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MORAVA E-THEORY OF SYMMETRIC GROUPS

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We compute the completed $E(n)$ cohomology of the classifying spaces of the symmetric groups, and relate the answer to the theory of finite subgroups of formal groups.

1. INTRODUCTION

Fix a prime p and an integer $n > 0$. In this paper we study the $\widehat{E(n)}$-cohomology of the space

$$DS^0 = \bigvee_{k \geqslant 0} B\Sigma_{k+}$$

where Σ_k denotes the symmetric group on k letters. It will be technically convenient to work with a certain 2-periodic algebraic extension of $\widehat{E(n)}$ called Morava E-theory, which we denote by E. See Section 2 for details.

It will turn out that E^0DS^0 is related to the theory of subgroups of formal groups. This general theme has its roots in the work of Hopkins and Ando [1]. We will also make heavy use of results of Takuji Kashiwabara [13], and the generalised character theory of Hopkins *et al.* [9].

Hopkins has proved that E can be made into an E_∞ ring spectrum, but unfortunately no proof has yet appeared. Our calculation of E^0DS^0 is an important first step towards the exploitation of the E_∞ structure, as will be explained in a sequel to this paper. We have avoided using the E_∞ structure here, because of the lack of a published account.

1.1. General notation and conventions

We write $\mathbb{F}_{p^n}$ for the finite field with p^n elements; there is a unique such field up to unnatural isomorphism. In particular $\mathbb{F}_p = \mathbb{Z}/p$. We write $\mathbb{Z}_p$ for the ring of p-adic integers, and $\mathbb{Q}_p$ for its field of fractions, so $\mathbb{Q}_p/\mathbb{Z}_p \simeq \varinjlim_n \mathbb{Z}/p^n$.

Unless otherwise stated, all our vector bundles, group representations and tensor products are over the complex numbers. Given a representation V of a finite group G, we use the same symbol V to denote the associated vector bundle $EG \times_G V$ over BG. Given a vector bundle V over a space X, we write X^V for the associated Thom space, or for its suspension spectrum. Given an integer n, we sometimes write n for the trivial bundle of complex dimension n, so that $n \otimes V$ is the direct sum of n copies of V.

Because E^* is periodic, we can choose a degree-zero complex orientation $x \in \tilde{E}^0\mathbb{C}P^\infty$. Using this, we define Thom classes, Euler classes and Chern classes for complex bundles, in the usual way (all of them in degree zero). We write $e(V)$ for the Euler class of a bundle V.

We write V_m for the usual permutation representation of Σ_m on $\mathbb{C}^m$, or for the corresponding bundle $E\Sigma_m \times_{\Sigma_m} \mathbb{C}^m$ over $B\Sigma_m$. This contains an evident copy of the trivial representation of complex dimension one, so we can interpret $V_m - 1$ as an honest complex representation or bundle, and thus define the Euler class $c_m = e(V_m - 1) \in E^0 B\Sigma_m$.

1.2. Statement of results

The stable transfer maps $B\Sigma_{k+1} \to B\Sigma_k \times B\Sigma_l$ and the diagonal maps $B\Sigma_k \to B\Sigma_k \times B\Sigma_k$ give two products on $E^0 DS^0$, write $\times$ and $\bullet$. The evident maps $B\Sigma_k \times B\Sigma_l \to B\Sigma_{k+l}$ give rise to a coproduct ψ_* on $E^0 DS^0$ (provided that we use a completed tensor product).

The following theorem summarises most of our results.

THEOREM 1.1. *With the above structure, E^0DS^0 is a Hopf ring. It is a formal power series ring under $\times$. The $\times$-indecomposables are*

$$\mathrm{Ind}(E^0DS^0) = \prod_{k \geqslant 0} \bar{R}_k$$

where

$$\bar{R}_k = E^0 B\Sigma_{p^k}/\mathrm{transfer}(E^0 B\Sigma_{p^{k-1}}^p).$$

This ring is naturally isomorphic to the ring $\mathcal{O}_{\mathrm{Sub}_k(\mathbb{G})}$ studied in [18], *which classifies subgroup-schemes of degree p^k in the formal group $\mathbb{G}$ associated to $E^0\mathbb{C}P^\infty$. It is a Gorenstein local ring, and a free module over E^0, with rank given by the Gaussian binomial coefficient*

$$\begin{bmatrix} n+k-1 \\ n-1 \end{bmatrix}_p = \prod_{j=1}^{n-1} \frac{p^{k+j}-1}{p^j-1} = |\{\text{subgroups } A < (\mathbb{Q}_p/\mathbb{Z}_p)^n \text{ with } |A| = p^k\}|.$$

In Section 10, *we will give an explicit basis consisting of monomials in the Chern classes of V_{p^k}. We also have*

$$\mathrm{Prim}(E^0DS^0) = \prod_k \ker(\mathrm{res}\colon E^0 B\Sigma_{p^k} \to E^0 B\Sigma_{p^{k-1}}^p).$$

This is a free module over $\mathrm{Ind}(E^0DS^0)$ *on one generator c, whose component in $E^0B\Sigma_m$ is the Euler class $c_m = e(V_m - 1)$.*

We now give a brief outline of the methods used to prove this. Firstly, we appeal to the results of [20]. There, we showed that the maps of spectra that we used to define the products and coproduct on E^0DS^0 interact in the right way to make E^0DS^0 a Hopf ring; this is explained in Section 4. Also, following work of Hopkins *et al.* [9], we defined an E^0-algebra L which is a free module over $p^{-1}E^0$, and we gave a complete description of $L \otimes_E E^0 B\Sigma_k$. This essentially tells us everything we need to know about $p^{-1}E^0B\Sigma_k$. It is also known that $K^{\mathrm{odd}} B\Sigma_k = 0$ (see [9, Section 6], or [12]). We recall the proof in Section 3, because we need to extract from it the fact that $c_{p^k}^{(p^n-1)/(p-1)} \neq 0$ in $K^0 B\Sigma_{p^k}$, which turns out to be crucial later on. It is not hard to deduce from the fact that $K^{\mathrm{odd}} B\Sigma_k = 0$ that $E^* B\Sigma_k$ is a finitely generated free module over E^*, concentrated in even degrees. This of course means that the loss of information in passing from $E^0 B\Sigma_k$ to $p^{-1}E^0 B\Sigma_k$ is relatively small.

In Section 5 we recall Kashiwabara's result [13, Corollary 6.4] that $K_0 DS^0$ is a polynomial ring under the product dual to ψ_*, and we give a simplified proof. It follows that $E^\vee DS^0$ is a completed polynomial ring over E^0, and thus that $\mathrm{Ind}(E^\vee DS^0)$ is a completed free module and is a summand in $E^\vee DS^0$. Dually, we conclude that $\mathrm{Prim}(E^0DS^0)$ is a product of free modules and is a summand in E^0DS^0.

We next consider E^0QS^2. There is only one sensible product on E^0QS^2, which comes from the diagonal map $QS^2 \to QS^2 \times QS^2$, which is also the map obtained by applying Q to the pinch map $S^2 \to S^2 \vee S^2$. We also prove in Section 5 that E^0QS^2 is a formal power series ring under this product, using an unpublished argument of Kashiwabara.

Our next task is to transfer this result to E^0DS^0, using the Snaith splitting $\Sigma^\infty QS^2 = \Sigma^\infty DS^2$ and the Thom isomorphism $\tilde{E}^0D_k(S^2) = \tilde{E}^0B\Sigma_k^{V_k} = E^0B\Sigma_k$. We need a property of the Snaith splitting that does not seem to appear in the literature; in order to prove it, we recall the details of the construction in Section 6. We find that our isomorphism converts the product on E^0QS^2 to the $\times$-product on E^0DS^0, so we conclude that E^0DS^0 is a formal power series ring under the $\times$-product. It follows from this that $\mathrm{Ind}(E^0DS^0)$ is a product of free modules over E^0, and a retract of E^0DS^0.

In Section 8, we study the indecomposables and primitives more closely. We distinguish between two different kinds of primitives, and we define indecomposables and primitives in more elementary terms. We find that the double suspension map $E^*QS^2 \to E^*QS^0$ corresponds to multiplication by $c_k = e(V_k - 1)$ in $E^0B\Sigma_k$. This implies that $\mathrm{Prime}(E^0DS^0)$ is free of rank one over $\mathrm{Ind}(E^0DS^0)$.

In Section 9, we use generalised character theory to show that the representation V_{p^k} defines a subgroup scheme of the formal group $\mathbb{G}$ over the scheme $\mathrm{spf}(\bar{R}_k)$. This gives us a classifying map $\mathcal{O}_{\mathrm{Sub}_k(\mathbb{G})} \to \bar{R}_k$, and we prove that this is an isomorphism. All this makes heavy use of [18].

In Section 10, we give a basis for $\mathcal{O}_{\mathrm{Sub}_k(\mathbb{G})}$, consisting of monomials in the Chern classes of V_{p^k}. This is a piece of pure algebra that should really have appeared in [18].

2. MORAVA K-THEORY AND E-THEORY

In this section, we give more precise definition of the (co)homology theories which we need.

We will consider a completed and extended version of $E(n)$, which we call Morava E-theory. We start with the graded ring

$$E_* = W\mathbb{F}_{p^n}[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}]$$

(where $u_i \in E_0$ and $u \in E_2$). We take $u_0 = p$, $u_n = 1$ and $u_k = 0$ for $k > n$. We make E_* into an algebra over BP_* by sending the Hazewinkel generator v_k to $u^{p^k-1}u_k$. This makes E_* into a Landweber exact BP_*-module, so we have a homology theory

$$E_*X = E_* \otimes_{BP_*} BP_*X = E_* \otimes_{MU_*} MU_*X.$$

This is represented by a spectrum E.

There is also a ring spectrum K, with

$$K_* = E_*/(u_0, \ldots, u_{n-1}) = \mathbb{F}_{p^n}[u^{\pm 1}] = \mathbb{F}_{p^n} \otimes K(n)_*[u]/(u^{p^n-1} - v_n).$$

We shall call this the *Morava K-theory* spectrum. There is a ring map $E \to K$ with the obvious effect on homotopy groups. If $p > 2$ then K is commutative. If $p = 2$ then there is a well-known formula

$$ab - ba = uQ_{n-1}(a)Q_{n-1}(b)$$

where $Q_{n-1}: K \to \Sigma K$ is a derivation. (A proof is given for $K(n)$ in [19, Theorem 2.13], for example, and the statement for K follows easily.) It follows that K^0X is commutative whenever X is a space with $K^1X = 0$. This will be the case for all spaces that we consider.

We also write $\hat{L}X$ for the Bousfield localisation of a spectrum X with respect to $K(n)$ (or equivalently, with respect to K). We write

$$E^{\vee}(X) = \pi_0 \hat{L}(E \wedge X).$$

This is a more natural thing to consider than E_0X when X is K-local. In that case E_0X is analogous to something like $\mathbb{Z}_p \otimes \mathbb{Z}_p$, which contains a large divisible subgroup, whereas $E^{\vee}X$ is analogous to the p-completion of $\mathbb{Z}_p \otimes \mathbb{Z}_p$, which is just $\mathbb{Z}_p$. It would lead us too far afield to justify these remarks here; we refer the interested reader to [11] for some discussion of related points.

3. E-THEORY OF SYMMETRIC GROUPS

In this section, we approach the calculation of $E^*B\Sigma_k$ using fairly traditional methods. We begin with a definition.

Definition 3.1. Define $\Lambda = (\mathbb{Q}_p/\mathbb{Z}_p)^n$, and $\Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Q}_p/\mathbb{Z}_p) = \mathbb{Z}_p^n$. Let $\mathbb{B}_k$ be the set of isomorphism classes of sets of order k with an action of Λ^*, and write $d(k) = |\mathbb{B}_k|$. Let $\mathbb{L}_k$ be the set of lattices of index p^k in Λ^*, which bijects with the set of transitive Λ^*-sets of order p^k, or with the set of subgroups of Λ of order p^k. Write $\bar{d}(k) = |\mathbb{L}_k|$. It is shown in [18, Section 10.1] that

$$\bar{d}(k) = \begin{bmatrix} n+k-1 \\ n-1 \end{bmatrix}_p = \prod_{j=1}^{n-1} \frac{p^{k+j}-1}{p^j-1}.$$

Our conclusions are as follows.

THEOREM 3.2. *$E^0B\Sigma_k$ is a Noetherian local ring and a free module over E^0 of rank $d(k)$. Moreover, we have*

$$\begin{aligned} E^{\vee}B\Sigma_k &= \mathrm{Hom}_{E^0}(E^0B\Sigma_k, E^0) \\ K^0B\Sigma_k &= \mathbb{F}_{p^n} \otimes_E E^0B\Sigma_k \\ K^0B\Sigma_k &= \mathrm{Hom}_{\mathbb{F}_{p^n}}(K^0B\Sigma_k, \mathbb{F}_{p^n}) \\ E^1B\Sigma_k &= E_1^{\vee}B\Sigma_k = K^1B\Sigma_k = K_1B\Sigma_k = 0. \end{aligned}$$

Finally, if $k = p^m$ then $c_k^{(p^n-1)/(p-1)}$ is nonzero in $K^0B\Sigma_k$ (this will be important later).

Proof. Assemble Propositions 3.4 and 3.6, and Corollary 3.8. □

Definition 3.3. Let W_m be the m-fold iterated wreath product $C_p \wr \dots \wr C_p$. It is well-known that this is the Sylow p-subgroup of Σ_{p^m}, and that for any k the Sylow subgroup of Σ_k is a product of W_m's.

PROPOSITION 3.4. *The group K^*BW_m is generated by transfers of Euler classes of complex representations of subgroups, and thus is concentrated in even degrees. Moreover, the element $b_m = c_{p^m}^{(p^n-1)/(p-1)}$ is nonzero.*

Proof. The first sentence is proved in [9]; we need to recall the details in order to prove the second sentence.

Suppose that the proposition holds for W_m, this being trivial for $m = 0$. We can then choose a basis $\{e_1, \dots, e_d\}$ for K^0BW_m over $\mathbb{F}_{p^n}$, where $e_1 = b_m$ and each e_k is a transferred Euler class. There is a spectral sequence

$$H^*(C_p; K^*(BW_m)^{\otimes p}) \Rightarrow K^*(BW_{m+1}).$$

Let z be a generator of $H^1(C_p; \mathbb{F}_p)$. Write $y = \beta z \in H^2(C_p; \mathbb{F}_p)$, so that y is the Euler class of the representation L of W_{m+1} associated to the character $W_{m+1} \to C_p \subset S^1$. Let $x \in K^0BW_{m+1}$ be the Morava K-theory Euler class for the same representation.

Given a sequence of indices $I = (i_0 \leqslant \cdots \leqslant i_{p-1})$ with $i_0 < i_{p-1}$, let e_I be the sum of the orbit under C_p of $e_{i_0} \otimes \cdots \otimes e_{i_{p-1}}$. We also write $e'_i = e_i \otimes \cdots \otimes e_i$. The E_2 term is

$$E[z] \otimes P[y] \otimes \mathbb{F}_p\{e_I, e'_i\}/(xe_I, ye_I).$$

We shall exhibit elements $\tilde{e}_I$ and $\tilde{e}'_i$ in K^0BW_{m+1} that hit e_I and e'_i under the restriction map $K^*BW_{m+1} \to BW_m^p$, which is the edge map of the spectral sequence. First, we take

$$\tilde{e}_I = \mathrm{tr}_{W_m^p}^{W_{m+1}}(e_{i_0} \otimes \cdots \otimes e_{i_{p-1}}) \in K^0BW_{m+1}.$$

It is easy to check that this hits e_I, and also that $x\tilde{e}_I = 0 = y\tilde{e}_I$. Next, suppose that $e_i = \mathrm{tr}_H^{W_m}(e(V))$ for some complex representation V of some subgroup $H \leqslant W_m$. There is an obvious way to extend the action of H^p on $p \otimes V$ to get an action of $C_p \wr H$. Write

$$\tilde{e}'_i = \mathrm{tr}_{C_p \wr H}^{W_{m+1}}(e(p \otimes V)) \in K^0(BW).$$

This hits e'_i under the edge map.

Our spectral sequence is a module over the Atiyah–Hirzebruch spectral sequence

$$H^*(BC_p; K^*) = E[z] \otimes P[y] \Rightarrow K^*BC_p = P[x]/x^{p^n}.$$

This has only one differential, viz. $d_{p^n-1}(z) = y^{p^n}$. We conclude that $d_{p^n-1}(ze'_i) = y^{p^n}e'_i$, and after this differential the spectral sequence is concentrated in even bidegrees so nothing else can happen. Thus, we have

$$K^*BW_{m+1} = K^*\{\tilde{e}_I, x^j\tilde{e}'_i\}$$

where j runs from 0 to $p^n - 1$.

We still need to show that b_{m+1} is nonzero. In fact, we will show that it is a unit multiple of $x^{p^n-1}\tilde{e}'_1$.

Let U be the pull back of the regular representation of C_p along the projection $W_{m+1} \twoheadrightarrow C_p$. Note that $U \simeq \bigoplus_{r=0}^{p-1} L^{\otimes r}$. We have $e(L^{\otimes r}) = [r](x)$, where $[r](t)$ is the r-series for the formal group law associated to E. This has the form $[r](t) = rt \pmod{t^2}$, so when $0 < r < p$ we see that $[r](x)$ is a unit multiple of x. It follows that x^{p-1} is a unit multiple of $e(U - 1)$, and thus that x^{p^n-1} is a unit multiple of $e(U-1)^N = e(N \otimes (U - 1))$, where $N = (p^n - 1)/(p - 1)$.

Next, recall that $b_m = e(V_{p^m} - 1)^N = e(N \otimes V_{p^m} - N)$. The restriction of the representation $N \otimes V_{p^{m+1}} - N \otimes U$ of W_{m+1} to the subgroup W_m^p is isomorphic to $p \otimes (N \otimes V_{p^m} - N)$, so $\tilde{e}'_1 = e(V_{p^{m+1}} - U)^N$. It follows that $x^{p^n-1}\tilde{e}'_1$ is a unit multiple of $e(V_{p^{m+1}} - 1)^N = b_{m+1}$, as required. □

PROPOSITION 3.5. *If Z is a spectrum such that E^*Z is finitely generated over E^* and K^*Z is concentrated in even degrees then E^*Z is free and concentrated in even degrees.*

Proof. One can construct ring spectra E/I_k (for $k > 0$) with

$$\pi_*(E/I_k) = E_*/I_k = \mathbb{F}_{p^n}[\![u_k, \ldots, u_{n-1}]\!][u^{\pm 1}]$$

and cofibrations $E/I_k \xrightarrow{u_k} E/I_k \to E/I_{k+1}$. In particular, we have $E/I_n = K$, and we put $E/I_0 = E$. The confibrations give short exact sequences

$$(E/I_k)^*(X)/u_k \rightarrowtail (E/I_{k+1})^*(X) \twoheadrightarrow \mathrm{ann}(u_k, (E/I_k)^*(\Sigma^{-1}X)).$$

By induction on k, using the fact that E^* is Noetherian, we see that $(E/I_k)^*X$ is finitely generated for all k. We claim that it is actually free, and concentrated in even degrees. This is trivially true for $(E/I_n)^*X$. If it is true for E/I_{k+1}, we see from the short exact sequence that $(E/I_k)^{\mathrm{odd}}(X)/u_k = 0$, so $(E/I_k)^{\mathrm{odd}}(X) = 0$ by Nakayama's lemma. We also see that $\mathrm{ann}(u_k, (E/I_k)^{\mathrm{even}}(X)) = 0$, so u_k acts regularly on $(E/I_k)^*X$. Finally, we see that $(E/I_{k+1})^*X = (E/I_k)^*(X)/u_k$. It follows by elementary commutative algebra that $(E/I_k)^*(X)$ is free, as required. □

PROPOSITION 3.6. *$E^*B\Sigma_k$ is a free module of rank $d(k)$ over E^*, concentrated in even degrees.*

Proof. Because the Sylow subgroup of $B\Sigma_k$ is a product of W_m's, we know from Proposition 3.4 and a transfer argument that $K^*B\Sigma_k$ is concentrated in even degrees. We also know from [8, Corollary 5.4] (for example) that E^*BG is finitely generated as an E^*-module, for any finite group G. It follows using Proposition 3.5 that $E^*B\Sigma_k$ is free and concentrated in even degrees. The rank is necessarily the same as the rank of $p^{-1}E^*B\Sigma_k$ over $p^{-1}E^*$, and the main theorem of [9] shows that this is the same as the number of conjugacy classes of homomorphisms $\Lambda^* = \mathbb{Z}_p^n \to B\Sigma_k$. (A version of this theorem is proved in [8, Appendix A], which may be easier to get hold of than [9].) It is easily seen that these biject with isomorphism classes of Λ^*-sets of order k, so the rank is $d(k)$. This is explained in more detail in [20]. □

It turns out that whenever E^*Z is a finitely generated free module over E^*, the module $E^\vee_*Z$ is simply the dual of E^*Z. However, we shall avoid having to prove this by giving a special results for classifying spaces.

PROPOSITION 3.7. *If G is a finite group then there is a weak equivalence*

$$\hat{L}(E \wedge BG_+) = F(BG_+, E)$$

and thus an isomorphism $E^*BG = E^\vee_*BG$.

Proof. Let X be a spectrum. The Greenlees–May theory of Tate spectra [5] gives a cofibration of G-spectra

$$EG_+ \wedge X \to F(EG_+, X) \to t_G(X).$$

We now apply the Lewis–May fixed point functor and use Adams' isomorphism $(EG_+ \wedge X)^G = EG_+ \wedge_G X$ (see [14, Theorem 7.1]). This gives a nonequivariant cofibration

$$BG_+ \wedge X \to F(BG_+, X \to P_G(X) = t_G(X)^G$$

and thus a cofibration

$$\hat{L}(BG_+ \wedge X) \to \hat{L}F(BG_+, X) \to \hat{L}P_G(X).$$

Moreover, all three terms here are exact functors of X. Let Y be a generalized Moore spectrum to type n, so that $E \wedge Y$ lies in the thick subcategory generated by K. It is shown in [7, 6] that $P_G K = 0$, so we must have $Y \wedge P_G E = 0$. As $K_* Y \neq 0$ we conclude that $K_* P_G E = 0$ and thus $\hat{L}P_G E = 0$. Note also that E is K-local, and thus the same is true of $F(BG_+, E)$. We thus conclude that $\hat{L}(BG_+ \wedge E) = F(BG_+, E)$, as claimed. □

COROLLARY 3.8. *If E^*BG is free then $E^\vee_* BG$ is its dual, and $K^*BG = K^* \otimes_{E^*} E^*BG$.*

Proof. Proposition 3.7 tells us that $E^\vee_* BG$ is free over E^*. The cofibrations in the proof of Proposition 3.5 show us that $K^*BG = K^* \otimes_{E^*} E^*BG$ and $K_*BG = K_* \otimes_{E_*} E^\vee_* BG$. Both of these are finitely generated over the graded field K^*, so they are dual to each other. Thus, the natural map $d_K : K_*BG \to \mathrm{Hom}_{K^*}(K^*BG, E^*)$ is an isomorphism. We also have a natural map $d_E : E^\vee_* BG \to \mathrm{Hom}_{E^*}(E^*BG, E^*)$, and one can check that $d_K = 1_{K_*} \otimes_{E_*} d_E$. As d_K is an isomorphism and the source and target of d_E are free of finite rank over the local ring E_*, we conclude that d_E is an isomorphism as required. □

4. THE EXTENDED POWER FUNCTOR

We next recall some of the ideas developed in [20]. For any based space X, we can define the total extended power

$$DX = \bigvee_{k \geqslant 0} E\Sigma_{k+} \wedge_{\Sigma_k} X^{(k)}.$$

In [14], it is shown how to define DX when X is a spectrum, so that $D(\Sigma^\infty X) = \Sigma^\infty DX$. We are mainly interested in $DS^0 = \bigvee_{k \geqslant 0} B\Sigma_{k+}$. We will sometimes think of this as a space, and sometimes as a spectrum. It can also be described as $\Sigma^\infty B\mathscr{C}_+$, where $\mathscr{C}$ is the category of finite sets. We observed in [20] that it admits two products and two coproducts:

$$\sigma\colon\ DS^0 \wedge DS^0 \to DS^0, \qquad \mu\colon\ DS^0 \wedge DS^0 \to DS^0,$$

$$\theta\colon\ DS^0 \to DS^0 \wedge DS^0, \qquad \delta\colon\ DS^0 \to DS^0 \wedge DS^0.$$

The map σ is obtained by applying D to the fold map $S^0 \vee S^0 \to S^0$, or by applying B to the coproduct functor $\coprod : \mathscr{C}^2 \to \mathscr{C}$. The map μ is obtained by applying B to the product functor $\times : \mathscr{C}^2 \to \mathscr{C}$. The map θ only exists stably. It is obtained by applying D to the stable pinch map $S^0 \to S^0 \vee S^0$, or by taking the transfer associated to the covering map σ. Finally, δ is just the diagonal map of the space $B\mathscr{C}$. The interaction between these maps is studied in detail in [20].

In order to transfer our results about $E^\vee_* B\Sigma_k$ to DS^0, we need to following lemma.

LEMMA 4.1. *Let M be an E-module spectrum such that the sequence $\{u_0, \ldots, u_{n-1}\}$ is regular on $\pi_* M$. Then $\pi_* \hat{L}M = \pi_*(M)_{\hat{I}_n}$.*

Proof. By [10, Theorem 2.1 and Corollary 2.2], we see that $\hat{L}M$ is the homotopy inverse limit of spectra $S/I \wedge L_n M$, as S/I runs over a suitable tower of generalized Moore spectra of type n. As M is a module over the $E(n)$-local spectrum E, we see that $L_nM = M$. If $I = (v_0^{a_0}, \ldots, v_{n-1}^{a_{n-1}})$, then regularity implies that $\pi_*(S/I \wedge M) = \pi_*(M)/I$. The Milnor sequence now implies that $\pi_*\hat{L}M = \pi_*(M)_{\hat{I}_n}$, as claimed. □

In particular, the lemma applies when $\pi_*(M)$ is a free module over E_*. To understand what it says in that case, observe that the completion of $\bigoplus_k E^0$ is the set of sequences $\underline{a} \in \prod_k E^0$ such that $a_k \to 0$ in the I-adic topology as $k \to \infty$.

It is now easy to deduce the following.

Proposition 4.2. *E^0DS^0 and $E^0(DS^0 \wedge DS^0)$ are products of countably many copies of E^0, and $E^0(DS^0 \wedge DS^0) = E^0DS^0 \hat{\otimes}_E E^0DS^0$. Also, $E^\vee DS^0$ and $E^\vee(DS^0 \wedge DS^0)$ are completions of countably generated free modules over E^0, and $E^\vee(DS^0 \wedge DS^0) = E^\vee DS^0 \hat{\otimes}_E E^\wedge DS^0$. Moreover, we have $E^0DS^0 = \mathrm{Hom}_E(E^\wedge DS^0, E^0)$.*

We therefore get (co)products as follows:

$$\begin{aligned}
* &: E^\vee DS^0 \hat{\otimes}_E E^\vee DS^0 \to E^\vee DS^0 \quad \text{induced by } \sigma\\
\psi_\times &: E^\vee DS^0 \to E^\vee DS^0 \hat{\otimes}_E E^\vee DS^0 \quad \text{induced by } \theta\\
\psi_\bullet &: E^\vee DS^0 \to E^\vee DS^0 \hat{\otimes}_E E^\vee DS^0 \quad \text{induced by } \delta\\
\psi_* &: E^0DS^0 \to E^0DS^0 \hat{\otimes}_E E^0DS^0 \quad \text{induced by } \sigma\\
\times &: E^0DS^0 \hat{\otimes}_E E^0DS^0 \to E^0DS^0 \quad \text{induced by } \theta\\
\bullet &: E^0DS^0 \hat{\otimes}_E E^0DS^0 \to E^0DS^0 \quad \text{induced by } \delta.
\end{aligned}$$

Theorem 4.3. *E^0DS^0 is a Hopf ring with products $\times$ and $\bullet$, and coproduct ψ_*.*

Proof. We need various diagrams of cohomology groups to commute. Each one is obtained by applying E^0 to an obvious diagram of spectra. These diagrams are proved to commute in [20, Theorem 3.2]. □

5. MORAVA HOMOLOGY OF DS^0

Kashiwabara proved in [13] that K_0DS^0 is polynomial under $*$. It seems worthwhile to give the following simplification of part of his argument.

Proposition 5.1 (Kashiwabara). *K_0DS^0 is a polynomial algebra under $*$.*

Proof. We can make K_0DS^0 into a graded vector space by putting $K_0B\Sigma_m$ in degree m. Because σ sends $B\Sigma_k \times B\Sigma_l$ to $B\Sigma_{k+l}$ and θ send $B\Sigma_m$ to $\bigvee_{m=k+l}(B\Sigma_k \times B\Sigma_l)_+$, we see that K_0DS^0 becomes a graded Hopf algebra over $\mathbb{F}_{p^n}$ using $*$ and $\psi_\times$. It is clearly connected, in the sense that the degree zero part is just $\mathbb{F}_{p^n}$. It follows from Theorem 3.2 that K_0DS^0 has finite type. It follows in turn from a theorem of Borel that K_0DS^0 is isomorphic as a graded ring to a tensor product of polynomial and truncated polynomial algebras. The proposition will therefore follow if we show that K_0DS^0 has no nontrivial nilpotent elements.

To see this, recall that we have a group-completion map $DS^0 \to QS^0$ and a unit map $QS^0 = \Omega^\infty S^0 \to \Omega^\infty BP$. The composite map $DS^0 \to \Omega^\infty BP$ induces a ring map

$K_0DS^0 \to K_0\Omega^\infty BP$. It is shown in [13] that the dual map $K^0\Omega^\infty BP \to K^0DS^0$ is surjective, so that $K_0DS^0 \to K_0\Omega^\infty BP$ is injective. It is also known from work of Wilson that $K_0\Omega^\infty BP$ is a tensor product of polynomial rings and Laurent series rings (use [21, Theorem 3.3] and a degenerate Atiyah–Hirzebruch spectral sequence, or see [17]). It follows that K_0DS^0 has no nontrivial nilpotent elements, as required. □

COROLLARY 5.2. *K_0QS^0 is the tensor product of a polynomial algebra with the group ring*

$$\mathbb{F}_{p^n}[\mathbb{Z}] = \mathbb{F}_{p^n}[\pi_0 QS^0] \simeq \mathbb{F}_{p^n}[u, u^{-1}].$$

Proof. It is well known that QS^0 is the group-completion of CS^0. Let Q_0S^0 be the component of the basepoint in QS^0, so that QS^0 is a disjoint union of copies of Q_0S^0 indexed by $\pi_0 QS^0 = \mathbb{Z}$. There is an evident self-map $s: QS^0 \to QS^0$ which sends the kth copy to the $(k+1)$th. It is not hard to construct a self-map $s: DS^0 \to DS^0$ compatible with this, and to check that the induced map from the telescope $s^{-1}DS^0$ to QS^0 is a weak equivalence. Moreover, for $a \in K_0DS^0$ we have $s_*(a) = [1] * a \in K_0DS^0$. It follows that K_0QS^0 is obtained from K_0DS^0 by inverting the element $[1]$. Also, $[1]$ is indecomposable in K_0DS^0, so we can take it as one of our polynomial generators. The claim follows. □

The next corollary is a straightforward consequence of Propositions 4.2 and 5.1.

COROLLARY 5.3. *$E^\vee DS^0$ is the completion of a polynomial algebra over E^0, and*

$$K_0DS^0 = \mathbb{F}_{p^n} \hat{\otimes}_E E^\vee DS^0.$$

We next prove that K^0QS^2 is a formal power series ring, using an unpublished argument of Kashiwabara.

Remark 5.4. Here we use the usual product on K^0QS^2, coming from the diagonal map. One might think that there were two products on K^0QS^2, by analogy with the two products on K^0DS^0. Indeed, the two projections $S^2 \vee S^2 \to S^2$ give two maps $Q(S^2 \vee S^2) \to QS^2$, and the resulting map $Q(S^2 \vee S^2) \to QS^2 \times QS^2$ is an equivalence. We can thus use the pinch map $S^2 \to S^2 \vee S^2$ to get a map $\theta: QS^2 \to QS^2 \times QS^2$, giving a product on K^0QS^2. However, it is easy to check that θ is homotopic to the diagonal map, so this is the same product as we considered before. This argument does not show that the two products on K^0DS^0 are the same, because the map $\theta: DS^0 \to DS^0 \wedge DS^0$ only exists stably.

Remark 5.5. For our purposes, the ring of formal power series over $\mathbb{F}_{p^n}$ on a countable set of indeterminates $x_0, x_1, x_2, \ldots$ consists of all sums $\sum_\alpha c_\alpha x^\alpha$. Here α runs over multi-indices $\alpha = (\alpha_0, \alpha_1, \ldots)$ such that $\alpha_i = 0$ for $i \gg 0$, and we impose no condition at all on the coefficients $c_\alpha \in \mathbb{F}_{p^n}$. In particular, $\sum_i x_i$ is allowed.

PROPOSITION 5.6 (Kashiwabara). *K^0QS^2 is a formal power series algebra over $\mathbb{F}_{p^n}$. Moreover, the double suspension map induces isomorphisms*

$$\mathrm{Ind}(K_0QS^0) \simeq \mathrm{Prim}(K_0QS^2) \quad \textit{and} \quad \mathrm{Ind}(K^0QS^2) \simeq \mathrm{Prim}(K^0QS^0).$$

Proof. We use the bar spectral sequence. Let Y be a connected infinite loop space, and $X = \Omega Y$, so that $Y = BX$. We then have a spectral sequence of Hopf algebras

$$E^2_{st} = \mathrm{Tor}^{K_*X}_{s,t}(K_*, K_*) \Rightarrow K_{s+t}(Y)$$

$$d^r : E^r_{st} \to E^r_{s-r,t+r-1}.$$

Note that $E^r_{st} = 0$ where $s < 0$; it follows that everything on the line $s = 1$ survives to E^∞. The resulting edge map

$$\mathrm{Ind}(K_t X) = \mathrm{Tor}^{K_*X}_{1,t}(K_*, K_*) \Rightarrow K_{1+t}(Y)$$

is just the homology suspension map.

We first apply this with $Y = QS^1$, so that $X = QS^0$. We know that K_*QS^0 is a polynomial algebra on generators $\{x_i\}$ in degree zero, with one generator inverted. It follows that $\mathrm{Tor}^{K_*QS^0}_{st}(K_*, K_*)$ is exterior on generators $y_i \in E^2_{10}$. These are represented by the homology suspensions of the generators of K_0QS^0. Because the spectral sequence is multiplicative and the generators survive, there can be no differentials at all. Thus K_*QS^1 is the exterior algebra on primitive classes in odd degree. Moreover, the suspension map induces an isomorphism $\mathrm{Ind}(K_*QS^0) \simeq \mathrm{Prim}(K_*QS^1)$, and the evident composite $\mathrm{Prim}(K_*QS^1) \rightarrowtail K_*QS^1 \twoheadrightarrow \mathrm{Ind}(K_*QS^1)$ is also an isomorphism.

We now repeat the process, and consider the bar spectral sequence

$$\mathrm{Tor}^{K_*QS^1}_{st}(K_*, K_*) \Rightarrow K_*(QS^2) \simeq K_*(DS^2).$$

The E^2 term is a divided power algebra on the suspensions of the elements y_i, which we shall call z_i. The coproduct is just the usual one:

$$\psi(z_i^{[m]}) = \sum_{m=j+k} z_i^{[j]} \otimes z_i^{[k]}.$$

The differential d^r maps E^r_{st} to $E^r_{s-r,t+r-1}$ and the E^2 page is concentrated in the lines where $s + t$ is even, so all the differentials vanish.

It follows that $K_*(QS^2)$ admits a filtration whose associated graded is a divided power Hopf algebra. Dually, we see that $K^*(QS^2)$ admits a filtration whose associated graded is a polynomial algebra on primitive generators in even degrees (without loss of generality, in degree zero). If we choose elements $z_i' \in K^0(QS^2)$ lifting these generators, we discover in the usual way that $K^0(QS^2) \simeq \mathbb{F}_{p^n}[\![z_i']\!]$. Moreover, the suspension map induces an isomorphism

$$\mathrm{Ind}(K_*QS^1) \simeq \mathrm{Prim}(K_*QS^2).$$

By combining this with information from our first spectral sequence, we see that the double suspension induces an isomorphism

$$\mathrm{Ind}(K_*QS^0) \simeq \mathrm{Prim}(K_*QS^2).$$

Dually, it induces an isomorphism

$$\mathrm{Ind}(K^*QS^2) \simeq \mathrm{Prim}(K^*QS^0). \qquad \square$$

Using Proposition 4.2, we deduce the following corollary.

COROLLARY 5.7. *E^0QS^2 is a formal power series ring over E^0.* □

6. THE SNAITH SPLITTING

In this section, we recall the fundamental theorem of Snaith, which gives a splitting $\Sigma^\infty QX \simeq \Sigma^\infty DX$ for connected spaces X. We will need to prove an additional property of this map (see Proposition 6.3) and thus we will need to give some details of its construction.

We will need to use the usual combinatorial approximation to QX, which we now describe briefly. First, we define

$$F_k = F_k\mathbb{R}^\infty = \{\text{injective maps } a:\{1, \dots, k\} \rightarrowtail \mathbb{R}^\infty\}.$$

It is well-known that this is a contractible free Σ_k-space, in other words, a model for $E\Sigma_k$. Next, we define

$$B_k(\mathbb{R}^\infty; X) = F_k \times_{\Sigma_k} X^k.$$

The disjoint union $\coprod_k B_k(\mathbb{R}^\infty; X)$ can be thought of as the set of pairs (A, x), where A is a finite subset of $\mathbb{R}^\infty$ and $x: A \to X$. We can impose an equivalence relation on this set by identifying (A, x) with $(B, x|_B)$ if $B \subseteq A$ and x sends $A \backslash B$ to the basepoint of X. We define

$$C(X) = C(\mathbb{R}^\infty; X) = \left(\coprod_k B_k(\mathbb{R}^\infty; X)\right)/\sim .$$

THEOREM 6.1. *There is a natural map $t: C(X) \to Q(X)$, which is a weak equivalence when X is connected. On the other hand, when $X = Y_+$ there is a natural homeomorphism*

$$C(Y_+) = \coprod_{k \geqslant 0} B_k(\mathbb{R}^\infty; X) = \coprod_{k \geqslant 0} E\Sigma_k \times_{\Sigma_k} Y^k = D(Y_+).$$

In that case, the resulting map $E\Sigma_k \times_{\Sigma_k} Y^k \to Q(Y_+)$ is adjoint to the composite

$$\Sigma^\infty E\Sigma_k \times_{\Sigma_k} Y_+^k \xrightarrow{\text{transfer}} \Sigma^\infty E\Sigma_k \times_{\Sigma_{k-1}} Y_+^k \xrightarrow{\text{proj}} \Sigma^\infty Y_+.$$

Proof. For the first sentence, see [15, 6.3] (for example). The rest follows easily from the constructions given there. □

We next define

$$C_k(X) = \text{image}(B_k(\mathbb{R}^\infty; X) \to C(X)).$$

This gives a filtration

$$0 = C_0(X) \subseteq X = C_1(X) \subseteq C_2(X) \subseteq \cdots \subseteq C(X)$$

and one can easily see that

$$C_k(X)/C_{k-1}(X) = E\Sigma_{k+} \wedge_{\Sigma_k} X^{(k)} = D_k(X).$$

We can now define the Snaith map. For this, we note that the polynomial rings $\mathbb{R}[u]$ and $\mathbb{R}[u, v]$ can both be used as models of $\mathbb{R}^\infty$. A point of $a \in C(\mathbb{R}[u]; X)$ is a finite set A of polynomials in u, with each polynomial $f \in A$ given a label $x_f \in X$. For any subset $B \subseteq A$, we have a polynomial

$$g_B = \prod_{f \in B} (v - f) \in \mathbb{R}[u, v].$$

Clearly, if $B \neq B'$ then $g_B \neq g_{B'}$, by unique factorisation. Note also that $(B, x|_B)$ defines a point of $C_{|B|}(\mathbb{R}[u]; X)$; we write y_B for its image in $D_{|B|}(\mathbb{R}[u]; X)$. By labelling each g_B with the point y_B, we obtain a point $s'(a) = (\{g_B | B \subseteq A\}, y) \in C(\mathbb{R}[u, v]; DX)$. It can be shown that this gives a well-defined and continuous map

$$s' : C(\mathbb{R}[u]; X) \to C(\mathbb{R}[u, v]; DX).$$

After composing with the map $t : C(\mathbb{R}[u, v]; DX) \to QDX$ and taking adjoints, we obtain a map $s : \Sigma^\infty CX \to \Sigma^\infty DX = D\Sigma^\infty X$. One finds that this sends $\Sigma^\infty C_k X$ into $\Sigma^\infty \bigvee_{j \leqslant k} D_j X$ and thus that it induces a map $\Sigma^\infty(C_k X / C_{k-1} X) \to \Sigma^\infty D_k X$, which is an equivalence. This is essentially the proof of the following theorem; for more details, see [3].

THEOREM 6.2. *The Snaith map $s : \Sigma^\infty CX \to \Sigma^\infty DX = D\Sigma^\infty X$ is a weak equivalence. In the case $X = Y_+$, it can also be obtained by applying Σ^∞ to the homeomorphism $C(Y_+) = D(Y_+)$ mentioned in Theorem* 6.1.

There is an obvious equivalence $QS^0 \simeq \Omega^2 QS^2$, which gives by adjunction a map $e : \Sigma^2 QS^0 \to QS^2$. We will need to see how this interacts with the Snaith splitting. Firstly, we have a map $e' : \Sigma^2 CS^0 \to CS^2$, defined as follows: if $z \in S^2$ and $A \in CS^0$ is a finite subset of $\mathbb{R}^\infty$, then $e'(z, A)$ is just the subset A with each point labelled by z.

Next, recall that V_k is the bundle over $B\Sigma_k$ corresponding to the usual representation of Σ_k on $\mathbb{C}^k$. We write $B\Sigma_k^{V_k}$ for the corresponding Thom space. One can show directly (see [14, Section IX.5]) that

$$D_k(S^2) = B\Sigma_k^{V_k} = D_k(S^0)^{V_k}.$$

There is an evident vector $v = (1, \ldots, 1) \in V_k$ that is fixed under the action of Σ_k. Write $L = \mathbb{C}v = V_k^{\Sigma_k} < V_k$, so that $B\Sigma_k^{V_k} = \Sigma^2 B\Sigma_k$. The inclusion $L \to V_k$ thus gives a map $\Delta : \Sigma^2 D_k(S^0) \to D_k(S^2)$ and thus a map $\Delta : \Sigma^2 DS^0 \to D(S^2)$.

We will also need a map $\varepsilon_1 : DS^0 \to S^0$. This is the counit map for the coproduct map $\delta : DS^0 \to DS^0 \wedge DS^0$. It sends each space $B\Sigma_k$ in $DS^0 = \bigvee_{k \geqslant 0} B\Sigma_{k+}$ to the non-basepoint in S^0.

PROPOSITION 6.3. *The following diagram commutes.*

$$\begin{tikzcd}
\Sigma^{\infty+2} QS^0 \arrow[rrr, "e"] & & & \Sigma^\infty QS^2 \\
\Sigma^{\infty+2} CS^0 \arrow[u, "\Sigma^{\infty+2} t"] \arrow[rrr, "e'"] \arrow[d, equal, "\Sigma^2 s"'] & & & \Sigma^\infty CS^2 \arrow[u, "\simeq"', "\Sigma^\infty t"] \arrow[d, "\simeq"', "s"] \\
\Sigma^{\infty+2} DS^0 \arrow[r, "\Sigma^2 \theta"'] & \Sigma^{\infty+2} DS^0 \wedge DS^0 \arrow[r, "1 \wedge \epsilon_1"'] & \Sigma^{\infty+2} DS^0 \arrow[r, "\Sigma^\infty \Delta"'] & \Sigma^\infty D(S^2)
\end{tikzcd}$$

Proof. It is immediate from the definition of t that the top rectangle commutes. We next observe that the connectivity of $D_m(S^2)$ is at least $2m - 1$, so that $\bigoplus_m \pi_t \Sigma^\infty D_m(S^2) = \prod_m \pi_t \Sigma^\infty D_m(S^2)$ for all t. It follows that the spectrum $D(S^2)$ is the product of the spectra $D_m(S^2)$, and of course $\Sigma^{\infty+2} DS^0$ is the coproduct of the spectra $\Sigma^{\infty+2} B\Sigma_{k+}$. It is thus enough to check that the bottom rectangle commutes after composing with the inclusion $\Sigma^2 B\Sigma_{k+} \to \Sigma^2 DS^0$ and the projection $D(S^2) = \prod_m D_m(S^2) \to D_m(S^2)$.

The two ways around the rectangle give two maps $\Sigma^{\infty+2} B\Sigma_{k+} \to \Sigma^\infty D_m(S^2)$. We shall assume that $m \leqslant k$, leaving it to the reader to see that both maps are zero when $m > k$. We can also take adjoints and identify $QD_m(S^2)$ with $C(\mathbb{R}[u, v];\ D_m(S^2))$ to get two maps of spaces $\Sigma^2 B\Sigma_{k+} \to C(\mathbb{R}[u, v]; D_m(S^2))$. We write F for the one which comes from $s \circ e' : \Sigma^{\infty+2} DS^0 \to \Sigma^\infty D(S^2)$, and G for the one which comes from $\Delta \circ (1 \wedge \varepsilon_1) \circ \theta \circ t$.

Consider a point $(z, A) \in \Sigma^2 B\Sigma_{k+}$, so $z \in S^2$ and A is a finite subset of $\mathbb{R}[u]$ of order k. We see easily from our definition of the Snaith map that $F(z, A) \in C(\mathbb{R}[u, v];\ D_m(S^2))$ is the collection of polynomials g_B (where $B \subseteq A$ and $|B| = m$) labelled with the points $\Delta(z, B) \in D_m(S^2)$.

Next, recall that θ is built from the transfer maps $\Sigma^\infty B\Sigma_{(i+j)+} \to \Sigma^\infty (B\Sigma_i \times B\Sigma_j)_+$. It follows that we have a commutative diagram

$$\begin{tikzcd}
\Sigma^\infty B\Sigma_{k+} \arrow[r, "\text{transfer}"] \arrow[d] & \Sigma^\infty (B\Sigma_m \times B\Sigma_{k-m})_+ \arrow[r, "\text{proj}"] & \Sigma^\infty B\Sigma_{m+} \\
DS^0 \arrow[r, "\theta"'] & DS^0 \wedge DS^0 \arrow[r, "1\wedge\epsilon_1"'] & DS^0 \arrow[u]
\end{tikzcd}$$

To understand the top composite, we need to recall something about transfers. Let $q : X \to Y$ be a finite covering map, and let $i : X \to \mathbb{R}^\infty$ be a map that is injective on each fibre of q. We then get a map $r : Y_+ \to C(X_+)$ by sending a point $y \in Y$ to the set $iq^{-1}\{y\}$, with each point $i(x) \in iq^{-1}\{y\}$ labelled by x. It is easy to see from the standard constructions that $t \circ r : Y_+ \to Q(X_+)$ is adjoint to the transfer map $q' : \Sigma^\infty Y_+ \to \Sigma^\infty X_+$.

We now apply this with $X = F_k(\mathbb{R}[u])/(\Sigma_m \times \Sigma_{k-m})$ (which is a model for $B\Sigma_m \times B\Sigma_{k-m}$) and $Y = F_k(\mathbb{R}[u])/\Sigma_k$ (which is a model for $B\Sigma_k$). We can also think of Y as the set of subsets $A \subset \mathbb{R}[u]$ of order k, and X as the set of pairs (B, C) of disjoint subjects of $\mathbb{R}[u]$ with $|B| = m$ and $|C| = k - m$. The map q just sends (B, C) to $B \cup C$. For our map $i : X \to \mathbb{R}^\infty = \mathbb{R}[u, v]$, we take $i(B, C) = g_B = \prod_{f \in B}(v - f)$. The adjoint to the transfer is thus the map $Y_+ \to C(\mathbb{R}[u, v]; X_+)$ that sends A to the set of polynomials g_B (for all subsets $B \subseteq A$ of order m), with g_B labelled by (B, C). When we compose with the projection $X = B\Sigma_m \times B\Sigma_{k-m} \to B\Sigma_m$, we just replace the label $(B, C) \in X$ by $B \in B\Sigma_m$.

It now follows that adjoint of the double suspension of the top composite in the above diagram sends a point $(z, A) \in \Sigma^2 B\Sigma_k$ to the set of polynomials g_B (for $B \subseteq A$ with $|B| = k$), labelled by (z, B). To get $G(z, A)$, we just have to apply Δ to each label. By comparing this with our analysis of F, we see that $G(z, A) = F(z, A)$ as claimed. $\square$

7. THE THOM ISOMORPHISM

We now discuss the Thom isomorphism. We saw above that $D_m(S^2) = B\Sigma_m^{V_m}$, so we have a Thom isomorphism $\tilde{E}^0 D_m(S^2) = E^0 B\Sigma_m$ and thus $E^0 D(S^2) = E^0 D(S^0)$. We need to show that this respects ring structures in a suitable sense. Just as in Remark 5.4, we find that there is only one sensible product on $E^0 D(S^2)$, which comes both from the diagonal map $D(S^2) \to D(S^2) \times D(S^2)$ and from the pinch map $S^2 \to S^2 \vee S^2$. We give $E^0 D(S^0)$ the $\times$-product, coming from the stable pinch map $S^0 \to S^0 \vee S^0$.

LEMMA 7.1. *With the above product structures, the Thom isomorphism* $E^0 D(S^2) = E^0 D(S^0)$ *is a ring map.*

Proof. For any spectra X and Y, we have a natural map $\delta: D_k(X \wedge Y) \to D_k(X) \wedge D_k(Y)$, which is induced in an evident way by the diagonal map $E\Sigma_k \to E\Sigma_k \times E\Sigma_k$ when X and Y are suspension spectra. See [2, Section I.2] for discussion of this, and [14, Section VII.1] for proofs. In particular, there is a natural map $\delta: D_k(\Sigma^2 X) \to D_k(S^2) \wedge D_k(X)$. Let $u \in E^0 D_k(S^2)$ be the Thom class, and define $\phi: E^0 D_k(X) \to E^0 D_k(\Sigma^2 X)$ by $\phi(x) = \delta^*(u \otimes x)$. When X has the form $\Sigma^\infty Y_+$ we notice that $D_k(\Sigma^2 Y_+)$ is the Thom space for the pullback of V_k to $D_k(Y_+)$, and that ϕ is the Thom isomorphism. The upshot is that this Thom isomorphism is natural for stable maps of $\Sigma^\infty Y_+$, not merely for unstable maps of Y. The lemma follows by considering the stable pinch map $S^0 \to S^0 \vee S^0$. □

COROLLARY 7.2. *E^0DS^0 is a formal power series ring under $\times$.*

Proof. By the lemma, E^0DS^0 under $\times$ is isomorphic to E^0DS^2 under the product coming from the (unstable) pinch map $S^2 \to S^2 \vee S^2$. By the Snaith splitting, E^0DS^2 is isomorphic to E^0QS^2. By naturality, this converts our product to the product on E^0QS^2 that comes from the pinch map. By Proposition 5.6 and Remark 5.4, E^0QS^2 is a formal power series ring under this product. □

8. PRIMITIVES AND INDECOMPOSABLES

In this section we consider the primitives and indecomposables in the Hopf ring E^0DS^0. We need to interpret the indecomposables in a completed sense.

Definition 8.1. We topologise $E^0B\Sigma_k$, $E^\vee B\Sigma_k$ and $E^\vee DS^0$ using the $\mathfrak{m}$-adic topology, where $\mathfrak{m} = (u_0, \ldots, u_{n-1})$ is the maximal ideal in E^0. We give $E^0DS^0 = \prod_k E^0B\Sigma_k$ the product topology. Given an augmented topological algebra A over E^0, we define $\mathrm{Ind}(A)$ to be the quotient of the augmentation ideal by the closure of its square.

Remark 8.2. With this definition, $\mathrm{Ind}(E^0[\![x_0, x_1, \ldots]\!])$ is a countable product of copies of E^0, indexed by the variables x_i.

We need to distinguish between two kinds of primitives in E^0DS^0. Recall that we have two unit elements $[0]$ and $[1]$ in $E^0DS^0 = \prod_k E^0B\Sigma_k$. The component of $[0]$ in $E^0B\Sigma_k$ is 1 if $k=0$ and 0 if $k>0$, whereas the component of $[1]$ is 1 for all k. We have $[0] \times a = a = [1] \bullet a$ for all $a \in E^0DS^0$. Note also that $[1]$ is invertible under the $\times$-product. Its inverse is $[-1]$, which is the image of $[1]$ under the map $D(-1)$: $DS^0 \to DS^0$.

Definition 8.3. We write

$$\mathrm{Prim}(E^0DS^0) = \{x \in E^0DS^0 \mid \psi_*(x) = x \otimes [0] + [0] \otimes x\}$$

$$\mathrm{Prim}'(E^0DS^0) = \{x \in E^0DS^0 \mid \psi_*(x) = x \otimes [1] + [1] \otimes x\}.$$

The first of these is the more usual definition for primitives in a Hopf ring. It is easy to see from general thoughts about Hopf rings that the $\bullet$-product makes $\mathrm{Ind}(E^0DS^0)$ into a ring and $\mathrm{Prim}(E^0DS^0)$ into a module over $\mathrm{Ind}(E^0DS^0)$.

On the other hand, one would like to say that the primitives in E^0DS^0 are dual to $\mathrm{Ind}(E^\vee DS^0)$. Here $E^\vee DS^0$ is a Hopf algebra with coproduct ψ_*, and it is natural to use the counit for ψ_* as our augmentation on $E^\vee DS^0$. This counit is $[1] \in E^0DS^0$ rather than $[0]$, and because of this the natural duality is between $\mathrm{Prim}'(E^0DS^0)$ and $\mathrm{Ind}(E^\vee DS^0)$. This becomes clearer by examining the explicit formulae in [20, Section 6]. As mentioned there, the map $a \mapsto a \times [1]$ gives an isomorphism $\mathrm{Prim}(E^0DS^0) \to \mathrm{Prim}'(E^0DS^0)$.

We will need to relate our primitives and indecomposables in terms of transfers and restrictions, which leads us to our next definitions.

Definition 8.4. Write $R_m = E^0B\Sigma_m$. We shall call the subgroups $B\Sigma_i \times B\Sigma_j \leqslant B\Sigma_{i+j}$ (with $i, j > 0$) *partition subgroups*. For each such subgroup $H \leqslant B\Sigma_m$, we have a transfer map $E^0BH \to R_m$; let $I = I_m \leqslant R_m$ be the sum of the images of these maps. We also have restriction maps $R_m \to E^0BH$; let $J = J_m \leqslant R_m$ be the intersection of the kernels of these maps.

The following two results summarise our conclusions. They are proved after Lemma 8.11.

THEOREM 8.5. *The star-indecomposables and primitives in the Hopf ring E^0DS^0 are as follows*:

$$\mathrm{Ind}(E^0DS^0) = \prod_m R_m/I_m = \prod_k R_{p^k}/I_{p^k}$$

$$\mathrm{Prim}(E^0DS^0) = \prod_m J_m = \prod_k J_{p^k}.$$

Moreover, $\mathrm{Prim}(E^0DS^0)$ *is a free module over* $\mathrm{Ind}(E^0DS^0)$ *on one generator c, whose component in $R_m = E(B\Sigma_{m+})$ is the Euler class $c_m = e(V_m - 1)$.*

THEOREM 8.6. *Suppose that $m = p^k$. Then $I_m = \mathrm{ann}(c_m) = \mathrm{ann}(J_m)$, and this is retract of R_m as an E^0-module. Moreover, J_m is a free module on one generator c_m over R_m/I_m (and thus is generated as an ideal by c_m). The ideal J_m is also a retract of R_m as an E^0-module. The rank of R_m/I_m over E^0 is $\bar{d}(k)$ (see Definition* 3.1.).

In the light of the above, it is natural to make the following definition.

Definition 8.7. $\bar{R}_k = R_{p^k}/I_{p^k}$.

We now prove a series of lemmas leading to the above theorems.

LEMMA 8.8. *I_m and J_m are ideals in R_m, and $I_mJ_m = 0$.*

Proof. This follows from the fact that res is a ring map, and the formula $x\,\mathrm{tr}(y) = \mathrm{tr}(\mathrm{res}(x)y)$. □

LEMMA 8.9. *The Euler class $c_m = e(V_m - 1)$ lies in J_m.*

Proof. Suppose that $m = i + j$ with $i, j > 0$. Then the restriction of V_m to $\Sigma_i \times \Sigma_j$ is isomorphic to $V_i \oplus V_j$, which has two trivial summands. Thus, the restriction of $V_m - 1$ has a trivial summands and thus $e(V_m - 1)$ maps to zero. □

LEMMA 8.10. *If m does not have the form $m = p^k$ then $I_m = R_m$ and $J_m = 0$.*

Proof. In this case it is well known that $(a+b)^m \not\equiv a^m + b^m \pmod p$, and thus that there exist $i, j > 0$ with $i + j = m$ and $d = |\Sigma_m/H|$ not divisible by p, where $H = \Sigma_i \times \Sigma_j$. The rest of the argument is well-known: we note that the composite

$$E^0 B\Sigma_m \xrightarrow{\text{res}} E^0 BH \xrightarrow{\text{tr}} E^0 B\Sigma_m$$

is just multiplication by $\text{tr}(1)$. The double coset formula shows that $\varepsilon(\text{tr}(1)) = d \in E^0$, and this is a unit. Because $E^0 B\Sigma_m$ is a local ring, we conclude that $\text{tr}(1)$ is a unit. This implies that $\text{tr}: E^0 BH \to E^0 B\Sigma_m$ is surjective, and $\text{res}: E^0 B\Sigma_m \to E^0 BH$ is injective. □

LEMMA 8.11. *If $m = p^k$ then*

$$I_m = \text{image}(\text{tr}: E^0 B\Sigma_{p^{k-1}}^p \to E^0 B\Sigma_{p^k})$$

$$J_m = \ker(\text{res}: E^0 B\Sigma_{p^k} \to E^0 B\Sigma_{p^{k-1}}^p).$$

Proof. Write $H = \Sigma_{p^{k-1}}^p$ and $G = \Sigma_{p^k}$. Consider a partition subgroup $L = \Sigma_i \times \Sigma_j$, where $i + j = p^k$ and $i, j > 0$. Using the argument of Lemma 8.10, it will be enough to show that $|L/L \cap H|$ is not divisible by p, or equivalently that H contains a Sylow p-subgroup of L. There are various ways to see this. The most direct is to use the following well-known formulae for the p-adic valuations of factorials and binomials:

$$v_p(p^k!) = (p^k - 1)/(p-1)$$

$$v_p\binom{p^k}{i} = k - v_p(i) \quad \text{if } 0 < i < p^k.$$

If $i = p^{k-1} r + s$ with $0 \leqslant s < p^{k-1}$ then

$$H \cap L = \Sigma_{p^{k-1}}^{p-1} \times \Sigma_s \times \Sigma_{p^{k-1}-s}$$

and we calculate directly that

$$v_p|H \cap L| = (p^k - 1)/(p-1) - k + v_p(s) = v_p|L|$$

as required. □

Proof of Theorem 8.5. The augmentation for the Hopf ring $E^0 DS^0$ is the counit for the coproduct ψ_*, which is just the restriction map

$$E^0 DS^0 \to E(D_0 S) = E(B\Sigma_0) = E.$$

Thus, the augmentation ideal is $\prod_{m>0} E(B\Sigma_m)$. The star product is the product $\times$, which is derived from the map $\theta = D(\Delta): DS^0 \to DS^0 \wedge DS^0$. As mentioned in Section 4, this sends $\Sigma^\infty B\Sigma_{m+}$ to $\bigvee_{m=i+j} \Sigma^\infty (B\Sigma_i \times B\Sigma_j)_+$ by the sum of the transfer maps. It follows easily (using Lemma 8.10) that

$$\text{Ind}(E^0 DS^0) = \prod_m R_m/I_m = \prod_k R_{p^k}/I_{p^k}.$$

Similarly, the unit for the star product is the element $[0]$, whose component in $E(B\Sigma_m)$ is 1 if $m = 0$ and 0 otherwise. The coproduct ψ_* is derived from the map $\sigma: DS^0 \wedge DS^0 \to DS^0$, which sends $B\Sigma_i \times B\Sigma_j$ to $B\Sigma_{i+j}$ by the usual map, which induces the restriction map in

E-cohomology. It follows that an element $u \in E^0 DS^0$ is primitive (i.e. $\psi_*(u) = [0] \otimes u + u \otimes [0]$) if and only if the component u_m lies in J_m for all m. In other words (using Lemma 8.10 again)

$$\mathrm{Prim}(E^0 DS^0) = \prod_m J_m = \prod_k J_{p^k}.$$

The circle product in $E^0 DS^0$ is just the ordinary product $\bullet$ induced by the diagonal map of the space DS^0. As I_m is an ideal in R_m, we see that the circle product induces a ring structure on $\mathrm{Ind}(E^0 DS^0)$. We observed above that $I_m J_m = 0$, which implies that $\mathrm{Prim}(E^0 DS^0)$ is a module over $\mathrm{Ind}(E^0 DS^0)$. (Both of these facts hold in an arbitrary Hopf ring, and can be proved by elementary manipulations.)

This proves all of Theorem 8.5 except for the last sentence. This is a trivial consequence of Theorem 8.6, which we prove next. □

Proof of Theorem 8.6. Firstly, we known from Corollary 7.2 that $E^0 DS^0$ is a formal power series ring over E^0. It follows that the ideal R_+^2 of decomposables is a retract of the augmentation ideal R_+ as an E^0-module. We also know that both ideals are homogeneous with respect to the splitting $E^0 DS^0 = \prod_m R_m$. By choosing homogeneous bases we conclude that $I_m = R_+^2 \cap R_m$ is a retract of R_m, as claimed.

Similarly, we know that $E^\vee DS^0$ is a completed polynomial ring; it follows that $\mathrm{Ind}(E^\vee DS^0)$ is a retract of $E^\vee DS^0$. Dually, we see that $\mathrm{Prim}'(E^0 DS^0)$ is a retract of $E^0 DS^0$. By applying the map $x \mapsto (-1) \times x$ (which is an automorphism of $E^0 DS^0$), we conclude that $\mathrm{Prime}(E^0 DS^0)$ is a retract of $E^0 DS^0$. Moreover, $\mathrm{Prime}(E^0 DS^0)$ is homogeneous with respect to the splitting $E^0 DS^0 = \prod_m R_m$ (although $\mathrm{Prim}'(E^0 DS^0)$ is not). It follows as in the previous paragraph that J_m is a retract of R_m.

In [20, Section 6] it is shown that there is a natural isomorphism $L \otimes_E R_m/I_m \simeq F(\mathbb{L}_m, L)$, where L is a certain extension ring of E^0, $\mathbb{L}_m$ is the set of lattices of index m in $\mathbb{Z}_p^n$, and $F(\mathbb{L}_m, L)$ is the ring of functions from the finite set $\mathbb{L}_m$ to L (with pointwise operations). In particular, we see that the rank of R_m/I_m over E^0 is the number of lattices of index m in $\mathbb{Z}_p^n$, which is one definition of $\bar{d}(k)$ (see Definition 3.1). It is also shown in [20] that $L \otimes_E J_m$ has rank $\bar{d}(k)$, and we conclude in the same way that J_m has rank $\bar{d}(k)$ over E^0.

Next, we show that $\mathrm{Prime}(E^0 DS^0)$ is freely generated by c over $\mathrm{Ind}(E^0 DS^0)$. We apply the functor $E^0(-)$ to the diagram in Proposition 6.3 and use various Thom isomorphisms to obtain the following diagram.

$$\begin{tikzcd}
E(QS^2) \arrow[rr, "\sigma^{*2}"] \arrow[d] & & E(QS^0) \arrow[d] \\
E(DS^2) \arrow[r, "\bullet c"'] & E(DS^0) \arrow[r, "\times[1]"'] & E(DS^0)
\end{tikzcd}$$

(The first map in the bottom row is induced by the inclusions $\Sigma^2 B\Sigma_m = B\Sigma_m^{\mathbb{C}} \xrightarrow{\Delta} B\Sigma_m^{V_m}$ and the Thom isomorphisms $E(B\Sigma_m^{\mathbb{C}}) \simeq E(B\Sigma_m) \simeq E(B\Sigma_m^{V_m})$. It is standard that the resulting map is just multiplication by the Euler class of $V_m - 1$, which is just c_m. This shows that the first map on the bottom row is just $x \mapsto x \bullet c$, as marked in the diagram.)

The right hand vertical map is an isomorphism (by the Snaith splitting theorem). The top arrow is the double suspension map, so the image is contained in $\mathrm{Prime}'(E^0 DS^0)$.

Consider the analogous map in K-cohomology. Proposition 5.6 shows that the image of this map is precisely $\mathbb{F}_{p^n} \hat{\otimes}_E \mathrm{Prim}'(E^0DS^0) = \mathrm{Prim}'(K^0DS^0)$. As all the modules in question are products of free modules, we conclude that the double suspension $E(QS^2) \to \mathrm{Prim}'(E^0DS^0)$ is surjective.

Recall that $x \mapsto x \times [1]$ is an E^0-module automorphism of E^0DS^0 which carries $\mathrm{Prim}(E^0DS^0)$ isomorphically to $\mathrm{Prim}'(E^0DS^0)$. It follows from this and the discussion above that the map $x \mapsto x \bullet c$ is an epimorphism $\mathrm{Ind}(E^0DS^0) \to \mathrm{Prim}(E^0DS^0)$. We therefore have an epimorphism $R_m/I_m \xrightarrow{c} J_m$. Both source and target are finitely generated projective (and hence free) modules over E^0, and we have seen that they have the same rank. It follows that our map $R_m/I_m \xrightarrow{c_m} J_m$ is an isomorphism. This means in particular that $\mathrm{ann}(c_m) = I_m$ as claimed. □

9. SUBGROUPS OF FORMAL GROUPS

In this section, we identify the functor represented by the ring $\bar{R}_k$. We shall assume that results and terminology of [18], and the discussion of the E-cohomology of Abelian groups in [8, Section 4]. We shall write $X = \mathrm{spf}(E^0)$ and $\mathbb{G} = \mathrm{spf}(E^0\mathbb{C}P^\infty)$, so that $\mathbb{G}$ is a formal group over X. The special fibre of X is $X_0 = \mathrm{spf}(\mathbb{F}_{p^n})$, and the restriction of $\mathbb{G}$ over X_0 is $\mathbb{G}_0 = \mathrm{spf}(K^0\mathbb{C}P^\infty)$. Moreover, $\mathbb{G}$ is the universal deformation of $\mathbb{G}_0$.

Now let Z be a space, and W a complex vector bundle over Z. We can define the associated projective bundle

$$P(W) = \{(z, L) \mid z \in Z,\ L \leqslant W_z,\ \dim(L) = 1\}.$$

There is tautological line bundle $L(W)$ over $P(W)$, defined by

$$L(W)_{(z,L)} = L.$$

This is classified by a map $P(W) \to \mathbb{C}P^\infty$. We thus obtain a map $P(W) \to \mathbb{C}P^\infty \times Z$ and hence $\mathrm{spf}(E^0P(W)) \to \mathbb{G} \times_X \mathrm{spf}(E^0Z)$. If x is the Euler class of $L(W)$ and a_i is the i'th Chern class of W, then a well-known lemma says that

$$E^*P(W) = E^*Z[\![x]\!] / \sum_{k=0}^{m} a_{m-k}x^k.$$

(This is clear when W is trivialisable, and thus by a Mayer–Vietoris argument when Z is a finite union of reasonable subsets over which W is trivialisable, and thus for arbitrary Z by a limit argument. See [4, Theorems 7.4 and 7.6] or [16] for more discussion.) This shows that $\mathbb{D}(W) = \mathrm{spf}(E^0P(W))$ is a divisor of degree m on $\mathbb{G}$ over $\mathrm{spf}(E^0Z)$.

We can apply this process to get a divisor $\mathbb{D}(V_k)$ over $\mathrm{spf}(E^0B\Sigma_k)$. We write $Y_k = \mathrm{spf}(\bar{R}_k)$, which is a closed subscheme of $\mathrm{spf}(E^0B\Sigma_{p^k})$, and we let H_k denote the restriction of $\mathbb{D}(V_{p^k})$ to Y_k.

Recall that there is a scheme $\mathrm{Div}_{p^k}(\mathbb{G})$ over X that classifies divisors of degree p^k on $\mathbb{G}$ over X. (It is an enlightening exercise to identify $\mathrm{Div}_{p^k}(\mathbb{G})$ with $\mathrm{spf}(E^0BU(p^k))$.) We thus have a map $\mathrm{spf}(E^0B\Sigma_{p^k}) \to \mathrm{Div}_{p^k}(\mathbb{G})$, which classifies $\mathbb{D}(V_{p^k})$. We also recall from [18, Theorem 42] that there is a closed subscheme $\mathrm{Sub}_k(\mathbb{G}) \subseteq \mathrm{Div}_{p^k}(\mathbb{G})$ that classifies subgroup divisors. We define

$$Z_k = \mathrm{spf}(E^0B\Sigma_{p^k}) \times_{\mathrm{Div}_{p^k}(\mathbb{G})} \mathrm{Sub}_k(\mathbb{G}).$$

This is the largest closed subscheme of $\mathrm{spf}(E^0 B\Sigma_{p^k})$ over which $\mathbb{D}(V_{p^k})$ is a subgroup divisor.

PROPOSITION 9.1. *We have $Y_k \subseteq Z_k$, so that H_k is a subgroup divisor of $\mathbb{G}$ over Y_k.*

Proof. Let A be an Abelian p-subgroup of Σ_{p^k} that acts transitively on $\{1, \dots, p^k\}$. It is not hard to see that the restriction of V_{p^k} to A is the regular representation. This can also be written as $\bigoplus_{L \in A^*} L$, the direct sum of all one-dimensional representations of A. Recall from [8, Proposition 4.12] that $\mathrm{spf}(E^0 BA)$ can be naturally identified with $\mathrm{Hom}(A^*, \mathbb{G})$. Over this scheme we have a tautological map $\phi : A^* \to \Gamma(\mathrm{Hom}(A^*, \mathbb{G}), \mathbb{G})$ and the identification is such that $\mathbb{D}(L) = [\phi(L)]$ for each $L \in A^*$. It follows that the restriction of $\mathbb{D}(V_{p^k})$ to $\mathrm{Hom}(A^*, \mathbb{G})$ is the divisor $\sum_{L \in A^*}[\phi(L)]$.

If we choose a degree zero complex orientation for E, we get Euler classes $e(M) \in E^0 Z$ for each complex line bundle M over a space Z. In particular, a character $L \in A^*$ gives a line bundle over BA (which we also call L) and thus an Euler class $e(L) \in E^0 BA$. We also write x for the Euler class of the tautological line bundle over $\mathbb{C}P^\infty$, so $x \in \mathcal{O}_{\mathbb{G}}$ can be thought of as a coordinate on $\mathbb{G}$. All these identifications work out so that $x(\phi(L)) = e(L) \in E^0 BA = \mathcal{O}_{\mathrm{Hom}(A^*, \mathbb{G})}$.

In [18, Proposition 22], we defined a closed subscheme $\mathrm{Level}(A^*, \mathbb{G})$ of $\mathrm{Hom}(A^*, \mathbb{G})$. It follows from [18, Proposition 32] that the restriction of $\mathbb{D}(V_{p^k})$ to $\mathrm{Level}(A^*, \mathbb{G})$ is a subgroup divisor. This means that the image of the map

$$\mathrm{Level}(A^*, \mathbb{G}) \rightarrowtail \mathrm{Hom}(A^*, \mathbb{G}) = \mathrm{spf}(E^0 BA) \to \mathrm{spf}(E^0 B\Sigma_{p^k})$$

is contained in Z_k. It is thus enough to show that the union of these images is precisely Y_k. This is essentially clear from the rational description of $E^0 B\Sigma_{p^k}$ given in [20], but some translation is required, so we adopt a slightly different approach.

Recall from [18, Theorem 23] that $\mathrm{Level}(A^*, \mathbb{G})$ is a smooth scheme, and thus that $D(A) = \mathcal{O}_{\mathrm{Level}(A^*, \mathbb{G})}$ is an integral domain. Using [18, Proposition 26], we see that when $L \in A^*$ is nontrivial, we have $\phi(L) \neq 0$ as sections of $\mathbb{G}$ over $\mathrm{Level}(A^*, \mathbb{G})$, and thus $e(L) = x(\phi(L)) \neq 0$ in $D(A)$. It follows that $c_{p^k} = \prod_{L \neq 1} e(L)$ is not a zero-divisor in $D(A)$. On the other hand, if A' is an Abelian p-subgroup of Σ_{p^k} which does not act transitively on $\{1, \dots, p^k\}$, then the restriction of $V_{p^k} - 1$ to A' has a trivial summand, and thus c_{p^k} maps to zero in $D(A')$. Next, we recall the version of generalised character theory described in [8, Appendix A]. It is proved there that for any finite group G, there is a natural isomorphism of rings

$$p^{-1}E^0 BG = \left(\prod_A p^{-1} D(A)\right)^G$$

where A runs over all Abelian p-subgroups of G. As $\bar{R}_k = E^0(B\Sigma_{p^k})/\mathrm{ann}(c_{p^k})$ and everything in sight is torsion-free, we see that $p^{-1}\bar{R}_k$ is the quotient of $p^{-1}E^0 B\Sigma_{p^k}$ by the annihilator of the image of c_{p^k}. Using our analysis of the images of c_{p^k} in the rings $D(A)$, we conclude that

$$p^{-1}\bar{R}_k = \left(\prod_A p^{-1} D(A)\right)^{\Sigma_{p^k}},$$

where the product is now over all transitive Abelian p-subgroups. This implies that for such A, the map $E^0 B\Sigma_{p^k} \to D(A)$ factors through $\bar{R}_k$, and that the resulting maps $\bar{R}_k \to D(A)$ are jointly injective. This means that Y_k is the union of the images of the corresponding schemes $\mathrm{Level}(A^*, \mathbb{G})$, as required. □

The above proposition gives us a classifying map $Y_k = \mathrm{spf}(\bar{R}_k) \to \mathrm{Sub}_k(\mathbb{G})$, such that the pullback of the universal subgroup divisor over $\mathrm{Sub}_k(\mathbb{G})$ is H_k. Equivalently, we have a classifying map $\mathcal{O}_{\mathrm{Sub}_k(\mathbb{G})} \to \bar{R}_k$.

THEOREM 9.2. *The classifying map $\mathcal{O}_{\mathrm{Sub}_k(\mathbb{G})} \to \bar{R}_k$ is an isomorphism.*

Proof. By Theorem 8.6 above and [18, Theorem 42], we know that $\mathcal{O}_{\mathrm{Sub}_k(\mathbb{G})}$ and $\bar{R}_k$ are free modules of the same rank over E^0. We shall show that the induced map

$$\mathcal{O}_{\mathrm{Sub}_k(\mathbb{G}_0)} = \mathbb{F}_{p^n} \otimes_E \mathcal{O}_{\mathrm{Sub}_k(\mathbb{G})} \to \mathbb{F}_{p^n} \otimes_E \bar{R}_k$$

is injective. We will conclude that it is an isomorphism by dimension count; it follows easily that the original map $\mathcal{O}_{\mathrm{Sub}_k(\mathbb{G})} \to \bar{R}_k$ is an isomorphism.

Recall from [18, Proposition 56] that the socle of $\mathcal{O}_{\mathrm{Sub}_k(\mathbb{G}_0)}$ is generated by a certain element a'^N, where $N = p + \cdots + p^{n-1}$. We claim that a' maps to c_{p^k} in $\mathbb{F}_{p^n} \otimes_E \bar{R}_k$. To see this, let K be the universal subgroup divisor over $\mathrm{Sub}_k(\mathbb{G})$. We have $\mathcal{O}_K = \mathcal{O}_{\mathrm{Sub}_k(\mathbb{G})}[\![x]\!]/f_K(x)$ for a uniquely determine monic polynomial f_K of degree p^k, called the equation of K, and a' is defined to be the coefficient of x in $f_K(x)$. As H_k is the pullback of K, we see that the image of f_K is the equation of $H_k = \mathbb{D}(V_{p^k})$. This is just the polynomial $\sum_{i=0}^{p^k} a_i x^{p^k - i}$, where a_i is the i'th Chern class of V_{p^k}. We thus find that the image of a' is $a_{p^k - 1} = e(V_{p^k} - 1) = c_{p^k}$, as claimed.

We showed in Theorem 3.2 that $c_{p^k}^{N+1} \neq 0$ in $\mathbb{F}_{p^n} \otimes_E E^0 B\Sigma_{p^k}$, but $c_{p^k} I_{p^k} = 0$ so $c_{p^k}^N \neq 0$ in $\mathbb{F}_{p^n} \otimes_E E^0 B\Sigma_{p^k}/I_k = \mathbb{F}_{p^n} \otimes_E \bar{R}_k$. Thus, the socle of $\mathcal{O}_{\mathrm{Sub}_k(\mathbb{G}_0)}$ is mapped injectively. As any nonzero ideal meets the socle, we see that the whole ring is mapped injectively, as required. □

10. A BASIS FOR $\mathcal{O}_{\mathrm{Sub}_m(\mathbb{G})}$

We finish by finding a basis for $\bar{R}_m = \mathcal{O}_{\mathrm{Sub}_m(\mathbb{G})}$ over E^0.

Definition 10.1 Write

$$\sigma(u, v) = \sum_{i=u}^{v-1} p^i = (p^v - p^u)/(p - 1).$$

Let a_j be the $(p^m - p^j)$th Chern class of V_{p^m}. Let μ and ν be sequences of the form

$$1 = \mu_0 < \cdots < \mu_r \leqslant n$$

$$0 \leqslant \nu_0 < \cdots < \nu_r = m.$$

Write

$$b = \prod_{i=0}^{r-1} a_{\nu_i}^{\sigma(\mu_{i+1}, \mu_i)}.$$

We also define ρ_j for $0 \leqslant j < m$ as follows: find the unique i such that $\nu_{i-1} \leqslant j < \nu_j$, and then set

$$\rho_j = \begin{cases} p^{\mu_i} & \text{if } \mu_i < n \\ 1 & \text{if } \mu_i = n. \end{cases}$$

Now write

$$C(\mu, \nu) = \{ b a^\alpha \mid \forall j \ 0 \leqslant \alpha_j < \rho_j \}$$

and

$$C = \coprod_{\mu, \nu} C(\mu, \nu).$$

THEOREM 10.2. *C is basis for $\bar{R}_m$ over E^0.*

In order to prove this, we need yet more definitions. In [18, Proposition 49] we introduced certain quotient rings $E'(k, l)$ of $\mathbb{F}_{p^n} \otimes_E \bar{R}_m$, and we need to find bases for these in order to make an inductive argument.

Definition 10.3. Suppose that $0 \leqslant k \leqslant m$ and $0 < l \leqslant n$. Let μ and ν be sequences of the form

$$l = \mu_0 < \cdots < \mu_r \leqslant n$$

$$k \leqslant \nu_0 < \cdots < \nu_r = m.$$

Write

$$b = \prod_{i=0}^{r-1} a_{\nu_i}^{\sigma(\mu_{i+1}, \mu_i)}.$$

We also define ρ_j for $0 \leqslant j < m$ as follows. If $j < k$ we put $\rho_j = 0$. If $k \leqslant j < m$, we find the unique i such that $\nu_{i-1} \leqslant j < \nu_j$, and then set

$$\rho_j = \begin{cases} p^{\mu_i} & \text{if } \mu_i < n \\ 1 & \text{if } \mu_i = n. \end{cases}$$

Now write

$$C_{kl}(\mu, \nu) = \{ b a^\alpha \mid \forall j \quad 0 \leqslant \alpha_j < \rho_j \}$$

and

$$C_{kl} = \coprod_{\mu, \nu} C_{kl}(\mu, \nu).$$

Proof of Theorem 10.2. We use the notation of [18, Proposition 49]. There we introduced certain quotient rings $E'(k, l)$ of $\mathbb{F}_{p^n} \otimes_E \bar{R}_m$ and elements $v, a \in E'(k, l)$ such that

$$E'(0, 1) = \mathbb{F}_{p^n} \otimes_E \bar{R}_m, \qquad E'(m, l) = \mathbb{F}_{p^n}, \qquad E'(k, n) = \mathbb{F}_{p^n}$$

$$E'(k, l)/a = E'(k+1, l), \qquad E'(k, l)/v = E'(k, l+1), \qquad v a^{p^l} = 0 \quad \text{if } l < n.$$

From the definitions, it is not hard to check that a is (up to sign) the image of our Chern class a_k in $E'(k, l)$. Because of the first property, it will suffice to show that C_{kl} is a basis for $E'(k, l)$, for all $0 \leqslant k \leqslant m$ and $0 < l \leqslant n$. This is easy to check when $k = m$ or $l = n$. Suppose that we know that $C_{k+1,l}$ is a basis for $E'(k+1, l) = E'(k, l)/a_k$ and that $C_{k,l+1}$ is a basis for $E'(k, l+1) = E'(k, l)/v$. Using the relation $v a_k^{p^l} = 0$, we conclude that

$$C'_{kl} = \{ a_k^j t \mid t \in C_{k+1,l} \text{ and } 0 \leqslant j < p^l \} \coprod \{ a_k^{p^l} t \mid t \in C_{k,l+1} \}$$

is a spanning set for $E'(k, l)$. This gives an upper bound for $\dim_{\mathbb{F}_{p^n}} E'(k, l)$ and thus by induction a bound for $\dim_{\mathbb{F}_{p^n}} E'(0, 1) = \dim_{E^0} \bar{R}_m$. This is the conclusion of [18, Proposition 49]. It is proved in [18, Corollary 53] that this bound is sharp, and it follows easily that the bounds used at each stage must be sharp, so that C'_{kl} is a basis for $E'(k, l)$. Thus, we need only check the purely combinatorial fact that $C'_{kl} = C_{kl}$. In fact, one can check that

$$\{a_k^j t \mid t \in C_{k+1,l}(\mu, \nu) \text{ and } 0 \leqslant j < p^l\} = C_{kl}(\mu, \nu).$$

This shows tht the first piece of C'_{kl} is the disjoint union of the sets $C_{kl}(\mu, \nu)$ for which $\nu_0 > k$. Next, consider a piece $C_{k,l+1}(\mu, \nu)$ of $C_{k,l+1}$. If $\nu_0 = k$, we define

$$\nu' = (\nu_0 < \nu_1 < \cdots < \nu_r = m)$$

$$\mu' = (l < \mu_1 < \cdots < \mu_r \leqslant n).$$

If $\nu_0 > k$, we define instead

$$\nu' = (k < \nu_0 < \nu_1 < \cdots < \nu_r = m)$$

$$\mu' = (l < l + 1 < \mu_1 < \cdots < \mu_r \leqslant n).$$

Either way, one can check that

$$\{a_k^{p^l} t \mid t \in C_{k,l+1}(\mu, \nu)\} = C_{kl}(\mu', \nu').$$

Using this, one can conclude that $C'_{kl} = C_{kl}$ as required. □

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