h}[(\tau\bar{k}, \rho\bar{l})] = \phi(\bar{\theta}) + \tfrac{1}{2}(\delta_1 + \delta_2).$$

Hence

$$|\phi(\bar{\theta}) - \bar{h}(\tau\bar{k}, \rho\bar{l})| = \frac{\delta_1 + \delta_2}{2}. \tag{7$_E$}$$

From (6_E), (7_E) we have, then,

$$\epsilon_\phi(\bar{\pi}) \equiv \sup_{\theta \in \bar{E}} \sup_{m \in \bar{\mu}(\theta)} |\phi(\theta) - \bar{h}(m)| = \frac{\delta_1 + \delta_2}{2} \tag{8$_E$}$$

If $\bar{\pi}$ is indeed to be a mechanism covering $\bar{E}$, then (δ_1, δ_2) must be chosen so as to achieve coverage; i.e., for every $(\theta_1, \theta_2) \in \bar{E}$, there must exist integers (l, k) such that (3_E), (4_E) hold. We now ask: (i) Does the set of pairs (δ_1, δ_2) achieving coverage of $\bar{E}$ possess a minimizer of the error $\frac{1}{2}(\delta_1 + \delta_2)$? (ii) If so, what is the minimum of $\frac{1}{2}(\delta_1 + \delta_2)$ on the set? It will turn out that there is indeed a minimizer of $\frac{1}{2}(\delta_1 + \delta_2)$ and that the minimum value of $\frac{1}{2}(\delta_1 + \delta_2)$ *exceeds* the lower bound $\tau/2$ for all ρ, τ $(\rho > 0, \tau > 0)$. So if one wants to bring the error arbitrarily close to $\tau/2$ then one *has to* introduce a rescaling factor into the discrete mechanism.

To proceed, let $\tilde{S}_{\rho\tau}$ denote the set of all pairs (δ_1, δ_2) which achieve coverage of $\bar{E}$. That is to say

$$\tilde{S}_{\rho\tau} \equiv \{(\delta_1, \delta_2)\text{: for every } (\theta_1, \theta_2) \text{ with } \theta_1 \geq 0,\ \theta_2 \geq 0, \text{ there exist integers } l, k \text{ such that } l > 0,\ |\rho l + \tau k - \theta_1| \leq \delta_1,\ |\rho l - \tau k - \theta_2| \leq \delta_2\}.$$

Now write $p + x = \theta_1$, $p - x = \theta_2$ (as in the equilibrium conditions for the continuum Walrasian mechanism), so that $p = \frac{1}{2}(\theta_1 + \theta_2)$, which is nonnegative for $(\theta_1, \theta_2) \in \bar{E}$. Then the set $\tilde{S}_{\rho\tau}$ can alternatively be written

$\tilde{S}_{\rho\tau} = [(\delta_1, \delta_2)$: for every (x, p) with $p \geq 0$, there exist integers k, l, such that $l \geq 0$, $|\rho l + \tau k - (p + x)| \leq \delta_1$, $|\rho l - \tau k - (p - x)| \leq \delta_2]$.

It is easily verified that

$$\tilde{S}_{\rho\tau} = \tau\tilde{S}_{\rho/\tau,1} \qquad \text{for all } \rho > 0,\ \tau > 0. \tag{9_E}$$

(For a set J and for $a \in \mathbb{R}$, the symbol aJ denotes the set $\{aj : j \in J\}$.) Now the result obtained in Appendix 1 (see footnote §) implies that

$$\begin{aligned}&\text{The minimum of } \tfrac{1}{2}(\delta_1 + \delta_2) \text{ on the set } S_{\rho/\tau,1} \text{ equals } \tfrac{1}{2}(\rho/\tau + \tfrac{1}{2})\\ &\qquad \text{if } \rho \geq \tau, \text{ and equals } \tfrac{1}{2}(1 + (\rho/\tau)/2) \text{ if } \rho \leq \tau.\end{aligned} \tag{10_E}$$

Hence, in view of (9_E), the minimum of $\frac{1}{2}(\delta_1 + \delta_2)$ on $\tilde{S}_{\rho\tau}$ equals

$$\tau\left[\frac{1}{2}\left(\rho/\tau + \frac{1}{2}\right)\right] = \frac{\rho}{2} + \frac{\tau}{4} \geq \frac{\tau}{2} + \frac{\rho}{4} \qquad \text{if } \rho \geq \tau,$$

and equals

$$\tau\left[\frac{1}{2}\left(1 + \frac{\rho/\tau}{2}\right)\right] = \frac{\tau}{2} + \frac{\rho}{4} \qquad \text{if } \rho \leq \tau.$$

In order to establish (10_E), we do not—in Appendix 1—deal with the set $\tilde{S}_{\rho/\tau,1}$ directly but rather with the more easily analyzed and *not smaller* set $S_{\rho/\tau,1}$ which is obtained from $\tilde{S}_{\rho/\tau,1}$ by deleting the "nonnegative-price" requirements $p \geq 0$, $l \geq 0$. Thus we define, for any $\gamma > 0$,

$$\begin{aligned}S_{\gamma 1} \equiv [(\delta_1, \delta_2)&: \text{for each } (x, p), \text{ there exist integers } k, l \text{ such that}\\ &|\gamma l + k - (p + x)| \leq \delta_1,\ |\gamma l - k - (p - x)| \leq \delta_2].\end{aligned}$$

The minimum of $\frac{1}{2}(\delta_1 + \delta_2)$ on $S_{\rho/\tau,1}$ cannot be less on $\tilde{S}_{\rho/\tau,1}$ than on $S_{\rho/\tau,1}$. To further simplify notation, write

$$S_\gamma \equiv S_{\gamma 1}.$$

The precise result shown in Appendix 1 is that

$$\begin{aligned}&\min\{\delta_1 + \delta_2 : (\delta_1, \delta_2) \in S_\rho\} \text{ equals } \rho + \tfrac{1}{2} \text{ when } \rho \geq 1,\\ &\qquad \text{and equals } 1 + \rho/2 \text{ when } \rho \leq 1.\end{aligned} \tag{11_E}$$

Clearly (11_E) implies (10_E).

Remark E1. Return now to Proposition II, concerning the *nonrescaled* discrete price mechanism. For the case $\rho = \tau = 1$, (10_E) implies that the minimum error of the nonrescaled price mechanism on $\bar{E}$ is $\frac{3}{4}$. (To attain that error, while achieving coverage, we chose $\delta_1 = 1$, $\delta_2 = \frac{1}{2}$.) Since $E_{\epsilon\eta} \subset \bar{E}$, we therefore have, in the language of Proposition II, that the performance correspondence O^*_ϵ is realized on $E_{\epsilon\eta}$ for $\epsilon \geq \frac{1}{4}$. That yields conclusion (ii) of the proposition.

F. Approximating the Continuum Walrasian Mechanism on the Set of Linear-Quadratic Exchange Economies When the Parameters (β_1, β_2) Are "Unbounded"; the Problem of Coverage; Fixed versus Variable Tolerances

1. *Mechanisms with Rescaling and Constant Round-off Tolerances*

Consider trying to apply Proposition III to a class E of a priori admissible environments in which (β_1, β_2) ranges over a set of the form $B_1 \times B_2$ where at least one of the sets B_i is either not bounded away from zero or not bounded from above. (By abuse of language, we refer to such situations by calling (β_1, β_2) "unbounded.") An extreme example is that of E^*, where (β_1, β_2) ranges over the whole positive quadrant. Other examples are those where (β_1, β_2) ranges over such sets as $(0, 1] \times [1, 2]$, $[1, \infty] \times [1, 2]$, or $(0, \infty) \times [1, \infty)$.

In such cases, for the continuous Walrasian mechanism, Condition A (iv) assumed in the proposition is violated. Here the solution

$$m = f(y, e), \qquad m = (x, p)$$

of the equation system $g(m, e) = y$ is

$$x = \frac{(\theta_1 + y_1) - (\theta_2 + y_2)}{\beta_1 + \beta_2}, \qquad p = \frac{(\theta_1 + y_1)\beta_2 + (\theta_2 + y_2)\beta_1}{\beta_1 + \beta_2}.$$

For x to be continuous with regard to y uniformly over $Y \times E$, it is necessary that at least one of the β_i be bounded away from zero. For p to be continuous in y uniformly over $Y \times E$, it is necessary that the ratio β_1/β_2 be bounded away from zero and also from above. (However, if the β-domain is not a Cartesian product, the β_i's need not be bounded from above. For example, $(\beta_1, \beta_2) \in B = \{(\beta_1, \beta_2) : \beta_1 = \beta_2, \beta_1 \geq 1\}$ satisfies A (iv). More generally, this would be true for $B \subseteq \{(\beta_1, \beta_2) : \lambda^* \leqq \beta_1/\beta_2 \leqq \lambda^{**}, \beta_i \geqq \beta_i^*, i = 1, 2\}$, where λ^*, λ^{**}, β_1^*, β_2^* are fixed positive numbers.)

There is a further problem in applying Proposition III when (β_1, β_2) is unbounded. For it turns out that in such situations a rescaled discrete approximation mechanism of the round-off type postulated in Proposition III may fail to cover the class E of a priori admissible environments. That is, there may exist some e in E such that, no matter what rescaling factor s and tolerances δ_i, $i \in I$, are chosen, the system $\|g_i(z/s, e_i)\| \leqq \delta_i$, $i \in I$, of inequalities has no solutions for z that are scalar combinations of integers. (That is, there are no *fixed* positive reals ζ_1, ζ_2 such that, for every e in E, the above system of inequalities has a solution of the form $z = \zeta_1 n_1 + \zeta_2 n_2$, where n_1 and n_2 are integers.)

We have constructed one such example where the a priori class of environments, denoted by $E^{*\circ}$, is defined by

$$E^{*\circ} = \{[(\theta_1, \beta_1), (\theta_2, \beta_2)] : \theta_i \geqq 0, i = 1, 2; \beta_1 > 0, \beta_2 = 1\}.$$

As shown in Appendix 2, it turns out that, whatever the chosen rescaling factor s and tolerances δ_1, δ_2, there is at least one point, namely $(x, p) = (\frac{1}{2}, 0)$, which is not "covered" when $\beta_1 = 2(\delta_1 + \delta_2)$. (Note that it would not have helped to have β_2 bounded from above.)

Explicitly, this means that, for any $s > 0, \rho > 0, \tau > 0$, there is no pair of positive numbers (δ_1, δ_2) and no integers k, l such that the inequalities

$$\left| p + \beta_1 x - \left(\frac{\rho l}{s} + \frac{\beta_1 \tau k}{s}\right) \right| \leq \delta_1,$$

$$\left| p - x - \left(\frac{\rho l}{s} - \frac{\tau k}{s}\right) \right| \leq \delta_2,$$

are satisfied when $x = \frac{1}{2}$, $p = 0$, and $\beta_1 = 2(\delta_1 + \delta_2)$. (As in Section E above, ρ and τ denote, respectively, the "mesh" of the "price" and "trade" grids.)

Admittedly, the choice of $p = 0$ may be somewhat artificial in its economic interpretation. But if, as appears to be the case, the failure of coverage is a more general feature of "unbounded" (β_1, β_2), we are led to consider a class of mechanisms more general than those treated in earlier sections. It turns out that, at least in certain cases of "unbounded" (β_1, β_2), we can avoid the failure of coverage and retain the conclusion of Proposition III (that the error can be pushed arbitrarily close to $\tau/2$), by permitting the tolerances δ_i to depend on e_i rather than be fixed as in previous sections. It should be noted that using δ_i dependent on e_i does not violate the requirement of informational decentralization (the privacy-preserving property), since a condition of the form

$$|g_i(m, e_i)| \leqq \hat{\delta}_i(e_i),$$

where $\hat{\delta}_i(\cdot)$ is a functional relation specified by the designer, is still of the form

$$m \in \mu_i(e_i).$$

That is, the ith individual need only know his/her own characteristic e_i to verify whether a given m satisfies the ith equilibrium condition. The details are given in the next section.

2. *Mechanisms with Variable Round-off Tolerances and with Rescaling*

In this section we show how coverage and asymptotically minimal error can be achieved on the class of environments denoted by $\hat{E}^2(\overline{\beta}_2)$, or $\hat{E}$ for short, defined by

$$\begin{aligned}\hat{E} \equiv \hat{E}^2(\overline{\beta}_2) &= \{[(\theta_1, \beta_1), (\theta_2, \beta_2)] : \theta_1 \geq 0,\ \theta_2 \geq 0,\ \beta_1 \geq 0,\ \beta_2 \geq \overline{\beta}_2\} \\ &= \hat{E}_1^2 \times \hat{E}_2^2(\overline{\beta}_2) \equiv \hat{E}_1 \times \hat{E}_2 \qquad \text{(for short)},\end{aligned}$$

for some fixed $\overline{\beta}_2 > 0$. Here (β_1, β_2) ranges over $B_1 \times B_2 = (0, \infty) \times [\overline{\beta}_2, \infty)$, so B_2 is bounded away from zero but not from above and B_1 is unbounded in both directions; so (β_1, β_2) is "unbounded." (Of course, we could equally use $\hat{E}^1(\overline{\beta}_1)$, where $\beta_1 \geq \overline{\beta}_1 > 0$ while β_2 is only required to be positive.)

We shall use a discrete mechanism $\overline{\pi} = (\overline{M}, \overline{\mu}_1, \overline{\mu}_2, \overline{h})$, a discrete approximation to the continuous Walrasian mechanism $\pi = (M, g_1, g_2, h)$ on $\hat{E}$. Here, for $i = 1, 2$, $\overline{\mu}_i$ is defined by

$$(x, p) \in \overline{\mu}_i(e_i) \qquad \text{if and only if } |g_i(x, \rho/s, e_i)| \leqq \hat{\delta}_i(e_i), \tag{1_F}$$

where $e_i = (\theta_i, \beta_i)$, (x, p) is a message (a lattice point) in $\overline{M}$; $\overline{M}$ is the set $\{\tau k, \rho l) : k, l \text{ are integers}\}$; $\rho > 0$; $\tau > 0$; $s > 0$ is a rescaling factor; and $\hat{\delta}_i$ is a positive real-valued function on i's local-environment set $\hat{E}_i$.

The rescaling factor s and the tolerance functions $\hat{\delta}_i(\cdot)$, as well as h and the functions g_i, are chosen by the designer (of mechanisms) who only knows the set $\hat{E}$ and the desired-performance function ϕ: $E \rightarrow \mathbb{R}$ which is to be realized; the designer does *not* know the prevailing e. (The number ρ may be either given to the designer or chosen by him/her.)

The explicit form of the equilibrium inequalities is

$$\left|\frac{\rho l}{s} + \beta_1 \tau k - \theta_1\right| \leq \hat{\delta}_1(\theta_1, \beta_1)$$

$$\left|\frac{\rho l}{s} - \beta_2 \tau k - \theta_2\right| \leq \hat{\delta}_2(\theta_2, \beta_2),$$

where k and l are integers.

The outcome function in $\overline{\pi}$ is defined for any $(x, p) \in \overline{M}$ by

$$\overline{h}(x, p) = x, \qquad \text{i.e., } \overline{h}(\tau k, \rho l) = \tau k.$$

We have shown (details in Appendix 3, see footnote §) that, for any $s \geq \rho(\tau\bar{\beta}_2)$ and $\hat{\delta}_1(\cdot)$, $\hat{\delta}_2(\cdot)$ specified below, $\bar{\pi}$ covers $\hat{E}$; furthermore that, with the same $\hat{\delta}_i$'s and s sufficiently large, error $\epsilon_\phi(\bar{\pi})$ on $\hat{E}$ can be brought arbitrarily close to $\tau/2$.

An appropriate choice of the tolerance functions turns out to be

$$\hat{\delta}_1(e_1) = \frac{\tau\beta_1 + \rho/s}{2}, \qquad \hat{\delta}_2(e_2) = \frac{\tau\beta_2}{2}.$$

It can be shown that the equilibrium correspondence $\bar{\mu}$ is pseudo-Lipschitzian ("smooth") on the set $E'_{\epsilon\eta}$ which has the uniqueness property for ϕ.[18] It then follows by Proposition I[19] that, among discrete round-off mechanisms that realize the ϵ-accurate Walrasian correspondence $O^*_\epsilon\tau$ and are pseudo-Lipschitzian on $E'_{\epsilon\eta}$, none has a message space of discrete dimension[20] less than that of $\bar{M}$, which is 2. Thus $\bar{\pi}$ can be said to have a minimal size message space, a property analogous to that of the continuum Walrasian process in Euclidean spaces.

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[18] $E'_{\epsilon\eta}$ is the counterpart of $E_{\epsilon\eta}$ in a generalization of Proposition I. Explicitly, $E'_{\epsilon\eta} = E^{1'}_{\epsilon\eta} \times E^{2'}_{\epsilon\eta}$, where $E^{i'}_{\epsilon\eta} = \{\theta_i, \beta_i) : \beta_i = \beta'_i, \theta_i = n \cdot (\eta + 2\epsilon(\beta'_1 + \beta'_2)), n = 0, 1, 2, \ldots\}$, $i = 1, 2$, the β'_i are fixed positive numbers, and $\eta > \tau(\beta'_1 + \beta'_2)$. $\bar{\mu}$ is not pseudo-Lipschitzian on all of $\hat{E}$, but that is not needed to justify our claim of minimality—just as, in Proposition I, it is only required that μ be pseudo-Lipschitzian on $E_{\epsilon\eta}$. (The situation is analogous in minimality results for continuum mechanisms discussed in Section A2.)

[19] More precisely, by a generalization of Proposition I in which $E_{\epsilon\eta}$ is replaced by $E'_{\epsilon\eta}$; the correspondence O^*_ϵ is replaced by $O^*_{\epsilon\tau}$, where $x \in O^*_{\epsilon\tau}(e)$ if and only if $|x - \phi(e)| \leqq \tau/2 + \epsilon$; and the sequence of "candidate" message spaces $\{0\}, N, N^2, \ldots$ is replaced by $\{0\}, L_1, L_2, \ldots$ where L_k is a product of k terms, each isomorphic to N.

[20] A message space is said to have discrete dimension k, where $k = 0, 1, 2, \ldots$, if it has the same Lipschitzian size as N^k_z, the set of k-tuples of integers, with $N^0 = \{0\}$.

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