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**Journal of
Combinatorial
Theory**

Series A

Journal of Combinatorial Theory, Series A 114 (2007) 1394–1406

www.elsevier.com/locate/jcta

Base subsets of polar Grassmannians

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Received 1 December 2006

Available online 3 March 2007

Communicated by Alexander Ivanov

Abstract

Let Δ be a thick building of type $X_n = C_n, D_n$. Let also \mathcal{G}_k be the Grassmannian of k -dimensional singular subspaces of the associated polar space Π (of rank n). We write \mathfrak{G}_k for the corresponding shadow space of type $X_{n,k}$. Every bijective transformation of \mathcal{G}_k which maps base subsets to base subsets (the shadows of apartments) is a collineation of \mathfrak{G}_k , and it is induced by a collineation of Π if $n \neq 4$ or $k \neq 1$.
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Keywords: Building; Polar space; Polar Grassmannian; Base subset

1. Introduction

Let us consider a building of type X_n [10] as an incidence geometry $(\mathcal{G}, *, d)$, where \mathcal{G} is a set of subspaces, $*$ is an incidence relation, and $d: \mathcal{G} \rightarrow \{0, \dots, n-1\}$ is a dimension function. Then apartments of the building are certain subgeometries of rank n . Denote by \mathcal{G}_k the Grassmannian consisting of all k -dimensional subspaces. The *shadow space* \mathfrak{G}_k is the partial linear space whose point set is \mathcal{G}_k and lines are defined in some natural way (cf. Subsection 2.3). The intersection of an apartment with \mathcal{G}_k is known as the *shadow* of this apartment.

Every apartment preserving (in both directions) bijective transformation of the chamber set (the set of maximal flags of our geometry) can be extended to a bijective transformation of \mathcal{G} preserving the incidence relation [1].

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Conjecture. Every bijective transformation of \mathcal{G}_k sending the shadows of apartments to the shadows of apartments is a collineation of \mathfrak{G}_k .

This conjecture holds for the shadow spaces of type $A_{n,k}$ [7] and all shadow spaces of symplectic buildings [8]. In the present paper we show that this is true for all buildings of types C_n and D_n (the buildings associated with polar spaces). Our proof is based on certain characterizations of the collinearity relation of the shadow space \mathfrak{G}_k in terms of the shadows of apartments. Some characterizations of the shadows of apartments in terms of the collinearity relation (roughly speaking, results of “opposite nature”) were given in [3] for the shadow spaces of type $A_{n,k}$, $C_{n,n}$, and $D_{n,n}$, and a more large class of shadow spaces was considered in [6].

Now suppose that our building is of type C_n or D_n . Let Π be the associated polar space (of rank n). We say that Π is of type C or D if the correspondent possibility is realized:

- (C) every $(n - 2)$ -dimensional (second maximal) singular subspace is contained in at least 3 distinct maximal singular subspaces,
- (D) every $(n - 2)$ -dimensional singular subspace is contained in precisely 2 maximal singular subspaces.

A subset of \mathcal{G}_k is said to be *inexact* if it is contained in the shadows of at least two distinct apartments. In the case (C), there are precisely two possibilities for a maximal inexact subset; as in [8], we use maximal inexact subsets of first or second types to characterize the collinearity relation of \mathfrak{G}_k for $k \leq n - 2$ or $k = n - 1$, respectively. In the case (D), the first possibility does not realize and we characterize the collinearity relation in terms of maximal inexact subsets of second type.

2. Polar geometry

2.1. Partial linear spaces

Let us consider a pair $\Pi = (P, \mathcal{L})$, where P is a non-empty set and \mathcal{L} is a family of proper subsets of P . Elements of P and \mathcal{L} will be called *points* and *lines*, respectively. Two or more points are said to be *collinear* if there is a line containing them. We suppose that Π is a *partial linear space*: each line contains at least two points, for each point there is a line containing it, and for any two distinct collinear points $p, q \in P$ there is precisely one line containing them, this line will be denoted by pq . We say that $S \subset P$ is a *subspace* of Π if for any two distinct collinear points $p, q \in S$ the line pq is contained in S . By this definition, any set consisting of non-collinear points is a subspace. A subspace is said to be *singular* if any two points of the subspace are collinear (the empty set and one-point subspaces are singular).

A bijective transformation of P is a *collineation* of Π if \mathcal{L} is preserved in both directions.

2.2. Polar spaces

A *polar space of rank n* can be defined as a partial linear space $\Pi = (P, \mathcal{L})$ satisfying the following axioms:

- (1) each line contains at least 3 points,

- (2) if $p \in P$ and $L \in \mathcal{L}$ then p is collinear with one or all points of L ,
- (3) there is no point collinear with all others,
- (4) every maximal flag of singular subspaces consists of n elements.

It follows from the equivalent axiom system [10] that every singular subspace is contained in a maximal singular subspace and the restrictions of Π to all maximal singular subspaces are $(n - 1)$ -dimensional projective spaces (throughout the paper the dimension is always assumed to be projective). The complete list of polar spaces of rank greater than 2 can be found in [10].

The collinearity relation will be denoted by \perp : we write $p \perp q$ if p and q are collinear points, and $p \not\perp q$ otherwise. More general, if X and Y are subsets of P then $X \perp Y$ means that every point of X is collinear with all points of Y . The minimal subspace containing X is called *spanned* by X and denoted by \overline{X} ; in the case when $X \perp X$, it is singular. For a subset $X \subset P$ we denote by X^\perp the set of points collinear with each point of X ; it follows from the axiom (1) that X^\perp is a subspace. If $p \perp q$ then

$$\{p, q\}^{\perp\perp} = p q^{\perp\perp} = p q$$

and we get the following.

Fact 1. *Every bijective transformation of P preserving the collinearity relation (in both directions) is a collineation of Π .*

Remark 1. If Π is defined by a sesquilinear or quadratic form then every collineation of Π can be extended to a collineation of the corresponding projective space [2,4], see also [5].

2.3. Grassmann spaces

Let $\Pi = (P, \mathcal{L})$ be a polar space of rank $n \geq 3$. For each $k \in \{0, 1, \dots, n - 1\}$ we denote by \mathcal{G}_k the *Grassmannian* consisting of all k -dimensional singular subspaces. Thus $\mathcal{G}_0 = P$ and $\mathcal{G}_1 = \mathcal{L}$.

We say that a subset of \mathcal{G}_{n-1} is a *line* if it consists of all maximal singular subspaces containing a certain $(n - 2)$ -dimensional singular subspace. In the case when $1 \leq k \leq n - 2$, a subset of \mathcal{G}_k is a *line* if there exist incident $S \in \mathcal{G}_{k-1}$ and $U \in \mathcal{G}_{k+1}$ such that it is the set of all k -dimensional singular subspaces incident with both S and U . The family of all lines in \mathcal{G}_k will be denoted by \mathcal{L}_k . The pair

$$\mathfrak{G}_k := (\mathcal{G}_k, \mathcal{L}_k)$$

is a partial linear space for each $k \in \{1, \dots, n - 1\}$. In what follows we will suppose that $\mathfrak{G}_0 = \Pi$.

Two distinct maximal singular subspaces are collinear points of \mathfrak{G}_{n-1} if and only if their intersection belongs to \mathcal{G}_{n-2} . In the case when $k \leq n - 2$, two distinct k -dimensional singular subspaces S and U are collinear points of \mathfrak{G}_k if and only if $S \perp U$ and the subspace spanned by them belongs to \mathcal{G}_{k+1} . We say that two elements of \mathcal{G}_k ($k \leq n - 2$) are *weak-adjacent* if their intersection is an element of \mathcal{G}_{k-1} ; it is trivial that any two distinct collinear points of \mathfrak{G}_k are weak-adjacent, the converse fails.

Every collineation of Π induces a collineation of \mathfrak{G}_k . Conversely,

Fact 2. *Every bijective transformation of \mathcal{G}_k ($1 \leq k \leq n - 1$) preserving the collinearity relation (in both directions) is a collineation of \mathfrak{G}_k ; moreover, it is induced by a collineation of Π if $n \neq 4$ or $k \neq 1$. Every bijective transformation of \mathcal{G}_k ($1 \leq k \leq n - 2$) preserving the weak-adjacency relation (in both directions) is the collineation of \mathfrak{G}_k induced by a collineation of Π .*

Remark 2. It was proved in [2,4] for $k = n - 1$ (see also [5]), and we refer [9] for the general case.

Now suppose that $n \geq 4$ and the case (D) is realized. Then Π is the polar space of a non-degenerate quadratic form q defined on a certain $2n$ -dimensional vector space (if the characteristic of the field is not equal to 2 then the associated bilinear form is symmetric and non-degenerate, and the corresponding polar space coincides with Π). In this case, each line of \mathcal{G}_{n-1} consists of precisely 2 points. Let \mathcal{O}_+ and \mathcal{O}_- be the orbits of the action of the orthogonal group $O^+(q)$ on \mathcal{G}_{n-1} . Then

$$n - \dim(S \cap U) \text{ is } \begin{cases} \text{odd} & \text{if } S, U \in \mathcal{O}_\delta, \delta = +, -, \\ \text{even} & \text{if } S \in \mathcal{O}_+, U \in \mathcal{O}_-. \end{cases}$$

A subset of \mathcal{O}_δ is called a *line* if it consists of all elements of \mathcal{O}_δ containing a certain $(n - 3)$ -dimensional singular subspace. We denote by \mathcal{L}_δ the family of all lines and get the partial linear space

$$\mathfrak{D}_\delta = (\mathcal{O}_\delta, \mathcal{L}_\delta).$$

Two distinct elements of \mathcal{O}_δ are collinear points of \mathfrak{D}_δ if and only if their intersection is an element of \mathcal{G}_{n-3} . Every collineation of Π induces collineations of \mathfrak{D}_+ and \mathfrak{D}_- .

Fact 3. (See [2,4].) *Every bijective transformation of \mathcal{O}_δ ($\delta = +, -$) preserving the collinearity relation (in both directions) is a collineation of \mathfrak{D}_δ induced by a collineation of Π .*

2.4. Base subsets

We say that $B = \{p_1, \dots, p_{2n}\}$ is a *base* of Π if for each $i \in \{1, \dots, 2n\}$ there exists unique $\sigma(i) \in \{1, \dots, 2n\}$ such that

$$p_i \not\perp p_{\sigma(i)}.$$

In this case, the set of all k -dimensional singular subspaces spanned by points of B is said to be the *base subset* of \mathcal{G}_k associated with (defined by) B ; it coincides with B if $k = 0$.

Every base subset of \mathcal{G}_k consists of precisely

$$2^{k+1} \binom{n}{k+1}$$

elements (Proposition 1 in [8]).

Proposition 1. *For any two k -dimensional singular subspaces of Π there is a base subset of \mathcal{G}_k containing them.*

Proof. Easy verification. \square

We define *base subsets* of \mathcal{O}_δ as the intersections of \mathcal{O}_δ with base subsets of \mathcal{G}_{n-1} . Every base subset of \mathcal{O}_δ consists of 2^{n-1} elements and it follows from Proposition 1 that for any two elements of \mathcal{O}_δ there is a base subset of \mathcal{O}_δ containing them.

2.5. Buildings associated with polar spaces

In the case (C), the incidence geometry of singular subspaces is a thick building of type C_n , where each apartment is the subgeometry consisting of all singular subspaces spanned by points of a certain base of Π ; the shadow spaces are \mathfrak{G}_k ($0 \leq k \leq n - 1$), and the shadows of apartments are base subsets.

Suppose that $n \geq 4$ and Π is of type D. The *oriflamme* incidence geometry consists of all singular subspaces of dimension distinct from $n - 2$; two singular subspaces S and U are incident if $S \subset U$, or $U \subset S$, or the dimension of $S \cap U$ is equal to $n - 2$. This is a thick building of type D_n ; as above, apartments are the subgeometries defined by bases of Π . The shadow spaces are \mathfrak{G}_k ($0 \leq k \leq n - 3$) and \mathfrak{D}_δ ($\delta = +, -$), the shadows of apartments are base subsets.

3. Results

Theorem 1. *Let f be a bijective transformation of \mathfrak{G}_k ($0 \leq k \leq n - 1$) which maps base subsets to base subsets.¹ Then f is a collineation of \mathfrak{G}_k . If $k \geq 1$ and Π is of type C then this collineation is induced by a collineation of Π . In the case when $k \geq 1$ and Π is of type D, it is induced by a collineation of Π if $n \neq 4$ or $k \neq 1$.*

Theorem 2. *If $n \geq 4$ and Π is of type D then every bijective transformation of \mathfrak{O}_δ ($\delta = +, -$) which maps base subsets to base subsets² is the collineation of \mathfrak{D}_δ induced by a collineation of Π .*

4. Inexact and complement subsets

4.1. Inexact subsets

Let $B = \{p_1, \dots, p_{2n}\}$ be a base of Π and \mathcal{B} be the associated base subset of \mathfrak{G}_k . Recall that \mathcal{B} consists of all k -dimensional singular subspaces spanned by $p_{i_1}, \dots, p_{i_{k+1}}$, where

$$\{i_1, \dots, i_{k+1}\} \cap \{\sigma(i_1), \dots, \sigma(i_{k+1})\} = \emptyset.$$

If $k = n - 1$ then every element of \mathcal{B} contains precisely one of the points p_i or $p_{\sigma(i)}$ for each i .

We write $\mathcal{B}(+i)$ and $\mathcal{B}(-i)$ for the sets of all elements of \mathcal{B} which contain p_i or do not contain p_i , respectively. For any i_1, \dots, i_s and j_1, \dots, j_u belonging to $\{1, \dots, 2n\}$ we define

$$\mathcal{B}(+i_1, \dots, +i_s, -j_1, \dots, -j_u) := \mathcal{B}(+i_1) \cap \dots \cap \mathcal{B}(+i_s) \cap \mathcal{B}(-j_1) \cap \dots \cap \mathcal{B}(-j_u).$$

It is trivial that

$$\mathcal{B}(+i) = \mathcal{B}(+i, -\sigma(i))$$

and in the case when $k = n - 1$ we have

$$\mathcal{B}(+i) = \mathcal{B}(-\sigma(i)).$$

Let $\mathcal{R} \subset \mathcal{B}$. We say that \mathcal{R} is *exact* if there is only one base subset of \mathfrak{G}_k containing \mathcal{R} ; otherwise, \mathcal{R} will be called *inexact*. If $\mathcal{R} \cap \mathcal{B}(+i)$ is not empty then we define $S_i(\mathcal{R})$ as the

¹ We do not require that f preserves the class of base subsets in both directions.

² As in Theorem 1, we do not require that our transformation preserves the class of base subsets in both directions.

intersection of all subspaces belonging to \mathcal{R} and containing p_i , and we define $S_i(\mathcal{R}) := \emptyset$ if the intersection of \mathcal{R} and $\mathcal{B}(+i)$ is empty. If

$$S_i(\mathcal{R}) = p_i$$

for all i then \mathcal{R} is exact; the converse fails.

Proposition 2. *If $k = n - 1$ then $\mathcal{B}(-i)$ is inexact, but this inexact subset is not maximal. In the case when $k \in \{0, \dots, n - 2\}$, the following assertions are fulfilled:*

- if Π is of type C then $\mathcal{B}(-i)$ is a maximal inexact subset,
- $\mathcal{B}(-i)$ is exact if Π is of type D.

Proof. If $k = n - 1$ then for each $U \in \mathcal{B} \setminus \mathcal{B}(-i)$ the subset

$$\mathcal{B}(-i) \cup \{U\}$$

is inexact (see proof of Proposition 3 in [8] for details); this means that $\mathcal{B}(-i)$ is a non-maximal inexact subset.

Let $k \leq n - 2$. Then

$$S_j(\mathcal{B}(-i)) = p_j \quad \text{for } j \neq i$$

(proof of Proposition 3 in [8]). Therefore, if $\mathcal{B}(-i)$ is contained in the base subset of \mathcal{G}_k associated with a base B' of Π then

$$B' = (B \setminus \{p_i\}) \cup \{p\}. \tag{1}$$

It is clear that p is collinear with all points of $B \setminus \{p_i, p_{\sigma(i)}\}$ and non-collinear with $p_{\sigma(i)}$. If $p \neq p_i$ then $p \not\perp p_i$ and each $(n - 2)$ -dimensional singular subspace spanned by points of $B \setminus \{p_i, p_{\sigma(i)}\}$ is contained in 3 distinct maximal singular subspaces which contradicts (D).

Suppose that Π is of type C. Let S and U be non-intersecting $(n - 2)$ -dimensional singular subspaces spanned by points of $B \setminus \{p_i, p_{\sigma(i)}\}$. By (C), there exists a point $p \in P$ non-collinear with $p_i, p_{\sigma(i)}$ and collinear with all points of S . Since any two flags are contained in a certain apartment, there is a point $q \in P$ such that

$$(B \setminus \{p_i, p_{\sigma(i)}\}) \cup \{p, q\}$$

is a base of Π whose points span U and $\overline{S \cup \{p\}}$. This means that p is collinear with all points of $B \setminus \{p_i, p_{\sigma(i)}\}$. The base subset of \mathcal{G}_k associated with the base (1) contains $\mathcal{B}(-i)$. So it is an inexact subset. As in [8] (proof Proposition 3), we show that this inexact subset is maximal. \square

Proposition 3. *If $j \neq i, \sigma(i)$ then*

$$\mathcal{R}_{ij} := \mathcal{B}(+i, +j) \cup \mathcal{B}(+\sigma(i), +\sigma(j)) \cup \mathcal{B}(-i, -\sigma(j))$$

is inexact; moreover, it is a maximal inexact subset except the case when $k = 0$ and Π is of type C.

Proof. As in [8] (proof of Proposition 4), we establish that \mathcal{R}_{ij} is a maximal inexact subset if $k \geq 1$. If $k = 0$ then

$$\mathcal{R}_{ij} = B \setminus \{p_i, p_{\sigma(j)}\}.$$

We choose a point p'_i on the line $p_i p_j$ distinct from p_i and p_j , the line $p_{\sigma(j)} p_{\sigma(i)}$ contains the unique point $p'_{\sigma(j)}$ collinear with p'_i . Then

$$(B \setminus \{p_i, p_{\sigma(j)}\}) \cup \{p'_i, p'_{\sigma(j)}\}$$

is a base of Π and \mathcal{R}_{ij} is inexact. If Π is of type C then $\mathcal{B}(-i) = B \setminus \{p_i\}$ is an inexact subset containing \mathcal{R}_{ij} , hence the inexact subset \mathcal{R}_{ij} is not maximal. \square

A direct verification shows that

$$\mathcal{R}_{ij} = \mathcal{B}(+i, +j) \cup \mathcal{B}(-i)$$

if $k = n - 1$.

As in [8], the inexact subsets considered in Propositions 2 and 3 will be called of *first* and *second* type, respectively.

Proposition 4. *Every maximal inexact subset is of first or second type. In particular, each maximal inexact subset is of second type if $k = n - 1$ or Π is of type D, and each maximal inexact subset is of first type if $k = 0$ and Π is of type C.*

Proof. The case $k = 0$ is trivial. Let $k \geq 1$ and \mathcal{R} be a maximal inexact subset of \mathcal{B} . First, we consider the case when all $S_i(\mathcal{R})$ are non-empty. Denote by I the set of all i such that the dimension of $S_i(\mathcal{R})$ is non-zero. Since \mathcal{R} is inexact, I is non-empty. Suppose that for certain $l \in I$ the subspace $S_l(\mathcal{R})$ is spanned by $p_l, p_{j_1}, \dots, p_{j_u}$ and

$$M_1 := S_{\sigma(j_1)}(\mathcal{R}), \quad \dots, \quad M_u := S_{\sigma(j_u)}(\mathcal{R})$$

do not contain $p_{\sigma(l)}$. Then p_l belongs to $M_1^\perp, \dots, M_u^\perp$; on the other hand,

$$p_{j_1} \notin M_1^\perp, \quad \dots, \quad p_{j_u} \notin M_u^\perp$$

and we have

$$M_1^\perp \cap \dots \cap M_u^\perp \cap S_l(\mathcal{R}) = p_l.$$

If this holds for all $l \in I$ then our subset is exact. Therefore, there exist $i \in I$ and $j \neq i, \sigma(i)$ such that

$$p_j \in S_i(\mathcal{R}) \quad \text{and} \quad p_{\sigma(i)} \in S_{\sigma(j)}(\mathcal{R}).$$

Then \mathcal{R} is contained \mathcal{R}_{ij} . We have $\mathcal{R} = \mathcal{R}_{ij}$, since the inexact subset \mathcal{R} is maximal.

Suppose that $S_i(\mathcal{R}) = \emptyset$ for certain i . Then \mathcal{R} is contained in $\mathcal{B}(-i)$ and we get a maximal inexact subset of first type if Π is of type C. Otherwise, one of the following possibilities is realized:

- (1) $S_j(\mathcal{R}) = \emptyset$ for certain $j \neq i, \sigma(i)$. Then $\mathcal{R} \subset \mathcal{B}(-j, -i)$ is a proper subset of $\mathcal{R}_{j\sigma(i)}$.
- (2) All $S_l(\mathcal{R}), l \neq i, \sigma(i)$ are non-empty and there exists $j \neq i, \sigma(i)$ such that $S_j(\mathcal{R})$ contains $p_{\sigma(i)}$. Then

$$\mathcal{R} \subset \mathcal{B}(+j, +\sigma(i)) \cup \mathcal{B}(-j, -i);$$

and as in the previous case, \mathcal{R} is a proper subset of $\mathcal{R}_{j\sigma(i)}$.

- (3) Each $S_j(\mathcal{R})$, $j \neq i, \sigma(i)$ is non-empty and does not contain $p_{\sigma(i)}$. As in the case when $S_j(\mathcal{R}) \neq \emptyset$ for all $j \in \{1, \dots, 2n\}$, we show that \mathcal{R} is contained in certain \mathcal{R}_{lm} with $l, m \neq i, \sigma(i)$. It is clear that

$$\mathcal{R} \subset \mathcal{B}(-i) \cap \mathcal{R}_{lm}$$

is a proper subset of \mathcal{R}_{lm} .

In each of the cases considered above, \mathcal{R} is a proper subset of a maximal inexact subset; this contradicts the fact that our inexact subset is maximal. \square

4.2. Complement subsets

Let \mathcal{B} be as in the previous subsection. We say that $\mathcal{R} \subset \mathcal{B}$ is a *complement subset* if $\mathcal{B} \setminus \mathcal{R}$ is a maximal inexact subset. A complement subset is said to be of *first* or *second* type if the corresponding maximal inexact subset is of first or second type, respectively. The complement subsets for the maximal inexact subsets from Propositions 2 and 3 are

$$\mathcal{B}(+i) \quad \text{and} \quad \mathcal{C}_{ij} := \mathcal{B}(+i, -j) \cup \mathcal{B}(+\sigma(j), -\sigma(i)).$$

If $k = n - 1$ then the second subset coincides with

$$\mathcal{B}(+i, +\sigma(j)) = \mathcal{B}(+i, +\sigma(j), -j, -\sigma(i)).$$

In the case when $k \geq 1$, each complement subset is of second type if $k = n - 1$ or Π is of type D. If $k = 0$ and Π is of type D then all complement subsets are of second type (pairs of collinear points). The case when $k = 0$ and Π is of type C is trivial: each complement subset consists of a single point.

Lemma 1. *Suppose that $1 \leq k \leq n - 2$ and Π is of type C. Let \mathcal{R} be a complement subset of \mathcal{B} . If \mathcal{R} is of first type then there are precisely $4n - 3$ distinct complement subsets of \mathcal{B} which do not intersect \mathcal{R} . If \mathcal{R} is of second type then there are precisely 4 distinct complement subsets of \mathcal{B} which do not intersect \mathcal{R} .*

Proof. See proof of Lemma 5 in [8]. \square

Lemma 2. *Suppose that $k = n - 1$. Then $S, U \in \mathcal{B}$ are collinear points of \mathfrak{G}_k if and only if there are precisely*

$$\binom{n-1}{2}$$

distinct complement subsets of \mathcal{B} containing S and U .

Proof. Let $\dim(S \cap U) = m$. The complement subset $\mathcal{B}(+i, +j)$ contains S and U if and only if the line $p_i p_j$ is contained $S \cap U$. Thus there are

$$\binom{m+1}{2}$$

distinct complement subsets of \mathcal{B} containing S and U . \square

Lemma 3. Let $1 \leq k \leq n - 2$ and Π be of type C. If \mathcal{R} is the intersection of $2n - k - 2$ distinct complement subsets of first type of \mathcal{B} then one of the following possibilities is realized:

- \mathcal{R} consists of $k + 2$ mutually collinear points of \mathfrak{G}_k ,
- \mathcal{R} consists of 2 weak-adjacent elements of \mathfrak{G}_k which are non-collinear.

Proof. Direct verification. \square

Let $c(S, U)$ be the number of complement subsets of second type of \mathcal{B} containing both $S, U \in \mathcal{B}$.

Lemma 4. Suppose that $1 \leq k \leq n - 2$. Let S and U be distinct elements of \mathcal{B} and $\dim(S \cap U) = m$. Then

$$c(S, U) \leq (m + 1)(2n - 2k + m - 2) + (k - m)^2;$$

moreover, if $k = n - 2$ then

$$c(S, U) \leq (m + 1)(m + 2) + 1.$$

If S and U are collinear points of \mathfrak{G}_k then

$$c(S, U) = k(2n - k - 3) + 1.$$

In the case when S and U are weak adjacent and non-collinear, we have

$$c(S, U) = k(2n - k - 3).$$

Proof. If the complement subset \mathcal{C}_{ij} contains S and U then one of the following possibilities is realized:

- (A) at least one of the subsets $\mathcal{B}(+i, -j)$ or $\mathcal{B}(+\sigma(j), -\sigma(i))$ contains both S, U ;
- (B) each of the subsets $\mathcal{B}(+i, -j)$ and $\mathcal{B}(+\sigma(j), -\sigma(i))$ contains precisely one of our subspaces.

The case (A). If S and U both are contained in $\mathcal{B}(+i, -j)$ then

$$p_i \in S \cap U \quad \text{and} \quad p_j \notin S \cup U.$$

Since $S \cup U$ contains precisely $2k - m + 1$ points of B and $j \neq \sigma(i)$, we have $2n - 2k + m - 2$ distinct possibilities for j if i is fixed. Thus there are precisely

$$(m + 1)(2n - 2k + m - 2)$$

complement subsets satisfying (A), it must be taken into account that \mathcal{C}_{ij} coincides with $\mathcal{C}_{\sigma(j)\sigma(i)}$.

The case (B). If

$$S \in \mathcal{B}(+i, -j) \setminus \mathcal{B}(+\sigma(j), -\sigma(i)) \quad \text{and} \quad U \in \mathcal{B}(+\sigma(j), -\sigma(i)) \setminus \mathcal{B}(+i, -j)$$

then

$$p_i \in S \setminus U \quad \text{and} \quad p_{\sigma(j)} \in U \setminus S$$

(since $p_{\sigma(j)} \in U$, we have $p_j \notin U$; then $U \notin \mathcal{B}(+i, -j)$ guarantees that $p_i \notin U$; similarly, we show that $p_{\sigma(j)} \notin S$). Hence there are at most

$$(k - m)^2$$

distinct complement subsets satisfying (B) (as above, we take into account that \mathcal{C}_{ij} coincides with $\mathcal{C}_{\sigma(j)\sigma(i)}$). Note that

$$p_i \perp U \quad \text{and} \quad p_{\sigma(j)} \perp S$$

(since $p_{\sigma(i)} \notin U$ and $p_j \notin S$). In the case when $k = n - 2$, there are at most one point of

$$B \cap (S \setminus U)$$

collinear with all points of U and at most one point of

$$B \cap (U \setminus S)$$

collinear with all points of S ; hence there is at most one complement subset satisfying (B).

Now suppose that $m = k - 1$. It was shown above that there are precisely $k(2n - k - 3)$ complement subsets of kind (A). Let

$$B \cap (S \setminus U) = \{p_u\} \quad \text{and} \quad B \cap (U \setminus S) = \{p_v\}.$$

If S and U are collinear points of \mathfrak{G}_k then $v \neq \sigma(u)$ and the complement subset $\mathcal{C}_{u\sigma(v)} = \mathcal{C}_{v\sigma(u)}$ contains S and U . Otherwise, we have $v = \sigma(u)$ and there are no complement subsets of kind (B). \square

Let us define

$$m_c := \max\{c(S, U) : S, U \in \mathcal{B}, S \neq U\}.$$

Lemma 5. *Distinct $S, U \in \mathcal{B}$ are collinear points of \mathfrak{G}_k if and only if $c(S, U) = m_c$.*

Proof. Since

$$g(x) := (x + 1)(2n - 2k + x - 2) + (k - x)^2, \quad x \in \mathbb{R},$$

is a parabola and the coefficient of x^2 is positive,

$$\max\{g(-1), g(0), \dots, g(k - 1)\} = \max\{g(-1), g(k - 1)\}.$$

Suppose that $k \leq n - 3$. Then

$$g(-1) < g(k - 1)$$

($g(k - 1) = k(2n - k - 2) + 1 \geq k(k + 3) + 1 > (k + 1)^2 = g(-1)$) and Lemma 4 guarantees that $c(S, U) = m_c$ if and only if S and U are collinear points of \mathfrak{G}_k . If $k = n - 2$ then

$$k(2n - k - 3) + 1 = k(k + 1) + 1 > (m + 1)(m + 2) + 1 \quad \text{for } m \leq k - 2$$

and Lemma 4 gives the claim. \square

4.3. Proof of Theorem 1

Let f be a bijective transformation of \mathcal{G}_k sending base subsets to base subsets.

First we consider the case when $k > 0$. Let $S, U \in \mathcal{G}_k$ and \mathcal{B} be a base subset of \mathcal{G}_k containing S and U . It is clear that f transfers an inexact subset of \mathcal{B} to inexact subset of $f(\mathcal{B})$. Since the base subsets \mathcal{B} and $f(\mathcal{B})$ have the same number of inexact subsets, every inexact subset of $f(\mathcal{B})$ is the f -image of a certain inexact subset of \mathcal{B} . This implies that an inexact subset of \mathcal{B} is maximal if and only if its f -image is a maximal inexact subset of $f(\mathcal{B})$. Thus f and f^{-1} map complement subsets to complement subsets. Lemma 1 guarantees that the type of a complement subset is preserved.³ Using the characterizations of the collinearity relation in terms of complement subsets of second type (Lemmas 2 and 5), we establish that S and U are collinear points of \mathcal{G}_k if and only if the same holds for their f -images. In the case (C), S and U are weak adjacent if and only if $f(S)$ and $f(U)$ are weak adjacent (this is a consequence of Lemma 3). The required statement follows from Fact 2.

Let $k = 0$ and $B = \{p_1, \dots, p_{2n}\}$ be a base of Π . Since for any two points of Π there is a base of Π containing them, we need to establish that two points of B are collinear if and only if their f -images are collinear. As above, we show that $X \subset B$ is a complement subset if and only if $f(X)$ is a complement subset of the base $f(B)$. In the case (D), each complement subset is a pair of collinear points and we get the claim.

Consider the case (C). For each $i \in \{1, \dots, 2n\}$ there exists a point p such that

$$(B \setminus \{p_i\}) \cup \{p\} \quad \text{and} \quad (B \setminus \{p_{\sigma(i)}\}) \cup \{p\}$$

are bases of Π (see proof of Proposition 2). Then

$$(f(B) \setminus \{f(p_i)\}) \cup \{f(p)\} \quad \text{and} \quad (f(B) \setminus \{f(p_{\sigma(i)})\}) \cup \{f(p)\}$$

are bases of Π . Since $f(B)$ is a base of Π , there are unique $u, v \in \{1, \dots, 2n\}$ such that

$$f(p_i) \not\ll f(p_u) \quad \text{and} \quad f(p_{\sigma(i)}) \not\ll f(p_v).$$

It is clear that $f(p)$ is non-collinear with $f(p_u)$ and $f(p_v)$, and there is no base of Π containing $f(p), f(p_u), f(p_v)$. This is possible only in the case when $u = \sigma(i), v = i$. Thus $f(p_i) \not\ll f(p_{\sigma(i)})$ and $f(p_i)$ is collinear with $f(p_j)$ if $j \neq \sigma(i)$.

4.4. Inexact and complement subsets of \mathcal{O}_δ

In this subsection we require that $n \geq 4$ and Π is of type D. Let $B = \{p_1, \dots, p_{2n}\}$ be a base of Π and \mathcal{B} be the associated base subset of \mathcal{O}_δ ($\delta = +, -$). Every element of \mathcal{B} contains precisely one of the points p_i or $p_{\sigma(i)}$ for each i .

As in Section 4.1, we write $\mathcal{B}(+i)$ and $\mathcal{B}(-i)$ for the sets of all elements of \mathcal{B} which contain p_i or do not contain p_i , respectively; for any i_1, \dots, i_s and j_1, \dots, j_u belonging to $\{1, \dots, 2n\}$ we define

$$\mathcal{B}(+i_1, \dots, +i_s, -j_1, \dots, -j_u) := \mathcal{B}(+i_1) \cap \dots \cap \mathcal{B}(+i_s) \cap \mathcal{B}(-j_1) \cap \dots \cap \mathcal{B}(-j_u).$$

It is clear that

$$\mathcal{B}(-i) = \mathcal{B}(-i, +\sigma(i)) = \mathcal{B}(+\sigma(i)).$$

³ This is trivial if $k = n - 1$, or Π is of type D, or complement subsets of distinct types have distinct cardinalities; however, for some pairs n, k the complement subsets of first and second types have the same number of elements.

Let $\mathcal{R} \subset \mathcal{B}$. We say that \mathcal{R} is *exact* if there is only one base subset of \mathcal{O}_δ containing \mathcal{R} ; otherwise, \mathcal{R} will be called *inexact*. If $\mathcal{R} \cap \mathcal{B}(+i)$ is not empty then we define $S_i(\mathcal{R})$ as the intersection of all subspaces belonging to \mathcal{R} and containing p_i , and we define $S_i(\mathcal{R}) := \emptyset$ if the intersection of \mathcal{R} and $\mathcal{B}(+i)$ is empty. If

$$S_i(\mathcal{R}) = p_i$$

for all i then \mathcal{R} is exact; the converse fails.

Proposition 5. *If $j \neq i, \sigma(i)$ then*

$$\mathcal{B}(-i) \cup \mathcal{B}(+i, +j) \tag{2}$$

is an inexact subset.

Proof. It is clear that

$$p_j \in S_i(\mathcal{B}(-i) \cup \mathcal{B}(+i, +j)).$$

Suppose that S belongs to (2) and contains $p_{\sigma(j)}$. Then S does not contain p_j , hence $p_i \notin S$. The latter means that $p_{\sigma(i)}$ belongs to S . Therefore,

$$p_{\sigma(i)} \in S_{\sigma(j)}(\mathcal{B}(-i) \cup \mathcal{B}(+i, +j)).$$

We choose a point p'_i on the line $p_i p_j$ distinct from p_i and p_j . There is the unique point $p'_{\sigma(j)} \in p_{\sigma(i)} p_{\sigma(j)}$ collinear with p'_i . Then

$$(\mathcal{B} \setminus \{p_i, p_{\sigma(j)}\}) \cup \{p'_i, p'_{\sigma(j)}\}$$

is a base of Π and the associated base subset of \mathcal{O}_δ contains (2). \square

Proposition 6. *If \mathcal{R} is a maximal inexact subsets of \mathcal{B} then*

$$\mathcal{R} = \mathcal{B}(-i) \cup \mathcal{B}(+i, +j)$$

for some i and j such that $j \neq i, \sigma(i)$.

Proof. Since \mathcal{R} is inexact, we have $S_i(\mathcal{R}) \neq p_i$ for certain i . If $S_i(\mathcal{R})$ is empty then \mathcal{R} is contained in $\mathcal{B}(-i)$ which contradicts the fact that the inexact subset \mathcal{R} is maximal. Thus there exists $p_j \in S_i(\mathcal{R})$ such that $j \neq i$. Then

$$\mathcal{R} \subset \mathcal{B}(-i) \cup \mathcal{B}(+i, +j).$$

The inexact subset \mathcal{R} is maximal and the inverse inclusion holds. \square

We say that $\mathcal{R} \subset \mathcal{B}$ is a *complement subset* if $\mathcal{B} \setminus \mathcal{R}$ is a maximal inexact subset. In this case,

$$\mathcal{B} \setminus \mathcal{R} = \mathcal{B}(-i) \cup \mathcal{B}(+i, +j)$$

(Proposition 6) and

$$\mathcal{R} = \mathcal{B}(+i, +\sigma(j)).$$

Lemma 6. *$S, U \in \mathcal{B}$ are collinear points of \mathfrak{D}_δ if and only if there are precisely*

$$\binom{n-2}{2}$$

distinct complement subsets of \mathcal{B} containing S and U .

Proof. See the proof of Lemma 2. \square

Let f be a bijective transformation of \mathcal{O}_δ which maps base subsets to base subsets. Consider $S, U \in \mathcal{O}_\delta$ and a base subset $\mathcal{B} \subset \mathcal{O}_\delta$ containing them. As in the previous subsection, we show that $\mathcal{R} \subset \mathcal{B}$ is a complement subset if and only if $f(\mathcal{R})$ is a complement subset $f(\mathcal{B})$. Lemma 6 guarantees that S and U are collinear points of \mathfrak{D}_δ if and only if the same holds for their f -images, and Fact 3 gives the claim.

5. Final remark

Using the arguments of [8] (Section 8) and Lemmas 1–3 we can establish the following analogue of the main result of [8]. Let Π and Π' be polar spaces of same rank $n \geq 3$. We suppose that they both are of type C and denote by \mathcal{G}_k and \mathcal{G}'_k the Grassmannians consisting of all k -dimensional singular subspaces of Π and Π' , respectively. Then every mapping of \mathcal{G}_k to \mathcal{G}'_k which sends base subsets to base subsets is induced by an embedding of Π to Π' .

Acknowledgment

I thank A. Pasini for useful information concerning the article [6].

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