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Hypergraph polytopes

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A B S T R A C T

We investigate a family of polytopes introduced by E.M. Feichtner, A. Postnikov and B. Sturmfels, which were named nestohedra. The vertices of these polytopes may intuitively be understood as constructions of hypergraphs. Limit cases in this family of polytopes are, on the one end, simplices, and, on the other end, permutohedra. In between, as notable members one finds associahedra and cyclohedra. The polytopes in this family are investigated here both as abstract polytopes and as realized in Euclidean spaces of all finite dimensions. The later realizations are inspired by J.D. Stasheff's and S. Shnider's realizations of associahedra. In these realizations, passing from simplices to permutohedra, via associahedra, cyclohedra and other interesting polytopes, involves truncating vertices, edges and other faces. The results presented here reformulate, systematize and extend previously obtained results, and in particular those concerning polytopes based on constructions of graphs, which were introduced by M. Carr and S.L. Devadoss.

1. Introduction

One key to understanding the permutohedron is that it is a truncated simplex. Our results here are a development of that idea. They present the abstract underpinnings of these truncations.

We investigate a family of polytopes that like permutohedra may be obtained by truncating the vertices, edges and other faces of simplices, in any finite dimension. The permutohedra are limit cases in that family, where all possible truncations have been made. The limit cases at the other end, where no truncation has been made, are simplices, like the tetrahedron in three dimensions.

As notable intermediate cases, with some truncations, whose principles we make manifest, we have in this family associahedra and cyclohedra (see Appendix B; for historical references concerning associahedra, cyclohedra and permutohedra see [30,31], [35, Lecture 0, Example 0.10], and [19]). There are also other interesting polytopes in the vicinity of these, which are not very well known, or are quite unknown. These other polytopes have an application in category theory and the theory of operads similar to that of associahedra (see [14], and the end of this introduction).

This family of polytopes was introduced as a family in [16, Section 3] and [26, Section 7]. The polytopes in it were named nestohedra in [27, Section 6].

The polytopes in this family are defined with respect to hypergraphs (in the sense of [2]; see the beginning of Section 1). These hypergraphs are essentially a special kind of building sets, which are defined in [15] with respect to arbitrary finite meet semilattices. For hypergraphs we have instead finite set lattices, with meet and join being respectively intersection and union (see [16] and [26]).

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A vertex of one of these polytopes may be identified with another hypergraph that may intuitively be understood as a construction of the original hypergraph. The faces of greater dimension of the polytope correspond to partial constructions. These constructions, partial and not partial, which we call constructs, are called nested sets in [16] and [26] (a term that was introduced with respect to the more general notion of building set in [15]; references concerning antecedents of this notion are in [15, beginning of the Introduction], and [27, beginning of Section 6]).

In most of our text, we deal with the hypergraph polytopes in an abstract manner, based on the definition of abstract polytope of [25, Section 2A]. We believe that this point of view is novel.

We devote however one part of our work (Section 9) to the Euclidean realizations of hypergraph polytopes. This approach to realizing these polytopes, which is inspired by [31, Appendix B], and is based on truncating simplices, may be found in the literature in cases where the hypergraphs can be identified with graphs (see [6,10,11] and further references that may be found in these papers; the polytopes in question are called there graph-associahedra). Here this approach is extended to all hypergraph polytopes. The approach to realizing hypergraph polytopes of [16] and [26], which is based on the Minkowsky sum of simplices, is different (see, however, also Remark 6.6 of [27], [33,5] and references in there).

Another difference of our approach is that for us inductive definitions play a more important role than it is the case in the other approaches. We find that these definitions enable us to clarify and simplify matters; for proving some results they have a clear advantage. We present several alternative views on the same subject matter—in particular, three equivalent notions of constructions. (These notions are closely related to notions in [16, end of Section 3], and [26, Definition 7.7]; we studied them first in [13].) When we restrict ourselves to graphs, then the notion of construct, which is for us a secondary notion, derived from the primary, more basic, notion of construction, amounts to the notion of tubing of [11] (see Appendix A, where these matters are treated in detail; this appendix provides a bridge between the approach through nested sets and the approach through tubings).

Another novelty that we give may be an inductive definition of an abstract polytope (see Section 8), equivalent to the definition of [25]. We find this inductive definition useful for showing that abstract hypergraph polytopes are indeed abstract polytopes. For that, we rely on the results of Section 7, which are closely related to the results of [34], though the presentation is different.

We survey all the hypergraph polytopes up to and including dimension 3 in Appendix B. To obtain intuitive pictures, the reader may consult this appendix while going through the previous exposition.

Our investigation of the matters covered here started in [13]. In [14], which is about a problem in category theory and the theory of operads, one finds an application of the ideas of [13]. In general, in these two preceding papers we were less concerned with the theory of polytopes, abstract or realized.

In [13] we worked in the direction from the permutohedra towards other hypergraph polytopes, which is the direction of [32] and [26]. We could not reach simplices, because we stuck to graphs only, and did not envisage hypergraphs. We were collapsing different vertices of a permutohedron into a single vertex (which is akin to what is done in [32]). Now we work in the opposite direction, by truncating, starting from the simplices, as in [31, Appendix B]. The two approaches, with two opposite directions, collapsing and truncating, cover however essentially the same ground (provided that by introducing hypergraphs we allow collapsing to go all the way up to simplices). They have an identical basic core, and our goal here is to present clearly this core.

As one can base an alternative proof of Mac Lane's coherence theorem for monoidal categories of [23] (see also [24, Section VII.2]) on Stasheff's results of [29] concerning associahedra (see also [30,31] and references therein), so one can base an alternative proof of a categorial coherence result of [14], concerning operads, on the results presented here. We will however deal with these matters of category theory on another occasion.

The first version of this paper posted in the arXiv, which differs unessentially from the present one, was written without our being aware of [15,16,26,27,34]. We were also not aware of the papers [5,33,8], dealing with matters related to our truncations. We would probably have presented matters differently if we knew about these references from the outset, but perhaps our independent approach, for which we believe that it is sometimes simpler, sheds a new light on the matter.[1]

2. Connected hypergraphs

In this section we define the basic notions that we need concerning hypergraphs.

For C a finite (possibly empty) set, consider families of sets H such that $H \subseteq \mathcal{P}C$, i.e. families of subsets of C. When $\emptyset \notin H$ and C is the union $\bigcup H$ of all the members of H, the family H is a *hypergraph* on C (see [2, Section 1.1]). The members of C correspond to the vertices of a graph, and the members of H that are pairs, i.e. two-element sets, correspond to the edges of a graph. It is not necessary to mention always the *carrier* C of a hypergraph, since $C = \bigcup H$, and every hypergraph H is a hypergraph on $\bigcup H$; sometimes however mentioning C is useful, and clarifies matters. A hypergraph on $\{x, y, z, u, v\}$ is, for example, the family

$$E = \big\{\{x, y\}, \{x, y, z\}, \{y, z\}, \{u\}, \{v\}\big\}.$$

[1] We are grateful to an anonymous referee for remarking that already in Coxeter's book [7] it is explained how very symmetric truncations of regular polytopes, including the simplices, can be derived by Wythoff's construction being applied to their symmetry group.

The *empty hypergraph* is the hypergraph $\emptyset$ on $\emptyset$. There is no other hypergraph on $\emptyset$, since $\{\emptyset\}$ is not a hypergraph on $\emptyset$; we have that $\emptyset \in \{\emptyset\}$. So $H = \emptyset$ iff $\bigcup H = \emptyset$.

We allow the empty hypergraph, and spend some time in explaining limit matters pertaining to it, but the reader should not imagine that this is extremely important. In much of our text the empty hypergraph fits nicely into the picture, but in some parts (see the end of Section 7 and the beginning of Section 9) we treat it separately. Our main interest is in nonempty hypergraphs, and the limit case of the empty hypergraph could as well have been omitted.

A *hypergraph partition* of a hypergraph H is a partition $\{H_1, \dots, H_n\}$, with $n \geqslant 1$, of H such that $\{\bigcup H_1, \dots, \bigcup H_n\}$ is a partition of $\bigcup H$. For example, the sets

$$E' = \big\{\big\{\{x,y\},\{x,y,z\},\{y,z\}\big\},\big\{\{u\}\big\},\big\{\{v\}\big\}\big\},$$
$$E'' = \big\{\big\{\{x,y\},\{x,y,z\},\{y,z\},\{u\}\big\},\big\{\{v\}\big\}\big\},$$
$$E''' = \{E\}$$

are hypergraph partitions of E. The partition

$$\big\{\big\{\{x,y\},\{x,y,z\}\big\},\big\{\{y,z\},\{u\},\{v\}\big\}\big\}$$

of E is not a hypergraph partition of E, because we have that $\bigcup\{\{x,y\},\{x,y,z\}\} = \{x,y,z\}$ and $\bigcup\{\{y,z\},\{u\},\{v\}\} = \{y,z,u,v\}$, and $\{\{x,y,z\},\{y,z,u,v\}\}$ is not a partition of $\{x,y,z,u,v\}$.

The *trivial* partition $\{H\}$ of H exists if H is nonempty, and it is a hypergraph partition. If $H = \emptyset$, then $\{\emptyset\}$ is not a partition of $\emptyset$, because all the members of a partition must be nonempty. The empty hypergraph has however one, and only one, partition; this is the empty partition $\emptyset$, which is a hypergraph partition of $\emptyset$.

A hypergraph H is *connected* when it has only one hypergraph partition; if H is nonempty, then this unique hypergraph partition is the trivial partition $\{H\}$, and if $H = \emptyset$, then this hypergraph partition is $\emptyset$. The hypergraph E above is not connected; the family

$$\big\{\{x,y\},\{x,y,z\},\{y,z\},\{z,u\}\big\}$$

is a connected hypergraph on $\{x,y,z,u\}$.

For a hypergraph H, let the *intersection graph* of H be the graph $\Omega(H)$ whose vertices are the elements of H, which are connected by an edge when they have a nonempty intersection (see [20, Chapter 2]). For example, $\Omega(E)$ is

$$\begin{tikzcd}
\{x,y\} \arrow[rr, dash] \arrow[dr, dash] & & \{y,z\} \arrow[dl, dash] & \{u\} & \{v\} \\
& \{x,y,z\} & & &
\end{tikzcd}$$

A *path* of a hypergraph H is a sequence $X_1, \dots, X_n$, with $n \geqslant 1$, of distinct elements of H that make a path in $\Omega(H)$ (see [20, Chapter 2], for the notion of path in a graph; this is a sequence of distinct vertices such that consecutive vertices are joined by edges). Three examples of paths in E are the sequences

$$\{x,y\}, \quad \{y,z\},$$
$$\{y,z\}, \quad \{x,y\}, \quad \{x,y,z\},$$
$$\{x,y\}.$$

If $n = 1$, then the path X_1 is just the element X_1 of H. In the third example, the element $\{x,y\}$ of E is a path of E.

For $x, y \in \bigcup H$, we say that a path $X_1, \dots, X_n$ of H *joins* x with y when $x \in X_1$ and $y \in X_n$. So the path $\{x,y\},\{y,z\}$ of E joins x with z, but it also joins x with y, y with z and y with y. The path $\{u\}$ of E joins u with u.

One can verify the following.

Remark 2.1. A nonempty hypergraph H is connected

iff $\Omega(H)$ is connected;
iff for every $x, y \in \bigcup H$ there is a path of H that joins x with y.

This shows that our notion of connected hypergraph is the same as the notion in [2, Section 1.2]. We have in this remark the assumption that H is nonempty because otherwise the graph $\Omega(H)$ would be without vertices (and edges), and this presumably goes counter to common usage in graph theory (see [20, Chapter 2], and [21]; cf. Appendix A). Otherwise, if $\Omega(H)$ is allowed to be without vertices, then we may lift the assumption of nonemptiness for H.

For every nonempty hypergraph H there is a unique hypergraph partition $\{H_1, \dots, H_n\}$, with $n \geqslant 1$, of H such that for every $i \in \{1, \dots, n\}$ we have that H_i is a connected hypergraph on $\bigcup H_i$. We call this hypergraph partition the *finest* hypergraph partition of H. For example, E' above is the finest hypergraph partition of E.

The trivial partition $\{H\}$ is the *coarsest* hypergraph partition of a nonempty hypergraph H. If H is nonempty and connected, then the finest and coarsest hypergraph partitions of H coincide. (The empty hypergraph, which also happens to be connected by our definition, has only one partition, namely $\emptyset$, which may be taken as the finest and coarsest hypergraph partition of this hypergraph.)

3. Constructions

In this section we introduce three equivalent notions that may intuitively be understood as constructions of hypergraphs. They are called construction, f-construction and s-construction. The first two notions are based on sets, while the last is based on words, i.e. finite sequences. Of the first two notions, the notion of f-construction ("f" comes from "forest") is perhaps more intuitive—it involves a more direct record of constructing. But the equivalent notion that we call simply construction is the notion on which we rely in the remainder of the text, and to which for this reason we give prominence. (Since our constructions are hypergraphs, we could call them h-constructions, but it would be onerous to write "h" all the time.) The third equivalent notion, the notion of s-construction ("s" may be associated with "syntax"), is based on a notion investigated in [13]. It provides the most economical notation.

For $F \subseteq \mathcal{P}C$ and $Y \subseteq C$ let

$$F_Y =_{df} \{X \in F \mid X \subseteq Y\}.$$

We are interested in this definition in particular when F is a hypergraph H and $Y \subseteq \bigcup H$.

We call a hypergraph H *atomic* when for every x in $\bigcup H$ we have that $\{x\} \in H$ (cf. Lemma 3.9 of [16] and Definition 7.1 of [26]). Note that the empty hypergraph is atomic, for trivial reasons. One can verify the following.

Remark 3.1. The hypergraph H is atomic iff for every subset Y of $\bigcup H$ we have that H_Y is a hypergraph on Y.

With the help of this remark we establish easily the following.

Remark 3.2. If H is an atomic hypergraph and $Y \subseteq \bigcup H$, then H_Y is an atomic hypergraph on Y.

For H an atomic hypergraph, we define families of subsets of $\bigcup H$ that we call *constructions* of H. This definition is by induction on the cardinality $|\bigcup H|$ of $\bigcup H$:

(0) if $|\bigcup H| = 0$, then H is the empty hypergraph $\emptyset$, and $\emptyset$ is the only construction of $\emptyset$;
(1) if $|\bigcup H| \geqslant 1$, and H is connected, and K is a construction of $H_{\bigcup H - \{x\}}$ for some $x \in \bigcup H$, then $K \cup \{\bigcup H\}$ is a construction of H;
(2) if $|\bigcup H| \geqslant 2$, and H is not connected, and $\{H_1, \dots, H_n\}$, where $n \geqslant 2$, is the finest hypergraph partition of H, and for every $i \in \{1, \dots, n\}$ we have that K_i is a construction of H_i, then $K_1 \cup \dots \cup K_n$ is a construction of H.

This concludes our inductive definition of a construction of H. Note that $\bigcup H$ in clause (1) is $\bigcup K \cup \{x\}$, and $x \notin \bigcup K$.

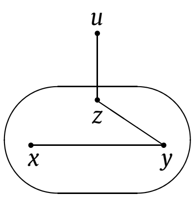
For this definition to be correct, in clause (1) we must verify that $H_{\bigcup H - \{x\}}$ is an atomic hypergraph for every x in $\bigcup H$, and in clause (2) we must verify that H_i is an atomic hypergraph for every $i \in \{1, \dots, n\}$. For both of these verifications we use Remark 3.2. The present remark about correctness of definition applies also to the inductive definitions of f-constructions and w-constructions, to be given later in this section.

It is easy to verify that a construction of an atomic hypergraph H on $\bigcup H$ is itself a hypergraph on $\bigcup H$ (though not necessarily atomic). In particular cases, it will be a subfamily of H (see Section 4).

To give examples of constructions, consider the atomic hypergraph

$$A = \big\{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, u\}, \{x, y, z\}\big\}.$$

We could draw this hypergraph in the following manner:



In such drawings, the circles corresponding to singletons are taken for granted, and instead of the circles corresponding to two-element sets, we draw edges between the two elements of such sets, as for graphs.

A construction of A is the hypergraph

$$L = \big\{\{u\}, \{z, u\}, \{y, z, u\}, \{x, y, z, u\}\big\}$$

on $\{x, y, z, u\}$. Here is how L was obtained by our definition. We had first $L_0 = \emptyset$ as a construction of the atomic hypergraph $\emptyset$. Then $L_1 = L_0 \cup \{\{u\}\} = \{\{u\}\}$ was a construction of the atomic hypergraph $\{\{u\}\}$. Next we had $L_2 = L_1 \cup \{\{z, u\}\} = \{\{u\}, \{z, u\}\}$ as a construction of the atomic hypergraph $\{\{z\}, \{u\}, \{z, u\}\}$. Then we had $L_3 = L_2 \cup \{\{y, z, u\}\} = \{\{u\}, \{z, u\}, \{y, z, u\}\}$ as a construction of the atomic hypergraph

$$A' = \big\{\{y\}, \{z\}, \{u\}, \{y, z\}, \{z, u\}\big\}.$$

Finally, we have $L = L_3 \cup \{\{x, y, z, u\}\}$ as a construction of A. In all that, we applied only clauses (0) and (1) of our definition of a construction.

Here is another construction of A:

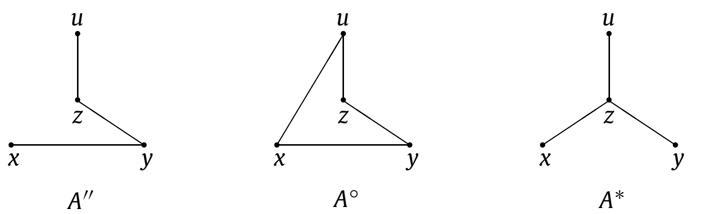
$$M = \big\{\{y\}, \{u\}, \{y, z, u\}, \{x, y, z, u\}\big\}.$$

It was obtained from the construction $M_1 = \{\{u\}\}$ of $\{\{u\}\}$ and the construction $M_2 = \{\{y\}\}$ of $\{\{y\}\}$ by applying clause (2) of the definition, which yields $M_1 \cup M_2 = \{\{y\}, \{u\}\}$ as a construction of the atomic hypergraph $\{\{y\}, \{u\}\}$, which is not connected. Then we had by clause (1) that $M_3 = M_1 \cup M_2 \cup \{\{y, z, u\}\}$ is a construction of A', mentioned above, and, finally, $M = M_3 \cup \{\{x, y, z, u\}\}$ is a construction of A.

Note that L and M would be constructions also of the atomic hypergraph A'' obtained from A by rejecting $\{x, y, z\}$ (we deal with that matter in Section 4). They would also be constructions of atomic hypergraphs more different from A than A''. Such are, for example, the hypergraphs

$$A^\circ = A'' \cup \big\{\{u, x\}\big\},$$
$$A^* = \big(A'' - \big\{\{x, y\}\big\}\big) \cup \big\{\{x, z\}\big\},$$

which, together with A'', may be drawn as follows:



By removing x from the sets of these three hypergraphs we obtain the hypergraph A'. (These three hypergraphs should be compared with Examples 5.15, 5.13 and 5.16 of [13], and with H'_{4321}, H°_{4441} and H^*_{4331} in Appendix B.)

We will make a comment on the intuitive meaning of our constructions after introducing the equivalent notion of f-construction, and after giving analogous examples of f-constructions.

The definition of an *f-construction* of an atomic hypergraph H is again by induction on $|\bigcup H|$, and its clauses (0) and (2) are exactly as in the definition of a construction above, with “construction” replaced by “f-construction”. For clause (1) of the new definition we make that replacement, and moreover $K \cup \{\bigcup H\}$ is replaced by $\{K \cup \{x\}\}$. This concludes the definition of an f-construction.

To give examples of f-constructions, consider again the atomic hypergraph A above. An f-construction of A is $L^f = \{\{x, \{y, \{z, \{u\}\}\}\}\}$. (Note that L^f is a singleton.) We will show exactly later how this f-construction corresponds to the construction L above. Here is how L^f was obtained by our definition. We had first $L^f_0 = \emptyset = L_0$ as a construction of $\emptyset$. Then $L^f_1 = \{L^f_0 \cup \{u\}\} = \{\{u\}\} = L_1$ was an f-construction of $\{\{u\}\}$. Next we had $L^f_2 = \{L^f_1 \cup \{z\}\} = \{\{z, \{u\}\}\}$ as a construction of $\{\{z\}, \{u\}, \{z, u\}\}$. Then we had $L^f_3 = \{L^f_2 \cup \{y\}\} = \{\{y, \{z\{u\}\}\}\}$ as a construction of A' above. Finally, we have $L^f = \{L_3 \cup \{x\}\}$ as a construction of A. In this example, we applied only the new clauses (0) and (1).

Another f-construction of A is $M^f = \{\{x, \{z, \{y\}, \{u\}\}\}\}$, which corresponds to M. In obtaining M^f by the definition of an f-construction we apply also clause (2).

These examples should explain the denomination “construction” in our constructions and f-constructions. The hypergraphs L and M, as well as the sets L^f and M^f, may be understood as constructions of A in time. Within a connected part of A the construction proceeds by adding in L^f and M^f a chosen vertex, and this choice induces a temporal order. Clause (1) serves for that. Connected parts of A that are mutually disconnected are added simultaneously, without order, and clause (2) serves for that.

We define next another notion equivalent to the notion of construction. For an atomic hypergraph H we define first words in the alphabet $\bigcup H$, which we call *w-constructions*. This definition is again by induction on $\bigcup H$:

(0) if $|\bigcup H| = 0$, then the empty word e is the only w-construction of the hypergraph $\emptyset$;
(1) if $|\bigcup H| \geqslant 1$, and H is connected, and t is a w-construction of $H_{\bigcup H - \{x\}}$ for some $x \in \bigcup H$, then xt is a w-construction of H;

(2) if $|\bigcup H| \geqslant 2$, and H is not connected, and $\{H_1, \dots, H_n\}$, where $n \geqslant 2$, is the finest hypergraph partition of H, and for every $i \in \{1, \dots, n\}$ we have that t_i is a w-construction of H_i, then $(t_1 + \dots + t_n)$ is a w-construction of H.

This concludes our inductive definition of a w-construction of H.

Consider equivalence classes of w-constructions of H obtained by factoring through the commutativity of $+$. We call these equivalence classes *s-constructions*. We refer to an s-construction by any w-construction that belongs to it. Our s-constructions are analogous to the **S**-forests of [13, Section 5].

Here are two examples of s-constructions of the hypergraph A above. These are L^s, which is $xyzu$, and M^s, which is $xz(y+u)$; they correspond to L and M respectively. The s-construction $xz(u+y)$ is the same as M^s.

Our task now is to show that the notions of construction, f-construction and s-construction are all equivalent. By this we mean that there are structure-preserving bijections between the sets $\mathcal{C}(H)$, $\mathcal{C}^f(H)$ and $\mathcal{C}^s(H)$ of, respectively, constructions, f-constructions and s-constructions of an atomic hypergraph H.

We define first a map $f : \mathcal{C}(H) \to \mathcal{C}^f(H)$ by induction on $\bigcup H$, in parallel with the clauses of the inductive definitions of construction and f-construction:

(0) $f(\emptyset) = \emptyset$,
(1) $f(K \cup \{\bigcup H\}) = \{f(K) \cup \{x\}\}$,
(2) $f(K_1 \cup \dots \cup K_n) = f(K_1) \cup \dots \cup f(K_n)$.

The conditions concerning H, K, $K_1, \dots, K_n$ are taken from the clauses (1) and (2) of the definition of a construction. For (1) we have that H is connected and K is a construction of $H_{\bigcup H - \{x\}}$ for some $x \in \bigcup H$. For (2) we have that H is not connected and $K_1, \dots, K_n$, with $n \geqslant 2$, are constructions of respectively $H_1, \dots, H_n$ for $\{H_1, \dots, H_n\}$ being the finest hypergraph partition of H. We proceed analogously for the two other maps below.

Next we define analogously a map $s : \mathcal{C}^f(H) \to \mathcal{C}^s(H)$:

(0) $s(\emptyset) = e$,
(1) $s(\{K \cup \{x\}\}) = xs(K)$,
(2) $s(K_1 \cup \dots \cup K_n) = s(K_1) + \dots + s(K_n)$.

Finally, we define analogously a map $c : \mathcal{C}^s(H) \to \mathcal{C}(H)$:

(0) $c(e) = \emptyset$,
(1) $c(xt) = c(t) \cup \{\bigcup H\}$,
(2) $c(t_1 + \dots + t_n) = c(t_1) \cup \dots \cup c(t_n)$.

Then to verify that f, s and c are bijections it is enough to verify the following three equations:

$$c\big(s\big(f(K)\big)\big) = K, \qquad f\big(c\big(s(K)\big)\big) = K, \qquad s\big(f\big(c(t)\big)\big) = t,$$

which is quite straightforward.

Constructions, f-constructions and s-constructions bear a forest structure (a forest is a disjoint union of trees, with a tree being a limit case). This structure is clearer in f-constructions and s-constructions. In f-constructions, the x added in clause (1) is the *root* of the tree $\{K \cup \{x\}\}$. Here K, if it is nonempty, is equal to a forest $K_1 \cup \dots \cup K_n$, with $n \geqslant 1$, and the roots of the trees $K_1, \dots, K_n$ are the immediate successors into which x branches.

4. Saturation and cognate hypergraphs

Two different atomic hypergraphs may have the same constructions. Such are, for example, A and A'' of Section 3. In this section we concentrate on atomic hypergraphs that have the same constructions in order to find among them a representative that is easiest to work with.

In the set of all atomic hypergraphs on the same carrier that have the same constructions there is a greatest one, which has a property we will call saturation. We will characterize the equivalence relation that the hypergraphs in this set bear to each other in terms of a relation where their difference is reduced to atomic differences. Such an atomic difference consists in one hypergraph having a member that is not in the other hypergraph, but this one member—a dispensable member—does not, roughly speaking, increase connectedness. Two hypergraphs are called cognate when they differ only with respect to dispensable members, and a hypergraph is saturated when all possible dispensable members are in it.

It will simplify the exposition later if we concentrate on saturated hypergraphs (see Section 6). The results of the present section justify this simplification; they show that it makes no difference with respect to constructions.

We call a hypergraph H *saturated* when for every $X_1, X_2 \in H$ if $X_1 \cap X_2 \neq \emptyset$, then $X_1 \cup X_2 \in H$ (cf. Lemma 3.9 of [16] and Definition 7.1 of [26]). One can verify the following.

Remark 4.1. The hypergraph H is saturated

iff for every $Y \subseteq \bigcup H$, if $H_Y - \{Y\}$ is a connected hypergraph on Y, then $Y \in H$;
iff for every $Y \subseteq \bigcup H$, if H_Y is a connected hypergraph on Y, then $Y \in H$;
iff for every $Y \subseteq \bigcup H$ we have that H_Y is a connected hypergraph on Y iff $Y \in H$.

The hypergraph E of Section 2 is saturated, and $E - \{\{x, y, z\}\}$, which is $\{\{x, y\}, \{y, z\}, \{u\}, \{v\}\}$, is not saturated. The hypergraph A of Section 3 is not saturated. The empty hypergraph is saturated, for trivial reasons.

For a hypergraph H, we say that a subset Y of $\bigcup H$ is *dispensable* in H when $H_Y - \{Y\}$ is a connected hypergraph on Y. For example, $\{x, y, z\}$ is dispensable in the hypergraph E. It is also dispensable in $E - \{\{x, y, z\}\}$. Note that singleton members of a hypergraph are never dispensable.

By Remark 4.1, we have that the hypergraph H is saturated iff every subset of its carrier dispensable in H is an element of H. In terms of dispensability we can also formulate a notion dual to saturation. We will say that the hypergraph H is *bare* when no subset of its carrier dispensable in H is an element of H.

We can prove the following.

Proposition 4.2. *Suppose Y is dispensable in H. Then Z is dispensable in H iff Z is dispensable in $H \cup \{Y\}$.*

Proof. The equivalence of the proposition from left to right is trivial. For the other direction, suppose Z is dispensable in $H \cup \{Y\}$.

So $(H \cup \{Y\})_Z - \{Z\}$ is a connected hypergraph on Z. If $Y \nsubseteq Z$, then $(H \cup \{Y\})_Z = H_Z$, and we are done.

Suppose $Y \subseteq Z$. Then, by Remark 2.1, for every $x, y \in Z$ there is a path $X_1, \dots, X_n$ of $(H \cup \{Y\})_Z - \{Z\}$ that joins x with y; so $x \in X_1$ and $y \in X_n$. If for every $i \in \{1, \dots, n\}$ we have that $X_i \neq Y$, then $X_1, \dots, X_n$ is a path of $H_Z - Z$, and we are done.

Suppose for some i we have that $X_i = Y$. Then we have the following four cases: (1) $1 < i < n$, (2) $1 = i < n$, (3) $1 < i = n$ and (4) $1 = i = n$.

In case (1) our path is of the form

$$X_1, \dots, X_{i-1}, Y, X_{i+1}, \dots, X_n$$

with $x' \in X_{i-1} \cap Y$ and $y' \in Y \cap X_{i+1}$, where $X_1, \dots, X_{i-1}$ is a path of $H_Z - \{Z\}$ that joins x with x' and $X_{i+1}, \dots, X_n$ is a path of $H_Z - \{Z\}$ that joins y' with y. Since Y is dispensable in H, we have that $H_Y - \{Y\}$ is a connected hypergraph on Y. By Remark 2.1, this means that there is a path $Y_1, \dots, Y_m$ of $H_Y - \{Y\}$ that joins x' with y'. For every $j \in \{1, \dots, m\}$ we have that $Y_j \subset Y$. Since $Y \subseteq Z$, we obtain that $Y_j \subset Z$, and hence $Y_1, \dots, Y_m$ is a path of $H_Z - \{Z\}$. So either

$$X_1, \dots, X_{i-1}, Y_1, \dots, Y_m, X_{i+1}, \dots, X_n$$

is a path of $H_Z - \{Z\}$ that joins x with y, or it can easily be transformed into such a path by contracting, if for some $l \in \{1, \dots, n\} - \{i\}$ and some $j \in \{1, \dots, m\}$ we have that X_l is Y_j.

We take that x' is x in cases (2) and (4), and we take that y' is y in cases (3) and (4). In all these three remaining cases we proceed analogously to case (1). This is enough to conclude that Z is dispensable in H. $\square$

We will say that a hypergraph $H \cup \{Y\}$ *enhances* the hypergraph H when Y is dispensable in H and $Y \notin H$.

As a corollary of Proposition 4.2 we have that Y is dispensable in H iff Y is dispensable in $H \cup \{Y\}$. (It is easy to prove this corollary directly.) This corollary shows that in the definition of enhancement the dispensability of Y in H amounts to the dispensability of Y in $H \cup \{Y\}$, and the later dispensability could serve for the definition.

Consider the equivalence relation on the set of hypergraphs on the same carrier C obtained as the reflexive, symmetric and transitive closure of the relation of enhancement. When two hypergraphs on C are in this relation we say that they are *cognate*. As a corollary of Proposition 4.2 we obtain the following.

Proposition 4.3. *Suppose H and J are cognate hypergraphs on C. Then for every $Z \subseteq C$ we have that Z is dispensable in H iff Z is dispensable in J.*

A *cognate set* of hypergraphs is an equivalence class of hypergraphs with respect to the cognation equivalence relation. With the help of Proposition 4.3, we establish that a cognate set is a lattice with respect to intersection and union. It has a greatest element, the union of all its members, which is a saturated hypergraph, and it has a least element, the intersection of all its members, which is a bare hypergraph.

We need the following remark.

Remark 4.4. For $Y \subseteq Z \subseteq \bigcup H$ we have that Y is dispensable in H iff Y is dispensable in H_Z.

This is because $(H_Z)_Y = H_Y$. We also need the following lemma.

Lemma 4.5. *Suppose Y is dispensable in H. Then H is connected iff $H \cup \{Y\}$ is connected.*

Proof. This proof will be quite similar to the proof of Proposition 4.2. The equivalence of the lemma from left to right is trivial. For the other direction suppose there is a path $X_1, \ldots, X_n$ of $H \cup \{Y\}$ that joins x with y, for $x, y \in \bigcup H$. If for every $i \in \{1, \ldots, n\}$ we have that $X_i \neq Y$, then $X_1, \ldots, X_n$ is a path of H, and we are done.

Suppose for some i we have that $X_i = Y$. Then we have the following four cases: (1) $1 < i < n$, (2) $1 = i < n$, (3) $1 < i = n$ and (4) $1 = i = n$.

In case (1) we proceed as in the proof of Proposition 4.2 until we reach the path $Y_1, \ldots, Y_m$ of $H_Y - \{Y\}$ that should replace Y. This path is made of elements of H, and after the replacement we have a path of H that joins x with y. In the remaining cases we proceed analogously. □

Then we can prove the following.

Proposition 4.6. *Suppose H and J are cognate hypergraphs on C. Then for every $Z \subseteq C$ we have that H_Z is connected iff J_Z is connected.*

Proof. It is enough to prove this proposition when J is $H \cup \{Y\}$ for Y dispensable in H. It is clear that if H_Z is connected, then $(H \cup \{Y\})_Z$ is connected.

Suppose $(H \cup \{Y\})_Z$ is connected. We have that $(H \cup \{Y\})_Z$ is different from H_Z only when $Y \subseteq Z$. Then by Remark 4.4 we have that Y is dispensable in H_Z. Since $Y \subseteq Z$, we also have that $(H \cup \{Y\})_Z = H_Z \cup \{Y\}$. It suffices to apply Lemma 4.5 to obtain that H_Z is connected. □

Remember that for an atomic hypergraph H the set $\mathcal{C}(H)$ is the set of all constructions of H. We will prove the following.

Proposition 4.7. *If H and J are cognate atomic hypergraphs, then $\mathcal{C}(H) = \mathcal{C}(J)$.*

Proof. It is enough to prove this proposition when J is $H \cup \{Y\}$ for Y dispensable in H. We establish first that every construction of H is a construction of $H \cup \{Y\}$.

We proceed by induction on the cardinality of $\bigcup H$, as in the inductive definition of a construction. If $\bigcup H = \emptyset$, then $H = H \cup \{Y\} = \emptyset$.

Suppose $|\bigcup H| \geqslant 1$ and H is connected. By Lemma 4.5, we have that $H \cup \{Y\}$ is connected. Then a construction of H is of the form $K \cup \{\bigcup H\}$ for K a construction of $H_{\bigcup H - \{x\}}$, where $x \in \bigcup H$. If $x \in Y$, then $(H \cup \{Y\})_{\bigcup H - \{x\}} = H_{\bigcup H - \{x\}}$. So K is a construction of $(H \cup \{Y\})_{\bigcup H - \{x\}}$. If $x \notin Y$, then we have that $Y \subseteq \bigcup H - \{x\} \subseteq \bigcup H$, and by Remark 4.4 and the induction hypothesis, K is a construction of $(H \cup \{Y\})_{\bigcup H - \{x\}}$. Hence $K \cup \{\bigcup H\}$ is a construction of $H \cup \{Y\}$.

Suppose $|\bigcup H| \geqslant 2$ and H is not connected. By Lemma 4.5, we have that $H \cup \{Y\}$ is not connected. Suppose $\{H_1, \ldots, H_n\}$, where $n \geqslant 2$, is the finest hypergraph partition of H. Then a construction of H is of the form $K_1 \cup \cdots \cup K_n$ for $K_1, \ldots, K_n$ constructions of $H_1, \ldots, H_n$ respectively. For some $i \in \{1, \ldots, n\}$ we must have that $Y \subseteq \bigcup H_i$. Hence Y is dispensable in H_i by Remark 4.4 (we have that $Y \subseteq \bigcup H_i = Z \subseteq \bigcup H$). Then, by the induction hypothesis, K_i is a construction of $H_i \cup \{Y\}$, and $K_1 \cup \cdots \cup K_n$ is a construction of $H \cup \{Y\}$.

We proceed analogously to establish in the converse direction that every construction of $H \cup \{Y\}$ is a construction of H. □

The *saturated closure* $\bar{H}$ of a hypergraph H is the saturated hypergraph in the cognate set of hypergraphs to which H belongs. We can prove the following.

Proposition 4.8. *For an atomic hypergraph H we have that $\bigcup \mathcal{C}(H) = \bar{H}$.*

Proof. We prove this proposition first for $H = \bar{H}$. From left to right, suppose that for some construction K of H we have that $Y \in K$. Then $Y \subseteq \bigcup H$ and H_Y is a connected hypergraph on Y. So, by Remark 4.1, we have that $Y \in H$.

From right to left, suppose that $Y \in H$. We show by induction on $k = |\bigcup H - Y|$ that there is a construction K of H such that $Y \in K$. If $k = 0$, then $Y = \bigcup H$, and H is a connected hypergraph on Y. For an arbitrary construction K of H we must have that $Y \in K$. If $k > 0$ and $x \in \bigcup H - Y$, then by the induction hypothesis we have a construction K' of $H_{\bigcup H - \{x\}}$ such that $Y \in K'$. For $\{H_1, \ldots, H_n\}$ being the finest hypergraph partition of H, and $x \in \bigcup H_i$, we define the construction K of H by $K = K' \cup \{\bigcup H_i\}$, and we have that $Y \in K$.

It remains to remark that $\mathcal{C}(H) = \mathcal{C}(\bar{H})$ for an arbitrary atomic hypergraph H, which we have by Proposition 4.6. □

The following proposition, which completes Proposition 4.7, characterizes cognate classes in terms of constructions. Atomic hypergraphs are cognate iff they have the same constructions.

Proposition 4.9. *Suppose H and J are atomic hypergraphs. Then H and J are cognate iff $\mathcal{C}(H)=\mathcal{C}(J)$.*

Proof. From left to right we have Proposition 4.7. For the other direction, suppose $\mathcal{C}(H)=\mathcal{C}(J)$. Then, with the help of Proposition 4.8, we obtain that $\bar{H}=\bar{J}$. So H and J are in the same cognate set. □

5. Constructs and abstract polytopes of hypergraphs

In this section we define the abstract polytopes of atomic hypergraphs with the help of the notion of construct, which is a notion derived from our notion of construction. The notion of construction is the notion on which all the burden rests. The proof that the polytopes so defined are indeed abstract polytopes will be given in Section 8.

We start with the following remark.

Remark 5.1. For every construction K of an atomic hypergraph H we have that $|K|=|\bigcup H|$.

This is established in a straightforward manner by induction on the size of K.

Note also that if $\{H_1,\dots,H_n\}$, with $n\geqslant 1$, is the finest hypergraph partition of the atomic hypergraph H, then for every $i\in\{1,\dots,n\}$ we have that $\bigcup H_i\in K$. We say that H_i is a *connected component* of H, and $\bigcup H_i$ is a *connected component* of the carrier $\bigcup H$ of H. The number n is the *connectedness number* of H; it is the number of connected components of H, or of $\bigcup H$. The atomic hypergraph $\emptyset$ and its carrier $\emptyset$ have just one partition $\emptyset$, with 0 connected components; so the connectedness number of this hypergraph is 0.

A *construct* of an atomic hypergraph H is a subfamily (not necessarily proper) of a construction of H that contains every connected component of the carrier $\bigcup H$ of H. For example,

$$\big\{\{u\},\{y,z,u\},\{x,y,z,u\}\big\}$$

is a construct of the hypergraph A of Section 3. It is a subfamily of both of the constructions L and M of A.

It is clear that every construct of H is a hypergraph on $\bigcup H$, as H is. It is also clear, in accordance with Remark 5.1, that for C a construct of H we have that $|C|\leqslant|\bigcup H|$.

The constructs of the atomic hypergraph H serve to define as follows the *abstract polytope* of H, which we designate by $\mathcal{A}(H)$ (for the definition of an abstract polytope in general, and related notions used below, see [25, Section 2A], and our Section 8 below).

The elements of $\mathcal{A}(H)$, i.e. the faces of $\mathcal{A}(H)$, are all the constructs of H plus the set $\bar{H}^*$, which is $\bar{H}\cup\{*\}$, for $\bar{H}$ being the saturated closure of H (see Section 4) and $*$ a new element that is not in $\bigcup H$.

We take $\mathcal{A}(H)$ as a partial order with the inverse of the subset relation; i.e. for the faces C_1 and C_2 of $\mathcal{A}(H)$ we have that $C_1\leqslant C_2$ when $C_2\subseteq C_1$. So the incidence relation of $\mathcal{A}(H)$ is the symmetric closure of the subset relation. In the partial order $\mathcal{A}(H)$ the element $\bar{H}^*$ is the least element.

Proposition 4.8 states that the union of all the constructions of H is $\bar{H}$. Hence the union of all the constructs of H is $\bar{H}$. So it seems we could take simply $\bar{H}$ instead of $\bar{H}^*$ as the least element. We did not do that for the following reason.

If all the elements of H are singletons (which means that H can also be empty), then $\bar{H}$ coincides with H, which is the only construct of H, and the only construction of H. We want however to distinguish even in that case the construct $\bar{H}$ from $\bar{H}^*$.

The choice of $\bar{H}^*$ is also dictated by our wish to base the incidence relation in $\mathcal{A}(H)$ on the subset relation at every level. We could however obtain the same effect by having instead of $\bar{H}^*$ any set in which $\bar{H}^*$ is included; for example, the power set of $\bigcup H$, with $\emptyset$ being $*$, or even a universal set in which all the sets $\bar{H}^*$ are included. We could also replace $\bar{H}^*$ by anything different from the other elements of $\mathcal{A}(H)$ if we do not insist that incidence with respect to it must be based on the subset relation.

If n is the connectedness number of H, then the rank r of $\mathcal{A}(H)$ is $|\bigcup H|-n$. In general, we have that $r\geqslant 0$. The rank is the dimension of the realization of the abstract polytope as a convex polytope in space (see Section 9). In our example with the hypergraph A of Section 2, we have that $|\bigcup A|=4$ and $n=1$; so the rank of $\mathcal{A}(A)$ is 3. The abstract polytope $\mathcal{A}(A)$ corresponds to the three-dimensional associahedron K_5 (see (H'_{4321}) in Appendix B, and references therein).

The least face F_{-1} of $\mathcal{A}(H)$ is $\bar{H}^*$, and the greatest face F_r is the set $\{\bigcup H_1,\dots,\bigcup H_n\}$ of the connected components of the carrier $\bigcup H$ of H. (If all the elements of H are singletons, then $F_r\cup\{*\}=F_{-1}$.) With the hypergraph A, we have that F_{-1} of $\mathcal{A}(A)$ is $A\cup\{\{y,z,u\},\{x,y,z,u\},*\}$, while F_3 of $\mathcal{A}(A)$ is $\{\{x,y,z,u\}\}$.

The vertices, i.e. the faces of rank 0, of $\mathcal{A}(H)$ are the constructions of H. By Remark 5.1, the cardinality of every vertex is $|\bigcup H|$. With the hypergraph A, we have 14 vertices in $\mathcal{A}(A)$, among which we find L and M of Section 3.

Besides L, there are seven more vertices of the same type. These eight vertices correspond to the s-constructions

$$xyzu,\quad xyuz,\quad xuyz,\quad xuzy,$$
$$uzyx,\quad uzxy,\quad uxzy,\quad uxyz,$$

the first s-construction $xyzu$ corresponding exactly to L. How these eight vertices are distributed in K_5 may be seen in the second picture of K_5 in Appendix B (see (H'_{4321})), which is based on a picture in [13] (Example 5.15, where $xyzu$ is written $x \cdot y \cdot z \cdot u$; we omit $\cdot$ now). There are two vertices of type M, which correspond to the s-constructions

$$xz(y+u), \qquad uy(x+z),$$

with the first s-construction corresponding exactly to M.

There are four vertices of another type, corresponding to the s-constructions

$$y\big(x+(zu)\big), \qquad y\big(x+(uz)\big),$$
$$z\big((xy)+u\big), \qquad z\big((yx)+u\big),$$

where, for example, the first s-construction $y(x+(zu))$ corresponds to the construction

$$N = \big\{\{x\}, \{u\}, \{z,u\}, \{x,y,z,u\}\big\}.$$

(Here it is clear how much the notation of s-constructions is more economical.) With that we have obtained all the 14 vertices of $\mathcal{A}(A)$, whose distribution may be seen in the picture of K_5 of Appendix B, mentioned above.

The edges, i.e. the faces of rank 1, of $\mathcal{A}(H)$ are all the constructs of H of cardinality $|\bigcup H|-1$. For example, the edge joining L and M in $\mathcal{A}(A)$ is $L \cap M = \{\{u\}, \{y,z,u\}, \{x,y,z,u\}\}$, while the edge joining L and N is $L \cap N = \{\{u\}, \{z,u\}, \{x,y,z,u\}\}$. We will ascertain later that for every edge of $\mathcal{A}(H)$ there are exactly two different vertices such that our edge is their intersection. (This follows from property (P4) when $i=0$; see Section 8.)

In general, for $k \geqslant 0$, the faces of rank k of $\mathcal{A}(H)$ are all the constructs of H of cardinality $|\bigcup H|-k$, and if $k=-1$, then the unique face of rank -1 is $\bar{H}^*$, whose cardinality is $|\bar{H}|-k = |\bar{H}|+1$.

So the facets, i.e. the faces of rank $r-1$, of $\mathcal{A}(H)$, where r is the rank of $\mathcal{A}(H)$, are all the constructs of H of cardinality $|\bigcup H|-(r-1) = |\bigcup H|-(|\bigcup H|-n-1) = n+1$. Besides the n connected components of $\bigcup H$ we find in each facet a single additional member. This member is from $\bar{H}$ if $r>0$, and it is $*$ if $r=0$.

We have that $r=0$ for $\mathcal{A}(H)$ in the following two cases. The first case is when $H = \{\{x_1\}, \dots, \{x_n\}\}$, for $n \geqslant 1$. Then

$$\mathcal{A}(H) = \big\{\big\{\{x_1\}, \dots, \{x_n\}\big\}, \big\{\{x_1\}, \dots, \{x_n\}, *\big\}\big\},$$

with F_0 being the vertex $\{\{x_1\}, \dots, \{x_n\}\}$, and F_{-1} being the facet $\{\{x_1\}, \dots, \{x_n\}, *\}$. The second case is when $H = \bigcup H = \emptyset$. Then $\mathcal{A}(H) = \{\emptyset, \{*\}\}$, with F_0 being the vertex $\emptyset$, and F_{-1} being the facet $\{*\}$. Both situations are anomalous for having a vertex strictly above a facet. When $r=1$, the vertices and facets coincide, and when $r>1$, the vertices are strictly below the facets.

We have the following.

Proposition 5.2. *If $r>0$ is the rank of $\mathcal{A}(H)$, then every vertex of $\mathcal{A}(H)$ is incident with r facets.*

Proof. Every vertex of $\mathcal{A}(H)$, i.e. every construction K of H, is of cardinality $|\bigcup H|$, by Remark 5.1. The facets of $\mathcal{A}(H)$ have each besides the n connected components of $\bigcup H$ a single additional member. Those facets with which K is incident have as this additional member a member of K different from the n connected components of $\bigcup H$. There are $|\bigcup H|-n$ such members in K. $\square$

If $r=0$, then the additional member of a facet mentioned in this proof is not from K, but it is $*$. If the rank of $\mathcal{A}(H)$ is 0, then every vertex of $\mathcal{A}(H)$ is incident with 1 facet (in this case there is a single vertex strictly above the single facet).

In our example with the hypergraph A, we have as the facets of $\mathcal{A}(A)$ the two-element sets that besides $\{x,y,z,u\}$ have as an additional member one of

$$\{x\}, \quad \{y\}, \quad \{z\}, \quad \{u\}, \quad \{x,y,z\}, \quad \{y,z,u\}, \quad \{x,y\}, \quad \{y,z\}, \quad \{z,u\}.$$

The first six facets in the ensuing list correspond to pentagons, while the last three correspond to squares. The facet incident with the vertices L, M and N is $\{\{u\}, \{x,y,z,u\}\}$.

If the atomic hypergraph H is not connected, and its connected components are $H_1, \dots, H_n$, for $n \geqslant 2$, then $\mathcal{A}(H)$ may be obtained out of $\mathcal{A}(H_1), \dots, \mathcal{A}(H_n)$ in the following manner. Let $C_1, \dots, C_n$ be constructs, i.e. faces of rank at least 0, of $\mathcal{A}(H_1), \dots, \mathcal{A}(H_n)$ respectively. Let the constructs $C_1, \dots, C_n$ be respectively of cardinalities $k_1, \dots, k_n$. Then $C_1 \cup \dots \cup C_n$ of cardinality $k_1 + \dots + k_n$ is a face of $\mathcal{A}(H)$; it is a face of rank k when $k_1 + \dots + k_n = |\bigcup H|-k$. This is how we obtain all the faces of $\mathcal{A}(H)$ of rank at least 0. The face F_{-1} of $\mathcal{A}(H)$ is, as always, $\bar{H}^*$.

We have that $\mathcal{A}(H) - \{\bar{H}^*\}$, i.e. $\mathcal{A}(H) - \{F_{-1}\}$, is isomorphic to the Cartesian product $(\mathcal{A}(H_1) - \{\bar{H}_1^*\}) \times \dots \times (\mathcal{A}(H_n) - \{\bar{H}_n^*\})$ (cf. the product $\cdot$ of Section 8). Isomorphism means here the existence of an order-preserving bijection. We may conceive of $\mathcal{A}(H)$ as obtained from $\mathcal{A}(H_1), \dots, \mathcal{A}(H_n)$ by an operation $\otimes$ related to $\times$. Binary $\otimes$ differs from binary $\times$ by having instead of an ordered pair the union of the two members of the ordered pair; these members are disjoint, and their disjoint union corresponds bijectively to the ordered pair. We conflate moreover the least faces F_{-1} into a single one. Hence we have $\mathcal{A}(H) = \mathcal{A}(H_1) \otimes \dots \otimes \mathcal{A}(H_n)$. (For an example, see $\mathcal{A}(H_{4200})$ in Appendix B.)

It remains to verify that $\mathcal{A}(H)$ for an arbitrary atomic hypergraph H is indeed an abstract polytope of rank $|\bigcup H|-n$, in the sense of [25, Section 2A], and this will be done in Section 8. In the remainder of this section, and in the next two sections, we consider various properties of abstract polytopes. This will help us for the results of Section 8, and also for those of Section 9, where we deal with the realizations of our abstract polytopes in Euclidean spaces.

The partial order $\mathcal{A}(H)$ is a lattice, with join being intersection. The meet of two faces of $\mathcal{A}(H)$ is their union if this union is a construct of H, and otherwise it is $\bar{H}^*$, i.e. F_{-1}. For example, for α being $\{x,y,z,u\}$ the union of the facets $\{\{x\},\alpha\}$ and $\{\{y\},\alpha\}$ of $\mathcal{A}(A)$ is not a construct of A; so the meet of these two facets is $\bar{A}^*$.

In general, the lattice $\mathcal{A}(H)$ is not distributive. For example, $\mathcal{A}(A)$ is not distributive because

$$\big\{\{y\},\alpha\big\}\wedge\big(\big\{\{x\},\alpha\big\}\vee\big\{\{z\},\alpha\big\}\big)=\big\{\{y\},\alpha\big\},$$
$$\big(\big\{\{y\},\alpha\big\}\wedge\big\{\{x\},\alpha\big\}\big)\vee\big(\big\{\{y\},\alpha\big\}\wedge\big\{\{z\},\alpha\big\}\big)=\bar{A}^*$$

(here the join $\vee$ is intersection, while the meet $\wedge$ is either union or its result is $\bar{A}^*$, as explained above).

A more natural lattice than $\mathcal{A}(H)$ is the dual lattice (namely, $\mathcal{A}(H)$ upside down). In the dual lattice the meet would be intersection, and the join would be the other operation involving union. As an abstract polytope, we need however $\mathcal{A}(H)$ as it is, and not the dual lattice, which would give another polytope.

For a face F of $\mathcal{A}(H)$ different from F_{-1}, consider the section F_r/F of $\mathcal{A}(H)$, i.e. the set of all the constructs of H of which the greatest face F_r is a subset, and which are subsets of the construct F. This lattice is isomorphic to the lattice $\langle\mathcal{P}(F-F_r),\cup,\cap\rangle$, with $\mathcal{P}(F-F_r)$ being the power set of $F-F_r$, meet being $\cup$, join being $\cap$; the greatest element of this lattice is $\emptyset$, and the least element is $F-F_r$. Hence, by Proposition 2.16 of [35, Section 2.5], we may conclude that a geometric realization of $\mathcal{A}(H)$ whose face lattice is isomorphic to $\mathcal{A}(H)$ must be a simple polytope. (This means that each of its vertex figures, which are figures obtained by truncating a vertex, is a simplex; we deal with these matters in Section 9.) Another way to reach the same conclusion is to rely on Proposition 5.2, and appeal again to Proposition 2.16 of [35].

We can prove the following.

Proposition 5.3. *If for some $k\in\{-1,\dots,r-1\}$ all the faces of rank k of $\mathcal{A}(H_1)$ and $\mathcal{A}(H_2)$ are the same, then $\mathcal{A}(H_1)=\mathcal{A}(H_2)$.*

Proof. Let $k\in\{0,\dots,r-1\}$. Then the union of all the faces of rank k of $\mathcal{A}(H)$ is $\bar{H}$; this follows from Proposition 4.8. Hence if the faces of rank k of $\mathcal{A}(H_1)$ and $\mathcal{A}(H_2)$ are the same, then $\bar{H}_1=\bar{H}_2$, and hence $\mathcal{A}(H_1)=\mathcal{A}(H_2)$. When $k=-1$, we reason similarly, with $\bar{H}^*$ instead of $\bar{H}$. $\square$

So, in particular, $\mathcal{A}(H)$ is completely determined by its vertices, or by its facets.

6. Constructions of ASC-hypergraphs

Let an *ASC-hypergraph* be a hypergraph that is atomic (see the beginning of Section 3), saturated (see the beginning of Section 4) and connected (see Section 2). Concentrating on these hypergraphs will simplify our exposition. With their help, we will give in this section noninductive characterizations of constructions (Proposition 6.11) and of constructs (Proposition 6.13), which presents an alternative to our inductive definition of a construction in Section 3. These alternative characterizations will serve for the results of Section 9, which are about the realizations of polytopes of hypergraphs. They will also serve in Appendix A to explain the relationship between constructs and tubings.

Putting atomicity in the definition of an ASC-hypergraph is essential, as it was essential up to now when we dealt with constructions, and the derived notions of construct and abstract polytope of a hypergraph. Putting in this definition the other two properties—saturation and connectedness—is however only a matter of convenience.

Saturation could be omitted at the cost of having a little bit more complicated formulations, in which "Y belongs to the hypergraph H" is replaced by "Y is a connected subset of the hypergraph H", which should mean that Y is a subset of $\bigcup H$ such that H_Y is connected. By Proposition 4.7, an arbitrary atomic hypergraph and its saturated closure have the same constructions. So they do not differ essentially when we deal with constructions; they yield the same result.

Connectedness too is assumed to organize reasonably the exposition. It covers what we need most when we deal with abstract polytopes of hypergraphs. The abstract polytopes of arbitrary atomic hypergraphs may be derived from the abstract polytopes of ASC-hypergraphs. The connected components $H_1,\dots,H_n$ of the saturated closure of an atomic hypergraph H are ASC-hypergraphs, and $\mathcal{A}(H)$ is equal to $\mathcal{A}(H_1)\otimes\cdots\otimes\mathcal{A}(H_n)$ (see Section 5 for the operation $\otimes$). We will return to this matter at the end of Section 7.

First we have the following, which is proved in a straightforward manner.

Remark 6.1. If H is an ASC-hypergraph and $Y\in H$, then H_Y is an ASC-hypergraph.

For that we rely on Remarks 3.1 and 3.2.

Our goal in this section is to characterize noninductively constructions of ASC-hypergraphs. Before we start going towards this goal, we will see in the next proposition how we could define inductively constructions of ASC-hypergraphs. This is an adaptation of our inductive definition of a construction of Section 3 to ASC-hypergraphs specifically. Constructions of hypergraphs that are not connected do not occur separately any more—they are incorporated into constructions of connected hypergraphs. In the proof of this proposition we rely on the fact that the empty hypergraph is an ASC-hypergraph (see the remarks concerning the empty hypergraph before Remark 3.1, after Remark 4.1, and in Section 2).

Proposition 6.2. *If K is a construction of an ASC-hypergraph H, then either $K = \emptyset$, or $K = \{\{x\}\}$, or*

(K) $K = K_1 \cup \dots \cup K_n \cup \{\bigcup H\}$, *where* $n \geqslant 1$, *and* $\bigcup H = \bigcup H_1 \cup \dots \cup \bigcup H_n \cup \{x\}$ *for* $\{H_1, \dots, H_n\}$ *being the finest hypergraph partition of* $H_{\bigcup H - \{x\}}$, *and for every* $i \in \{1, \dots, n\}$ *we have that* K_i *is a construction of the ASC-hypergraph* H_i.

Proof. If $|\bigcup H| = 0$, then $K = \emptyset$ by clause (0) of the definition of a construction. If $|\bigcup H| > 0$, then K is obtained by applying clause (1) of our definition as the last clause.

If clause (0) preceded immediately this application of clause (1), then $\bigcup H = \{x\}$, and $K = \{\{x\}\}$. If another application of clause (1) or an application of clause (2) preceded immediately this application of clause (1), then we have (K); if that was another application of (1), then $n = 1$, and if that was an application of (2), then $n \geqslant 2$.

That H_i, which is equal to $H_{\bigcup H_i}$, is atomic follows from the atomicity of H, Remark 3.1 and $\bigcup H_i \in H$, which holds because H is saturated. We can then conclude easily that H_i is an ASC-hypergraph. $\square$

Next we introduce some terminology, and we give some preliminary lemmata, that lead towards the main goal of this section.

Let H be an ASC-hypergraph and let $M \subseteq H$. We say that a subset S of M is an *M-antichain* when $|S| \geqslant 2$ and for every two distinct members X and Y of S neither $X \subseteq Y$ nor $Y \subseteq X$.

An M-antichain S *misses* H when $\bigcup S \notin H$. (This notion may be found in Definition 2.7 of [15].) An M-antichain is *pairwise disjoint* when every pair of distinct members of it is disjoint.

For the series of lemmata that follows we make all the time the following assumption:

(M) M is a subset of the ASC-hypergraph H such that every M-antichain misses H.

We can prove the following.

Lemma 6.3. *Every M-antichain is pairwise disjoint.*

Proof. Suppose that for some distinct members X and Y of an M-antichain we had $X \cap Y \neq \emptyset$. From $X, Y \in H$ and the assumption that H is saturated we could then infer that $X \cup Y \in H$. So $\{X, Y\}$ would make an M-antichain that does not miss H, which contradicts (M). $\square$

We can also infer that if (M) holds, then every pair of members X and Y of M is non-overlapping, which means that either $X \cap Y = \emptyset$ or $X \subseteq Y$ or $Y \subseteq X$, and non-adjacent, which means that if $X \cap Y = \emptyset$, then $X \cup Y \notin H$ (see Lemmata A.1 and A.2 of Appendix A). Note however that non-overlapping and non-adjacency are binary, and are tied to tubings, which are subsets of constructions of *graphs* (see Appendix A). These two properties do not suffice for the constructions of hypergraphs in general.

For $X \in M$, an element x of X is called *X-superficial* when for every Y in M that is a proper subset of X we have $x \notin Y$. The notion of X-superficial element is relative to M, but we need not mention that when M is fixed, as it will be for us most of the time. This notion corresponds to the notion of root of an f-construction (see the end of Section 3). We can prove the following.

Lemma 6.4. *Every X in M has at least one X-superficial element.*

Proof. Suppose that some X in M has no X-superficial element. So for every element y of X there is at least one Y in M that is a proper subset of X such that $y \in Y$. Let $Y_1, \dots, Y_m$ be all these sets Y, for all the elements y of X. Here $m \geqslant 1$, because X is nonempty (it is an element of the hypergraph H), and we cannot have that $m = 1$, because Y_1 is a proper subset of X; so $m \geqslant 2$.

Eliminate from $Y_1, \dots, Y_m$ every Y_i that is a proper subset of another Y_j in the sequence, and let the resulting sequence be $X_1, \dots, X_n$. Here $n \geqslant 2$ for the same reasons that gave $m \geqslant 2$ above. Since $\{X_1, \dots, X_n\}$ is an M-antichain, by (M) it should miss H, but $X_1 \cup \dots \cup X_n = X$, and $X \in H$, which is contradictory. $\square$

We have established with this lemma that there is a map φ that assigns to every X in M a nonempty set $\varphi(X)$ of X-superficial elements. Next we establish the following.

Lemma 6.5. *For every X and Y in M, if $X \neq Y$, then $\varphi(X) \cap \varphi(Y) = \emptyset$.*

Proof. Suppose that for X and Y in M we have an x in $\bigcup H$ such that $x \in \varphi(X) \cap \varphi(Y)$. If we have that $X \neq Y$, then we have the M-antichain $\{X, Y\}$; we cannot have that $X \subseteq Y$, because then x would not be Y-superficial, and analogously we cannot have that $Y \subseteq X$.

The M-antichain $\{X, Y\}$ does not miss H. This is because from $\varphi(X) \cap \varphi(Y) \neq \emptyset$ we infer $X \cap Y \neq \emptyset$, and hence $X \cup Y \in H$, by the saturation of H. This contradicts (M). $\square$

Let $\varphi(M)$ be $\{\varphi(X) \mid X \in M\}$. With Lemmata 6.4 and 6.5 we have established that $\varphi(M)$ is a family of nonempty pairwise disjoint subsets of $\bigcup H$.

From Lemmata 6.4 and 6.5 we also infer the following.

Lemma 6.6. *The map φ is injective.*

We need the following two general remarks for the proof of the two lemmata that follow them. Consider a family Φ of nonempty pairwise disjoint subsets of a set W. We have the following.

Remark 6.7. $|\Phi| \leqslant |W|$.

Remark 6.8. For W finite, we have that $|\Phi| = |W|$ iff $\Phi = \{\{w\} \mid w \in W\}$.

Note that Remark 6.8 does not hold for W infinite. If W is the set of natural numbers $\mathbf{N}$, then Φ may be $\{\{0,1\}, \{2\}, \{3\}, \dots\}$ or $\{\{0\}, \{2\}, \{3\}, \dots\}$ with $|\Phi| = |\mathbf{N}|$, but $\Phi \neq \{\{n\} \mid n \in \mathbf{N}\}$.

Assume (M) as above, and let the map φ and $\varphi(M)$ be defined as above. We can prove the following.

Lemma 6.9. $|M| \leqslant |\bigcup H|$.

Proof. As we said above, with Lemmata 6.4 and 6.5 we have established that $\varphi(M)$ is a family of nonempty pairwise disjoint subsets of $\bigcup H$. So, by Remark 6.7, we have that $|\varphi(M)| \leqslant |\bigcup H|$. By using Lemma 6.6, we have that $|M| = |\varphi(M)|$, and the lemma follows. $\square$

Lemma 6.10. *We have that $|M| = |\bigcup H|$ iff $\varphi(M) = \{\{x\} \mid x \in \bigcup H\}$.*

Proof. With Lemmata 6.4 and 6.5 we have established that $\varphi(M)$ is a family of nonempty pairwise disjoint subsets of $\bigcup H$. By using Lemma 6.6, we have that $|M| = |\bigcup H|$ iff $|\varphi(M)| = |\bigcup H|$. Since moreover $\bigcup H$ is finite, we obtain the desired conclusion by Remark 6.8. $\square$

The following proposition gives our noninductive characterization of constructions of ASC-hypergraphs.

Proposition 6.11. *We have* (M) *and $|M| = |\bigcup H|$ iff M is a construction of H.*

Proof. From left to right we proceed as follows. If $|\bigcup H| = 0$, then $H = \emptyset$ and $M = \emptyset$. If $|\bigcup H| = 1$, then $H = \{\{x\}\}$ and $M = \{\{x\}\}$.

We establish next that if $|\bigcup H| \geqslant 1$, then $\bigcup H \in M$. For $|\bigcup H| = 1$, we established that in the preceding paragraph. Suppose $|\bigcup H| = k \geqslant 2$ and $\bigcup H \notin M$. Then by omitting from M members that are proper subsets of other members we can obtain an M-antichain $Y_1, \dots, Y_m$. We must have that $m \geqslant 2$ because of Lemma 6.10 and $\bigcup H \notin M$. However, by Lemma 6.10 and the saturation and connectedness of H, we have that $Y_1 \cup \dots \cup Y_m = \bigcup H \in H$, and hence our M-antichain does not miss H, which contradicts (M).

We proceed next by induction on $|\bigcup H|$, the basis being the case above when $|\bigcup H| = 1$. So let $|\bigcup H| \geqslant 2$, and let x be the $\bigcup H$-superficial element, which exists by Lemma 6.10. Then for $\{H_1, \dots, H_n\}$, where $n \geqslant 1$, being the finest hypergraph partition of $H_{\bigcup H - \{x\}}$ we have that $M = M_1 \cup \dots \cup M_n \cup \{\bigcup H\}$, where for every $i \in \{1, \dots, n\}$ the set M_i is $M_{\bigcup H_i}$. We show that

(M_i) M_i is a subset of the ASC-hypergraph H_i such that every M_i-antichain misses H_i.

It follows immediately that $M_i \subseteq H_i$, and H_i is an ASC-hypergraph by Remark 6.1. Every M_i-antichain is an M-antichain, and misses H by assumption; hence it must also miss H_i. So we have (M_i).

We have that $|M_i| = |\bigcup H_i|$ for the following reason. For every $i \in \{1, \dots, n\}$ we must have that $|M_i| \leqslant |\bigcup H_i|$ by Lemma 6.9, and if for some $j \in \{1, \dots, n\}$ we had $|M_i| < |\bigcup H_i|$, then we could not secure that

$$|M_1| + \dots + |M_n| + 1 = |M| = \left|\bigcup H\right| = \left|\bigcup H_1\right| + \dots + \left|\bigcup H_n\right| + 1.$$

By the induction hypothesis, we may conclude that M_i is a construction of the ASC-hypergraph H_i, and it follows that M is a construction of H.

From right to left we proceed by induction on the size of the construction K of the ASC-hypergraph H. Consider what K may be according to Proposition 6.2. If $K=\emptyset$ or $K=\{\{x\}\}$, then it is trivial that every K-antichain misses H (there are no K-antichains) and that $|K|=|\bigcup H|$. If we have (K) as in Proposition 6.2, then the induction hypothesis applies to the constructions K_i for every $i\in\{1,\dots,n\}$, and by it we conclude that

(K_i) K_i is a subset of the ASC-hypergraph H_i such that every K_i-antichain misses H_i

and $|K_i|=|\bigcup H_i|$.

If a K-antichain is a K_i-antichain, then (K_i) applies to it; hence it misses H_i, and it follows that it misses H.

Suppose we have a K-antichain S that is not a K_i-antichain for any $i\in\{1,\dots,n\}$. The $\bigcup H$-superficial element x relative to K cannot be in $\bigcup S$, because otherwise $\bigcup H$ would have to belong to S, and this is impossible (every member of S is a subset of $\bigcup H$).

Since S is not a K_i-antichain, there must be two distinct elements y and z of $\bigcup S$ such that $y\in\bigcup H_p$ and $z\in\bigcup H_q$ for two distinct members H_p and H_q of the finest hypergraph partition $\{H_1,\dots,H_n\}$ of $H_{\bigcup H-\{x\}}$, which we have according to (K). We conclude that $\bigcup S\notin H_{\bigcup H-\{x\}}$, and since $x\notin\bigcup S$, and hence $\bigcup S\subseteq\bigcup H-\{x\}$, it follows that $\bigcup S\notin H$.

So every K-antichain misses H, and $|K|=|\bigcup H|$, by Remark 5.1. $\square$

The following proposition will help us to obtain a characterization of the notion of construct in the style of Proposition 6.11.

Proposition 6.12. *We have* (M) *iff for some construction K of H we have that $M\subseteq K$.*

Proof. The direction from right to left is obtained easily as follows from Proposition 6.11. Suppose for some construction K of H we have that $M\subseteq K$. By Proposition 6.11 from right to left we have that every K-antichain misses H. Every M-antichain is however a K-antichain.

For the other direction, suppose (M). If we could prove that

(R) there is a subset K of H such that $M\subseteq K$, $|K|=|\bigcup H|$ and every K-antichain misses H,

then by Proposition 6.11 from left to right we would have that K is a construction of H, and we would obtain the right-hand side of the proposition we are proving. The remainder of our proof is an inductive proof of (R).

We have that $|M|\leqslant|\bigcup H|$ by Lemma 6.9. Our proof of (R) will proceed by induction on $|\bigcup H|-|M|$. In the basis, when this number is zero, and hence $|M|=|\bigcup H|$, we take $K=M$.

Suppose for the induction step that $|M|<|\bigcup H|$. Let $M^+=M\cup\{\bigcup H\}$. From (M) we easily infer that every M^+-antichain misses H, since $\bigcup H$ is not a member of any M^+-antichain, and hence M^+-antichains are M-antichains. If $\bigcup H\notin M$, then $|M|<|M^+|\leqslant|\bigcup H|$, and we may apply the induction hypothesis to M^+; namely, we have (R) for M replaced by M^+. For the set K that this yields we have that $M^+\subseteq K$, and hence $M\subseteq K$, which gives (R).

Suppose that $\bigcup H\in M$. For every x in $\bigcup H$ we have then a member of M, namely $\bigcup H$, to which x belongs. We easily infer that there is hence a member X of M such that x is X-superficial. If that is not $\bigcup H$, then we pass to a proper subset Y of $\bigcup H$ such that $Y\in M$ and $x\in Y$, and continue in this manner until we reach X (officially, a trivial induction on the number of members of M to which x belongs is here at work; this number is, of course, finite). So for every x in $\bigcup H$ there is an X in M such that $x\in\varphi(X)$.

Since $|M|<|\bigcup H|$, we have that $\bigcup H$ is not empty, and for an x in $\bigcup H$ there is an X in M such that $x\in\varphi(X)$ and $|\varphi(X)|\geqslant 2$. If we always had $|\varphi(X)|=1$, then, by Lemma 6.10, we would obtain $|M|=|\bigcup H|$, which contradicts $|M|<|\bigcup H|$. So we have that $x,y\in\varphi(X)$ and $x\neq y$.

Let $\{H_1,\dots,H_n\}$, for $n\geqslant 1$, be the finest hypergraph partition of $H_{X-\{x\}}$. This partition is nonempty, because for some $i\in\{1,\dots,n\}$ we must have that $y\in\bigcup H_i$. Note that $\bigcup H_i\notin M$; otherwise, $y\notin\varphi(X)$, i.e. y would not be X-superficial. Let M' be $M\cup\{\bigcup H_i\}$. Since $\bigcup H_i\notin M$, we have that $|M|<|M'|$.

We prove next that every M'-antichain misses H. (This will occupy us for most of the remainder of the proof.) Suppose there is an M'-antichain S that does not miss H. We must have that $\bigcup H_i\in S$; otherwise, S would be an M-antichain, and we would contradict M.

Let $S'=\{Y\in S\mid Y\nsubseteq X\}$. If $S'\neq\emptyset$, then we show that (M) does not hold. Take $S'\cup\{X\}$; this is an M-antichain, because, first, all its members are from M (we have that $\bigcup H_i\subseteq X$), and, secondly, X cannot be a subset of a Y in S'; otherwise, we would have that $\bigcup H_i\subseteq Y$, and S would not be an M'-antichain.

We show next that $\bigcup(S'\cup\{X\})\in H$, which will contradict (M). Since every Z in $S-S'$ is a subset of X, we have that $\bigcup(S'\cup\{X\})=\bigcup(S\cup\{X\})=\bigcup S\cup X$. We have that $y\in\bigcup H_i\subseteq\bigcup S\cap X$, and so by the saturation of H, we have that $\bigcup S\cup X\in H$; so $\bigcup(S'\cup\{X\})\in H$, which contradicts (M).

So we have that $S'=\emptyset$, and hence $\bigcup S\subseteq X$. We have that $X\notin S$; otherwise, S would not be an M'-antichain (we have that $\bigcup H_i\subseteq X$). We also have that $x\notin\bigcup S$. Otherwise, since $x\notin\bigcup H_i$, we would have a Z in M such that $Z\in S$ and

$x \in Z$. But $Z \subseteq X$ and $Z \neq X$, as we have just shown, and hence x is not X-superficial relative to M, which is a contradiction.

So $\bigcup S \subseteq X - \{x\}$, and since $\bigcup S \in H$, we have that $\bigcup S \in H_{X-\{x\}}$. Since $\{H_1, \ldots, H_n\}$ is the finest hypergraph partition of $H_{X-\{x\}}$ and $\bigcup H_i \in S$, we obtain $\bigcup S \in H_i$. Hence $\bigcup H_i \subseteq \bigcup S$ and $\bigcup S \subseteq \bigcup H_i$, which means that $\bigcup S = \bigcup H_i$, and since $\bigcup H_i \in S$, this contradicts the assumption that S is an M'-antichain.

So every M'-antichain misses H, and we may apply the induction hypothesis to M'; namely, we have (R) for M replaced by M'. For the set K that this yields we have that $M' \subseteq K$, and hence $M \subseteq K$, which gives (R).

So we have established (R), and, as explained above, with that we may end our proof. □

As an immediate corollary of Proposition 6.12 we have the following.

Proposition 6.13. *We have* (M) *and* $\bigcup H \in M$ *iff* M *is a construct of* H.

We conclude this section with the following technical lemma, which we need for the proof of Lemma 9.5 in Section 9.

Lemma 6.14. *If K is a construction of the ASC-hypergraph H and $Y \in H - K$, then there is an X in K such that $Y \subset X$ and for x being X-superficial we have $x \in Y$.*

Proof. If $K = \emptyset$ or $K = \{\{x\}\}$, then $H - K$ is empty, and the lemma holds trivially.

Suppose K is of the form specified by (K) of Proposition 6.2 and $Y \in H - K$. Then either we find in Y the $\bigcup H$-superficial element x, and we are done, or there is an $i \in \{1, \ldots, n\}$ such that $Y \in H_i - K_i$; otherwise, $\{H_1, \ldots, H_n\}$ would not be the finest hypergraph partition of $H_{\bigcup H - \{x\}}$. We may then proceed by induction. □

7. Continuations of constructions

In this section we describe the vertices of the abstract polytope of a hypergraph in terms of a partial operation on constructions that we call continuation, which may be understood intuitively as indeed the continuation of one construction by another. This description leads to a characterization of all the faces of abstract polytopes of hypergraphs, and in particular of their facets. This characterization of facets will play an important role in the next section, where we prove that abstract polytopes of hypergraphs are indeed abstract polytopes. The results of this section are closely related to the results of [34], though the presentation is different. (In Definition 3.1 of [34], which introduces a notion that plays a role analogous to our $\bigcup_{H-Y} L$ below, one should require in (3.3) that I is nonempty.)

We start with some preliminary matters. Assume for the proposition below that L is a construction of the ASC-hypergraph H and that $Y \in L$. Then we can prove the following.

Proposition 7.1. *The set L_Y is a construction of the ASC-hypergraph H_Y.*

Proof. We know that H_Y is an ASC-hypergraph by Remark 6.1. It is clear that $L_Y \subseteq H_Y$. We show next that every L_Y-antichain misses H_Y (see Section 6). If for an L_Y-antichain S we had that $\bigcup S \in H_Y$, then we would also have $\bigcup S \in H$. Since S is also an L-antichain, we would have, according to Proposition 6.11 from right to left, that L is not a construction of H, which contradicts our assumption. So every L_Y-antichain misses H_Y.

It remains to show that $|L_Y| = |\bigcup H_Y|$ in order to obtain the proposition by applying Proposition 6.11 from left to right. We know by Lemma 6.10 and $|L| = |\bigcup H|$ that $\varphi(L) = \{\{x\} \mid x \in \bigcup H\}$. Let $\varphi(L_Y) = \{\varphi(X) \mid X \in L_Y\}$. It is easy to see that $\varphi(L_Y) = \{\{y\} \mid y \in Y\}$ since $\bigcup H_Y = Y$. So $|\varphi(L_Y)| = |Y|$, and, since φ is injective, we have $|L_Y| = |Y| = |\bigcup H_Y|$. □

For H a hypergraph and $X \subseteq \bigcup H$ let

$$X \cup_H Y = \begin{cases} X \cup Y & \text{if } X \cup Y \in H, \\ X & \text{if } X \cup Y \notin H. \end{cases}$$

Then, with assumptions as before the preceding proposition, we have the following lemma.

Lemma 7.2. *If $X \in L - L_Y$, then $(X - Y) \cup_H Y = X$.*

Proof. If $Y \subseteq X$, then we clearly have

$$(X - Y) \cup_H Y = (X - Y) \cup Y = X.$$

Suppose not $Y \subseteq X$. It is impossible that $X \subseteq Y$, because $X \in L - L_Y$. So $\{X, Y\}$ is an L-antichain (see Section 6). Since $X \cap Y = \emptyset$ (see Lemma 6.3), we have that $X - Y = X$, and since $X \cup Y$ cannot belong to H (our L-antichain misses H; see Section 6), we have that $X \cup_H Y = X$. □

For $F \subseteq \mathcal{P}C$ and $Z \subseteq C$ let

$$_Z F =_{df} \{X \cap Z \mid X \in F \;\&\; X \cap Z \neq \emptyset\}.$$

If F is a hypergraph H, and $Z \subseteq \bigcup H$, then it is easy to check that $_Z H$ is a hypergraph on Z. We have always that H_Z (which is defined at the beginning of Section 3) is a subset of $_Z H$. The converse need not hold, and here is a counterexample for that.

Consider the ASC-hypergraph

$$H = \big\{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y, z\}, \{y, z, u\}, \{x, y, z, u\}\big\}$$

on $\{x, y, z, u\}$, and let $Z = \{x, y, z\}$ (this hypergraph H is the hypergraph H_{4021} of Appendix B). Then we have

$$H_Z = \big\{\{x\}, \{y\}, \{z\}, \{x, y, z\}\big\},$$
$$_Z H = H_Z \cup \big\{\{y, z\}\big\}.$$

There are simpler counterexamples when H is not an ASC-hypergraph, but we wanted to have a counterexample with such a hypergraph.

We can verify the following in a straightforward manner.

Remark 7.3. If H is an ASC-hypergraph and $Z \subseteq \bigcup H$, then $_Z H$ is an ASC-hypergraph on Z.

This remark should be compared with Remark 6.1.

Let us assume, as before Proposition 7.1, that L is a construction of the ASC-hypergraph H and that $Y \in L$. It is then easy to see that $\bigcup_{H-Y} L = \bigcup_{H-Y}(L - L_Y)$, and we will rely on this equation without notice in the rest of this section. We have the following.

Proposition 7.4. *The set $\bigcup_{H-Y} L$ is a construction of the ASC-hypergraph $\bigcup_{H-Y} H$.*

Proof. We know that $\bigcup_{H-Y} H$ is an ASC-hypergraph by Remark 7.3. It is clear that $\bigcup_{H-Y} L \subseteq \bigcup_{H-Y} H$. We show next that every $\bigcup_{H-Y} L$-antichain misses $\bigcup_{H-Y} H$. Suppose for such an antichain $\{X_1, \dots, X_k\}$, where $k \geqslant 2$, we had that $X_1 \cup \dots \cup X_k \in \bigcup_{H-Y} H$.

For every $Z \in \bigcup_{H-Y} L$ we have that $Z = X - Y$ for some $X \in L - L_Y$. Then by Lemma 7.2 we have that $Z \cup_H Y = X$.

Consider then the set $S = \{X_1 \cup_H Y, \dots, X_k \cup_H Y\}$. Since for every $i \in \{1, \dots, k\}$ we have that $X_i \in \bigcup_{H-Y} L$, we can conclude as above that $X_i \cup_H Y = X$ for some $X \in L - L_Y$, and hence $X_i \cup_H Y \in L$. It follows that S is an L-antichain.

We have assumed above that $X_1 \cup \dots \cup X_k = W - Y$ for some $W \in H$. Suppose (1) for every i we have that $X_i \cup_H Y = X_i$ and (2) $W - Y = W$. Then S is an L-antichain that does not miss H, which together with Proposition 6.11 from right to left contradicts the assumption that L is a construction of H.

Suppose (1) and not (2), i.e. $W - Y \neq W$. Then

$$\begin{aligned} \bigcup S \cup Y &= X_1 \cup \dots \cup X_k \cup Y \\ &= (W - Y) \cup Y \\ &= W \cup Y. \end{aligned}$$

Since $W \cap Y \neq \emptyset$, as we supposed above, we have that $W \cup Y \in H$, because H is saturated. Then $S \cup \{Y\}$ is an L-antichain that does not miss H, which is contradictory, as above.

Suppose not (1), i.e. for some i we have that $X_i \cup_H Y = X_i \cup Y \in H$. Then $\bigcup S = W \cup Y$. Since $X_i \subseteq W$ and $X_i \neq \emptyset$, we have that $W \cap (X_i \cup Y) \neq \emptyset$. So $W \cup (X_i \cup Y) \in H$, because H is saturated, and hence $W \cup Y \in H$. So S is an L-antichain that does not miss H, which is contradictory, as above. Thereby, we have shown that every $\bigcup_{H-Y} L$-antichain misses $\bigcup_{H-Y} H$.

We have shown in the proof of Proposition 7.1 that $|L_Y| = |Y|$. It remains to show that $|\bigcup_{H-Y} L| = |\bigcup H - Y|$ (we have that $\bigcup H - Y = \bigcup \bigcup_{H-Y} H$) in order to obtain the proposition by applying Proposition 6.11 from left to right. We have that $|\bigcup_{H-Y} L| = |L - L_Y| = |L| - |L_Y| = |\bigcup H| - |Y|$, because L is a construction of H, and hence $|L| = |\bigcup H|$ (see Remark 5.1), and because $|L_Y| = |Y|$, as we showed in the proof of Proposition 7.1. We have that $|\bigcup H| - |Y| = |\bigcup H - Y|$, which concludes our proof. □

Let H be an ASC-hypergraph. For $Y \in H$, let K and J be respectively constructions of the ASC-hypergraphs H_Y and $\bigcup_{H-Y} H$. Then we define the *continuation* $K * J$ of K by J in the following way:

$$K * J = K \cup \{X \cup_H Y \mid X \in J\}.$$

Here is an example of continuation.

Take the ASC-hypergraph

$$\bar{A} = \big\{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}, \{y, z\}, \{z, u\}, \{x, y, z\}, \{y, z, u\}, \{x, y, z, u\}\big\},$$

which is the saturated closure of the hypergraph A of Section 3. (This hypergraph is called H'_{4321} in Appendix B.) Let K be the construction $\{\{u\}, \{z, u\}\}$ of the ASC-hypergraph $\bar{A}_{\{z,u\}} = \{\{z\}, \{u\}, \{z, u\}\}$, and let J be the construction $\{\{x\}, \{x, y\}\}$ of the ASC-hypergraph ${}_{\{x,y\}}\bar{A} = {}_{\{x,y,z,u\}-\{z,u\}}\bar{A} = \{\{x\}, \{y\}, \{x, y\}\}$. Then the continuation $K * J$ is $\{\{u\}, \{z, u\}, \{x\}, \{x, y, z, u\}\}$. (The continuation $K * J$ is the construction N of Section 5, which corresponds to the s-construction $y(x + (zu))$; with this s-construction we have labeled one of the vertices of the associahedron K_5 in Appendix B.)

Another example of continuation with the same ASC-hypergraph $\bar{A}$ is obtained by taking Y to be $\{y, z, u\}$, with K being the construction $\{\{y\}, \{u\}, \{y, z, u\}\}$ of the ASC-hypergraph

$$\bar{A}_{\{y,z,u\}} = \big\{\{y\}, \{z\}, \{u\}, \{y, z\}, \{z, u\}, \{y, z, u\}\big\},$$

and J being the construction $\{\{x\}\}$ of the ASC-hypergraph

$${}_{\{x\}}\bar{A} = {}_{\{x,y,z,u\}-\{y,z,u\}}\bar{A} = \big\{\{x\}\big\}.$$

Then $K * J$ is $\{\{y\}, \{u\}, \{y, z, u\}, \{x, y, z, u\}\}$. (This is the construction M of Section 3, which corresponds to the s-construction $xz(y + u)$.)

A third example of continuation is provided by taking $\bar{A}$, Y and K as in the first example and J as $\{\{y\}, \{x, y\}\}$. Then $K * J$ is $\{\{u\}, \{z, u\}, \{y, z, u\}, \{x, y, z, u\}\}$. (This is the construction L of Section 3, which corresponds to the s-construction $xyzu$.)

A fourth and final example is with everything being as in the second example except for K, which is now $\{\{u\}, \{z, u\}, \{y, z, u\}\}$. Then $K * J$ is equal to the $K * J$ of the preceding example (namely, to the construction L).

Note that if $Y = \bigcup H$, then $\bigcup H - Y = \emptyset$, and we have that ${}_{\emptyset}H = \emptyset$ and $J = \emptyset$. In that case $K * \emptyset = K$. We shall however need $*$ mostly when J is not $\emptyset$.

It is easy to see that $|K * J| = |K| + |J|$. This matches the fact that $|\bigcup H| = |Y| + |\bigcup H - Y|$, as the propositions below will show. We can then prove the following.

Proposition 7.5. *The set $K * J$ is a construction of H.*

Proof. We prove first that $K * J \subseteq H$. Let $X \in K * J$. If $X \in K$, then $X \in H_Y$, and hence $X \in H$. Suppose $X = X' \cup_H Y$ for $X' \in J$. If $X' \cup Y \in H$, then we are done. If $X' \cup Y \notin H$, then we reason as follows.

Since $X' \in {}_{\bigcup H-Y}H$, we have that $X' = W - Y$ for some $W \in H$. If $W - Y \neq W$, then $W \cap Y \neq \emptyset$, and $W \cup Y \in H$, because H is saturated; so $X' \cup Y = W \cup Y$, which contradicts the assumption that $X' \cup Y \notin H$. So we must have that $W - Y = W$, and hence $X' = W$. From $X' \cup Y \notin H$, we have that $X' \cup_H Y = X'$, and so $X \in H$. This proves that $K * J \subseteq H$.

We show next that every $K * J$-antichain misses H. Suppose not, and let S be a $K * J$-antichain such that $\bigcup S \in H$.

If $|{}_{\bigcup H-Y}S| = 0$, then S is a K-antichain that does not miss H_Y, which together with Proposition 6.11 from right to left contradicts our assumption that K is a construction of H_Y.

If $|{}_{\bigcup H-Y}S| \geqslant 2$, then ${}_{\bigcup H-Y}S$ is a J-antichain that does not miss ${}_{\bigcup H-Y}H$, because $\bigcup({}_{\bigcup H-Y}S) = \bigcup S \cap (\bigcup H - Y)$. To show that ${}_{\bigcup H-Y}S$ is a J-antichain it is sufficient to show that if $X_1 \subseteq X_2$, then $X_1 \cup_H Y \subseteq X_2 \cup_H Y$. This holds because it is impossible that $X_1 \cup_H Y = X_1 \cup Y$, while $X_2 \cup_H Y = X_2$; if $X_1 \cup Y \in H$, then $X_2 \cup Y \in H$, since $X_1 \subseteq X_2$ and H is saturated.

If $|{}_{\bigcup H-Y}S| = 1$, then S would not be a $K * J$-antichain for the following reason. There would be a unique member X of S of the form $X' \cup_H Y$ for $X' \in J$. We wish to show that $X' \cup_H Y = X' \cup Y$, and that will be the case when $X' \cup Y \in H$. We must have that $S - \{X\} \neq \emptyset$, because S must have at least two members. Since $\bigcup(S - \{X\})$ is a nonempty subset of Y, we must have that $(\bigcup(S - \{X\}) \cup X) \cap Y \neq \emptyset$ while $\bigcup(S - \{X\}) \cup X = \bigcup S \in H$ and $Y \in H$. We then obtain that $\bigcup(S - \{X\}) \cup X \cup Y = X \cup Y \in H$, because H is saturated. So $(X' \cup_H Y) \cup Y = X' \cup Y \in H$, and so $X' \cup_H Y = X' \cup Y$. Every member Z of $S - \{X\}$ is a subset of Y, and so $Z \subseteq X' \cup Y$. So S is not a $K * J$-antichain.

It remains to appeal to $|K * J| = |\bigcup H|$. This follows from the equations we mentioned before stating the proposition, together with $|K| = |Y|$ and $|J| = |\bigcup H - Y|$, which hold because K and J are constructions of H_Y and ${}_{\bigcup H-Y}H$ respectively. We conclude that $K * J$ is a construction by applying Proposition 6.11 from left to right. $\square$

With the assumptions stated before Proposition 7.1 we have the following lemma.

Lemma 7.6. *We have that $L_Y * {}_{\bigcup H-Y}L = L$.*

Proof. If $Y = \bigcup H$, then $L_Y = L$ and ${}_{\bigcup H-Y}L = \emptyset$. We then have that $L * \emptyset = L$.

Suppose $Y \subset \bigcup H$. To show that $L_Y * {}_{\bigcup H-Y}L \subseteq L$, suppose $X \in L_Y * {}_{\bigcup H-Y}(L - L_Y)$. If $X \in L_Y$, then $X \in L$. If, on the other hand, $X = X' \cup_H Y$ for some $X' \in {}_{\bigcup H-Y}L$, then $X' = X'' - Y \neq \emptyset$ for some $X'' \in L - L_Y$. So $X = (X'' - Y) \cup_H Y$, and, by Lemma 7.2, we obtain that $X = X''$. So $X \in L - L_Y$, and hence $X \in L$. Therefore we have established that $L_Y * {}_{\bigcup H-Y}L \subseteq L$.

For the converse inclusion, suppose $X \in L$. Then either $X \in L_Y$ or $X \in L - L_Y$. If $X \in L_Y$, then $X \in L_Y * \bigcup_{H-Y} L$. If $X \in L - L_Y$, then our purpose is to show that $X = X' \cup_H Y$ for some $X' \in \bigcup_{H-Y} L$, which means that $X' = X'' - Y \neq \emptyset$ for some $X'' \in L' - L_Y$. So it is enough to establish that $X = (X - Y) \cup_H Y$, which we have by Lemma 7.2. $\square$

With the assumptions stated before Proposition 7.5 we easily prove the following.

Lemma 7.7. *We have that* $(K * J)_Y = K$ *and* $\bigcup_{H-Y}(K * J) = J$.

For the second equation we rely on the equation $(X \cup_H Y) - Y = X$ for $X \subseteq \bigcup H - Y$.

We can then prove the following.

Proposition 7.8. *For a given construction L of the ASC-hypergraph H, and a given $Y \in L$, the constructions L_Y and $\bigcup_{H-Y} L$ are the unique constructions K and J of the ASC-hypergraphs H_Y and $\bigcup_{H-Y} H$ respectively such that $K * J = L$.*

Proof. We rely on Propositions 7.1 and 7.4, and on Lemma 7.6, to obtain that for the constructions $K = L_Y$ and $J = \bigcup_{H-Y} L$ of H_Y and $\bigcup_{H-Y} H$ respectively we have that $K * J = L$. For uniqueness suppose that $K * J = K' * J'$. Then, by relying on Lemma 7.7, we obtain that $K = K'$ and $J = J'$. $\square$

For every ASC-hypergraph H, every facet of the abstract polytope $\mathcal{A}(H)$ of H (see Section 5) is of the form $\{Y, \bigcup H\}$ for Y a member of $H - \{\bigcup H\}$, provided the rank of $\mathcal{A}(H)$ is at least 1. Take the section $\{Y, \bigcup H\}/F_{-1}$ of $\mathcal{A}(H)$, i.e. the set of all the faces of $\mathcal{A}(H)$ below, i.e. including, $\{Y, \bigcup H\}$ and above, i.e. included in, F_{-1}. The least face F_{-1} of $\mathcal{A}(H)$ is $\bar{H}^*$, but here $\bar{H} = H$, since H is saturated.

The vertices of $\mathcal{A}(H)$ in $\{Y, \bigcup H\}/F_{-1}$ are all the constructions of H in which Y is a member. These are the vertices in which the facet $\{Y, \bigcup H\}$ is included, i.e. the vertices incident with this facet. Each such vertex L is equal to $L_Y * \bigcup_{H-Y} L$, and, according to Proposition 7.8, this is the only way to represent L as a continuation of constructions of H_Y and $\bigcup_{H-Y} H$.

The faces of $\mathcal{A}(H)$ in $\{Y, \bigcup H\}/F_{-1}$ are all the constructs of H in which Y is a member, together with $\bar{H}^*$ as an additional face. These are the faces in which the facet $\{Y, \bigcup H\}$ is included, i.e. the faces incident with this facet.

The abstract polytope $\{Y, \bigcup H\}/F_{-1}$ is isomorphic to $\mathcal{A}(H_Y) \otimes \mathcal{A}(\bigcup_{H-Y} H)$ (see Section 5 for the product $\otimes$). We may take $\{Y, \bigcup H\}/F_{-1}$ of $\mathcal{A}(H)$ as being the result of a partial operation $*$ applied to $\mathcal{A}(H_Y)$ and $\mathcal{A}(\bigcup_{H-Y} H)$, akin to $\otimes$, but different from it.

We define $\mathcal{A}(H_Y) * \mathcal{A}(\bigcup_{H-Y} H)$ as the set of all the constructs of H in which Y is a member, together with $\bar{H}^*$ as an additional element. The reason for writing this operation $*$ is that every construction L in $\mathcal{A}(H_Y) * \mathcal{A}(\bigcup_{H-Y} H)$ is of the form $L_Y * \bigcup_{H-Y} L$ for L_Y in $\mathcal{A}(H_Y)$ and $\bigcup_{H-Y} L$ in $\mathcal{A}(\bigcup_{H-Y} H)$; this continuation $*$ operation on constructions induces analogous continuation $*$ operations for all the other constructs in $\mathcal{A}(H_Y) * \mathcal{A}(\bigcup_{H-Y} H)$. Each of these constructs C is the result of applying a $*$ to a construct C_1 in $\mathcal{A}(H_Y)$ and a construct C_2 in $\mathcal{A}(\bigcup_{H-Y} H)$. The presence of Y in C guarantees the existence of C_1, and the presence of $\bigcup H$ in C guarantees the existence of C_2, in which we have $\bigcup H - Y$. (The set Y, as a member of H, is nonempty, and since it is a proper subset of $\bigcup H$, we have that $\bigcup H - Y$ is nonempty too.)

For an ASC-hypergraph H with $\mathcal{A}(H)$ of rank $r \geqslant 1$, we can construct $\mathcal{A}(H)$ inductively in terms of abstract polytopes of lower rank in the following manner. For every member Y of $H - \{\bigcup H\}$, take the abstract polytopes $\mathcal{A}(H_Y)$ and $\mathcal{A}(\bigcup_{H-Y} H)$, which are of rank lower than r; the rank of $\mathcal{A}(H_Y)$ is $|Y| - 1$, and the rank of $\mathcal{A}(\bigcup_{H-Y} H)$ is $|\bigcup H - Y| - 1$. Then take the union of all the abstract polytopes $\mathcal{A}(H_Y) * \mathcal{A}(\bigcup_{H-Y} H)$ (note that they all have the same least face F_{-1}, which is $\bar{H}^*$), and add as a new face $\{\bigcup H\}$ as the greatest face. This is $\mathcal{A}(H)$.

The basis of this induction is given by the abstract polytopes $\mathcal{A}(H) = \{\{x\}, \{x, *\}\}$ of rank 0, where H is $\{\{x\}\}$. Note that (as we said at the end of Section 5) if $H = \emptyset$, then $\mathcal{A}(H) = \{\emptyset, \{*\}\}$, which is also of rank 0, but is not needed for the basis of our induction, because $\emptyset$ cannot be a member of a hypergraph.

For every atomic hypergraph H, we can obtain $\mathcal{A}(H)$ as $\mathcal{A}(H_1) \otimes \cdots \otimes \mathcal{A}(H_n)$, for $n \geqslant 1$, and $\{H_1, \ldots, H_n\}$ being the finest hypergraph partition of the saturated closure of H. (If $n = 1$, then $\mathcal{A}(H_1) \otimes \cdots \otimes \mathcal{A}(H_n)$ is, of course, just $\mathcal{A}(H_1)$.) The hypergraphs $H_1, \ldots, H_n$ are ASC-hypergraphs.

So with the help of the operation $\otimes$, and of the related operation $*$, on hypergraphs we can define inductively the abstract polytope, i.e. the hypergraph, $\mathcal{A}(H)$ for an arbitrary hypergraph H. If $n > 1$, then for every $i \in \{1, \ldots, n\}$, the rank of $\mathcal{A}(H_i)$ is strictly smaller than the rank of $\mathcal{A}(H)$. We deal with this matter in more detail in the next section.

8. Abstract polytopes of hypergraphs are abstract polytopes

In this section we verify that for an arbitrary atomic hypergraph H we have that the abstract polytope $\mathcal{A}(H)$ is indeed an abstract polytope of rank $|\bigcup H| - n$, where n is the connectedness number of H (see the beginning of Section 5). This result follows from the geometric representation of $\mathcal{A}(H)$ as a convex polytope in Euclidean space, with which we deal in Section 9, but, since that section is rather concise, we prefer to give an independent direct proof in the present section, and make our treatment of the abstract polytopes $\mathcal{A}(H)$ self-contained.

Our general notion of abstract polytope is as in [25, Section 2A]. The definition given there, which we summarize now, says that an abstract polytope of rank r, for $r \geqslant -1$, is a partially ordered set $\langle P, \leqslant\rangle$, with *carrier* P, that satisfies four properties (P1), ..., (P4).

Property (P1) is that P has a least and a greatest *face*, i.e. element, denoted by F_{-1} and F_r (they need not be distinct).

Property (P2) is that each *flag* of P, i.e. maximal linearly ordered subset of P, contains exactly $r+2$ faces including F_{-1} and F_r. A partially ordered set P that satisfies (P1) and (P2) is said to be a partially ordered set of rank r.

Both (P1) and (P2) are easily verified for $\mathcal{A}(H)$. The verification of the remaining two properties (P3) and (P4) is a more difficult task, for which we will first reformulate the properties within an inductive definition of abstract polytope of rank r, the induction being based on r. Let us first state the remaining properties as they are formulated in [25, Section 2A].

For property (P3) we need some preliminary notions. A partially ordered set P of rank r is said to be *connected* when either $r \leqslant 1$, or $r \geqslant 2$ and for any two *proper* faces F and G of P, i.e. faces distinct from F_{-1} and F_r, there exists a finite sequence of consecutively incident proper faces connecting F with G, i.e. a sequence, starting with F and terminating with G such that every two consecutive faces in the sequence are *incident*, i.e. either in the relation $\leqslant$ or in the converse relation (either $C_i \leqslant C_{i+1}$ or $C_{i+1} \leqslant C_i$).

For F and G two faces of P such that $F \leqslant G$, the *section* G/F of P is the set of faces H of P such that $F \leqslant H \leqslant G$. Note that P itself is a section of P, and note that each section of P is a partially ordered set of rank $r' \leqslant r$.

We can then state the next property:

(P3) P is *strongly connected*, which means that every section of P is a connected partially ordered set of rank $r' \leqslant r$.

The *rank* r' *of a face* F of P is the rank of the section F/F_{-1}, understood as a partially ordered set of rank $r' \leqslant r$. We can then state the last defining property of abstract polytopes:

(P4) For every $i \in \{0, \dots, r-1\}$, if for F and G of ranks $i-1$ and $i+1$ respectively we have that $F \leqslant G$, then there are exactly two faces H_1 and H_2 of rank i such that $F \leqslant H_1 \leqslant G$ and $F \leqslant H_2 \leqslant G$.

As we announced above, we will not check properties (P3) and (P4) for $\mathcal{A}(H)$ directly, but note that verifying (P4) when $i > 0$ is quite easy. This is because then $G - F = \{a_1, a_2\}$ with $a_1 \neq a_2$, and we have that $H_1 = F \cup \{a_1\}$ and $H_2 = F \cup \{a_2\}$.

When sometimes, for the sake of brevity, we say in the remainder of this section just polytope, we mean abstract polytope. We state now the clauses and notions we need for our inductive definition of an abstract polytope of rank $r \geqslant -1$:

(-1) $P_{-1} = \langle \{F_{-1}\}, \leqslant_{P_{-1}}\rangle$, where $\leqslant_{P_{-1}} = \{(F_{-1}, F_{-1})\}$, is a polytope of rank -1, with the unique face F_{-1} of P_{-1} being of rank -1.

(0) For any object $a \neq F_{-1}$, we have that $P_0^a = \langle \{F_{-1}, a\}, \leqslant_{P_0^a}\rangle$, where $\leqslant_{P_0^a} = \{(F_{-1}, F_{-1}), (F_{-1}, a), (a, a)\}$, is a polytope of rank 0, with the faces F_{-1} and a being respectively of rank -1 and 0.

For a polytope of rank $r \geqslant 0$, its faces of rank $r-1$ are its *facets*. (So P_{-1} has no facets.)

(P3′) Two distinct polytopes of the same rank are *close neighbors* when they have a common facet. A set of polytopes all of the same rank is *closely connected* when each pair of distinct polytopes in it is connected by a finite sequence of consecutively close neighbors (in other words, they are connected by the transitive closure of the close neighbors relation).[2]

(P4′) A set of polytopes is *bivalent* when every facet of a polytope in it belongs to exactly two polytopes in this set.

Our inductive definition of an abstract polytope has clause (-1) in the basis, and it has the following inductive clause:

If S is a closely connected bivalent set of polytopes of rank r, let $P = \langle \bigcup S \cup \{F^P_{r+1}\}, \leqslant_P\rangle$, where

$$\leqslant_P = \{(x, y) \mid (\exists Q \in S)\, x \leqslant_Q y \text{ or } y = F^P_{r+1}\},$$

be a polytope of rank $r+1$; the face F^P_{r+1}, which is not a face of any polytope in S, is the unique face of P of rank $r+1$, and the remaining faces have in P the rank they had in the polytopes of S.

This concludes the definition.

In P the greatest faces of the polytopes of S become facets, and the facets of these polytopes become *ridges*, i.e. faces of the rank of the polytope minus 2. Clause (0) is obtained by applying our inductive clause to clause (-1) in a trivial manner.

[2] That the two polytopes that are close neighbors have a common facet means that they have actually a common element, which is a facet in each of them. We are indebted to an anonymous referee for a remark in which (P3′) is generalized to all ranks $k \leqslant r-1$. Consider the incidence complex (see, for example, [28]) of rank $k+1$ obtained by removing all the faces of rank greater than k of an abstract polytope P and adding a dummy face of rank $k+1$. Then (P3′) is related to the connectedness of the edge graph of the dual of this incidence complex.

(There are no two distinct polytopes in $S=\{P_{-1}\}$, and there are no facets in P_{-1}.) We have however stated (0) for the sake of clarity.

We have indeed obtained in this manner clause (0), because it is possible that two different polytopes P_1 and P_2 of rank $r+1$ arise out of the same set S of polytopes of rank r. The faces $F^{P_1}_{r+1}$ and $F^{P_2}_{r+1}$ are then different objects.

To prevent confusion, we should understand our inductive clause in such a manner that if the sets S_1 and S_2 of polytopes of rank r are different, then the faces $F^{P_1}_{r+1}$ and $F^{P_2}_{r+1}$ of the polytopes P_1 and P_2 of rank $r+1$ with the carriers $S_1\cup\{F^{P_1}_{r+1}\}$ and $S_2\cup\{F^{P_2}_{r+1}\}$ must be different.

All our abstract polytopes have the same least face F_{-1}, but this is not an essential matter.

Our task now is to verify that our inductive definition is equivalent with the definition in terms of (P1), ..., (P4). (An equivalent inductive definition, different from ours, is mentioned in [25, Section 2A].)

It is very easy to see that every polytope of rank r defined by our inductive definition is a partial order of rank r, i.e. that it satisfies clauses (P1) and (P2). To show that it satisfies (P3), we proceed by induction on r.

If $r=-1$, then the clause is satisfied trivially. As a matter of fact, it is satisfied trivially, by the definition of connectedness, for every $r\leqslant 1$.

If $r>1$, then we have in P old sections, which occur in a polytope of the set S used for defining P inductively, which are connected as before, and new sections F^P_r/H. For F and G two proper faces in this new section we find the facets F' and G' of P with which they are respectively incident. We find that F' and G' are connected in P by applying (P3′). (The facets of P are the greatest elements of the polytopes in S.) So F and G are connected in P.

To show that our inductively defined polytopes of rank r satisfy (P4) we proceed again by induction. The cases where $r\leqslant 0$ are trivial (then $\{0,\dots,r-1\}$ is the empty set). For $r=1$ we have that all our polytopes have the following Hasse diagram:

$$\begin{array}{ccc} & F^P_1 & \\ a & & b \\ & F_{-1} & \end{array}$$

(The facet of P^a_0 and P^b_0 is F_{-1}.)

For $r>1$, we have as in the paragraph above old sections G/F of P, which are taken care of by the induction hypothesis, and new sections. The new sections are of the form F^P_r/F, and then for them we have by (P4′) that exactly two facets $F^{Q_1}_{r-1}$ and $F^{Q_2}_{r-1}$ of P are incident with a face G of rank $r-2$ (which is an old facet).

With that we have finished verifying that the inductively defined polytopes are abstract polytopes according to the old definition, in terms of (P1), ..., (P4). For the converse we have the following.

Note first that the old definition allows for our polytope P_{-1} and for another polytope P'_{-1} isomorphic to P_{-1}, which is in all respects like P_{-1} save that F_{-1} of P_{-1} is replaced by an object F'_{-1} different from F_{-1}. With our clause (-1) we have provided only for a single polytope P_{-1}. Two courses are now open to us. We may either consider that P_{-1} and P'_{-1}, since they are isomorphic, are in fact the same, or we may consider that F_{-1} in (-1) is in fact a variable. We will follow the first course, but this is not an essential matter.

To verify *close connectedness*, which corresponds to the property defined in (P3′), for a polytope P of rank $r+1$ defined in the old way means verifying that for two distinct facets (i.e. faces of rank r) F' and F'' of P there is a finite sequence of *close neighbors* connecting F' with F''; two close neighbors being now two facets incident with a common ridge of P, i.e. face of rank $r-1$. We will show that close connectedness is a consequence of essentially (P3).

Let a *proper path* of P from F' to F'' be, as in the definition of connectedness above, a finite sequence of consecutively incident proper faces of P connecting F' with F''.

For k_{l-2} being the number of faces of rank $r-l$, for $l\geqslant 2$, in a proper path Π let the *weight* of Π be the ordinal

$$k_0\omega^0+\dots+k_n\omega^n,\quad \text{for } n=r-2.$$

If $k_0=\dots=k_n=0$, then the weight of Π is 0, and in Π we have only facets and ridges, as required by close connectedness. The weight is an ordinal less than ω^ω, which is infinite if one of $k_1,\dots,k_n$ is greater than 0. We can prove the following.

Lemma 8.1. *For every proper path Π from F' to F'' of weight w greater than* 0, *there is a proper path Π' from F' to F'' of weight strictly less than w.*

Proof. Let Π be the sequence $F_1\dots F_m$, with F_1 being F' and F_m being F''; since $w>0$, we must have that $m\geqslant 5$. Let $\rho(F_i)$ be the rank of F_i in P, and consider a subsequence $F_i\dots F_{i+2}$ of Π such that $\rho(F_i)=\rho(F_{i+1})+1=\rho(F_{i+2})\leqslant r-1$. Such a subsequence must exist because $w>0$. For G being the greatest element of P consider the section G/F_{i+1}. Then we have that either $F_i=F_{i+2}$, or for $F_i\neq F_{i+2}$ by (P3) there is a proper path $F_iG_1\dots G_qF_{i+2}$, for $q\geqslant 1$, of the polytope G/F_{i+1}.

For Γ being either empty or $G_1\dots G_qF_{i+2}$, let Π' be $F_1\dots F_i\Gamma F_{i+3}\dots F_m$. The weight of Π' is less than w. $\Box$

From this lemma we can infer close connectedness by induction.

Suppose for P of rank r a face F of rank $r-2$ does not respect bivalence, which means that it is not incident with exactly two faces of P of rank $r-1$, i.e. two facets of P. It is clear that the section F_r^P/F does not then respects (P4).

With that we have finished verifying that the old definition and the new inductive definition are equivalent. We will verify below that $\mathcal{A}(H)$ is an abstract polytope by relying on the new inductive definition. For that verification we need some preliminary matters.

For P_1 and P_2 abstract polytopes, consider the partial order $P_1 \cdot P_2$ with carrier $((P_1 - \{F_{-1}\}) \times (P_2 - \{F_{-1}\})) \cup \{F_{-1}\}$, where $(F_1, F_2) \leqslant_{P \cdot Q} (G_1, G_2)$ iff $F_1 \leqslant_{P_1} G_1$ and $F_2 \leqslant_{P_2} G_2$, and moreover F_{-1} is the least element with respect to $\leqslant_{P_1 \cdot P_2}$. Note that $\mathcal{A}(H_1) \otimes \mathcal{A}(H_2)$, $\mathcal{A}(H_1) * \mathcal{A}(H_2)$ and $\mathcal{A}(H_1) \cdot \mathcal{A}(H_2)$ are all isomorphic, provided the first two products are defined (for $\otimes$ we must have $\bigcup H_1$ and $\bigcup H_2$ disjoint, and for $*$, the hypergraphs H_1 and H_2 should be H_Y and ${}_{\bigcup H - Y} H$ for $Y \in H$).

Then we can prove the following.

Proposition 8.2. *For P_1 and P_2 abstract polytopes of ranks r_1 and r_2 respectively, $P_1 \cdot P_2$ is an abstract polytope of rank $r_1 + r_2$.*

Proof. With the old definition of an abstract polytope, it is easy to check (P1) and (P2). For (P3) we make our connection via $\leqslant_{P \cdot Q}$ by moving first in one coordinate and then in the other. To check (P4) suppose that, for $\rho(F)$ being the rank of F in the appropriate partial order, we have that $(F_1, F_2) \leqslant_{P \cdot Q} (G_1, G_2)$ and $\rho(G_1, G_2) - \rho(F_1, F_2) = 2$. We have the last equation only if one of the following cases obtains:

(1) $\rho(G_1) - \rho(F_1) = 2$, $\rho(G_2) = \rho(F_2)$,
(2) $\rho(G_1) = \rho(F_1)$, $\rho(G_2) - \rho(F_2) = 2$,
(3) $\rho(G_1) - \rho(F_1) = 1$, $\rho(G_2) - \rho(F_2) = 1$.

In cases (1) and (2) we just rely on (P4) for P_1 and P_2 respectively. In case (3) we have (F_1, G_2) for H_1 and (F_2, G_1) for H_2. That the rank of $P_1 \cdot P_2$ is the sum of the ranks r_1 and r_2 of P_1 and P_2 is clear from the fact that we go up now in two coordinates, r_1 steps in the first coordinate, and r_2 steps in the second. □

We can then prove the following.

Theorem 8.3. *For every atomic hypergraph H we have that $\mathcal{A}(H)$ is an abstract polytope of rank $|\bigcup H| - n$, for n the connectedness number of H.*

Proof. We rely on the inductive definition of $\mathcal{A}(H)$ introduced at the end of Section 7. In the basis of our induction, it is clear that $\{\{x\}, \{x, *\}\}$ is an abstract polytope of rank 0.

Suppose for the induction step that, for an ASC-hypergraph H and for $Y \in H - \{\bigcup H\}$, we have that $\mathcal{A}(H_Y)$ and $\mathcal{A}({}_{\bigcup H - Y} H)$ are abstract polytopes of ranks $|Y| - 1$ and $|\bigcup H - Y| - 1$ respectively. To have both of these ranks at least 0, we must have that $|\bigcup H| \geqslant 2$. Then we have that $\mathcal{A}(H_Y) * \mathcal{A}({}_{\bigcup H - Y} H)$, which is isomorphic to $\mathcal{A}(H_Y) \cdot \mathcal{A}({}_{\bigcup H - Y} H)$, is an abstract polytope of rank $|\bigcup H| - 2$, by Proposition 8.2.

Let S be the set $\{\mathcal{A}(H_Y) * \mathcal{A}({}_{\bigcup H - Y} H) \mid Y \in H - \{\bigcup H\}\}$. We show that this set is closely connected. Take the polytopes $P_Y = \mathcal{A}(H_Y) * \mathcal{A}({}_{\bigcup H - Y} H)$ and $P_Z = \mathcal{A}(H_Z) * \mathcal{A}({}_{\bigcup H - Z} H)$ from S. Suppose that $Y \cup Z = \bigcup H$.

If $|\bigcup H| = 2$, then $Y = \{y\}$ and $Z = \{z\}$. In that case P_Y and P_Z are the only polytopes in S. They are closely connected because they have F_{-1} as a common facet, and S is bivalent.

If $|\bigcup H| > 2$, then either $|Y| \geqslant 2$ or $|Z| \geqslant 2$. Suppose $|Y| \geqslant 2$ (in case $|Z| \geqslant 2$ we proceed analogously). Then $Y \nsubseteq Z$, since Y and Z are both proper subsets of $\bigcup H$, and $Y \cup Z = \bigcup H$. So for some $y \in Y$ we have that $y \notin Z$. We have that $\{y\} \in H$, since H is atomic, and $y \in \bigcup H$, and so $P_{\{y\}} = \mathcal{A}(H_{\{y\}}) * \mathcal{A}({}_{\bigcup H - \{y\}} H)$ is a polytope in S. The polytopes P_Y and $P_{\{y\}}$ have as a common facet $\{\{y\}, Y, \bigcup H\}$; the polytopes $P_{\{y\}}$ and P_Z have as a common facet $\{\{y\}, Z, \bigcup H\}$ if $Z \cup \{y\} \notin H$, and they have as a common facet $\{Z, Z \cup \{y\}, \bigcup H\}$ if $Z \cup \{y\} \in H$. From that we may conclude that S is closely connected and bivalent.

Suppose that $Y \cup Z \subset \bigcup H$; in that case $|\bigcup H| > 2$. The following cases may arise.

(1) If $Y \subseteq Z$ or $Z \subseteq Y$, then P_Y and P_Z have as a common facet $\{Y, Z, \bigcup H\}$.
(2) Suppose neither $Y \subseteq Z$ nor $Z \subseteq Y$.
    (2.1) If $Y \cup Z \in H$, then since $Y \cup Z \subset \bigcup H$, we have $P_{Y \cup Z} = \mathcal{A}(H_{Y \cup Z}) * \mathcal{A}({}_{\bigcup H - (Y \cup Z)} H)$ as another polytope in S. The polytopes P_Y and $P_{Y \cup Z}$ have as a common facet $\{Y, Y \cup Z, \bigcup H\}$, and $P_{Y \cup Z}$ and P_Z have as a common facet $\{Z, Y \cup Z, \bigcup H\}$.
    (2.2) If $Y \cup Z \notin H$ and $Y \cap Z = \emptyset$, then, as for (1), the facet $\{Y, Z, \bigcup H\}$ is common to P_Y and P_Z.

Note that because of the saturation of H it is impossible that $Y \cup Z \notin H$ and $Y \cap Z \neq \emptyset$. From all that we conclude that S is closely connected. That S is bivalent follows from the fact that, if $|\bigcup H| > 2$, every facet of a polytope in S is of the form $\{Y, Z, \bigcup H\}$ for two polytopes P_Y and P_Z as above.

We add to S the new face $\{\bigcup H\}$ and obtain the polytope $\mathcal{A}(H)$ of rank $|\bigcup H|-1$, in accordance with the inductive clause of the new definition of an abstract polytope.

We have said at the end of Section 7 that for every atomic hypergraph H we can obtain $\mathcal{A}(H)$ as $\mathcal{A}(H_1)\otimes\cdots\otimes\mathcal{A}(H_n)$, for $n\geqslant 1$, where $\{H_1,\dots,H_n\}$ is the finest hypergraph partition of the saturated closure of H. The hypergraphs $H_1,\dots,H_n$ are ASC-hypergraphs, and so $\mathcal{A}(H_1),\dots,\mathcal{A}(H_n)$ are, according to the proof above, abstract polytopes of ranks $|\bigcup H_1|-1,\dots,|\bigcup H_n|-1$ respectively. That $\mathcal{A}(H_1)\otimes\cdots\otimes\mathcal{A}(H_n)$ is an abstract polytope of rank $|\bigcup H|-n$ follows then by Proposition 8.2, and by the isomorphism of the products $\cdot$ and $\otimes$. $\Box$

The proof we have just given shows that the facets of the abstract polytope $\mathcal{A}(H)$ for H atomic and connected are never very far from each other. They either share a common ridge, or there is in between them a facet with which each of them shares a ridge.[3]

9. Realizations

In this section we turn towards a geometrical approach to hypergraph polytopes. Our main concern in this work was the abstract approach of the preceding sections, and so in this section, which is in a related, but nevertheless different field, our exposition will be less detailed. We will strive to be concise in order not to make an already long text still longer. We will give no examples in this section; they can be worked out from Appendix B. We presume the reader is acquainted with some basic notions of the geometrical theory of polytopes, which may all be found in [35], whose terminology we will follow.

We define for every atomic hypergraph H a convex polytope in $\mathbf{R}^n$, for whose face lattice we will prove that it is isomorphic (as a partial order) to $\mathcal{A}(H)$. Most of the section is devoted to proving that for ASC-hypergraphs. The proof for the remaining atomic hypergraphs will then follow easily.

Let H be an ASC-hypergraph on the carrier $\bigcup H=\{1,\dots,d+1\}$ for $d\geqslant 0$. We take now the elements of $\bigcup H$ to be positive integers, because we want them to function as indices, but we may, however, take $\bigcup H$ to be an arbitrary finite nonempty set. We deal separately below with the case of the empty hypergraph $\emptyset$ on the carrier $\emptyset$.

For $\mathcal{S}$ being the set of hyperplanes in $\mathbf{R}^{d+1}$, let the map $\pi: H\to\mathcal{S}$ be defined by

$$\pi_X=\left\{(x_1,\dots,x_{d+1})\in\mathbf{R}^{d+1}\mid\sum_{i\in X}x_i=3^{|X|}\right\},$$

where π_X stands for $\pi(X)$. Note that π is injective. The function $f(x)=3^x$ in the definition of π_X is not the only one that could be chosen. Any function on natural numbers that would enable us to prove an analogue of Lemma 9.1 below would do. The function $f(x)=3^x$ is one of the "suitable" functions introduced in [30, Appendix B]. Intuitively, the choice of 3^x may be explained by the wish not to truncate too much. In a very simple case, this means that after truncating a one-dimensional edge at both ends, we are left with something; hence we divide the edge in three parts.

Consider the closed halfspace

$$\pi_X^+=\left\{(x_1,\dots,x_{d+1})\in\mathbf{R}^{d+1}\mid\sum_{i\in X}x_i\geqslant 3^{|X|}\right\},$$

whose boundary hyperplane is π_X. Let $\mathcal{G}(H)$ be the polytope

$$\bigcap\left\{\pi_X^+\mid X\in H-\left\{\bigcup H\right\}\right\}\cap\pi_{\bigcup H}$$

in the hyperplane $\pi_{\bigcup H}$.

That $\mathcal{G}(H)$ is indeed an $\mathcal{H}$-polytope, and not just an $\mathcal{H}$-polyhedron (see [35, Lecture 0, Definition 0.1]), is guaranteed by the atomicity of H. The set $\mathcal{G}(H)$ is bounded by the d-dimensional simplex $\bigcap\{\pi_{\{i\}}^+\mid i\in\bigcup H\}\cap\pi_{\bigcup H}$. Intuitively, we may assume that the polytope $\mathcal{G}(H)$ is obtained by truncating this simplex, which is a limit case with no truncation; in that case, H is just $\{\{i\}\mid i\in\bigcup H\}\cup\{\bigcup H\}$. The limit case at the other end, with all possible truncations, is with H being the set of all nonempty subsets of $\bigcup H$; in that case we obtain the d-dimensional permutohedron. This permutohedron is contained in $\mathcal{G}(H)$, which then proves that the dimension of $\mathcal{G}(H)$ is actually d, and not lower.

In the case of the empty hypergraph $\emptyset$ we define $\mathcal{G}(\emptyset)$ to be the polytope $\mathbf{R}^0=\{*\}$ in $\mathbf{R}^0$, whose face lattice is $\{\{*\},\emptyset\}$, with $\leqslant$ being $\subseteq$. This face lattice is isomorphic to $\mathcal{A}(\emptyset)=\{\emptyset,\{*\}\}$ (see the end of Section 5) by the bijection that assigns the vertex $\emptyset$ of $\mathcal{A}(\emptyset)$ to the vertex $\{*\}$ of $\mathcal{G}(\emptyset)$, and $\{*\}$, which is the face F_{-1} in $\mathcal{A}(\emptyset)$, to $\emptyset$, which is the face F_{-1} for $\mathcal{G}(\emptyset)$; the partial order $\leqslant$ in $\mathcal{A}(\emptyset)$ is the converse of $\subseteq$.

To make $\mathcal{G}(\emptyset)$ a limit case of the definition given above, take first $\{1,\dots,d+1\}$ to be the empty set when $d=-1$. In that case, we have $\mathbf{R}^{d+1}=\mathbf{R}^0=\{*\}$, and $\bigcap\{\pi_X^+\mid X\in\emptyset\}=\{*\}$. The face F_{-1} of any abstract polytope should be mapped to the empty subset of $\mathbf{R}^n$ (the empty set is a face of any geometrical polytope; see [35, Lectures 0 and 2], and [25, Section 5a]).

3 In other words, as an anonymous referee suggested, the dual polytope of $\mathcal{A}(H)$ has diameter 2; i.e. its edge graph has diameter 2.

So $\mathcal{G}(\emptyset)$ has one vertex $\{*\}$, and the face $\emptyset$, as the image of F_{-1}. We may extend π and π^+ to $\bar{H}^* = H \cup \{*\}$ so that $\pi_* = \pi_*^+ = \emptyset$; then $\bigcap\{\pi_X^+ \mid X \in \bar{H}^* - \{\bigcup H\}\} = \emptyset$.

We have however separated the case $\mathcal{G}(\emptyset)$ from the rest because it is degenerate; as $\mathcal{A}(\emptyset)$, the polytope $\mathcal{G}(\emptyset)$ has no important role to play. (Note that in the inductive definition of $\mathcal{A}(H)$, at the end of Section 7, we do not have $\mathcal{A}(H)$ in the basis.)

We will prove that $\mathcal{G}(H)$ is a polytope each of whose vertices lies in exactly $d+1$ boundary hyperplanes; these $d+1$ hyperplanes are the elements of $\{\pi_X \mid X \in K\}$ for some construction K of H (for a precise statement see Proposition 9.3). This will imply that $\mathcal{G}(H)$ is a simple polytope (which means that each of its vertex figures, which are figures obtained by truncating a vertex, is a simplex; see [35, Section 2.5, Proposition 2.16]). On the other hand, we will prove that for every construction K of H the intersection of the hyperplanes in $\{\pi_X \mid X \in K\}$ is a vertex of $\mathcal{G}(H)$ (see Proposition 9.6). From all that we will conclude that $\mathcal{G}(H)$ has a face lattice isomorphic to $\mathcal{A}(H)$.

The set $\mathcal{P}^+(\bigcup H)$ of all the nonempty subsets of $\bigcup H$ is an ASC-hypergraph on $\bigcup H$. The set $\mathcal{P}^+(\bigcup H)$ is a subset of itself, and we can consider all the $\mathcal{P}^+(\bigcup H)$-antichains, in accordance with our definition of an M-antichain in Section 6. We prove the following.

Lemma 9.1. *Let $\{X_1, \dots, X_n\}$ be a $\mathcal{P}^+(\bigcup H)$-antichain. If for every $i \in X_1 \cup \dots \cup X_n$ we have that $x_i \geqslant 0$, and for every $j \in \{1, \dots, n\}$ we have that $\sum_{i \in X_j} x_i \leqslant 3^{|X_j|}$, then $\sum_{i \in X_1 \cup \dots \cup X_n} x_i < 3^{|X_1 \cup \dots \cup X_n|}$.*

Proof. We proceed by induction on $n \geqslant 2$. If $n = 2$, then

$$\begin{aligned}
\sum_{i \in X_1 \cup X_2} x_i &\leqslant \sum_{i \in X_1} x_i + \sum_{i \in X_2} x_i, \quad \text{since } x_i \geqslant 0 \\
&\leqslant 3^{|X_1|} + 3^{|X_2|}, \quad \text{by the assumption} \\
&< 3^{\max(|X_1|, |X_2|)+1} \\
&\leqslant 3^{|X_1 \cup X_2|}, \quad \text{since } X_1, X_2 \subset X_1 \cup X_2.
\end{aligned}$$

If $n > 2$, then let $X = X_1 \cup \dots \cup X_{n-1}$. By the induction hypothesis, we have that $\sum_{i \in X} x_i < 3^{|X|}$. If $X_n \subseteq X$, i.e. $X_1 \cup \dots \cup X_n = X$, then we are done. If $X_n \nsubseteq X$, then, since $\{X_1, \dots, X_n\}$ is a $\mathcal{P}^+(\bigcup H)$-antichain, we cannot have that $X \subseteq X_n$; hence $\{X, X_n\}$ is a $\mathcal{P}^+(\bigcup H)$-antichain, and we may apply the induction hypothesis to it. □

The main arithmetical idea of this lemma is based on the following. If for every $i \in \{1, \dots, k\}$ we have that $m_i < m$, then we have the inequality

$$\sum_{i=1}^{k} (k+1)^{m_i} < (k+1)^m,$$

which yields as a particular case:

$$\text{if } m_1 < m \text{ and } m_2 < m, \quad \text{then } 3^{m_1} + 3^{m_2} < 3^m.$$

The idea of this inequality may be gathered from [18] (see also [9]).

Lemma 9.2. *For every $M \subseteq H$, if $\bigcap\{\pi_X \mid X \in M\} \cap \mathcal{G}(H) \neq \emptyset$, then every M-antichain misses H.*

Proof. Suppose $\{X_1, \dots, X_n\}$ is an M-antichain such that $X_1 \cup \dots \cup X_n \in H$, and let

$$(x_1, \dots, x_{d+1}) \in \bigcap\{\pi_X \mid X \in M\} \cap \mathcal{G}(H).$$

Since H is atomic, we have that

$$(x_1, \dots, x_{d+1}) \in \mathcal{G}(H) \subseteq \bigcap\big\{\pi^+_{\{i\}} \bigm| i \in \{1, \dots, d+1\}\big\},$$

and hence $x_1, \dots, x_{d+1} \geqslant 0$. Since

$$(x_1, \dots, x_{d+1}) \in \bigcap\big\{\pi_{X_j} \bigm| j \in \{1, \dots, n\}\big\},$$

we have for every $j \in \{1, \dots, n\}$ that $\sum_{i \in X_j} x_i = 3^{|X_j|} \leqslant 3^{|X_j|}$. Then by Lemma 9.1 we have that $\sum_{i \in X_1 \cup \dots \cup X_n} x_i < 3^{|X_1 \cup \dots \cup X_n|}$, and hence $(x_1, \dots, x_{d+1}) \notin \pi^+_{X_1 \cup \dots \cup X_n}$ (we have that $\pi^+_{X_1 \cup \dots \cup X_n}$ is defined since $X_1 \cup \dots \cup X_n \in H$). From $\mathcal{G}(H) \subseteq \pi^+_{X_1 \cup \dots \cup X_n}$ we conclude that $(x_1, \dots, x_{d+1}) \notin \mathcal{G}(H)$, which is a contradiction. □

The following proposition shows that $\mathcal{G}(H)$ is a polytope whose vertices correspond to constructions of H.

Proposition 9.3. *For every vertex $\{v\}$ of $\mathcal{G}(H)$ there is a construction K of H such that $\{v\} = \bigcap\{\pi_X \mid X \in K\}$ and for every Y in $H - K$ we have that $v \notin \pi_Y$.*

Proof. Since $\{v\}$ is a vertex of $\mathcal{G}(H)$, there are at least d members $X_1, \dots X_d$ of $H - \{\bigcup H\}$ such that

$$\pi_{X_1} \cap \dots \cap \pi_{X_d} \cap \pi_{\bigcup H} = \{v\} \subseteq \mathcal{G}(H).$$

Let $K = \{X_1, \dots, X_d, \bigcup H\}$. By Lemma 9.2, every K-antichain misses H, and so, by Proposition 6.11, we have that K is a construction of H. It remains to establish that if $Y \in H - K$, then $v \notin \pi_Y$.

Let $Y \in H - K$. By Lemma 6.9, for $M = K \cup \{Y\}$ there is an M-antichain that does not miss H. By Lemma 9.2, we have that $\bigcap\{\pi_X \mid X \in M\} \cap \mathcal{G}(H) = \emptyset$, and hence $v \notin \pi_Y$ since $v \in \bigcap\{\pi_X \mid X \in K\} \cap \mathcal{G}(H)$. □

From this proposition it is easy to derive the following two corollaries.

Corollary 9.3.1. *For every vertex $\{v\}$ of $\mathcal{G}(H)$ there is a unique construction K of H such that $\{v\} = \bigcap\{\pi_X \mid X \in K\}$.*

Corollary 9.3.2. *For every vertex $\{v\}$ of $\mathcal{G}(H)$ there are exactly d halfspaces π_X^+ such that $X \in H - \{\bigcup H\}$ and $v \in \pi_X$.*

For Corollary 9.3.2 we rely on the fact that every construction of H has $d+1$ members, one of which is $\bigcup H$, and that $\pi_{X_1} = \pi_{X_2}$ implies $\pi_{X_1}^+ = \pi_{X_2}^+$.

Now we need to show in the converse direction that for every construction K of H there is a vertex of $\mathcal{G}(H)$ such that $\{v\} = \bigcap\{\pi_X \mid X \in K\}$. This will be a consequence of the following two lemmata.

Lemma 9.4. *For every construction K of H there is a unique function $x : \bigcup H \to \mathbf{R}$ such that all the equations in the set $\{\sum_{i \in X} x(i) = 3^{|X|} \mid X \in K\}$ hold. Moreover, if for $X \in K$ we have that s is X-superficial, then $x(s) > 3^{|X|-1}$.*

Proof. By induction on d.

If $d = 0$, then $H = \{\{1\}\}$, and the only construction of H is H itself. Then we have a single equation $x(1) = 3$ that defines the function x, and $x(1) = 3 > 3^{1-1} = 1$.

Suppose $d > 0$. We have that there is an $s \in \bigcup H$ such that $H_1, \dots, H_n$, for $n \geqslant 1$, is the finest hypergraph partition of $H_{\bigcup H - \{s\}}$, and $K = K_1 \cup \dots \cup K_n \cup \{\bigcup H\}$, for K_j being a construction of the ASC-hypergraph H_j. By the induction hypothesis, for every $j \in \{1, \dots, n\}$ there is a unique function $x^j : \bigcup H_j \to \mathbf{R}$ such that all the equations in the following set $\{\sum_{i \in X} x^j(i) = 3^{|X|} \mid X \in K_j\}$ hold. Since $\bigcup H_j \in K_j$, we must have that

(a) $\sum_{i \in \bigcup H_j} x^j(i) = 3^{|\bigcup H_j|}$.

To obtain $x : \bigcup H \to \mathbf{R}$ with the desired properties, we form the union of the functions x^j (which is a function since the sets $\bigcup H_j$ are disjoint), and it remains to find the unique value of $x(s)$. (Note that $x(s)$ figures only in the equation $\sum_{i \in \bigcup H} x(i) = 3^{|\bigcup H|}$.) So, for $i \in \bigcup H_j$, let $x(i) = x^j(i)$, and let

$$x(s) = 3^{|\bigcup H|} - \sum_{i \in \bigcup H - \{s\}} x(i).$$

We have by the induction hypothesis that if s_j is the X-superficial element for $X \in K_j$, then $x(s_j) = x^j(s_j) > 3^{|X|-1}$. In particular, we have for every $i \in \bigcup H - \{s\}$ that $x(i) \geqslant 0$. It remains to check the analogous inequality concerning the $\bigcup H$-superficial element s; namely $x(s) > 3^{|\bigcup H|-1}$. In the case $n = 1$, we have by (a) that

$$\sum_{i \in \bigcup H - \{s\}} x(i) = \sum_{i \in \bigcup H_1} x(i) = 3^{|\bigcup H_1|} = 3^{|\bigcup H|-1},$$

and hence $x(s) = 3^{|\bigcup H|} - 3^{|\bigcup H|-1} > 3^{|\bigcup H|-1}$.

If $n > 1$, then we apply Lemma 9.1 to the $\mathcal{P}^+(\bigcup H)$-antichain $\{\bigcup H_1, \dots, \bigcup H_n\}$, relying on (a) and on the fact that $x(i) \geqslant 0$ for every $i \in \bigcup H - \{s\}$, in order to obtain

$$\sum_{i \in \bigcup H - \{s\}} x(i) = \sum_{i \in (\bigcup H_1) \cup \dots \cup (\bigcup H_n)} x(i) < 3^{|(\bigcup H_1) \cup \dots \cup (\bigcup H_n)|} = 3^{|\bigcup H|-1}.$$

With this we have that

$$x(s) = 3^{|\bigcup H|} - \sum_{i \in \bigcup H - \{s\}} x(i) > 3^{|\bigcup H|} - 3^{|\bigcup H|-1} > 3^{|\bigcup H|-1}. \qquad \square$$

This lemma says that the system of equations $\{\sum_{i\in X} x(i) = 3^{|X|} \mid X \in K\}$ has a unique solution, and this solution is the unique element of $\bigcap\{\pi_X \mid X \in K\}$. Moreover, it says something about the location of the coordinates of this solution, and hence about the location of this solution in the interior of $\bigcap\{\pi_Y^+ \mid Y \in H - K\}$, which will serve to determine that the solution is a vertex of $\mathcal{G}(H)$.

For K a construction of H and $x : \bigcup H \to \mathbf{R}$ the function obtained by Lemma 9.4, we can prove the following.

Lemma 9.5. *If $v = (x(1), \dots, x(d+1)) \in \mathbf{R}^{d+1}$, then $\{v\}$ is a vertex of $\mathcal{G}(H)$.*

Proof. By Lemma 9.4, we have that $\{v\} = \bigcap\{\pi_X \mid X \in K\}$. So for the rest of the proof it is sufficient to show that for every $Y \in H - K$ we have that $v \in \pi_Y^+$.

Let $Y \in H - K$. By Lemma 6.14, there is an $X \in K$ such that $Y \subset X$, and the X-superficial element s is in Y. By Lemma 9.4, we have that $x(s) > 3^{|X|-1} \geqslant 3^{|Y|}$, and since every other $x(i) \geqslant 0$ (see the proof of Lemma 9.4) we have that $\sum_{i\in Y} x(i) \geqslant x(s) > 3^{|Y|}$, and hence $v \in \pi_Y^+ - \pi_Y$. □

As a corollary of Lemmata 9.4 and 9.5, we have the following.

Proposition 9.6. *For every construction K of H there is a unique vertex $\{v\}$ of $\mathcal{G}(H)$ such that $\{v\} = \bigcap\{\pi_X \mid X \in K\}$.*

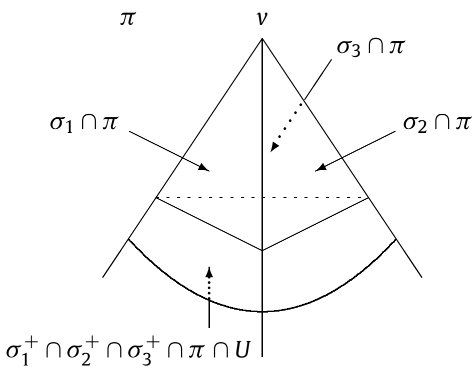
For the following two lemmata, which are not about $\mathcal{G}(H)$ specifically, but are more general, we have the following assumptions. Let $d \geqslant 1$ and let π be a hyperplane in $\mathbf{R}^{d+1}$. Let S^+ be a set of closed halfspaces in $\mathbf{R}^{d+1}$ whose boundary hyperplanes are collected in a set S. Let $P = (\bigcap S^+) \cap \pi \neq \emptyset$ be a polytope. For every vertex $\{v\}$ of P let there be exactly d halfspaces from S^+ such that $\{v\}$ is contained in their boundary hyperplanes. Then we can prove the following (for the notion of simple polytope see [35, Section 2.5, Proposition 2.16]).

Lemma 9.7. *P is a simple d-dimensional polytope.*

Proof. Since $P \neq \emptyset$, the set of vertices of P is not empty. Let $\{v\}$ be a vertex of P. We show that a vertex figure (see [35, Section 2.1]) of P at v is a $(d-1)$-dimensional simplex, from which the proposition follows.

Since $\{v\}$ is a vertex, there is a set V^+ of d halfspaces from S^+, whose boundary hyperplanes are collected in a set V, such that $(\bigcap V) \cap \pi = \{v\}$. By our assumption, for every halfspace in $S^+ - V^+$ (we have that $S^+ - V^+ \neq \emptyset$ since $d \geqslant 1$ and P is a polytope, and not just an $\mathcal{H}$-polyhedron) we have that v lies in the interior of this halfspace.

Since $S^+ - V^+$ is finite, we have that there is an open neighborhood U of v in $\mathbf{R}^{d+1}$ such that $(\bigcap V^+) \cap \pi \cap U \subseteq P$. Hence a vertex figure of P at v is a $(d-1)$-dimensional simplex. The following picture illustrates the case when $d = 3$:



□

Lemma 9.8. *There is a bijection β from the set of all the facets of P to the set of all the pairs $\{\sigma, \pi\}$, where σ is an element of S that contains a vertex of P. This bijection is such that for every facet φ and every vertex $\{v\}$ of P we have that $\{v\} \subseteq \varphi$ iff $\{v\} \subseteq \bigcap \beta(\varphi)$.*

Proof. Let β be a relation between the set of all the facets of P and the set of all the pairs $\{\sigma, \pi\}$, where σ is an element of S that contains a vertex of P, defined by

$$\big(\varphi, \{\sigma, \pi\}\big) \in \beta \quad \text{when the affine hull of } \varphi \text{ is } \sigma \cap \pi.$$

First we show that β is a function between these two sets. Let φ be a facet of P. Since $d \geqslant 1$, there is a vertex $\{v\}$ of P incident with φ. By reasoning as in the proof of Lemma 9.7, there must be a $\sigma \in S$ such that for aff(φ) being the affine hull of φ we have that aff$(\varphi) = \sigma \cap \pi$; so $v \in \sigma$, and we can take $\beta(\varphi) = \{\sigma, \pi\}$.

If $\mathrm{aff}(\varphi) = \sigma' \cap \pi$, then $v \in \sigma'$, and, since there are exactly d hyperplanes from S containing $\{v\}$, it must be that $\sigma' = \sigma$; otherwise, we would have that $\sigma' \cap \sigma \cap \dots \cap \pi = \{v\} = \sigma \cap \dots \cap \pi$, and hence we would have d hyperplanes in $\mathbf{R}^{d+1}$ intersecting in a point, which is impossible. So β is a function.

Since for all facets φ_1 and φ_2 of P we have that $\mathrm{aff}(\varphi_1) = \mathrm{aff}(\varphi_2)$ iff $\varphi_1 = \varphi_2$, we have that β is injective. It remains to show that β is onto.

Let $\sigma \in S$, and let $\{v\}$ be a vertex of P such that $v \in \sigma$. By reasoning as in the proof of Lemma 9.7, we have that $\sigma \cap P$ is a facet of P whose affine hull is $\sigma \cap \pi$. So this is the facet mapped by β to $\{\sigma, \pi\}$.

The equivalence of the proposition from left to right is trivial. For the other direction, we rely on the fact that for every facet φ of P we have that $\mathrm{aff}(\varphi) \cap P = \varphi$. □

For $d \geqslant 1$, by Corollary 9.3.2 and by Proposition 9.6 (which gives that $\mathcal{G}(H)$ is not empty), we have that $\mathcal{G}(H)$ satisfies the conditions of Lemmata 9.7 and 9.8 with π, S and S^+ being respectively $\pi_{\bigcup H}$, $\{\pi_X \mid X \in H - \{\bigcup H\}\}$ and $\{\pi_X^+ \mid X \in H - \{\bigcup H\}\}$. So there is a bijection from the set of facets of $\mathcal{G}(H)$ to the set Ψ of all pairs $\{\pi_X, \pi_{\bigcup H}\}$ such that

(1) $X \in H - \{\bigcup H\}$ and π_X contains a vertex of $\mathcal{G}(H)$.

Since for every X in H there is a construction of H to which it belongs (see Proposition 4.8), and since $\bigcup H$ belongs to every construction of H, from (1) we infer that

(2) $X \neq \bigcup H$ and there is a construction K of H such that $\{X, \bigcup H\} \subseteq K$.

(We could make the same inference with the help of Proposition 9.3 too.) We use Proposition 9.6 to show that (1) follows from (2), and hence $\Psi = \{\{\pi_X, \pi_{\bigcup H}\} \mid (2) \text{ holds}\}$.

Since π is injective, we obtain a bijection β^π from the set of facets of $\mathcal{G}(H)$ to the set of facets of $\mathcal{A}(H)$. From Corollary 9.3.1 and Proposition 9.6 we infer that there is a bijection γ from the set of vertices of $\mathcal{G}(H)$ to the set of vertices of $\mathcal{A}(H)$. The bijection γ may be explicitly defined by $\gamma(\{v\}) = \{X \in H \mid \{v\} \subseteq \pi_X\}$. For $\{v\}$ a vertex and φ a facet of $\mathcal{G}(H)$, these two bijections satisfy the following.

Lemma 9.9. *We have that $\{v\} \subseteq \varphi$ iff $\beta^\pi(\varphi) \subseteq \gamma(\{v\})$.*

Proof. Let $\beta(\varphi) = \{\pi_X, \pi_{\bigcup H}\}$. We have that

$$\begin{aligned} \{v\} \subseteq \varphi \quad & \text{iff} \quad \{v\} \subseteq \bigcap \beta(\varphi) = \pi_X \cap \pi_{\bigcup H}, \quad \text{by Lemma 9.8} \\ & \text{iff} \quad \beta^\pi(\varphi) = \Big\{X, \bigcup H\Big\} \subseteq \gamma\big(\{v\}\big), \quad \text{by the definition of } \gamma. \qquad \square \end{aligned}$$

Let $\mathcal{X} = \{X_1, \dots, X_n\}$, for $n \geqslant 1$. Consider a hypergraph $\mathcal{H} = \{\mathcal{X}_1, \dots, \mathcal{X}_m\}$ on $\mathcal{X}$. Here we must have $m \geqslant 1$, since $n \geqslant 1$. Let $\mathcal{L}$ be the lattice whose elements are in the set

$$\mathcal{P}(\mathcal{X}_1) \cup \dots \cup \mathcal{P}(\mathcal{X}_m) \cup \big(\mathcal{X} \cup \{*\}\big).$$

The join of $\mathcal{L}$ is set intersection $\cap$, while the meet $\wedge$ for $A, B \in \mathcal{L}$ is defined by

$$A \wedge B = \begin{cases} A \cup B & \text{if } (\exists i \in \{1, \dots, m\})\ A \cup B \subseteq \mathcal{X}_i, \\ \mathcal{X} \cup \{*\} & \text{otherwise.} \end{cases}$$

This lattice has a greatest element, namely $\emptyset$, and a least element, namely $\mathcal{X} \cup \{*\}$. (A more natural lattice is the dual lattice—namely, the same lattice taken upside down—but the present lattice $\mathcal{L}$ is analogous to the lattice $\mathcal{A}(H)$.) Consider a hypergraph $\mathcal{H}' = \{\mathcal{X}'_1, \dots, \mathcal{X}'_m\}$ on $\mathcal{X}' = \{X'_1, \dots, X'_n\}$, and let $\mathcal{L}'$ be defined exactly as $\mathcal{L}$; we just add the primes. Suppose that for every $i \in \{1, \dots, n\}$ and every $j \in \{1, \dots, m\}$ we have

(coinc) $X_i \in \mathcal{X}_j$ iff $X'_i \in \mathcal{X}'_j$.

Then it is trivial that the lattices $\mathcal{L}$ and $\mathcal{L}'$ are isomorphic.

This is the situation we have with, on the one hand, $\mathcal{X}$ being the set of facets of $\mathcal{G}(H)$, and the hypergraph $\mathcal{H}$ on $\mathcal{X}$ being the set obtained from the set of vertices of $\mathcal{G}(H)$ by representing each vertex with the set of facets of $\mathcal{G}(H)$ in which this vertex lies. It follows from Proposition 2.16 of [35, Section 2.5] and Birkhoff's representation theorem for finite Boolean algebras that for every $\mathcal{X}_i$ in $\mathcal{H}$ we have that $\mathcal{P}(\mathcal{X}_i)$ is isomorphic to the Boolean algebra of all the faces in which the vertex of $\mathcal{G}(H)$ corresponding to $\mathcal{X}_i$ lies. The set $\mathcal{H}$ above is clearly in a bijection with the set of vertices of $\mathcal{G}(H)$. The lattice $\mathcal{L}$ is isomorphic to the face lattice of $\mathcal{G}(H)$.

On the other hand, take that $\mathcal{X}'$ is $H - \{\bigcup H\}$ for an ASC-hypergraph H. This set is in a bijection with the set of facets of $\mathcal{A}(H)$ because each facet of $\mathcal{A}(H)$ is of the form $\{X, \bigcup H\}$ for an $X \in H - \{\bigcup H\}$. Let the hypergraph $\mathcal{H}'$ on $\mathcal{X}'$ be the set of vertices of $\mathcal{A}(H)$ with $\bigcup H$ removed. Each vertex of $\mathcal{A}(H)$ is a construction of H, and we consider here the set $\mathcal{H}' = \{K - \{\bigcup H\} \mid K \text{ is a construction of } H\}$, which happens to be a hypergraph on $H - \{\bigcup H\}$. It is clear that $\mathcal{H}'$ is in a bijection with the set of vertices of $\mathcal{A}(H)$ (just remove $\bigcup H$ from every vertex of $\mathcal{A}(H)$). It is easy to see that the lattice $\mathcal{L}'$ is isomorphic to $\mathcal{A}(H)$ (just remove $\bigcup H$ from every member of $\mathcal{A}(H)$).

The bijections β^π and γ above deliver two bijections between $\mathcal{X}$ and $\mathcal{X}'$ and $\mathcal{H}$ and $\mathcal{H}'$ respectively, and, due to Lemma 9.9, we have (coinc). So $\mathcal{L}$ and $\mathcal{L}'$ are isomorphic, and hence the face lattice of $\mathcal{G}(H)$ is isomorphic to $\mathcal{A}(H)$.

The foregoing covers the case when $d \geqslant 1$, and for $d < 1$ we obtain our isomorphism trivially. We have dealt with the case $d = -1$, i.e. the case when $H = \emptyset$, at the beginning of the section. When $d = 0$, i.e. when H is the singleton $\{\{1\}\}$, then $\bigcap\{\pi^+_X \mid X \in \emptyset\} = \mathbf{R}^1$, and $\pi_{\{1\}} = \{x_1 \in \mathbf{R}^1 \mid x_1 = 3\} = \{3\}$; so $\mathcal{G}(H) = \{3\}$. The face lattice of $\mathcal{G}(H)$ has one vertex $\{3\}$, and it has $\emptyset$ as the image of F_{-1}. The abstract polytope $\mathcal{A}(H)$ is now $\{\{\{1\}\}, \{\{1\}, *\}\}$, with the vertex $\{\{1\}\}$ being F_0 and $\{\{1\}, *\}$ being F_{-1}. It is clear that $\mathcal{G}(H)$ and $\mathcal{A}(H)$ are isomorphic.

When H is atomic and saturated, but not connected, and $\{H_1, \dots, H_n\}$, for $n \geqslant 2$, is the finest hypergraph partition of H, then $\mathcal{G}(H)$ is defined as $\mathcal{G}(H_1) \times \dots \times \mathcal{G}(H_n)$, with $\times$ being Cartesian product (polytopes are closed under $\times$; see [35, Lecture 0, pp. 9–10]). The face lattice of $\mathcal{G}(H)$ so defined is again isomorphic to $\mathcal{A}(H)$, which is defined as $\mathcal{A}(H_1) \otimes \dots \otimes \mathcal{A}(H_n)$ (see Sections 5 and 7).

When H is atomic, but not saturated, we define the polytope $\mathcal{G}(H)$ as the polytope $\mathcal{G}(\bar{H})$, for $\bar{H}$ being the saturated closure of H, and again we obtain that the face lattice of $\mathcal{G}(H)$ is isomorphic to $\mathcal{A}(H)$. So we may conclude the following.

Theorem 9.10. *For every atomic hypergraph H the face lattice of $\mathcal{G}(H)$ is isomorphic to $\mathcal{A}(H)$.*

Acknowledgements

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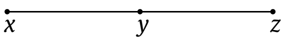
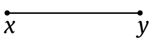
Appendix A. Constructs and tubings

A notion which, as we shall see, is closely related to the notion of construct of Section 5 was introduced under the name "tubing" in [6]; it was modified in [10], and modified further in [11] (which is posterior to [10]). In this section, we will determine with the help of the results of Section 6 the exact relationship between the tubings of [11] and constructs. For graphs, but not for hypergraphs in general, the two notions happen to be equivalent. Along the way, we obtain simpler characterizations of tubings than that given by the definition of [11].

Problems arise for the tubings of [6] and [10] with connected graphs like



We have that $\{\{x\}, \{y, z\}\}$ is a tubing because $\{x\}$ and $\{y, z\}$ are disjoint and are not adjacent (their union $\{x, y, z\}$ is not a tube in the sense of [6] and [10], because it is not a proper subset of vertices). This tubing is however rejected in Fig. 1(b) of [10]. A simpler problematic example is with the graph



and the tubing $\{\{x\}, \{y\}\}$ (in the sense of [6] and [10]). It is presumably because of such problems that the modifications of [11] were introduced in the definition of a tubing, and we will consider here only this last modified definition.

A tubing is defined relative to a *graph*, which we will identify with an ASC-hypergraph G that is the saturated closure $\bar{H}$ of a nonempty atomic hypergraph H such that $\bigcup H \in H$, and every member of H that is neither a singleton nor $\bigcup H$ is a two-element set; these two-element sets are the edges of the graph G, and its vertices are the members of $\bigcup G$, which is equal to $\bigcup H$. The remaining members of G are the connected subsets of the graph as they are usually conceived, except for $\bigcup G$, which is in G even if the underlying graph is not connected in the usual sense. We put always $\bigcup G$ in G to match what is in [11]; this assumption, which yields connectedness, enables us also to identify a graph with a kind of ASC-hypergraph. (The nonemptiness of G, which means that $\bigcup G \neq \emptyset$, is not stated explicitly in [11], but it is a common assumption for graphs, which seems to be made because of the treatment of tubes in [11], as we shall see in a moment; cf. the comment after Remark 2.1.)

A *tube* of a graph G is a member of G. Members of G are always nonempty, and in [6] tubes are said to be nonempty. In [11] this is not stated explicitly, but may be taken to follow from the nonemptiness of graphs.

Two nonempty sets X and Y are *overlapping* when $X \cap Y \neq \emptyset$ and neither $X \subseteq Y$ nor $Y \subseteq X$ (in [11] one finds "intersect" instead of "overlap"); they are said to be *adjacent* relative to a hypergraph H when $X \cap Y = \emptyset$ and $X \cup Y \in H$ (see the comments after Lemma 6.3).

We will say that an ASC-hypergraph is *loose* when $H - \{\bigcup H\}$ is not a connected hypergraph, which means that $\bigcup H$ is not dispensable in H (see Section 4). Loose graphs would normally be considered unconnected, but as hypergraphs they are connected.

For a loose hypergraph H, let $\{H_1, \dots, H_n\}$, with $n \geqslant 2$, be the finest hypergraph partition of $H - \{\bigcup H\}$, and let V_H be $\{\bigcup H_1, \dots, \bigcup H_n\}$, which is a partition of $\bigcup H$.

According to [11], a *tubing* of a graph G is a set T of tubes of G (i.e. a subset of G) such that every pair of tubes in T is neither overlapping nor adjacent relative to G; moreover, if G is loose, then V_G is not a subset of T, and, finally, $\bigcup G \in T$.

Consider the assumption stated in Section 6:

(M) M is a subset of the ASC-hypergraph H such that every M-antichain misses H.

We can prove the following.

Lemma A.1. *If* (M) *and $X, Y \in M$, then X and Y are not overlapping.*

Proof. If X and Y were overlapping, then $\{X, Y\}$ would be an M-antichain that would not miss H, because of the saturation of H. □

Lemma A.2. *If* (M) *and $X, Y \in M$, then X and Y are not adjacent.*

Proof. This holds simply because if $X \cap Y = \emptyset$, then $\{X, Y\}$ is an M-antichain, because X and Y are nonempty. □

Lemma A.3. *If* (M) *and H is loose, then V_H is not a subset of M.*

Proof. If $H - \{\bigcup H\}$ is not connected and $V_H \subseteq M$, then V_H is an M-antichain. This M-antichain does not miss H, since $\bigcup V_H = \bigcup H \in H$, which contradicts (M). □

As a corollary of these three lemmata we have the following.

Proposition A.4. *If* (M) *and $\bigcup H \in M$, then M is a tubing of H.*

We can also prove the converse for graphs H. For that we need the following lemmata.

Lemma A.5. *For H a graph that is not loose and $\{X_1, \dots, X_n\} \subseteq H$, with $n \geqslant 2$, if for every $i, j \in \{1, \dots, n\}$ such that $i \neq j$ we have that $X_i \cup X_j \notin H$, then $X_1 \cup \dots \cup X_n \notin H$.*

Proof. Since H is a graph and is not loose, it is equal to the saturated closure of an atomic hypergraph H' in which as members besides singletons we have only two-element sets. Suppose $X_1 \cup \dots \cup X_n \in H$. Choose two distinct elements x and y from respectively two different members of $\{X_1, \dots, X_n\}$; these elements exist because $n \geqslant 2$. There is a path of H connecting x with y, and since H and H' are cognate, by Proposition 4.6, there is a path $Y_1, \dots, Y_m$ of H' connecting x with y. Since x and y are distinct, all the members of $Y_1, \dots, Y_m$ may be taken as two-element sets, and for each of the two elements there is an $i \in \{1, \dots, n\}$ such that this element belongs to X_i.

Since x and y are from two different members of $\{X_1, \dots, X_n\}$, for some $l \in \{1, \dots, m\}$ we have that $Y_l = \{x_i, x_j\}$ for $x_i \in X_i$ and $x_j \in X_j$, with $i, j \in \{1, \dots, n\}$ and $i \neq j$. So we have that $X_i, Y_l, X_j \in H$, together with $X_i \cap Y_l \neq \emptyset$, $X_j \cap Y_l \neq \emptyset$ and $Y_l \subseteq X_i \cup X_j$. Since H is saturated, we may conclude that $X_i \cup X_j \in H$, which contradicts our assumption. □

This lemma is trivial when $n = 2$. What we need for the application in the next lemma are the cases $n \geqslant 3$.

Lemma A.6. *For H a graph that is not loose, an H-antichain such that every pair of its members is neither overlapping nor adjacent relative to H misses H.*

Proof. Take an H-antichain $\{X_1, \dots, X_n\}$, for $n \geqslant 2$. If $n = 2$, then we have that X_1 and X_2 are not overlapping, and since neither $X_1 \subseteq X_2$ nor $X_2 \subseteq X_1$, we have that $X_1 \cap X_2 = \emptyset$. Since X_1 and X_2 are not adjacent, it follows that $X_1 \cup X_2 \notin H$.

For $n \geqslant 3$, we have for every $i, j \in \{1, \dots, n\}$ such that $i \neq j$ that $\{X_i, X_j\}$ is an H-antichain. According to what we have just proved, $X_i \cup X_j \notin H$. By Lemma A.5, we may then conclude that $X_1 \cup \dots \cup X_n \notin H$. □

We can then prove for graphs the following converse of Proposition A.4.

Proposition A.7. *For H a graph, if M is a tubing of H, then every M-antichain misses H and $\bigcup H \in M$.*

Proof. Suppose $\{H_1, \dots, H_n\}$, for $n \geqslant 1$, is the finest hypergraph partition of $H - \{\bigcup H\}$. Take an M-antichain S of the form $S_1 \cup \dots \cup S_n$ such that for every $i \in \{1, \dots, n\}$ we have that $S_i \subseteq H_i$.

Note first that it is impossible that every member of $\{S_1, \dots, S_n\}$ is empty. If just one of these members is nonempty, then $S \subseteq H_i$ for some $i \in \{1, \dots, n\}$. Then we have that either $n \geqslant 2$ and H_i is a graph that is not loose, or $n = 1$ and $H_i = H_1 = H - \{\bigcup H\}$; in the second case we have that $S \subseteq H$ and H is not loose. So S is either an H_i-antichain or an H-antichain. In both cases, by Lemma A.6, we conclude that $\bigcup S \notin H$.

If at least two members of $\{S_1, \dots, S_n\}$ are nonempty and at least one member is empty, then $\bigcup S \subset \bigcup H$, and we may conclude that $\bigcup S \notin H$. It remains to consider the case when all the members of $\{S_1, \dots, S_n\}$ are nonempty and $n \geqslant 2$. In that case H is loose.

Since for every i we have that $S_i \subseteq H_i$, we must have that $\bigcup S_i \subseteq \bigcup H_i$. Suppose that for every i we have that $\bigcup S_i = \bigcup H_i$. We can conclude that $S_i = \{\bigcup H_i\}$. Otherwise, $|S_i| \geqslant 2$, and so S_i is an H_i-antichain; by Lemma A.6, it misses H_i, which contradicts $\bigcup S_i = \bigcup H_i$. So $S = \{\bigcup H_1, \dots, \bigcup H_n\} = V_H$, and this contradicts the assumption that V_H is not a subset of M, which we have when H is loose. So for some i we have that $\bigcup S_i \subset \bigcup H_i$, and then $\bigcup S \notin H$.

The condition $\bigcup H \in M$ is assumed for tubings. $\square$

As a corollary of Propositions A.4 and A.7, we have that for graphs H a subset M of H is a tubing of H iff every M-antichain misses H and $\bigcup H \in M$. This gives an alternative, simpler, definition of a tubing.

With this characterization of tubings, we can immediately infer from Proposition 6.13 that for graphs H a subset M of H is a tubing of H iff M is a construct of H. We can infer from Proposition 6.11 that for graphs H a subset M of H is a tubing of H and $|M| = |\bigcup H|$ iff M is a construction of H.

Note that we have these characterizations of constructs and constructions in terms of tubings only for graphs, i.e. for specific hypergraphs with which we have identified graphs. We do not have them for hypergraphs in general.

Another difference of our approach with the approach through tubings is that for us constructs, which for graphs amount to tubings, are a derived, secondary, notion. The basic, primary, notion is the notion of construction.

Appendix B. Hypergraph polytopes of dimension 3 and lower

In this appendix we survey the abstract polytopes $\mathcal{A}(H)$ of atomic saturated hypergraphs H with carriers $\bigcup H$ having no more than four elements. In this survey, we deal with the main types of these polytopes, and do not distinguish polytopes that would differ only up to renaming the elements of the carrier.

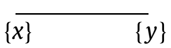
We name the hypergraphs in our survey by adding to H subscripts according to the following system (sometimes we also have superscripts). In $H_{i_1,\dots,i_k}$, the subscript i_j for $j \in \{1, \dots, k\}$ is the number of j-element members of $H_{i_1,\dots,i_k}$. So, for example, H_{21} below has 2 singletons and one pair. Since our hypergraphs are atomic, we always have that the first subscript i_1 is the cardinality of the carrier, while the last subscript i_k can be either 0 or 1.

(H_0, H_1) Let H_0 be the empty hypergraph $\emptyset$; then $\mathcal{A}(H_0) = \{H_0, H_0 \cup \{*\}\} = \{\emptyset, \{*\}\}$. Let H_1 be the hypergraph $\{\{x\}\}$; then $\mathcal{A}(H_1) = \{H_1, H_1 \cup \{*\}\}$. With that we have surveyed all we have with the carrier of the hypergraph having not more than one element.

(H_{20}, H_{21}) With the carrier having two elements, we have two atomic hypergraphs: $H_{20} = \{\{x\}, \{y\}\}$ and $H_{21} = H_{20} \cup \{\{x, y\}\}$. Analogously to what we had in (H_0, H_1), we have that $\mathcal{A}(H_{20}) = \{H_{20}, H_{20} \cup \{*\}\}$, and $\mathcal{A}(H_{21})$ has the following structure:

$$\begin{array}{lcc}
F_1 \text{ (edge)} & \multicolumn{2}{c}{\big\{\{x, y\}\big\}} \\
\text{vertices} & \big\{\{x\}, \{x, y\}\big\} & \big\{\{y\}, \{x, y\}\big\} \\
F_{-1} & \multicolumn{2}{l}{H_{21} \cup \{*\} = \big\{\{x\}, \{y\}, \{x, y\}, *\big\}.}
\end{array}$$

A realization of $\mathcal{A}(H_{21})$ may be pictured as



where in the labels of the vertices we have omitted $\{x, y\}$ and the outermost braces.

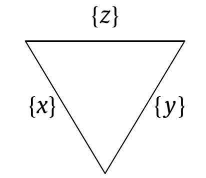
We pass next to atomic saturated hypergraphs whose carrier has three elements.

(H_{300}, H_{310}) We have first $H_{300} = \{\{x\}, \{y\}\}$, with $\mathcal{A}(H_{300}) = \{H_{300}, H_{300} \cup \{*\}\}$. Next we have $H_{310} = H_{300} \cup \{\{x, y\}\}$, with $\mathcal{A}(H_{310})$ being isomorphic to (i.e. being in an order-preserving bijection with) $\mathcal{A}(H_{21})$, which we write $\mathcal{A}(H_{310}) \cong \mathcal{A}(H_{21})$.

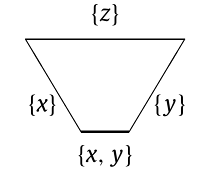
Cases like these with H_{300} and H_{310}, where the hypergraph is not connected (which happens when there is more than one subscript, and the last is 0), will be called *degenerate*. (So H_{20} above is degenerate.) In general, in a degenerate case the rank of $\mathcal{A}(H_{k\dots l})$ is lower than $k - 1$. So the rank of $\mathcal{A}(H_{310})$ is 1, while in the four non-degenerate cases with $k = 3$, which follow, it will be 2.

$(H_{301}, H_{311}, H_{321}, H_{331})$ As non-degenerate cases with the carrier having three elements, we have the hypergraphs H on the left, with the corresponding realizations of $\mathcal{A}(H)$ pictured on the right:

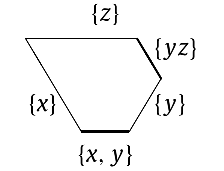
$$H_{301} = H_{300} \cup \big\{\{x, y, z\}\big\}$$



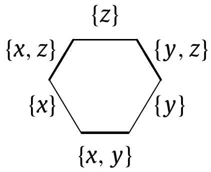
$$H_{311} = H_{301} \cup \big\{\{x, y\}\big\}$$



$$H_{321} = H_{311} \cup \big\{\{y, z\}\big\}$$

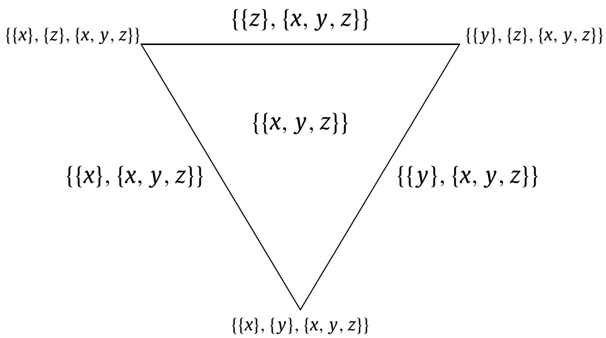


$$H_{331} = H_{321} \cup \big\{\{x, z\}\big\}$$



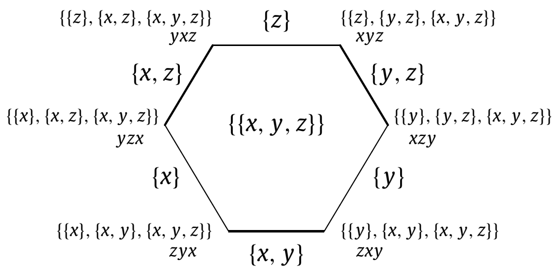
For the labels of the edges in the pictures on the right we have the convention that $\{x, y, z\}$ and the outermost braces are omitted; when they are restored, we obtain the edges of $\mathcal{A}(H)$. A vertex of $\mathcal{A}(H)$ is obtained by taking the set made of the labels of the edges that are incident with this vertex plus $\{x, y, z\}$. Finally, F_2 is here always $\{\{x, y, z\}\}$, which corresponds to the whole polygon.

Without all these abbreviations, the first picture would be



where the labels are the members of $\mathcal{A}(H_{301})$ without F_{-1}, which is $\bar{H}^{*}_{301} = \{\{x\}, \{y\}, \{z\}, \{x, y, z\}, *\}$ (we have $\bar{H}_{301} = H_{301}$). This is, of course, the picture of the two-dimensional simplex.

Without the abbreviations, for the vertices of the last, hexagonal, picture we would have the labels below, for which we also write underneath the corresponding s-constructions:



These s-constructions correspond, of course, to the six permutations of x, y and z.

For the third, pentagonal, picture, most of the labels would be the same; the difference would be only in the left upper corner, where we have a vertex labeled $\{\{x\},\{z\},\{x,y,z\}\}$, with the corresponding s-construction being $y(x+z)$.

This third picture is the picture of the two-dimensional associahedron, also known as Mac Lane's pentagon, and the last, fourth, picture is the picture of the two-dimensional permutohedron, also known as Mac Lane's hexagon (see [23] and [24, Sections VII.1 and VII.7], for Mac Lane's pentagon and hexagon; see (H'_{4321}) and (H_{4641}) below for references concerning associahedra and permutohedra). This comment, and the connection with the labels for vertices written as s-constructions, are explained in [12] and [13].

We pass from the triangle to the quadrilateral, the pentagon and the hexagon by truncating the vertices in succession. This truncating is explained in Section 9. Whereas here we truncate, in [13] we find the converse operation of collapsing several vertices into one. So the starting point would be not the simplex, but the permutohedron, and the direction would be in this case from the hexagon towards the triangle. Although the direction is reversed, this does not differ essentially from what we have here.

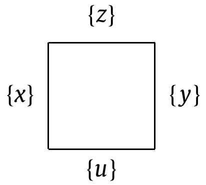
In the remainder of this survey we have atomic hypergraphs whose carrier has four elements.

$(H_{4000},\dots,H_{4310})$ As degenerate cases, we have first the following:

$$
\begin{aligned}
&H_{4000}=\big\{\{x\},\{y\},\{z\},\{u\}\big\}, && \mathcal{A}(H_{4000})=\big\{H_{4000},H_{4000}\cup\{*\}\big\},\\
&H_{4100}=H_{4000}\cup\big\{\{x,y\}\big\}, && \mathcal{A}(H_{4100})\cong\mathcal{A}(H_{21}),\\
&H_{4010}=H_{4000}\cup\big\{\{x,y,z\}\big\}, && \mathcal{A}(H_{4010})\cong\mathcal{A}(H_{301}),\\
&H_{4110}=H_{4010}\cup\big\{\{x,y\}\big\}, && \mathcal{A}(H_{4110})\cong\mathcal{A}(H_{311}),\\
&H_{4210}=H_{4100}\cup\big\{\{y,z\},\{x,y,z\}\big\}, && \mathcal{A}(H_{4210})\cong\mathcal{A}(H_{321}),\\
&H_{4310}=H_{4210}\cup\big\{\{x,z\}\big\}, && \mathcal{A}(H_{4310})\cong\mathcal{A}(H_{331})
\end{aligned}
$$

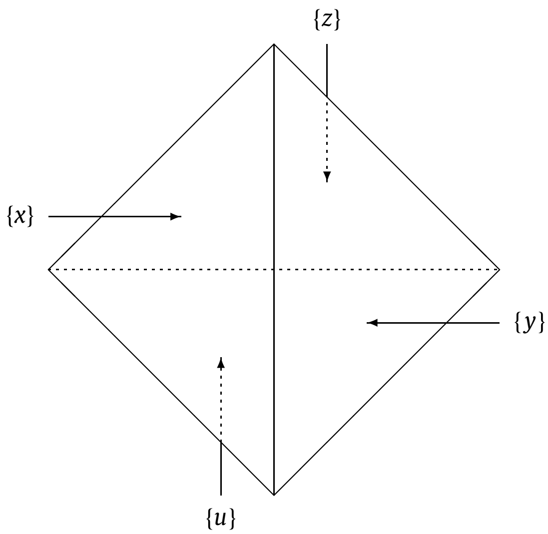
(the case with H_{4310} was investigated as Example 5.2 in [13]).

(H_{4200}) As the last degenerate case, we have $H_{4200}=H_{4100}\cup\{\{z,u\}\}$, and a realization of $\mathcal{A}(H_{4200})$ is pictured by



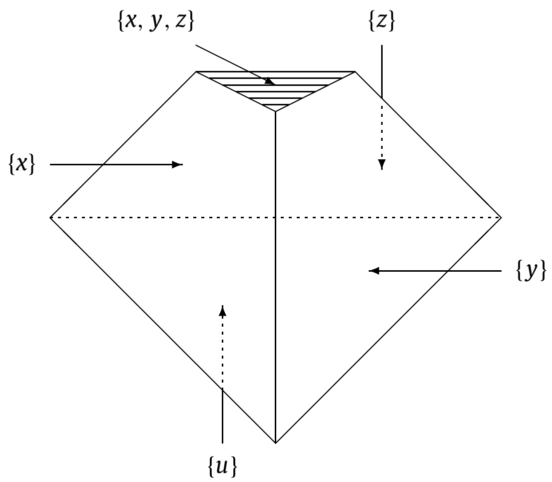
The polytope $\mathcal{A}(H_{4200})$ is obtained as the product with $\otimes$ of two copies of $\mathcal{A}(H_{21})$ (see Section 5). We pass next to non-degenerate cases.

(H_{4001}) As the first non-degenerate case with four elements in the carrier we have $H_{4001}=H_{4000}\cup\{\{x,y,z,u\}\}$, with $\mathcal{A}(H_{4001})$ being realized as the tetrahedron, i.e. the three-dimensional simplex. In general, for every $k\geqslant 3$ we have that $\mathcal{A}(H_{k0\dots01})$ may be realized as the $(k-1)$-dimensional simplex (see the realization of $\mathcal{A}(H_{301})$ above); $\mathcal{A}(H_{21})$ is realized as the one-dimensional simplex, and $\mathcal{A}(H_1)$ as the zero-dimensional simplex (see above). The tetrahedron is pictured by



We label here only the facets of the tetrahedron, with $\{x, y, z, u\}$ and the outermost braces omitted. The edges and vertices may be reconstructed out of these labels. We just look what facets are incident with the edge or the vertex. For example, the north–west edge is $\{\{x\}, \{z\}, \{x, y, z, u\}\}$, and the north vertex is $\{\{x\}, \{y\}, \{z\}, \{x, y, z, u\}\}$. The whole tetrahedron corresponds to $\{\{x, y, z, u\}\}$.

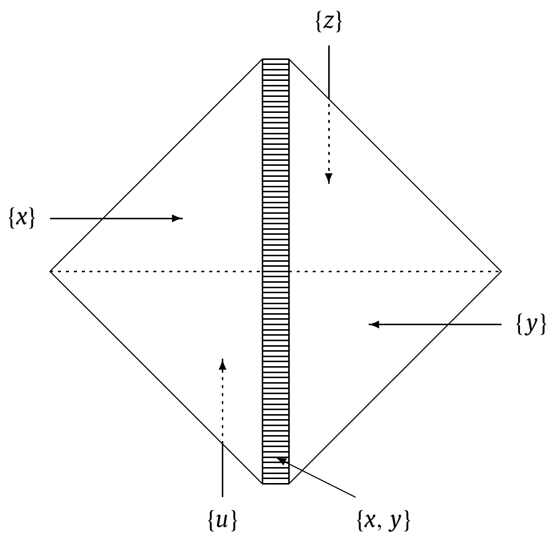
(H_{4011}) In this case we truncate a vertex. We have $H_{4011} = H_{4001} \cup \{\{x, y, z\}\}$, with a realization of $\mathcal{A}(H_{4011})$ obtained from our tetrahedron by truncating the vertex $\{\{x\}, \{y\}, \{z\}, \{x, y, z, u\}\}$, which is pictured by



This polytope may also be realized as a three-sided prism.

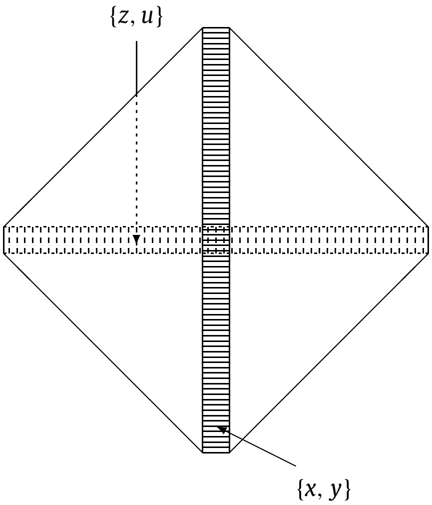
$(H_{4021}), (H_{4031}), (H_{4041})$ Next we have $H_{4021} = H_{4011} \cup \{\{y, z, u\}\}$, $H_{4031} = H_{4021} \cup \{\{x, z, u\}\}$ and $H_{4041} = H_{4031} \cup \{\{x, y, u\}\}$, with $\mathcal{A}(H_{4021})$, $\mathcal{A}(H_{4031})$ and $\mathcal{A}(H_{4041})$ being realized as the tetrahedron in which we have truncated two, three and four vertices respectively. (None of the last four cases is covered by the approach of [6] and [10], which is based on graphs, as explained in Appendix A; we will call such cases *essentially hypergraphical*.)

(H_{4101}) In this case we truncate an edge. We have $H_{4101} = H_{4001} \cup \{\{x, y\}\}$, with a realization of $\mathcal{A}(H_{4101})$ obtained from our tetrahedron by truncating the edge $\{\{x\}, \{y\}, \{x, y, z, u\}\}$, which is pictured by



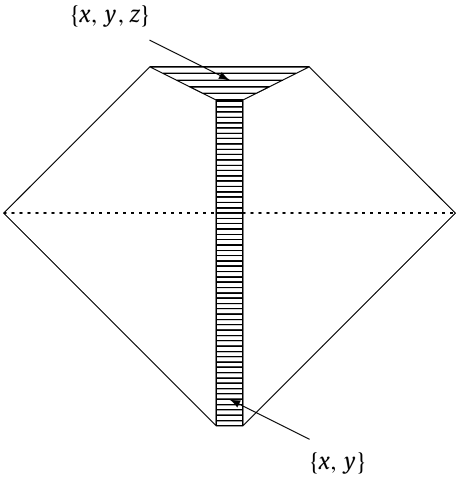
This polytope, as the preceding one, viz. $\mathcal{A}(H_{4011})$, may be realized as a three-sided prism.

(H_{4201}) With two opposite edges truncated, we have $H_{4201} = H_{4101} \cup \{\{z, u\}\}$, where a realization of $\mathcal{A}(H_{4201})$ is pictured by



with the labels for facets $\{x\}$, $\{y\}$, $\{z\}$ and $\{u\}$ omitted; they will mostly be omitted from now on. This polytope may also be realized as a cube.

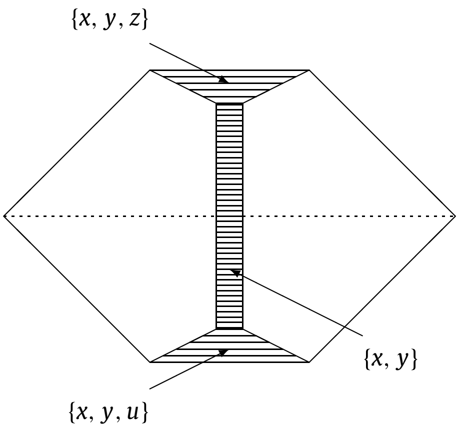
(H_{4111}) With one edge and one incident vertex truncated, we have $H_{4111} = H_{4011} \cup H_{4101}$, where a realization of $\mathcal{A}(H_{4111})$ is pictured by



This polytope, as the preceding one, viz. $\mathcal{A}(H_{4201})$, may be realized as a cube.

(H'_{4111}) With one edge and one non-incident vertex truncated, we have $H'_{4111} = H_{4101} \cup \{\{y, z, u\}\}$, with the picture of a realization of $\mathcal{A}(H'_{4111})$ obtained from that given for $\mathcal{A}(H_{4101})$ by truncating the east, i.e. right, vertex $\{\{y\}, \{z\}, \{u\}, \{x, y, z, u\}\}$.

(H_{4121}) With one edge and two incident vertices truncated, we have $H_{4121} = H_{4111} \cup \{\{x, y, u\}\}$, where a realization of $\mathcal{A}(H_{4121})$ is pictured by

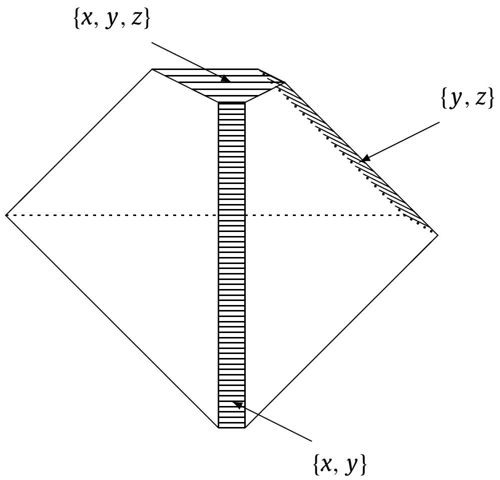


This polytope may also be realized as a five-sided prism.

(H'_{4121}) As a case where we truncate one edge and two vertices, one incident and the other not, we have $H'_{4121} = H_{4111} \cup H'_{4111}$. The picture of a realization of $\mathcal{A}(H'_{4121})$ is obtained from that given for H_{4111} by truncating the right vertex.

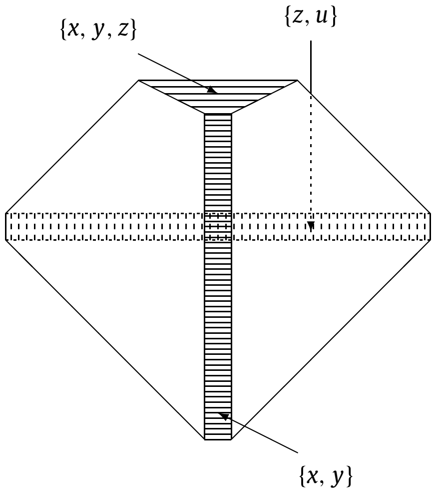
(H''_{4121}), (H_{4131}), (H'_{4131}), (H_{4141}) As a case where we truncate one edge and two vertices, none of them incident, we have $H''_{4121} = H_{4111} \cup \{\{x,z,u\}\}$. Next we have two cases where we truncate one edge and three vertices: $H_{4131} = H'_{4111} \cup H_{4121}$ and $H'_{4131} = H_{4111} \cup H''_{4121}$, and one case where we truncate one edge and all the four vertices: $H_{4141} = H_{4131} \cup H'_{4041}$. In all these cases, it should be clear by now how to picture a realization of $\mathcal{A}(H)$ starting from previous pictures. (The last eight cases are essentially hypergraphical.)

(H_{4211}) If we truncate two edges of our tetrahedron that are not opposite, but are incident with a common vertex, then we must truncate this vertex too (which is something related to saturation). This happens with $H_{4211} = H_{4111} \cup \{\{y,z\}\}$, with the picture of a realization of $\mathcal{A}(H_{4211})$ being



This polytope, as $\mathcal{A}(H_{4121})$, may be realized as a five-sided prism.

(H'_{4211}) As a case where we truncate two opposite edges and a vertex incident with just one of them, we have $H'_{4211} = H_{4111} \cup \{\{z,u\}\}$. The picture of a realization of $\mathcal{A}(H'_{4211})$ is



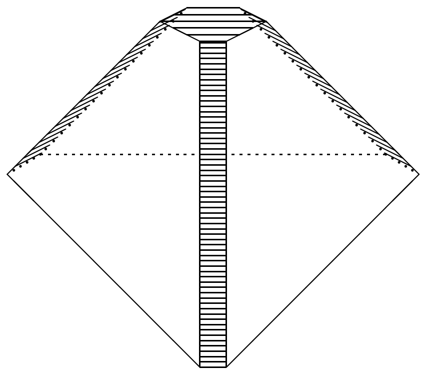
This polytope, as the preceding one, viz. $\mathcal{A}(H_{4211})$, and as $\mathcal{A}(H_{4121})$, may be realized as a five-sided prism.

(H_{4221}), ..., (H_{4241}) As remaining cases with two edges truncated we have

$$H_{4221} = H_{4211} \cup \big\{\{y,z,u\}\big\},$$
$$H'_{4221} = H_{4211} \cup \big\{\{x,z,u\}\big\},$$
$$H_{4231} = H_{4221} \cup H'_{4221},$$
$$H'_{4231} = H_{4221} \cup H_{4121},$$
$$H_{4241} = H_{4231} \cup H'_{4231},$$

with realizations of the corresponding polytopes having pictures easily obtained from the preceding ones. (The last six cases are essentially hypergraphical.)

(H_{4311}) If we truncate three edges and a common vertex with which they are all incident, we have the case of $H_{4311} = H_{4211} \cup \{\{x,z\}\} = H_{4310} \cup \{\{x,y,z,u\}\}$, with the picture of a realization of $\mathcal{A}(H_{4311})$ being



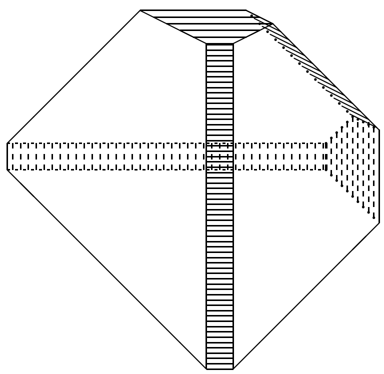
We have here omitted all the labels. This polytope may be realized also as a six-sided prism.

(H_{4321}), (H_{4331}), (H_{4341}) Next we have

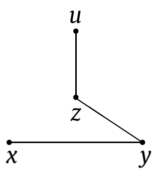
$$H_{4321} = H_{4311} \cup \big\{\{y,z,u\}\big\},$$
$$H_{4331} = H_{4321} \cup \big\{\{x,z,u\}\big\},$$
$$H_{4341} = H_{4331} \cup \big\{\{x,y,u\}\big\},$$

with the pictures of realizations of the corresponding polytopes obtained from the preceding picture by truncating the edges the base hexagon shares with the shaded quadrilaterals, so as to obtain one, two or three additional quadrilaterals. These edges originate from vertices, and we have in fact truncated these vertices. (The last three cases are essentially hypergraphical.)

(H'_{4321}) If we truncate the tetrahedron along a path of three edges and two vertices so as to obtain



we have the picture of a realization of $\mathcal{A}(H'_{4321})$ for $H'_{4321} = H_{4221} \cup \{\{u,z\}\}$. This polytope is the three-dimensional associahedron K_5 (see [30–32]), and H'_{4321} is the saturated closure of

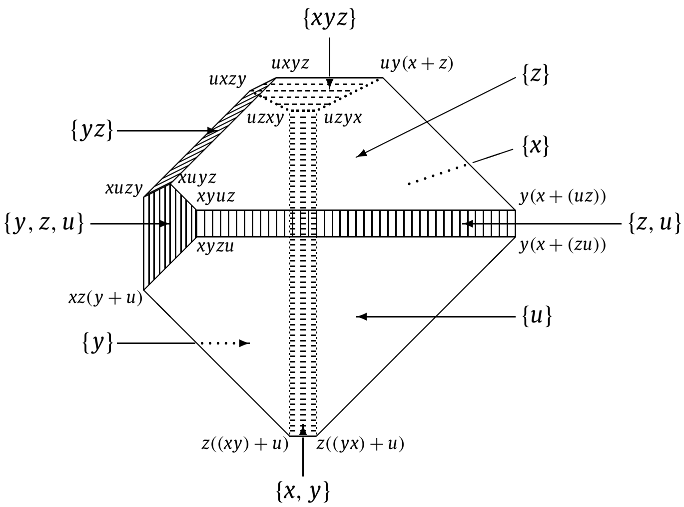


(see A'' in Section 3). It is also the saturated closure of the hypergraph A of Section 3.

This truncation should be compared with Example 5.15 of [13], where one reaches K_5 not by truncating the tetrahedron, but from the other end, by collapsing the vertices of the three-dimensional permutohedron (see (H_{4641}) below).

Note that we have truncated in the tetrahedron a chain made of three edges and two vertices. There is in the tetrahedron a complementary chain of exactly the same kind, with three edges and two vertices. Since for the three-dimensional permutohedron we will truncate all the edges and all the vertices of our tetrahedron, K_5 is located halfway.

To compare our picture of K_5 with the picture in [13], and with what we had in Section 5, we will turn it so as to obtain the following picture, with facets labeled as before in this survey, and vertices labeled with s-constructions (the edges are not labeled):

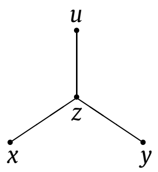


$(H'_{4331}), (H'_{4341}), (H^*_{4331})$ Next we have three more cases with three edges truncated:

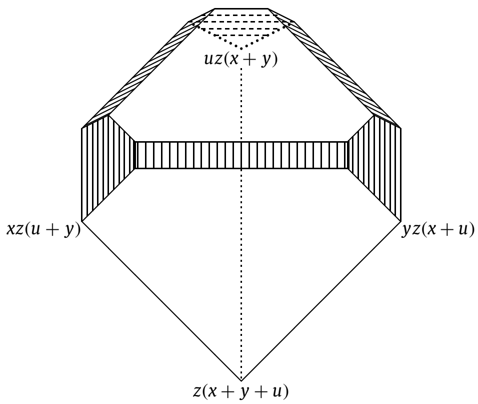
$$H'_{4331} = H'_{4321} \cup \big\{\{x, z, u\}\big\},$$
$$H'_{4341} = H'_{4331} \cup \big\{\{x, y, u\}\big\},$$
$$H^*_{4331} = H_{4031} \cup \big\{\{x, z\}, \{y, z\}, \{u, z\}\big\}.$$

The first of these cases, for which we will draw no picture, is interesting because H'_{4331} is the intersection of the hypergraphs of the hemiassociahedron (see (H_{4431}) below) and of the three-dimensional cyclohedron (see (H°_{4441}) below). We draw no picture for the second case either. Both of these pictures that we do not draw are obtained easily from our picture of K_5; we just extend with vertices the initial path of truncated edges and vertices of the tetrahedron. (These two cases are essentially hypergraphical.)

In the third case, we have that H^*_{4331} is the saturated closure of



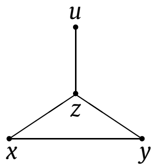
(see A^* in Section 3). The picture of a realization of $\mathcal{A}(H^*_{4331})$, turned in the manner of the preceding picture, is



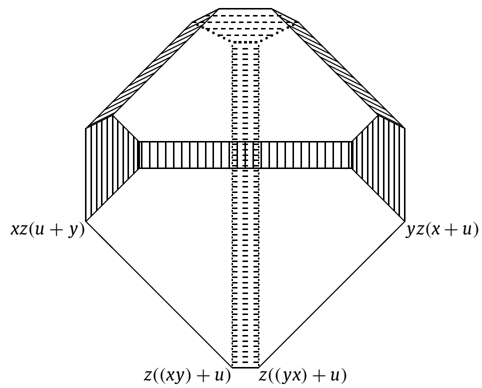
Here we label with s-constructions just those vertices whose s-constructions are not permutations of x, y, z and u. This picture permits a comparison with the picture of Example 5.16 of [13]. The pictured polytope is the three-dimensional stellohedron of [27] (Section 10.4; see also [26, Section 8.4]). This polytope is called $D4$ in [1, Fig. 17], and in [13] the entirely Greek name *astrohedron* is suggested.

(H^*_{4341}) Next we have $H^*_{4341} = H^*_{4331} \cup \{\{x, y, u\}\}$, with the picture for a realization of $\mathcal{A}(H^*_{4341})$ obtained from the preceding picture, given for $\mathcal{A}(H^*_{4331})$, by truncating the vertex labeled $z(x + y + u)$. (This is an essentially hypergraphical case.)

(H_{4431}) Next we have $H_{4431} = H_{4331} \cup \{\{z, u\}\} = H'_{4331} \cup \{\{x, z\}\} = H^*_{4331} \cup \{\{x, y\}\}$, which is the saturated closure of



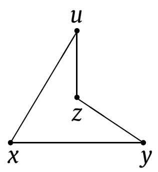
The picture for a realization of $\mathcal{A}(H_{4431})$, turned in the manner of the preceding two pictures, and labeled in the manner of the preceding one, is



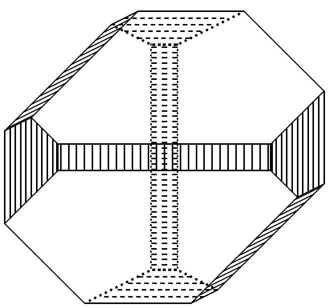
This picture permits a comparison with the picture of Example 5.14 of [13], where the pictured polytope was called *hemi-associahedron* (it is called X^a_4 in [1, Fig. 17], and $P_{1,2}$ in [3, Fig. 6]).

(H_{4441}) Next we have $H_{4441} = H_{4431} \cup \{\{x, y, u\}\}$, with the picture for a realization of $\mathcal{A}(H_{4441})$ obtained from the preceding picture of the hemiassociahedron, given for $\mathcal{A}(H_{4431})$, by truncating the bottom edge, so as to obtain a quadrilateral. This edge originates from a vertex, and in fact we truncate this vertex. (This is an essentially hypergraphical case.)

(H°_{4441}) Next we have $H^\circ_{4441} = H'_{4341} \cup \{\{x, u\}\}$, which is the saturated closure of

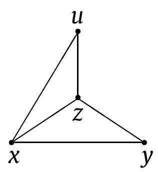


(see A° in Section 3). The picture for a realization of $\mathcal{A}(H^\circ_{4441})$, turned in the manner of the preceding three pictures, and with all labels omitted, is



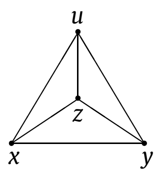
This picture permits a comparison with the picture of Example 5.13 of [13]. The pictured polytope is the three-dimensional cyclohedron (see [4] and [31, Section 4]).

(H_{4541}) Next we have $H_{4541} = H^{\circ}_{4441} \cup \{\{x, z\}\}$, which is the saturated closure of

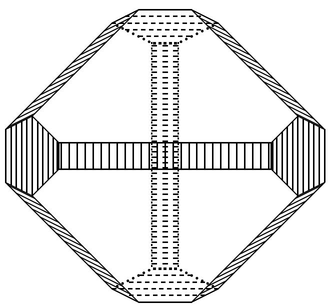


The picture for a realization of $\mathcal{A}(H_{4541})$ is obtained from the preceding picture of the three-dimensional cyclohedron, given for $\mathcal{A}(H_{4441})$, by truncating one more edge—in this case, the edge $\{\{x\}, \{z\}, \{x, y, z, u\}\}$, which is now in the north–east. This picture permits a comparison with the picture of Example 5.12 of [13], where the pictured polytope is called *hemicyclohedron* (see also [17, Fig. 10]).

(H_{4641}) Finally, we have $H_{4641} = H_{4541} \cup \{\{y, z\}\}$, which is the saturated closure of



Here we have selected for truncation all the vertices and all the edges of the tetrahedron. The picture for a realization of $\mathcal{A}(H_{4641})$, turned in the manner of the preceding four pictures, is

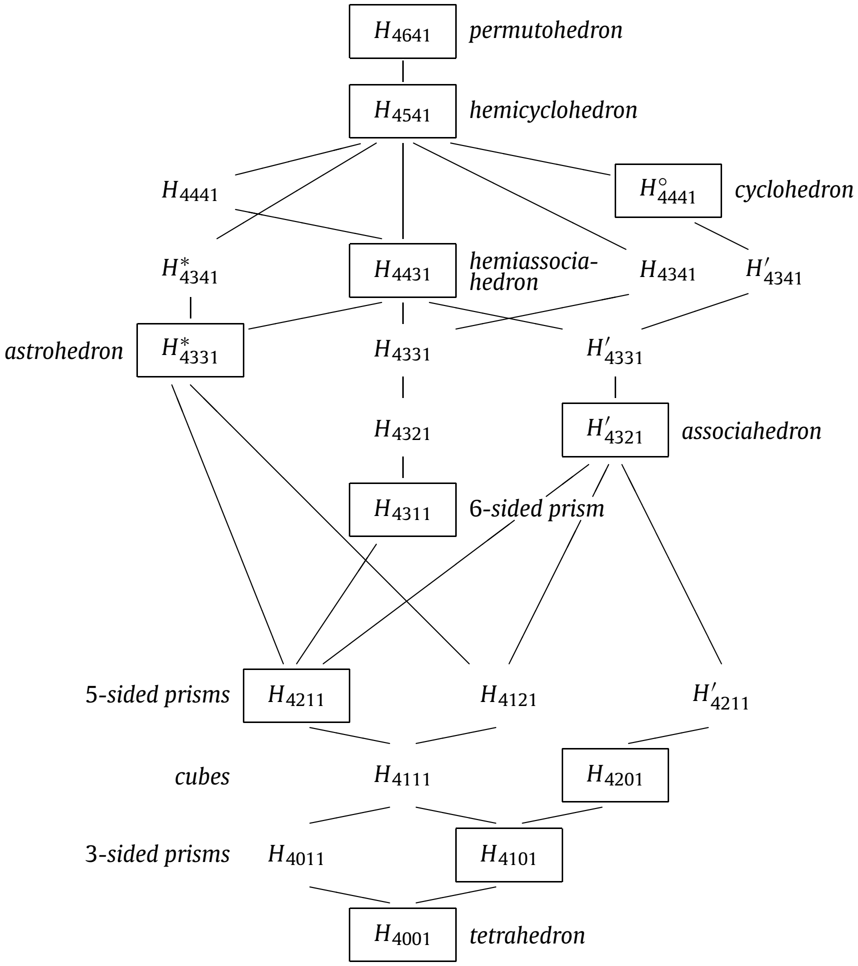


This picture permits a comparison with the picture of Example 5.11 of [13]. The pictured polytope is the three-dimensional permutohedron (see [35, Lecture 0, Example 0.10], and [19]). All its vertices are constructions of the type $\{\{x\}, \{x, y\}, \{x, y, z\}, \{x, y, z, u\}\}$; the corresponding s-construction is $uzyx$, i.e. a permutation of x, y, z and u.

As for every $k \geqslant 3$ we have that $\mathcal{A}(H_{k0\dots01})$ may be realized as the $(k-1)$-dimensional simplex, so, at the other end, with $\mathcal{A}(H_{kn_1\dots n_m1})$, where all the subscripts $n_1 \dots n_m$ are maximal (i.e. where $n_i = \binom{k}{i+1}$), we obtain the $(k-1)$-dimensional permutohedron. In between, with lesser values of $n_1, \dots, n_m$, but greater than the minimal values $0, \dots, 0$, we obtain the $(k-1)$-dimensional associahedron (the corresponding graph is a path of $k-1$ edges), the $(k-1)$-dimensional cyclohedron (the corresponding graph is a cycle of k edges and k vertices), and the $(k-1)$-dimensional astrohedron (the corresponding graph is a star-like graph with one vertex in the middle and $k-1$ vertices around joined by $k-1$ edges).

With $k = 3$, and dimension 2, the associahedron coincides with the astrohedron—both are the pentagon—and the cyclohedron with the permutohedron—both are the hexagon (see the cases of H_{321} and H_{331} above). In the degenerate case when k is 2, the simplex, the associahedron, the astrohedron and the permutohedron of dimension 1 all coincide; they are all a single edge with two incident vertices, which is the only polytope of dimension 1 (see the case of H_{21} above). If we take as in [20, Chapter 2] that a graph which is a cycle must have at least 3 vertices, then there is no one-dimensional cyclohedron; but we may stipulate by convention, as in [31, Section 4], that the one-dimensional cyclohedron is also a single edge with two incident vertices, and hence it coincides with the others. In the degenerate case when k is 1, we have again just one polytope of dimension 0; namely, a single vertex.

At the end, we give a chart of the types of hypergraphs corresponding to some of the polytopes encountered in this section, including those that are more interesting. (A chart with all the types would be too intricate.) A line in this chart is drawn when a hypergraph of one type is included in a hypergraph of another type. The labels of types in boxes are those of cases covered previously in [6,10,13] (these cases are not essentially hypergraphical). We have made complete the upper part of the chart above H^{*}_{4331}, H_{4311} and H'_{4321}, which involves the truncation of at least three edges. If H_{4311} and the seven points with labels not in boxes are omitted from this part of the chart, then we obtain (upside down) the chart of [13, after Example 5.16].



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