Bounds for the Perron root using max eigenvalues

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Abstract

Using the techniques of max algebra, a new proof of Al’pin’s lower and upper bounds for the Perron root of a nonnegative matrix is given. The bounds depend on the row sums of the matrix and its directed graph. If the matrix has zero main diagonal entries, then these bounds may improve the classical row sum bounds. This is illustrated by a generalized tournament matrix.

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1. Introduction

Our aim here is twofold. Firstly, we want to publicize bounds for the Perron root of a nonnegative matrix that have been proved by Al’pin [1]. If the matrix contains zero diagonal entries, then these bounds may be tighter than the classical bounds, namely that the Perron root lies between the minimum and maximum row sums; see, for example, [6, Theorem 8.1.22]. Secondly, we present a new proof of Al’pin’s bounds by using the techniques of max algebra; thus illustrating that this algebra is useful in proving results in the conventional algebra.

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The max algebra system \([2,4,5]\) consists of the set \(\mathbb{R}_+\) of nonnegative real numbers together with the sum and product defined for \(a, b \in \mathbb{R}_+\) as
\[
a \oplus b = \max\{a, b\}, \quad a \otimes b = ab.
\]
For an \(n\)-by-\(n\) nonnegative matrix \(A = (a_{ij})\) and an \(n\)-by-1 nonnegative vector \(x\)
\[
(A \otimes x)_i = \max_{j=1,\ldots,n}\{a_{ij}x_j\}, \quad i = 1, \ldots, n.
\]
Note that the max algebra system is isomorphic to the max-plus algebra system; for the latter see [3].

The eigenequation in the max algebra system is
\[
A \otimes x = \mu(A)x,
\]
where \(\mu(A)\) is the max eigenvalue of \(A\) with a max eigenvector \(x\). If \(A\) is irreducible, then \(\mu(A)\) is uniquely defined by this equation, and \(x\) is positive. However, if \(A\) is reducible, then \(\mu(A)\) is the maximum of the numbers satisfying this eigenequation with \(x\) nonnegative and nonzero. For \(A = (a_{ij})\) an \(n\)-by-\(n\) nonnegative matrix, we denote by \(G(A)\) the directed graph of \(A\), i.e., \(G(A) = (V, E)\) with \(V = \{1, \ldots, n\}\) and \((i, j) \in E \iff a_{ij} > 0\). The max eigenvalue \(\mu(A)\) is the maximal geometric circuit mean of \(G(A)\). In this directed graph, \(\gamma\) is a simple cycle of length \(q\) described by a sequence of distinct integers \(i_1, \ldots, i_q \in \{1, \ldots, n\}\) for which \(a_{i_1i_2}, \ldots, a_{i_{q-1}i_q}, a_{i_qi_1}\) are positive. Then with \(|\gamma| = q\)
\[
\mu(A) = \max_{\gamma} \left\{ (a_{i_1i_2} \cdots a_{i_{q-1}i_q} a_{i_qi_1})^{1/|\gamma|} \right\};
\]
see, for example [7, p. 367].

We begin by presenting Al’pin’s lower and upper bounds on the Perron root, and then proceed to our new proof. We illustrate the improvement of these bounds over the classical bounds by considering a generalized tournament matrix; see, for example [8].

2. Main results

Let \(A = (a_{ik})\) be a nonnegative \(n\)-by-\(n\) matrix with row sums \(r_i = r_i(A) = \sum_{k=1}^{n} a_{ik}\), for \(i = 1, \ldots, n\) and Perron root (spectral radius) denoted by \(\rho(A)\). With a simple cycle \(\gamma\) as defined in Section 1, the set of all simple cycles in \(G(A)\) is denoted by \(C(A)\).

In [1] the following result is proved by Al’pin.

**Theorem A.** If \(A\) is a nonnegative \(n\)-by-\(n\) matrix with no zero row, then
\[
\min \left\{ \left( \prod_{i \in \gamma} r_i(A) \right)^{1/|\gamma|} : \gamma \in C(A) \right\} \leq \rho(A) \leq \max \left\{ \left( \prod_{i \in \gamma} r_i(A) \right)^{1/|\gamma|} : \gamma \in C(A) \right\}.
\]

If \(A\) is irreducible, then either both inequalities hold with equality or both are strict.

In the following we give a new proof of the above theorem that, in our opinion, is easier and simpler than the proof in [1]. In particular it uses techniques from the theory of max algebra. For this we introduce the matrices
\[ S(A) = (r_i \text{ sgn}(a_{ij})), \quad i, j = 1, \ldots, n \]  \hspace{1cm} (2.2)

and if all \( r_i \) are positive

\[ S^{-}(A) = (r_i^{-1} \text{ sgn}(a_{ij})), \quad i, j = 1, \ldots, n. \]  \hspace{1cm} (2.3)

As stated in Section 1, the max eigenvalue \( \mu(S(A)) \) is the maximal circuit geometric mean, hence

\[ \mu(S(A)) = \max \left\{ \left( \prod_{i \in \gamma} r_i(A) \right)^{1/|\gamma|} : \gamma \in C(A) \right\}. \]  \hspace{1cm} (2.4)

Similarly

\[ \frac{1}{\mu(S^{-}(A))} = \min \left\{ \left( \prod_{i \in \gamma} r_i(A) \right)^{1/|\gamma|} : \gamma \in C(A) \right\}. \]  \hspace{1cm} (2.5)

For a positive vector \( x = (x_i) \) we denote its inverse vector by \( x^{-} = (x_i^{-1}) \).

**Lemma 2.1.** Let \( A \) be a nonnegative matrix with no zero row and \( S(A) \) and \( S^{-}(A) \) be defined as in (2.2) and (2.3). Then

\[ Ax \leq S(A) \otimes x \quad \forall x \geq 0 \]  \hspace{1cm} (2.6)

and

\[ (Ax)^{-} \leq S^{-}(A) \otimes x^{-} \quad \forall x > 0. \]  \hspace{1cm} (2.7)

**Proof.** Observe that \((Ax)_i = \sum_{k=1}^{n} a_{ik}x_k\) and this is obviously less than or equal to \((S(A) \otimes x)_i = \sum_{k=1}^{n} a_{ik} \max_{k : a_{ik} > 0} x_k\). Hence (2.6) holds.

Similarly, comparing

\[ (Ax)_i^{-1} = \frac{1}{\sum_{k=1}^{n} a_{ik}x_k} \]

with

\[ \frac{1}{r_i \min_{k : a_{ik} > 0} x_k} = r_i^{-1} \max_{k : a_{ik} > 0} x_k^{-1} = (S^{-}(A) \otimes x^{-})_i \]

gives the inequality (2.7). \( \Box \)

Using this lemma, we now prove the inequality part of Theorem A.

**Theorem 2.2.** If \( A \) is a nonnegative \( n \times n \) matrix with no zero row, then

\[ \frac{1}{\mu(S^{-}(A))} \leq \rho(A) \leq \mu(S(A)). \]  \hspace{1cm} (2.8)

**Proof.** First we assume that \( A \) and hence \( S(A) \) are irreducible. Then consider a max eigenvector \( x \) of \( S(A) \) with associated max eigenvalue \( \mu(S(A)) \), i.e., \( S(A) \otimes x = \mu(S(A))x \). By (2.6) \( Ax \leq S(A) \otimes x = \mu(S(A))x \). As \( x > 0 \), multiplying by \( y^T \), where \( y \) is the Perron vector of \( A^T \), implies that \( \rho(A) \leq \mu(S(A)). \) \hspace{1cm} (2.9)
Similarly, as also $S^{-}(A)$ is irreducible, a max eigenvector $z$ of $S^{-}(A)$ with the associated max eigenvalue $\mu(S^{-}(A))$ is strictly positive. Then by (2.7) with $y = z^{-}$ it follows that $(Ay) \leq S^{-}(A) \otimes z = \mu(S^{-}(A))z$, and hence $Ay \geq (\mu(S^{-}(A)))^{-1}z^{-} = (\mu(S^{-}(A)))^{-1}y$, which implies as above that

$$(\mu(S^{-}(A)))^{-1} \leq \rho(A). \quad (2.10)$$

Thus (2.9) and (2.10) give the result for $A$ irreducible.

In the general case, $\rho(A)$ is the Perron root of an irreducible principal submatrix $A_1$ of $A$. Then by construction of $S(A)$ it follows that $S(A_1)$ is entrywise not greater than the corresponding principal submatrix $S_1$ of $S(A)$. Hence

$$\rho(A) = \rho(A_1) \leq \mu(S(A_1)) \leq \mu(S_1) \leq \mu(S(A)).$$

Here the already proven irreducible version of (2.8) has been applied to $A_1$ and $S(A_1)$, and monotonicity of the max eigenvalue has been used. Similarly the left inequality of (2.8) follows by considering $A_1$ and $S^{-}(A_1)$.

The proof of the equality part of Theorem A for an irreducible matrix is contained in the following theorem, which gives some additional information. For its proof the next lemma is useful, which is proved by showing that statement 1 is equivalent to statement 2, and that statement 3 is equivalent to statement 2.

**Lemma 2.3.** For a nonnegative matrix $A$ with $S(A)$ and $S^{-}(A)$ defined in (2.2) and (2.3) and a positive vector $x$ the following are equivalent:

1. $(Ax)_i = (S(A) \otimes x)_i$,
2. $x_j = x_k$ if $a_{ik} > 0, a_{ij} > 0$,
3. $(Ax)^{-1}_i = (S^{-}(A) \otimes x^{-})_i$.

**Theorem 2.4.** Let $A$ be an irreducible nonnegative matrix. Then either

$$\frac{1}{\mu(S^{-}(A))} < \rho(A) < \mu(S(A)) \quad (2.11)$$

or

$$\frac{1}{\mu(S^{-}(A))} = \rho(A) = \mu(S(A)). \quad (2.12)$$

In the equality case the Perron vector of $A$ is also the max eigenvector of $S(A)$ and its inverse is the max eigenvector of $S^{-}(A)$.

**Proof.** We show that if (2.11) does not hold, then both equalities of (2.12) hold. First assume the right hand equality of (2.12), namely $\rho(A) = \mu(S(A))$. If $y$ is a max eigenvector of $S(A)$ then $Ay \leq S(A) \otimes y = \rho(A)y$ and hence, as $A$ is irreducible, it follows that $Ay = \rho(A)y$ and thus $y$ is a Perron vector of $A$. In particular $Ay = S(A) \otimes y$. Lemma 2.3 implies that $(Ay)^{-1} = S^{-}(A) \otimes y^{-}$ and hence $\rho(A)y = (S^{-}(A) \otimes y^{-})^{-1}$, implying that $\rho(A) = (\mu(S^{-}(A)))^{-1}$. This shows the left hand equality of (2.12).

Now assume this equality of (2.12) holds, namely $\rho(A) = (\mu(S^{-}(A)))^{-1}$. Define $y^{-}$ to be a max eigenvector of $S^{-}(A)$. By (2.7) it follows that $Ay \geq \rho(A)y$ and hence $Ay = \rho(A)y$, since $A$ is irreducible. So statement 3 of Lemma 2.3 holds. Hence by this lemma $Ay = S(A) \otimes y = \rho(A)y$.
and thus the right hand equality of (2.12) holds. The statement about the eigenvectors being equal
is obvious from this proof. □

We conclude this section with a remark about the optimality of the upper bound in Theorem 2.2. For the proof of Theorem 2.4 we only use Lemma 2.3. Hence, if $B$ is a nonnegative matrix satisfying

$$Ax \leq B \otimes x \quad \forall x \geq 0,$$

then $\rho(A) \leq \mu(B)$. The next theorem states that there is no matrix $B$ below $S(A)$ satisfying (2.13). Thus $S(A)$ is optimal in this sense.

**Theorem 2.5.** Let $A$ and $B = (b_{ij})$ be two nonnegative matrices with $S(A)$ defined as in (2.2). If (2.13) holds and if $B \leq S(A)$ then $B = S(A)$.

**Proof.** Assume that for some $i, j$ the strict inequality $b_{ij} < (S(A))_{ij}$ holds. Thus $(S(A))_{ij} > 0$ and hence is equal to $r_i = r_i(A)$. Without loss of generality take $i = j = 1$. Let $x = (1 + \epsilon, 1, \ldots, 1)$. Then for $\epsilon$ satisfying $\epsilon(b_{11} - a_{11}) < r_1 - b_{11}$ it follows that $(Ax)_1 = r_1 + \epsilon a_{11} > \max(b_{11}(1 + \epsilon), r_1) \geq (B \otimes x)_1$ and hence (2.13) does not hold. □

3. Example

We illustrate the improvement of the bounds given by Theorem 2.2 with the following example. Let $T = (t_{ij})$ be an $n$-by-$n$ generalized tournament matrix (see, for example, [8]) with

$$t_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \alpha, & \text{if } i < j, \\ \beta = 1 - \alpha, & \text{if } i > j \end{cases}$$

for $\alpha \in (1/2, 1)$. Here $T + T^T = J - I$, where $J$ is the matrix of all ones and $I$ is the identity matrix. The classical bounds for $\rho(T)$ using maximum and minimum row sums give

$$(n - 1)\beta \leq \rho(T) \leq (n - 1)\alpha.$$  

For the bounds in Theorem 2.2, with the $p$th row sum of $T$ denoted by $L_p = (n - p)\alpha + (p - 1)\beta$, set

$$M = \max \left\{ \prod_{p=1}^{2} L_p^{\frac{1}{2}}, \prod_{p=1}^{3} L_p^{\frac{1}{3}}, \ldots, \prod_{p=1}^{n} L_p^{\frac{1}{n}} \right\}$$

and

$$m = \min \left\{ \prod_{p=1}^{2} L_{n+1-p}^{\frac{1}{2}}, \prod_{p=1}^{3} L_{n+1-p}^{\frac{1}{3}}, \ldots, \prod_{p=1}^{n} L_{n+1-p}^{\frac{1}{n}} \right\}.$$

Note that each product term in $M$ and $m$ comes from a cycle in $G(T)$. Clearly, since $1 > \alpha > \beta = 1 - \alpha$,

$$M = (\alpha(n - 1)((n - 3)\alpha + 1))^\frac{1}{2}, \quad m = (\beta(n - 1)((n - 3)\beta + 1))^\frac{1}{2}.$$
Then by Theorem 2.2, it follows that
\[ m \leq \rho(T) \leq M, \]
which give an improvement on the classical bounds since \( \alpha > 1/2 \). For example, if \( n = 4 \), then Theorem 2.2 gives
\[ (3\beta(\beta + 1))^{1/2} \leq \rho(T) \leq (3\alpha(\alpha + 1))^{1/2}. \]

As a numerical example with \( n = 4 \), take \( \alpha = 0.75 \), thus \( \beta = 0.25 \). The classical bounds give \( 0.75 \leq \rho(T) \leq 2.25 \), whereas Al’pin’s bounds give \( 0.9682 \leq \rho(T) \leq 1.9843 \). Matlab gives \( \rho(T) = 1.3319 \).

Note added in proof

An earlier paper gives bounds for the Perron root of a nonnegative irreducible matrix \( A \) which depend only on the row sums and the graph of \( A \) [9]. We consider Corollary 4.6 and 4.8. If the diagonal entries vanish and the row sums \( r_i \) satisfy \( r_1 \leq r_2 \leq \ldots \leq r_n \) and if \( g \) is the girth of the directed graph of \( A \), i.e., the shortest length of a circuit, then
\[ (r_1 \ldots r_g)^{1/g} \leq \rho(A) \leq (r_{n-g+1} \ldots r_n)^{1/g}. \]

We note here that this result follows from Theorem 2.2: One shows easily the inequalities \( \mu(S(A)) \leq (r_{n-g+1} \ldots r_n)^{1/g} \), and \( (r_1 \ldots r_g)^{1/g} \leq (\mu(S^{-}(A)))^{-1} \). They hold even in the case of nonvanishing diagonal entries. Now apply Theorem 2.2 and get Brualdi’s result.

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References