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ON THE STIEFEL–WHITNEY NUMBERS OF COMPLEX MANIFOLDS AND OF SPIN MANIFOLDS

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THE FIRST section of this paper will characterize those cobordism classes in the Thom cobordism rings $\mathfrak{N}_*$ and Ω_* which contain complex manifolds†. The second section attempts to characterize those classes in $\mathfrak{N}_*$ which contain spin manifolds†. The attempt succeeds only through dimension 23.

§1. COMPLEX MANIFOLDS

Conner and Floyd have proved the following remarkable theorem [1]. *Let V_R be a 'real form' of the complex algebraic variety V_C, both being non-singular. Then V_C is non-oriented cobordant to $V_R \times V_R$.*

For example the complex projective space $P_k(C)$ is cobordant to the product $P_k(R) \times P_k(R)$.

An interesting consequence of this result is the following.

THEOREM (1). *A non-oriented cobordism class contains a complex manifold if and only if it contains a square $N \times N$.*

Proof. First consider the following example of the Conner–Floyd theorem. Let $H_{mn}(C)$ denote a non-singular hypersurface of degree (1,1) in the product $P_m(C) \times P_n(C)$. [In terms of homogeneous co-ordinates $(w_0, \dots, w_m)$ and $(z_0, \dots, z_n)$ with $m \leq n$ this hypersurface can be defined as the locus $w_0z_0 + w_1z_1 + \dots + w_mz_m = 0$. It can also be thought of as a $P_{n-1}(C)$-bundle over $P_m(C)$.] Then $H_{mn}(C)$ is non-oriented cobordant to the square $H_{mn}(R) \times H_{mn}(R)$ of the corresponding real variety.

These manifolds are of interest since it is known that the weakly complex cobordism ring Ω_*^U is generated (redundantly) by the complex cobordism classes of the manifolds $P_k(C)$ and $H_{mn}(C)$. (See [4, 6].)

This proves that the image of the natural ring homomorphism $j: \Omega_*^U \to \mathfrak{N}_*$ is generated by cobordism classes $(P_k(R))^2$ and $(H_{mn}(R))^2$ which are squares. But the collection of all squares $(N \times N) \in \mathfrak{N}_*$ forms a sub-ring which will be denoted briefly by $(\mathfrak{N}_*)^2$. Thus we have proved that $j(\Omega_*^U) \subset (\mathfrak{N}_*)^2$.

† All manifolds are to be smooth, compact, and without boundary. Connectedness is not required.

Now note that the cobordism ring $\mathfrak{N}_*$ is generated by the cobordism classes $(P_k(R))$ and $(H_{mn}(R))$. In fact:

LEMMA (1). *$\mathfrak{N}_*$ is a polynomial ring over Z_2 with independent generators* $(P_{2t}(R))$ and $(H_{2^k,\, 2t2^k}(R))$ *where* $t, k \geq 1$.

Proof. According to Thom [10] we must verify (1) that the manifolds listed include precisely one manifold M^p for each dimension p which is not of the form $2^i - 1$, and (2) that $s_p(w_1, \ldots, w_p)\,[M^p] \neq 0$, where s_p denotes the polynomial which expresses the symmetric function $\sum t_i^p$ in terms of the elementary symmetric functions. A straightforward computation shows that

$$s_{m+n-1}(w_1, \ldots, w_{m+n-1})[H_{mn}(R)] = (m+n)!/m!\,n!$$

(reduced modulo 2) for $m, n \geq 2$. (Compare [6] or the proof of Lemma (4)). The rest of the argument can easily be supplied by the reader.

Now let us return to the proof of Theorem [1]. Since $\mathfrak{N}_*$ is generated by the $(P_k(R))$ and $(H_{mn}(R))$, it follows that $(\mathfrak{N}_*)^2$ is generated by $(P_k(R))^2 = (P_k(C))$ and $(H_{mn}(R))^2 = H_{mn}(C)$. This shows that $j(\Omega_*^U) \supset (\mathfrak{N}_*)^2$; which completes the proof.

Remark. This argument suggests the conjecture that every smooth manifold is diffeomorphic to a real form of some complex algebraic variety. (Compare Nash [8].)

Now consider the Stiefel–Whitney numbers of a complex manifold V. If i is odd, then the Stiefel–Whitney class $w_i \in H^i(V; Z_2)$ is zero. (Compare [9, §41.8].) Hence any Stiefel–Whitney number $w_{i_1} \ldots w_{i_k}[V]$ which involves an odd w_i will be zero. Conversely:

THEOREM (2). *Let M be a manifold such that every Stiefel–Whitney number involving an odd w_i is zero. Then M is cobordant to a complex manifold. In fact if M can be oriented, then M is oriented cobordant to a complex manifold.*

The proof in the non-oriented case will be based on the following.

LEMMA (2). *The Stiefel–Whitney numbers of a product $N \times N$ are given by*

$$w_{2i_1} w_{2i_2} \ldots w_{2i_r}[N \times N] = w_{i_1} w_{i_2} \ldots w_{i_r}[N],$$

while the number $w_{j_1} \ldots w_{j_s}[N \times N]$ is zero if some j_h is odd.

The proof will be left to the reader.

Now let M be a manifold such that every 'odd' Stiefel–Whitney number of M is zero. We will construct a manifold N of half the dimension so that

$$w_{i_1} \ldots w_{i_r}[N] = w_{2i_1} \ldots w_{2i_r}[M]$$

for all $i_1 \ldots i_r$. This will imply that M is cobordant to $N \times N$ and hence, by Theorem [1], to a complex manifold.

Dold [2] has shown that a given collection of Stiefel–Whitney numbers actually corresponds to a manifold N^n if and only if the following *Wu relations* are satisfied. For each $i_1 + \ldots + i_k + p = n$ we must have

$$(Sq^p(w_{i_1} \ldots w_{i_k}))[N^n] = u_p w_{i_1} \ldots w_{i_k}[N^n].$$

[This is to be understood as follows. The expression $Sq^p(w_{i_1} \dots w_{i_k})$ is to be expressed as a polynomial in $w_1, w_2, \dots$, using the Wu formula

$$Sq^p w_n = w_p w_n + \binom{p-n}{1} w_{p-1} w_{n+1} + \dots + \binom{p-n}{p} w_0 w_{n+p};$$

where $\binom{a}{t} = a(a-1) \dots (a-t+1)/t!$. The Wu classes $u_p = u_p(w_1, w_2, \dots)$ are defined inductively by the conditions $u_1 = w_1$ and

$$u_n + Sq^1 u_{n-1} + Sq^2 u_{n-2} + \dots = w_n.]$$

Formulating this more invariantly, let $\mathscr{R}^n \subset H^n(B_O; Z_2)$ be the vector space generated by all elements of the form $Sq^p x + u_p x$. The Stiefel–Whitney numbers of each manifold N determine a homomorphism

$$h_N : H^n(B_O; Z_2) \longrightarrow Z_2,$$

where $h_N(x) = x[N]$. Dold's theorem says that a given homomorphism $H^n(B_O; Z_2) \to Z_2$ corresponds to a manifold if and only if it annihilates $\mathscr{R}^n$.

Define the 'doubling homomorphism' $d : H^*(B_O; Z_2) \to H^*(B_O; Z_2)$ by $d(w_i) = w_{2i}$. Let M, of dimension $2n$, satisfy the hypothesis of Theorem (2). Then we will show that $h_M d : H^n(B_O; Z_2) \to Z_2$ annihilates $\mathscr{R}^n$. This will prove the existence of the required manifold N.

Let $\mathscr{I} = (w_1, w_3, w_5, \dots) \subset H^*(B_O; Z_2)$ denote the ideal generated by the odd w_i.

LEMMA (3). *The doubling homomorphism d satisfies the congruence*

(1) $Sq^{2i} d(x) \equiv d(Sq^i x) \qquad \text{mod } \mathscr{I}.$

Furthermore

(2) $d(u_p) \equiv u_{2p} \qquad \text{mod } \mathscr{I}.$

Proof. In the special case $x = w_n$ we have

$$Sq^{2p} d(w_n) = Sq^{2p} w_{2n} = \sum_j \binom{2p-2n}{j} w_{2p-j} w_{2n+j}.$$

Deleting the odd terms, and using the congruence

$$\binom{2p-2n}{2s} \equiv \binom{p-n}{s} \qquad \text{(mod 2)},$$

this expression takes the required form

$$\sum_s \binom{p-n}{s} w_{2p-2s} w_{2n+2s} = d(Sq^p w_n).$$

Now assume that the congruence (1) has already been estalished for x and y. Then

$$Sq^{2p} d(xy) \equiv \sum_i (Sq^{2i} dx)(Sq^{2p-2i} dy)$$

$$\equiv \sum_i (dSq^i x)(dSq^{p-i} y) = dSq^p(xy) \qquad (\text{mod } \mathscr{I}).$$

Proceeding inductively, it follows that (1) is true for all elements of $H^*(B_O; Z_2)$.

The congruence $d(u_p) \equiv u_{2p}$ is proved by a straightforward induction on p. In order to carry out the induction, it is first necessary to verify that the ideal $\mathscr{I}$ is closed under the action of the squaring operations. Details will be left to the reader.

Now consider a manifold $M = M^{2n}$ which satisfies the condition $h_M(\mathscr{I}^{2n}) = 0$. For any generator $Sq^p x + u_p x$ of $\mathscr{R}^n$ we have

$$h_M d(Sq^p x + u_p x) = h_M(Sq^{2p}(dx) + u_{2p} dx + (\text{terms in } \mathscr{I}^{2n})) = 0.$$

Thus $h_M d$ satisfies the Wu relations, and hence is equal to h_N for some manifold N. This proves that M is cobordant to $N \times N$, and hence is cobordant to some complex manifold V.

Now suppose that M is an oriented manifold. Then the difference $M - V$ is an oriented manifold with all Stiefel–Whitney numbers zero. From the Rohlin exact sequence

$$\Omega_* \xrightarrow{2} \Omega_* \longrightarrow \mathfrak{N}_*$$

(compare [12]) it follows that $M - V$ is oriented cobordant to $2M_1$ for some oriented manifold M_1.

Recall that the ring Ω_* can be described as the direct sum of a polynomial ring $Z[Y^4, Y^8, Y^{12}, \ldots]$ and an ideal consisting of elements of order 2. The Y^{4i} are all complex manifolds. (For example we can take Y^{4i} equal to $P_{2i}(C)$ for $i = 1, 2, 3, 5, 6$; and equal to $9P_8(C) + H_{3,6}(C)$ for $i = 4$. Compare [5], [6].) Thus every oriented manifold is cobordant to a sum $V_1 - V_2 + T$ where V_1, V_2 are complex manifolds, and $2T \sim 0$. Replacing M_1 by such an expression, we see that $M - V \sim 2M_1 \sim 2V_1 - 2V_2 + 0$.

Thus M is cobordant to a difference of algebraic varieties, and hence (compare [4], [6]) is cobordant to an algebraic variety. This completes the proof of Theorem (2).

§2. SPIN MANIFOLDS

Let us start by considering some examples of spin manifolds.

LEMMA (4). *If* $m \equiv n \equiv 0 \pmod 2$ *then the complex hypersurface* $H_{mn}(C)$ *is a spin manifold.*

Proof. Let $a, b \in H^2(P_m(C) \times P_n(C); Z_2)$ be the standard generators. Then the Stiefel–Whitney class $w_2(P_m(C) \times P_n(C))$ is equal to $(m+1)a + (n+1)b$. The class $w_2(v)$ of the normal bundle of $H_{mn}(C)$ is equal to $(a + b) \mid H_{mn}(C)$, since $a + b$ is the cohomology class dual to this submanifold. Subtracting these two we obtain

$$w_2 H_{mn}(C) = (ma + nb) | H_{mn}(C)$$

which completes the proof.

A similar argument shows that the corresponding real variety $H_{mn}(R)$ is orientable, if m and n are even. Remembering that $H_{mn}(C)$ is cobordant to $H_{mn}(R) \times H_{mn}(R)$, we are tempted towards the following:

CONJECTURE. *If M is an orientable manifold then $M \times M$ is non-oriented cobordant to a spin manifold.* [Added in proof. This has been proved by P. G. Anderson: *Bull. Amer. Math. Soc.* **70** (1964), 818–819.]

As an example, consider the complex projective space $P_n(C)$.

LEMMA (5). *The product $P_n(C) \times P_n(C)$ is non-oriented cobordant to the quaternion projective space $P_n(H)$.*

Since $P_n(C)$ is orientable, and $P_n(H)$ is clearly a spin manifold, this tends to support the conjecture.

Proof. Both $P_n(C)$ and $P_n(H)$ have a mod 2 cohomology ring which has one generator a (of dimension 2 or 4 respectively) and one relation $a^{n+1} = 0$. In each case the total Stiefel–Whitney class is given by $w = (1 + a)^{n+1}$. (Compare Hirzebruch [3].) It follows that the Stiefel–Whitney numbers of these manifolds are given by the formula

$$w_{2i_1} \dots w_{2i_k}[P_n(C)] = w_{4i_1} \dots w_{4i_k}[P_n(H)] = \binom{n+1}{i_1} \dots \binom{n+1}{i_k}.$$

Together with Lemma (2), this completes the proof.

The cobordism ring Ω_* has been computed by Wall [12]. In dimensions less than 15 it follows from Wall's work that Ω_* is generated by the classes of the manifolds $P_2(C)$, $H_{2,4}(R)$, $P_4(C)$, $H_{2,8}(R)$, $H_{4,8}(R)$, $P_6(C)$, and $H_{2,12}(R)$ (of dimensions 4, 5, 8, 9, 11, 12, 13 respectively). But for each of these manifolds we have verified that the square is cobordant to a spin manifold. Thus:

THEOREM (3). *If M is orientable of dimension less than 15 then $M \times M$ is non-oriented cobordant to a spin manifold.*

I do not know what happens in dimensions ≥ 15.

Conversely we may ask whether every spin manifold is non-oriented cobordant to the square of an orientable manifold. It will turn out that this is true for spin manifolds of dimension ≤ 23. Again I do not know what happens in higher dimensions.

Let us look at Stiefel–Whitney numbers. Consider manifolds M which satisfy the following:

HYPOTHESIS (1). *Every Stiefel–Whitney number $w_{i_1} \dots w_{i_k}[M]$ which involves either w_1 or w_2 is zero.*

Clearly every spin manifold satisfies this hypothesis. Conversely we must ask:

PROBLEM. Does every non-oriented cobordism class which satisfies the hypothesis (1) contain a spin manifold?

We will verify that this is true in dimensions ≤ 23.

LEMMA (6). *If M^n satisfies (1) with $n \leq 23$, then every $w_{i_1} \dots w_{i_k}[M^n]$ involving an odd w_i is zero.*

The proof, which will be outlined presently, involves a tedious case by case application of the Wu relations. This lemma is definitely false for $n = 24$. (In slightly higher dimensions, the lemma is probably true for $24 < n < 29$; but false for $n = 29$.)

Assuming Lemma (6), it follows from Theorem (2) that M is non-oriented cobordant to a product $N \times N$. The identity

$$w_{i_1} \dots w_{i_k}[N] = w_{2i_1} \dots w_{2i_k}[M]$$

now implies that every Stiefel–Whitney number of N which involves w_1 is zero. But according to Wall [12, §9] this means that N is cobordant to an orientable manifold. Finally, using Theorem (3), it follows that M is cobordant to a spin manifold. Thus we have proved:

THEOREM (4). *For a non-oriented cobordism class* (M) *of dimension* ≤ 23 *the following three conditions are equivalent*

(1) *each* $w_{i_1} \dots w_{i_k}[M]$ *involving* w_1 or w_2 *is zero;*

(2) (M) *contains a spin manifold;*

(3) (M) *contains the square* $N \times N$ *of an orientable manifold.*

It follows that the natural homomorphism $h: \Omega_n^{\mathrm{Spin}} \to \mathfrak{N}_n$ is zero for odd values of n up to $n = 23$. The rank (over Z_2) of h can be tabulated as follows for even values of n. (Compare [7].)

n	2	4	6	8	10	12	14	16	18	20	22	24
rank $h(\Omega_n^{\mathrm{Spin}})$	0	0	0	1	1	0	0	2	2	1	1	(3 or 4)

The ambiguity in dimension 24 can be described as follows. There exists an orientable manifold X of dimension 24 such that every Stiefel–Whitney number involving w_2 is zero, but such that

$$w_4 w_6 w_7 w_7[X] \neq 0.$$

(This is proved by an exhaustive examination of the Wu relations: to be more precise X can be chosen so that $w_4 w_6 w_7^2 = w_6^4 = w_4^6 = w_4^3 w_6^2 = (w_4 w_8)^2 \neq 0$, but so that all other Stiefel–Whitney numbers are zero.) It is not known whether or not this X is cobordant to a spin manifold.

This description can be transformed into one involving Pontrjagin numbers as follows. Using the Wu relation $(Sq^2 + u_2)(w_4 w_6^3)[X] = 0$ one finds that

$$w_6^4[X] = w_4 w_6 w_7^2[X] \neq 0.$$

But w_6^2 is the mod 2 reduction of the Pontrjagin class p_3. Therefore $p_3^2[X] \equiv 1 \pmod 2$.

The description can be further transformed by considering the polynomial

$$s_6 = s_6(p_1, \dots, p_6) = p_1^6 - 6p_1^4 p_2 \pm \dots - 6p_6,$$

which expresses the symmetric function $\sum t_i^6$ in terms of elementary symmetric functions. Since

$$s_6 \equiv s_3^2 \equiv p_1^6 + p_1^2 p_2^2 + p_3^2 \qquad \pmod 2,$$

and since $p_1^6[X] \equiv w_2^{12}[X] = 0$, and similarly $p_1^2 p_2^2[X] \equiv 0$, we see that

$$s_6(p_1, \dots, p_6)[X] \equiv p_3^2[X] \equiv 1 \qquad \pmod 2.$$

Thus we are left with the following:

PROBLEM. Does there exist a spin manifold Σ of dimension 24 so that $s_6(p_1, \dots, p_6)[\Sigma] \equiv 1 \pmod 2$?

The rest of this paper will be concerned with the proof of Lemma (6). We first give two preliminary statements which are true in arbitrary dimensions.

LEMMA (7). *If M satisfies Hypothesis* (1) *then every $w_{i_1} \dots w_{i_k}[M]$ involving w_3, w_5 or w_9 is zero also.*

(These particular w_i presumably occur because w_3, w_5, and w_9 map into zero in $H^*(B^{\text{Spin}}; Z_2)$. Compare Thomas [11].)

Proof. Let $(w_a, w_b, \dots) \subset H^*(B_O; Z_2)$ denote the ideal generated by $w_a, w_b, \dots$. Let x stand for an arbitrary element of $H^*(B_O; Z_2)$.

The formulae

$$\begin{aligned} Sq^1(w_2 x) &\equiv w_3 x && \operatorname{mod}(w_1, w_2) \\ Sq^2(w_3 x) &\equiv w_5 x && \operatorname{mod}(w_1, w_2, w_3) \\ Sq^4(w_5 x) &\equiv w_9 x && \operatorname{mod}(w_1, w_2, w_3, w_5) \end{aligned}$$

are easily verified. Now if every $w_1 x'[M]$ and every $w_2 x''[M]$ is zero then the Wu relation

$$((Sq^1 + u_1)w_2 x)[M] = 0$$

implies that $w_3 x[M] = 0$. Hence the relation

$$((Sq^2 + u_2)w_3 x)[M] = 0$$

implies that $w_5 x[M] = 0$; and similarly with w_9.

LEMMA (8). *Let M satisfy* (1) *and suppose that the integers $i_1 \dots i_k$ are all either odd, or equal to* 4 *or* 8, *or occur in pairs; and that at least one of the i_j is odd. Then $w_{i_1} \dots w_{i_k}[M] = 0$.*

(For example $w_6^2 w_7 w_8[M^{27}] = 0$.)

Proof. Suppose that i_1 is odd. Note that

$$Sq^1(w_{i_1 - 1} w_{i_2} \dots w_{i_k}) \equiv w_{i_1} w_{i_2} \dots w_{i_k} \operatorname{mod}(w_1, w_5, w_9).$$

Thus the Wu relation $(Sq^1(w_{i_1 - 1} w_{i_2} \dots w_{i_k}))[M] = 0$ completes the proof.

We are now ready to prove Lemma (6). To avoid too much tedium, we will only consider the most difficult dimension, which happens to be 21. Consider then all partitions $i_1 + \dots + i_k = 21$ which are not excluded by Lemmas (7, 8). There turn out to be seven such partitions, namely:

10, 11; 4, 6, 11; 4, 7, 10; 6, 15; 7, 14; 6, 7, 8; and 4, 4, 6, 7.

To take care of the first, consider the Wu relation

$$(Sq^2(w_8 w_{11}) + u_2 w_8 w_{11})[M] = 0,$$

where

$$Sq^2(w_8 w_{11}) \equiv w_{10} w_{11} + w_8 w_{13}, \quad \text{and} \quad u_2 \equiv 0 \operatorname{mod}(w_1, w_2).$$

This proves that $w_{10} w_{11}[M]$ is equal to $w_8 w_{13}[M]$ which is zero by Lemma (8). Next consider the relation

$$(Sq^4(w_6 w_{11}) + u_4 w_6 w_{11})[M] = 0,$$

where

$$Sq^4(w_6 w_{11}) \equiv w_{10} w_{11}, \quad u_4 \equiv w_4 \operatorname{mod}(w_1, w_2, w_3)$$

This implies that $w_4w_6w_{11}[M]$ is equal to $w_{10}w_{11}[M]$, which we have just shown is zero. The remaining five partitions are handled similarly, using the Wu relations corresponding to $Sq^1(w_4w_6w_{10})$, $Sq^2(w_4w_{15})$, $Sq^2(w_7w_{12})$, $Sq^2(w_4w_7w_8)$ and $Sq^2(w_4{}^3w_7)$ respectively.

The reader who has enough patience should have no difficulty in carrying out the proof for other dimensions ≤ 23.

REFERENCES

1. P. Conner and E. Floyd: *Differentiable Periodic Maps*, *Ergebn. Math.* **33**, Springer, 1964. (See p. 63.)
2. A. Dold: Vollständigkeit der Wu-schen Relationen zwischen den Stiefel–Whitneyschen Zahlen differenzierbarer Mannigfaltigkeiten, *Math. Z.* **65** (1956), 200–206.
3. F. Hirzebruch: Uber die quaternionalen projektiven Räume, *S. B. bayer. Akad. Wiss.* (1953), 301–312.
4. F. Hirzebruch: Komplexe Mannigfaltigkeiten, *Proceedings* 1958 *International Congress of Mathematicians*, Cambridge (1960), 119–136.
5. J. Milnor: A survey of cobordism theory, *Enseign. Math.* **8** (1962), 16–23.
6. J. Milnor: On the cobordism ring Ω_* and a complex analogue II (in preparation).
7. J. Milnor: Remarks concerning spin manifolds (to be published).
8. J. Nash: Real algebraic manifolds, *Ann. Math., Princeton* **56** (1952), 405–421.
9. N. Steenrod: *The Topology of Fibre Bundles*. Princeton University Press, 1951.
10. R. Thom: Quelques propriétés globales des variétés différentiables, *Comment. Math. Helvet.* **28** (1954), 17–86.
11. E. Thomas: On the cohomology groups of the classifying space for the stable spinor group, *Bol. Soc Mat. Mex.* (1962), 57–69.
12. C. T. C. Wall, Determination of the cobordism ring, *Ann. Math., Princeton* 72 (1960), 292–311.
13. Wu Wen-Tsün: Les *i*-carrés dans une variété grassmannienne, *C.R. Acad. Sci., Paris* **230** (1950), 918–920.

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