ELSEVIER
Journal of Pure and Applied Algebra 164 (2001) 275–305
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

The cohomology of the Sylow 2-subgroup of the Higman–Sims group

A. Adem[a,*,1], J.F. Carlson[b,2], D.B. Karagueuzian[a,3], R. James Milgram[c,4]

[a] *Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA*
[b] *Department of Mathematics, University of Georgia, Athens, GA 30602, USA*
[c] *Department of Mathematics, Stanford University, Stanford, CA 94305, USA*

Received 15 September 1999; received in revised form 18 September 2000
Communicated by E.M. Friedlander

Abstract

In this paper we compute the mod 2 cohomology of the Sylow 2-subgroup of the Higman–Sims group *HS*, one of the 26 sporadic simple groups. We obtain its Poincaré series as well as an explicit description of it as a ring with 17 generators and 79 relations.

MSC: 20J06; 55R35

1. Introduction

Recently, there has been substantial progress towards computing the mod 2 cohomology of low rank sporadic simple groups. In fact the mod 2 cohomology of every sporadic simple group not containing $(\mathbb{Z}/2)^5$ has been computed, with the notable exceptions of *HS* (the Higman–Sims group) and Co_3 (one of the Conway groups). The reason for these exceptions is that these two groups have large and complicated Sylow 2-subgroups, with many conjugacy classes of maximal elementary abelian subgroups. The Higman–Sims group has order $44\,352\,000 = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$, and its largest elementary abelian 2-subgroup is of rank equal to four. It is a subgroup of Co_3 of index equal to $11178 = 2 \cdot 3^5 \cdot 23$, hence $Syl_2(HS)$ is an index 2 subgroup of $Syl_2(Co_3)$.

* Corresponding author.
E-mail addresses: adem@math.wisc.edu (A. Adem), jfc@math.uga.edu (J.F. Carlson), dikran@math.wisc.edu (D.B. Karagueuzian), milgram@math.stanford.edu (R.J. Milgram).
[1] Partially supported by the NSF, NSA and CRM-Barcelona.
[2] Partially supported by the NSF.
[3] Partially supported by an NSF postdoctoral fellowship and the NSA.
[4] Partially supported by the NSF and CRM-Barcelona.

PII: S0022-4049(00)00170-5

In this paper, we compute the mod 2 cohomology ring of $S = Syl_2(HS)$, obtaining explicit generators and relations. In a sequel, we will determine the necessary stability conditions for computing the cohomology of *HS* itself. This is a step towards obtaining a calculation of the mod 2 cohomology of Co_3. This latter group is of particular interest because of its relation to a homotopy–theoretic construction due to Dwyer and Wilkerson (see [2]) and it would seem that the most viable way of accessing this group is via *HS*.

The calculation we present is long and highly technical, involving techniques from topology, representation theory and computer algebra. Our main result is the following

Theorem 1.1. *The mod* 2 *cohomology of* $S = Syl_2(HS)$ *has Poincaré series*

$$\frac{(1+x)^2(1-x+x^2)(1+2x-x^5)}{(1-x)^2(1-x^4)(1-x^8)}.$$

As a ring, $H^*(S, \mathbb{F}_2)$ *has* 17 *generators*, *in degrees*

$$1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 5, 6, 7, 8,$$

and a (*minimal*) *set of* 79 *relations.*

Remark 1.2. Nine of the generators above can be given as Stiefel–Whitney classes associated to some of the irreducible representations of S (see Section 3), although the remaining eight had to be determined by other means. The complete set of relations and the Steenrod operations on a representative set of generators are described in two appendices at the end of the paper.

We briefly outline our method of proof. The group S can be expressed as a semidirect product $(\mathbb{Z}/4)^3 : U_3$, where $U_3 \cong D_8$ is the group of upper triangular 3×3 matrices over $\mathbb{F}_2$. Using a recent result in [7], we calculate the E_2-term of the Lyndon–Hochschild–Serre spectral sequence associated to this extension. This involves using an explicit decomposition of the symmetric algebra of a module.

The second ingredient is a computer-assisted verification of the cohomology of S through degree 10. Combined with the structure of the E_2 term we obtain the following crucial result

Theorem 1.3. *The mod* 2 *spectral sequence associated to* $S = 4^3 : U_3$ *collapses at* E_2, *and yields the Poincaré series above.*

A critical aspect of this is that the generators for the E_2 term occur in low degrees, where the computer can provide enough information to ensure a collapse. This method seems to be quite effective for many cohomology computations and gives a very easy calculation of the mod 2 cohomology of U_4. Moreover, with somewhat more effort it yields the mod 2 cohomology of U_5 as well. This last result is particularly interesting since U_5 is the Sylow 2-subgroup of the two sporadics M_{24} and *He*.

The next step consists of computing the restriction map from $H^*(S, \mathbb{F}_2)$ to the cohomology of centralizers of rank two elementary abelian subgroups, where we obtain an image ring having the same Poincaré series as the cohomology. From this we conclude

Theorem 1.4. *The cohomology of* $S = Syl_2(HS)$ *is detected by the centralizers of rank two elementary abelian subgroups, and its explicit image is the ring described above, with* 17 *generators and* 79 *relations.*

Remark 1.5. There are nine such subgroups up to conjugacy in S. Explicit generators are described in Section 3 via their restrictions to these nine centralizers.

As mentioned above, the results here will be used in a sequel to calculate the cohomology of HS, and from there obtain information on the cohomology of Co_3, thus bringing us closer to a complete understanding of rank 4 sporadic simple groups.

Throughout this paper, coefficients will be in $\mathbb{F}_2$, so they are suppressed. Occasionally, k will be used to denote this coefficient ring, especially, if it appears in a representation–theoretic context. We refer the reader to [1] for background on group cohomology.

2. The subgroup structure of $S = Syl_2(HS)$

The Sylow 2-subgroup of the Higman–Sims group (denoted S from here on) has a description as a semi-direct product $S = 4^3 : D_8$ with v_1, v_2, v_3 the generators of the 4^3 while $D_8 = \{t, s \mid t^2 = s^4 = (ts)^2 = 1\}$, and

$$v_1^t = v_3^{-1}, \quad v_2^t = v_2^{-1}, \quad v_1^s = v_2, \quad v_2^s = v_3, \quad v_3^s = v_2^{-1} v_1 v_3.$$

In this section, we will develop the required details about the subgroup structure of S to enable us to understand its cohomology. In our notation, if $x, y \in G$ are elements in a group, then $x^y = yxy^{-1}$.

To start we have

Lemma 2.1. $\langle v_1 v_3 \rangle = \mathbb{Z}/4$ *is a normal subgroup of* S *and* $S/\langle v_1 v_3 \rangle \cong 2\wr 2 \wr 2$.

Proof. In the quotient we have $v_1^t = v_3^{-1} \sim v_1$, while $v_1^{ts^2} = v_1^{-1}$. Also $v_2^t = v_2^{-1}$ while $v_2^{ts^2} = v_2 v_3^{-1} v_1^{-1}$, which is equal to v_2 in the quotient. Thus,

$$\langle v_1, ts^2 \rangle \cong D_8, \quad \langle v_2, t \rangle \cong D_8$$

and these two copies of D_8 commute with each other, giving a copy of $D_8 \times D_8$ in the quotient. Next, note that ts exchanges t, ts^2, and also exchanges v_1, v_2^{-1}, hence, the two copies of D_8 above, and the extension $(D_8 \times D_8)$: $\langle ts \rangle \cong 2\wr 2 \wr 2$. $\square$

Definition 2.2. The group K_β is the inverse image in S of the index two subgroup $D_8 \times D_8 \subset S/\langle v_1 v_3 \rangle$.

K_β is given explicitly as an extension

$$\langle v_1, v_2, v_3\rangle\colon \langle t, s^2\rangle = 4^3 : 2^2.$$

Moreover, S is the split extension $K_\beta\colon 2 = K_\beta\colon \langle ts\rangle$.

Next, we examine the maximal elementary abelian subgroups in S. We show that there are precisely eight copies of $2^4 \subset S$. To begin note that there are exactly five conjugacy classes of maximal two-elementaries in $2\wr 2\wr 2$, first three conjugacy classes of 2^4's:

$$2^4_{\mathrm{I}} = \langle v_1^2, ts^2, v_2^2, t\rangle, \quad 2^4_{\mathrm{II}} = \langle v_1^2, v_1ts^2, v_2^2, v_2^{-1}t\rangle$$

both normal, and $2^4_{\mathrm{I,II}} = \langle v_1^2, ts^2, v_2^2, v_2^{-1}t\rangle$ with Weyl group 2^2. Then, there are two conjugacy classes of 2^3's, each with Weyl group D_8,

$$2^3_{\mathrm{I}} = \langle (v_1v_2)^2, s^2, ts\rangle, \quad 2^3_{\mathrm{II}} = \langle (v_1v_2)^2, v_1v_2^{-1}s^2, ts\rangle,$$

which together generate a copy of $D_8 \times 2$,

$$\Delta(D_8) \times 2 = \langle v_1v_2^{-1}s^2, ts\rangle.$$

This is all standard and can be found in many references. Lifting these groups to S we find

Lemma 2.3. *The following list describes the groups above and their lifts*:

$2^4_{\mathrm{I}}, \quad D_8 * D_8 * 4,$

$2^4_{\mathrm{II}}, \quad D_8 * D_8 * 4,$

$2^4_{\mathrm{I,II}}, \quad (8\colon Aut(8)) * 4,$

$\Delta(D_8) \times 2, \quad D_8 \times D_8.$

Proof. We note that the lift of 2^4_{I} is given as

$$\langle v_1v_3, v_1^2, ts^2, v_2^2, t\rangle = \langle v_1v_3^{-1}, ts^2, t, v_1^2, v_1v_3^{-1}v_2^2\rangle.$$

On the other hand, note that $v_1v_3^{-1}v_2^2$ commutes with s^2, t, consequently, with each of the remaining four generators, while the first two generators commute with the third and fourth, and $\langle v_1v_3^{-1}, ts^2\rangle = D_8$, $\langle t, v_1^2\rangle = D_8$, and all three have the central element $(v_1v_3)^2$ in common. The verification for the second group is similar. For the third, note that $(v_1s^2)^2 = v_1v_3$ so that v_1s^2 has order 8. Also conjugation with v_2^2 takes the element to its fifth power, while conjugation by t takes it to its inverse. This gives the extension $8\colon Aut(8)$. The final generator can again be choosen as $v_1v_3^{-1}v_2^2$.

It remains to check the lift of $\Delta(D_8) \times 2$. Note, first that $\langle s^2, v_1v_2^{-1}\rangle = D_8$ as a subgroup of S, while v_1v_3, ts both commute with the elements of this D_8. On the other hand, $\langle v_1v_3, ts\rangle = D_8$ as well, and the final statement follows. $\square$

Now, to determine the structure of the set of 2^4's in S we check the inverse images of the conjugates of 2^3_{I} and 2^3_{II}. The normalizer of each of these groups has order 2^6 so there are a total of 4 such groups in $2\wr 2\wr 2$, with the remaining two given by conjugation with v_1,

$$(2^3_{\mathrm{I}})^{v_1} = \langle v_1v_2ts, v_2v_3^{-1}s^2, (v_1v_2)^2\rangle, \quad (2^3_{\mathrm{II}})^{v_1} = \langle v_1v_2ts, v_2v_3^{-1}s^2, (v_1v_2)^3\rangle.$$

The lift of each to S is a copy of $2^2 \times D_8$ which contains exactly two copies of 2^4, so S has at least eight copies of 2^4. In fact, we have

Corollary 2.4. *There are no 2^5's contained in S. There are exactly eight copies of 2^4 contained in S, breaking up into three conjugacy classes, two with two copies each and one with four.*

Proof. The only thing that needs to be pointed out is that if there were a 2^4 which did not contain the central element $(v_1v_3)^2$, (and hence a 2^5 obtained by adjoining $(v_1v_3)^2$), then it would project non-trivially to one of the four 2^4's in the quotient $2\wr 2\wr 2$. But we have identified the lifts of these groups as copies of groups having rank three! Hence, every possible 2^4 contains $(v_1v_3)^2$, and hence projects to one of the two conjugacy classes of extremal 2^3's. Consequently, it must lie in one or the other of the four copies of $2^2 \times D_8 \subset S$ that we have constructed. $\square$

We now list the eight 2^4's explicitly.

$$2_A = \langle (v_1v_3)^2, (v_1v_2)^2, s^2, ts\rangle,$$

$$2_A^{v_1} = \langle (v_1v_3)^2, (v_1v_2)^2, v_1^{-1}v_3s^2, v_1v_2ts\rangle,$$

$$2_B = \langle (v_1v_3)^2, (v_1v_2)^2, v_1v_2^{-1}s^2, ts\rangle,$$

$$2_B^{v_1} = \langle (v_1v_3)^2, (v_1v_2)^2, v_2v_3^{-1}s^2, v_1v_2ts\rangle,$$

$$2_B^{t} = \langle (v_1v_3)^2, (v_1v_2)^2, v_3v_2^{-1}s^2, v_3v_2^{-1}ts\rangle,$$

$$2_B^{v_1t} = \langle (v_1v_3)^2, (v_1v_2)^2, v_1v_2^{-1}s^2, v_1v_3ts\rangle,$$

$$2_C = \langle (v_1v_3)^2, (v_1v_2)^2, s^2, v_1v_3ts\rangle,$$

$$2_C^{v_1} = \langle (v_1v_3)^2, (v_1v_2)^2, v_1v_3^{-1}s^2, v_2v_3^{-1}ts\rangle.$$

Lemma 2.5. *There is an outer automorphism $\alpha : S \to S$ which exchanges 2_A and 2_C.*

Proof. We use the description of S as $4^3{:}\, D_8$ where we write

$$D_8 = 2^2 : 2 = \langle ts, ts^{-1}\rangle{:}\ \langle t\rangle.$$

But we can replace this copy of D_8 by

$$\langle v_2v_3^{-1}ts, v_1v_2^{-1}ts^{-1}\rangle{:}\ \langle t\rangle$$

and α is defined as the identity on 4^3 and the correspondence above on D_8. $\square$

We show that there are precisely two conjugacy classes of $D_8 \times D_8 \subset S$.

Remark 2.6. The group $D_8 \times D_8$ constructed in the proof above as the lift of $\Delta(D_8) \times 2$ is

$$\langle s^2, v_1 v_2^{-1} \rangle \times \langle ts, v_1 v_3 \rangle.$$

We can also construct a second copy of

$$D_8 \times D_8 \subset S$$

as

$$\langle ts, v_2 v_3^{-1} \rangle \times \langle ts^{-1}, v_1 v_2^{-1} \rangle.$$

It is direct to check that the two $(D_8)^2$ above are not conjugate in S. In fact, the intersection of the second copy of $(D_8)^2$ and $\langle v_1 v_3 \rangle$ is just $\langle (v_1 v_3)^2 \rangle$ and its image in $2 \wr 2 \wr 2$ is easily seen to be $D_8 * D_8$. Thus, there are at least two conjugacy classes of $D_8 \times D_8$'s contained in S. Shortly, we will show that there are exactly two.

Lemma 2.7. *The span* $\langle 2_I^{e_1}, 2_J^{e_2} \rangle$ *is always one of the three groups* $D_8 \times D_8$, $2^2 \times D_8$, *or* $2^{2+4} = Syl_2(L_3(4))$.

(a) *It is* 2^{2+4} *if and only if* $I = J$ *and* $e_2 = e_1 v_1$. *Consequently, there are exactly three conjugacy classes of* 2^{2+4}*'s in S, two normal and one containing two elements.*

(b) *It is* $D_8 \times D_8$ *in case*

$$\langle 2_A, 2_C^{v_1} \rangle = \langle 2_B, 2_B^t \rangle$$

and their conjugates by v, and also in the cases

$$\langle 2_A, 2_B^{v_1} \rangle = \langle 2_C, 2_B^t \rangle$$

and their four conjugates by $2^2 = \langle t, v_1 \rangle$. *Consequently, there are exactly two conjugacy classes of* $D_8 \times D_8 \subset S$, *one containing four groups and the other containing* 2.

(c) *In each of the remaining cases it is* $2^2 \times D_8$, *and each* $2^2 \times D_8$ *is contained in a* $D_8 \times D_8$.

Proof. First, we check the result for 2_A, $2_A^{v_1}$. We have

$$(s^2 v_1^{-1} v_3 s^2)^2 = (v_1 v_3)^2, \quad (s^2 v_1 v_2 ts)^2 = (v_2 v_3)^2,$$
$$(tsv_1^{-1} v_3 s^2)^2 = (v_2 v_3)^2, \quad (tsv_1 v_2 ts)^2 = (v_1 v_2)^2$$

and this is a presentation of 2^{2+4}. The same calculations result for 2_C using the automorphism above. Moreover, ts commutes with $v_1 v_2^{-1}$ while s^2 inverts it. So this change cancels out in the squares for the pair $\langle 2_B, 2_B^{v_1} \rangle$, and we have verified that the groups asserted to be 2^{2+4}'s in fact are.

The remaining statements are now easily checked by comparing with the $D_8 \times D_8$'s already constructed above. But this gives a complete list of possible pairs and the result follows. $\square$

We show there are exactly two conjugacy classes of $2\wr 2\wr 2 \subset S$. Note that t normalizes $\langle 2_A, 2_C^{v_1}\rangle$, exchanging the two copies of D_8, so

$$\langle 2_A, 2_C^{v_1}, t\rangle \cong 2\wr 2\wr 2.$$

However, since s^2, $v_i^{e_1}v_j^{e_2}ts$ are not conjugate in S, it follows that no $D_8\times D_8$ in the second conjugacy class is contained in a $2\wr 2\wr 2$.

Corollary 2.8. *There are exactly four copies of $2\wr 2\wr 2$ contained in S forming two conjugacy classes with conjugation by v_1 exchanging the groups in each class.*

Proof. The normalizer of $\langle 2_B, 2_C^{v_1}\rangle$ is obtained by adjoining t, v_2^2. Thus there are three degree two extensions of $\langle 2_B, 2_C^{v_1}\rangle$ in S. The extension by t is $2\wr 2\wr 2$. Clearly, the extension by v_2^2 does not give a $2\wr 2\wr 2$.

Finally, consider the extension by tv_2^2. Replace the second D_8 by

$$\langle (v_2v_3)^2ts^{-1}, v_1v_2^{-1}\rangle.$$

Then these two copies of D_8 commute with each other and their span is $D_8\times D_8$. Moreover, tv_2^2 exchanges them, giving an isomorphism of this group with a second copy of $2\wr 2\wr 2$. □

Remark 2.9. There is an automorphism of S fixing $\langle v_1, v_2, v_3\rangle$ and exchanging the two conjugacy classes of $2\wr 2\wr 2$'s constructed above. Indeed, such an automorphism can be given by setting

$$t \leftrightarrow tv_2^2, ts \leftrightarrow ts, \quad ts^{-1} \leftrightarrow ts^{-1}(v_2v_3)^2.$$

There are exactly six conjugacy classes of maximal $2^3 \subset S$.

A computer analysis of S using MAGMA or alternatively a direct analysis of the two copies of $D_8 * D_8 * 4$ and the $(8{:}\,Aut(8)) * 4$ obtained as the lifts of the three 2^4's in $2\wr 2\wr 2$ shows that the maximal elementary abelian subgroups of S consist of the three 2^4's discussed above and six copies of 2^3, all contained in the subgroup K_β: one given as $I = \langle v_1^2, v_2^2, v_3^2\rangle$ with centralizer $CI = 4^3$, and the other five all with centralizer of the form $2^2\times 4$. These centralizers are

$$C\mathrm{II} = \langle v_1v_3^{-1}v_2^2, ts^2, v_3^2\rangle,$$

$$C\mathrm{III} = \langle v_1v_3^{-1}v_2^2, t, s^2\rangle,$$

$$C\mathrm{IV} = \langle v_1v_3^{-1}v_2^2, s^2, v_1v_3t\rangle,$$

$$C\mathrm{V} = \langle v_1v_3^{-1}v_2^2, v_1^2, v_1ts^2\rangle,$$

$$C\mathrm{VI} = \langle v_1v_3^{-1}v_2^2, v_2t, v_2v_3^{-1}s^2\rangle$$

with the 2^3 subgroups denoted II–VI respectively.

The group-theoretic information which we have described in this section, will be used subsequently to establish a detection theorem for $H^*(S)$.

3. Explicit detection and Stiefel–Whitney classes

In this section, we describe the 17 generators of $H^*(S)$ mentioned in Theorem 1.1 in terms of their explicit restrictions to the cohomology of the nine detecting subgroups, which completely determines them in view of Theorem 1.4. It turns out that nine of the generators can be given as Stiefel–Whitney classes though the remaining eight had to be determined by the computer. For the nine Stiefel–Whitney classes we will be very explicit; for the remaining generators we simply give convenient representatives.

We use the following notation for the cohomology of the subgroups:

$$H^*(2^m) = \mathbb{F}_2[l_1, \ldots, l_m]$$

(where each l_i is one-dimensional);

$$H^*(4^3) = \mathbb{F}_2[b_1, b_2, b_3] \otimes \Lambda(e_1, e_2, e_3)$$

(where the e_i are one-dimensional, and b_i is the Bockstein of e_i); and

$$H^*(2^2 \times 4) = \Lambda(e) \otimes \mathbb{F}_2[l_2, l_3, b]$$

(where $|e| = 1,\ b = \beta(e)$ and $|l_2| = |l_3| = 1$).

One of the most powerful ways of constructing elements in cohomology is to use representations, i.e., homomorphisms $r_I : S \to GL_n(\mathbb{R})$, which, in turn, induce maps of classifying spaces,

$$B_{r_I} : B_S \to B_{GL_n(\mathbb{R})}$$

and pull back the Stiefel–Whitney classes. The restriction images of these cohomology classes in the detecting groups are then determined by taking the Stiefel–Whitney classes of the restrictions of the r_I. Here the determinations of the Stiefel–Whitney classes are standard (see [8,9]).

The real group-ring $\mathbb{R}(S)$ splits into 33 simple summands:

$$\mathbb{R}(S) = 8\mathbb{R} \oplus 12M_2(\mathbb{R}) \oplus M_2(\mathbb{C}) \oplus 8M_4(\mathbb{R}) \oplus 3M_8(\mathbb{R}) \oplus M_8(\mathbb{C}),$$

but we will only need a small number of these representations.

To begin consider the real representation r_4. Here r_4 is described by giving matrix images for the generators of the group:

$$v_1 \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad v_2 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad v_3 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$r_4(s) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad r_4(t) = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix},$$

where K is the 2×2 matrix

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

When we restrict to 2_A we have that the first two generators map to I, while

$$s^2 \mapsto \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Consequently, when we diagonalize the image of s^2 to

$$\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix},$$

we see that ts can also be diagonalized to

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so the representation becomes the sum of four one-dimensional representations, and the total Stiefel–Whitney class becomes

$$(1 + l_3 + l_4)(1 + l_3) = 1 + l_4 + l_3(l_3 + l_4).$$

Next, we consider CI. Here we see at once that the total Stiefel–Whitney class is

$$(1 + e_1)(1 + e_3)(1 + e_2 + e_3)(1 + e_1 + e_2) = 1 + e_2(e_1 + e_3).$$

As a final example of how to calculate these restrictions, consider CIII. Here, $v_1 v_3^{-1} v_2^2 \mapsto -I$ while the images of s^2 and t have already been discussed. Diagonalizing we have that the total Stiefel–Whitney class of the restriction is

$$\begin{aligned}&(1 + e)(1 + e + l_2)(1 + e + l_3)(1 + e + l_2 + l_3)\\ &\quad = 1 + d_2(l_2, l_3) + (1 + e)d_3(l_2, l_3).\end{aligned}$$

Here, $d_2(x, y) = x^2 + xy + y^2$ while $d_3(x, y) = x^2 y + xy^2 = xy(x + y)$ are the Dickson elements.

The other representations that will be needed are first, an eight-dimensional representation, r_8, which is very similar to the four-dimensional representation considered above:

$$v_1 \mapsto \begin{pmatrix} J & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & J \end{pmatrix}, \quad v_2 \mapsto \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & J & 0 \\ 0 & 0 & 0 & J \end{pmatrix}, \quad v_3 \mapsto \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & J & 0 & 0 \\ 0 & 0 & J & 0 \\ 0 & 0 & 0 & I \end{pmatrix},$$

$$s \mapsto \begin{pmatrix} 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 0 & K & 0 & 0 \\ K & 0 & 0 & 0 \\ 0 & 0 & 0 & K \\ 0 & 0 & K & 0 \end{pmatrix},$$

where I is the 2×2 identity matrix and

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The remaining representations we need are two dimensional: the first of these, $r_{2,1}$, has the form

$$v_i \mapsto J, \quad 1 \le i \le 3, \quad s \mapsto J, \quad t \mapsto K,$$

the second, $r_{2,2}$, is

$$v_i \mapsto J, \quad 1 \le i \le 3, \quad s \mapsto I, \quad t \mapsto K$$

and the third, $r_{2,3}$, is

$$v_1 \mapsto J, \quad v_2 \mapsto -J, \quad v_3 \mapsto J, \quad s \mapsto K, \quad t \mapsto K.$$

Finally, we should mention the one-dimensional representations which all factor in the form

$$S/S' = \langle v_1, s, t \rangle \cong 2^3 \to \mathbb{Z}/2 = \{\pm 1\}.$$

The first Stiefel–Whitney classes of $\langle v_1 \rangle$, $\langle s \rangle$, $\langle t \rangle$, respectively, give the one-dimensional generators for $H^*(S)$ while the second Stiefel–Whitney classes of the three 2-dimensional representations above, together with the four-dimensional representation, r_4, give the two-dimensional generators.

The four 3-dimensional generators do not occur as Stiefel–Whitney classes, and were obtained by a computer calculation using MAGMA as described previously.

The Stiefel–Whitney classes w_4 and w_8 for the eight-dimensional representation are generators.

The five-dimensional generator is Sq^2 of one of the computer generated three-dimensional generators. The remaining four-dimensional generator, n, as well as the seven-dimensional generator, i, also had to be determined by MAGMA, and the six-dimensional generator can be given as $Sq^2(n)$.

Tables 1–3 give the restrictions of the generators discussed above. where $d_2(2,3) = l_2^2 + l_2 l_3 + l_3^2$, $d_3(2,3) = l_2 l_3 (l_2 + l + 3)$, $d_4(2,3,4)$ is the fourth Dickson invariant in the three classes l_2, l_3, and l_4, while

$$L = b^4 + b^2 d_2(2,3)^2 + b d_3(2,3)^2,$$

$$M = l_1^8 + l_1^4 d_4(2,3,4) + l_1^2 Sq^2(d_4(2,3,4)) + l_1 Sq^3(d_4(2,3,4)).$$

Table 1
Restrictions of one-dimensional Stiefel–Whitney classes

	$w_1(v)$	$w_1(s)$	$w_1(t)$
CI	$e_1+e_2+e_3$	0	0
CII	0	0	l_2
CIII	0	0	l_2
CIV	0	0	l_3
CV	l_3	0	l_3
CVI	l_2	0	l_2
2_A	0	l_4	l_4
2_B	0	l_4	l_4
2_C	0	l_4	l_4

Table 2
Restrictions of two-dimensional Stiefel–Whitney classes

	$w_2(r_{2,1})$	$w_2(r_{2,2})$	$w_2(r_{2,3})$	$w_2(r_4)$
CI	$b_1+b_2+b_3$	$b_1+b_2+b_3$	$b_1+b_2+b_3$	$e_2(e_1+e_3)$
CII	$el_2+l_3(l_2+l_3)$	$el_2+l_3(l_2+l_3)$	$el_2+l_3(l_2+l_3)$	l_2^2
CIII	$el_2+l_3(l_2+l_3)$	$el_2+l_3(l_2+l_3)$	el_2	$d_2(2,3)$
CIV	$el_3+l_2(l_2+l_3)$	$el_3+l_2(l_2+l_3)$	el_3	$d_2(2,3)$
CV	$el_3+l_2(l_2+l_3)$	$el_3+l_2(l_2+l_3)$	$el_3+l_2(l_2+l_3)$	l_3^2
CVI	$el_2+l_3(l_2+l_3)$	$el_2+l_3(l_2+l_3)$	$el_2+l_3(l_2+l_3)$	$d_2(2,3)$
2_A	$l_3(l_3+l_4)$	0	0	$l_3(l_3+l_4)$
2_B	$l_3(l_3+l_4)$	0	l_3^2	$l_3(l_3+l_4)$
2_C	$l_3(l_3+l_4)$	0	l_4^2	$l_3(l_3+l_4)$

Table 3
Restrictions of w_4 and w_8 for the eight-dimensional representation

	w_4	w_8
CI	$(b_1+b_2+b_3)^2+b_2(b_1+b_3)$	$b_1b_3(b_1+b_2)(b_2+b_3)$
CII	$d_2(2,3)^2$	L
CIII	$d_2(2,3)^2$	L
CIV	$d_2(2,3)^2$	L
CV	$d_2(2,3)^2$	L
CVI	$d_2(2,3)^2$	L
2_A	$d_4(2,3,4)$	M
2_B	$d_4(2,3,4)$	M
2_C	$d_4(2,3,4)$	M

Next, we give the restrictions of the computer-generated indecomposables, s, r, p, and q, in dimension three.

$$\begin{array}{c} \\ C\mathrm{I} \\ C\mathrm{II} \\ C\mathrm{III} \\ C\mathrm{IV} \\ C\mathrm{V} \\ C\mathrm{VI} \\ 2_A \\ 2_B \\ 2_C \end{array} \begin{array}{c} \begin{array}{cc} \qquad\qquad s \qquad\qquad & \qquad\qquad r \qquad\qquad \end{array} \\ \begin{pmatrix} e_1e_2e_3 & e_1e_2e_3 \\ 0 & e_1l_2^2 \\ 0 & e_1l_2^2+e_1l_2l_3+e_1l_3^2+l_2^2l_3+l_2l_3^2 \\ 0 & e_1l_2^2+e_1l_2l_3+e_1l_3^2 \\ e_1l_3^2 & 0 \\ e_1l_2^2+e_1l_2l_3+e_1l_3^2+l_2l_3^2+l_3^3 & 0 \\ 0 & 0 \\ l_3^3+l_3^2l_4 & 0 \\ 0 & l_3^2l_4+l_3l_4^2 \end{pmatrix} \end{array},$$

$$\begin{array}{c} \\ C\mathrm{I} \\ C\mathrm{II} \\ C\mathrm{III} \\ C\mathrm{IV} \\ C\mathrm{V} \\ C\mathrm{VI} \\ 2_A \\ 2_B \\ 2_C \end{array} \begin{array}{c} \begin{array}{cc} \qquad\qquad q \qquad\qquad & \qquad\qquad p \qquad\qquad \end{array} \\ \begin{pmatrix} e_1e_2e_3+e_2b_3+e_3b_2 & e_1e_2e_3 \\ 0 & e_1l_2^2 \\ 0 & e_1l_2^2+l_2^2l_3+l_2l_3^2 \\ 0 & e_1l_3^2 \\ e_1l_3^2 & 0 \\ e_1l_2^2+e_1l_2l_3+e_1l_3^2+l_2l_3^2+l_3^3 & e_1l_2l_3+e_1l_3^2 \\ 0 & l_1^2l_4+l_1l_4^2+l_2^2l_3+l_2l_3^2+l_3^2l_4+l_3l_4^2 \\ l_3^3+l_3^2l_4 & l_1^2l_4+l_1l_4^2+l_2^2l_3+l_2l_3^2+l_3^2l_4+l_3l_4^2 \\ 0 & l_1^2l_4+l_1l_4^2+l_2^2l_3+l_2l_3^2 \end{pmatrix} \end{array}.$$

Note here that the computer has not always picked the simplest choices. For example, $s+q$ restricts to $e_2b_3+e_3b_2$ in CI and 0 in the remaining eight centralizers.

Here is the expansion of the computer-generated indecomposable, n in dimension four:

$$\begin{array}{c} \\ C\mathrm{I} \\ C\mathrm{II} \\ C\mathrm{III} \\ C\mathrm{IV} \\ C\mathrm{V} \\ C\mathrm{VI} \\ 2_A \\ 2_B \\ 2_C \end{array} \begin{array}{c} n \\ \begin{pmatrix} e_1e_2b_3+e_1e_3b_2+e_2e_3b_1+e_2e_3b_3 \\ e_1l_2^3+e_1l_2^2l_3+e_1l_2l_3^2 \\ e_1l_2^3+l_2^3l_3+l_2^2l_3^2 \\ e_1l_2^2l_3+e_1l_2l_3^2+e_1l_3^3 \\ e_1l_2^2l_3+e_1l_2l_3^2 \\ e_1l_2l_3^2+e_1l_3^3+l_2^2l_3^2+l_3^4 \\ 0 \\ l_1^2l_3l_4+l_1l_3l_4^2+l_2^2l_3^2+l_2l_3^3+l_3^4+l_3^3l_4+l_3^2l_4^2 \\ l_1^2l_4^2+l_1l_4^3+l_2^2l_3l_4+l_2l_3^2l_4+l_4^4 \end{pmatrix} \end{array}.$$

This element, together with w_4 for the eight-dimensional representation can be taken as the indecomposable generators in dimension four. Finally, for the generator i in degree seven the restrictions are given as follows. For CI the restriction is given as

$$\begin{aligned} &e_1e_2e_3b_3^2+e_1b_2^2b_3+e_1b_2b_3^2+e_2b_1^2b_3+e_2b_1b_3^2+e_2b_2^2b_3+e_2b_2b_3^2+e_3b_1^2b_2\\ &\quad+e_3b_1b_2^2+e_3b_2^3+e_3b_2^2b_3+e_3b_3^3, \end{aligned}$$

while for CII–CVI we have

$$\begin{array}{c} \\ C\mathrm{II} \\ C\mathrm{III} \\ C\mathrm{IV} \\ C\mathrm{V} \\ C\mathrm{VI} \end{array} \begin{array}{c} i \\ \left(\begin{array}{c} e_1l_2^6+e_1l_2^5l_3+e_1l_2^4l_3^2+l_2^4l_3^3+l_2^3l_3^4+l_2^2l_3^5+l_2l_3^6 \\ e_1l_2^6+e_1l_2^4l_3^2+e_1l_2^3l_3^3+e_1l_2l_3^5+e_1l_3^6+l_2^6l_3+l_2^4l_3^3+l_2^2l_3^5+l_2l_3^6 \\ e_1l_2^6+e_1l_2^5l_3+e_1l_2^3l_3^3+e_1l_2l_3^5+e_1l_3^6 \\ e_1l_2^2l_3^4+e_1l_2l_3^5+e_1l_3^6 \\ e_1l_2^6+e_1l_2^2l_3^4+e_1l_2l_3^5+l_2^2l_3^5+l_2l_3^6 \end{array}\right) \end{array}.$$

The restrictions to 2_A, 2_B, and 2_C are very long and complicated. For 2_A we obtain

$$\begin{aligned} &l_1^2l_2^4l_4+l_1^2l_2^2l_3^2l_4+l_1^2l_2^2l_3l_4^2+l_1^2l_2^2l_4^3+l_1^2l_2l_3^2l_4^2+l_1^2l_2l_3l_4^3+l_1l_2^4l_4^2+l_1l_2^2l_3^2l_4\\ &\quad+l_1l_2^2l_3l_4^3+l_1l_2^2l_4^4+l_1l_2l_3^2l_4^3+l_1l_2l_3l_4^4+l_2^6l_3+l_2^5l_3^2+l_2^4l_3^3+l_2^4l_3^2l_4+l_2^4l_3l_4^2\\ &\quad+l_2^4l_4^3+l_2^3l_3^4+l_2^2l_3^4l_4+l_2^2l_3^3l_4^2+l_2^2l_3^2l_4^3+l_2^2l_3l_4^4+l_2^2l_4^5+l_2l_3^2l_4^4+l_2l_3l_4^5. \end{aligned}$$

For 2_B the restriction is

$$\begin{aligned} &l_1^4l_3^2l_4+l_1^2l_2^4l_4+l_1^2l_2^2l_3^2l_4+l_1^2l_2^2l_3l_4^2+l_1^2l_2^2l_4^3+l_1^2l_2l_3^2l_4^2+l_1^2l_2l_3l_4^3+l_1^2l_3^4l_4\\ &\quad+l_1^2l_3^2l_4^3+l_1l_2^4l_4^2+l_1l_2^2l_3^2l_4^2+l_1l_2^2l_3l_4^3+l_1l_2^2l_4^4+l_1l_2l_3^2l_4^3+l_1l_2l_3l_4^4\\ &\quad+l_1l_3^4l_4^2+l_2^6l_3+l_2^5l_3^2+l_2^4l_3^3+l_2^4l_4^3+l_2^3l_3^4+l_2^2l_3^4l_4+l_2^2l_3^3l_4^2+l_2^2l_3^2l_4^3\\ &\quad+l_2^2l_4^5+l_2l_3^5l_4+l_2l_3^4l_4^2+l_2l_3l_4^5+l_3^6l_4+l_3^3l_4^4. \end{aligned}$$

For 2_C the restriction is

$$\begin{aligned} &l_1^4l_4^3+l_1^2l_2^4l_4+l_1^2l_2^2l_3^2l_4+l_1^2l_2^2l_3l_4^2+l_1^2l_2^2l_4^3+l_1^2l_2l_3^2l_4^2+l_1^2l_2l_3l_4^3+l_1^2l_3^2l_4^3+l_1^2l_3l_4^4\\ &\quad+l_1l_2^4l_4^2+l_1l_2^2l_3^2l_4^2+l_1l_2^2l_3l_4^3+l_1l_2^2l_4^4+l_1l_2l_3^2l_4^3+l_1l_2l_3l_4^4+l_1l_3^2l_4^4\\ &\quad+l_1l_3l_4^5+l_1l_4^6+l_2^6l_3+l_2^5l_3^2+l_2^4l_3^3+l_2^4l_3l_4^2+l_2^3l_3^4+l_2^2l_3^2l_4^3+l_2^2l_3l_4^4+l_2l_3^4l_4^2\\ &\quad+l_2l_3^3l_4^3+l_2l_3^2l_4^4+l_3^6l_4+l_3^5l_4^2+l_3^4l_4^3+l_3^3l_4^4. \end{aligned}$$

In the next section, we begin the proof of our main theorem.

4. Preliminaries on modules

We now turn to the proof of Theorem 1.1. As our intial step we have to determine $H^*(4^3)$ as a module over $\mathbb{F}_2(U_3)$.

Let $U_3 \cong D_8$ denote the Sylow 2-subgroup of $L_3(2)$. In this section we describe the symmetric algebra of the natural U_3 module M by giving a "factorization" of this algebra. This information will be used to describe the E_2 term of a spectral sequence converging to $H^*(S)$. In order to describe this factorization we must make a number of definitions and recall a few results. We will be using ideas and methods from [7]. We write the symmetric algebra as $k[x,y,z]$, where the action of U_3 on the homogeneous polynomials of degree one preserves the flag of subspaces $\langle x\rangle \subset \langle x,y\rangle \subset \langle x,y,z\rangle$. Let us denote by b,c,d the elements of U_3 which send $y \mapsto y+x$, $z \mapsto z+x$, and $z \mapsto z+y$, respectively. (b is supposed to fix z, c to fix y and d to fix x.) We write H_1 for the subgroup $\langle b,c\rangle$ and H_2 for the subgroup $\langle d\rangle$.

Remark 4.1. It is worth noting that the module M is *not* isomorphic to M^*, but that the two differ by an outer automorphism of the group U_3. This means that the symmetric algebras $S^*(M)$ and $S^*(M^*)$ also differ by an outer automorphism of the group; thus if we are only interested in Poincaré series it does not matter which one we study. However, it is *not* true that $S^*(M)^* \approx S^*(M^*)$.

Definition 4.2. $d_1 := x, d_2 := y^2 + xy, d_4 := z(z+y)(z+x)(z+y+x)$.

The following result is well known:

Proposition 4.3. $k[x,y,z]^{U_3} = k[d_1,d_2,d_4]$.

Definition 4.4. $\phi := z^2 + yz$, $\theta := z^2 + xz$. We denote by N, K, the U_3-submodule of the symmetric algebra generated by ϕ, θ, respectively.

Lemma 4.5. *N is isomorphic to the permutation module $k[U_3/H_2]$ and has socle d_1^2. K is isomorphic to the permutation module $k[U_3/H_1]$ and has socle d_2.*

We will now exhibit three U_3-submodules of $k[x,y,z]$.

Definition 4.6.

$$A := k[d_4], \quad B := k \oplus K \oplus d_2K \oplus d_2^2K \oplus \cdots ,$$

$$C := k \oplus M \oplus N \oplus d_1N \oplus d_1^2N \oplus d_1^3N \oplus \cdots .$$

Note that A is a $k[d_4][U_3]$-submodule of $k[x,y,z]$, while B is a $k[d_2][U_3]$-submodule and C a $k[d_1][U_3]$-submodule. (The only point to check is that $d_1M \subset N$, but this can be done by working directly with the generating polynomials.)

Now we can describe the "factorization" mentioned above:

Proposition 4.7. *There is an isomorphism of U_3-modules*

$$A \otimes B \otimes C \to k[x,y,z],$$

induced by multiplication in the symmetric algebra.

Proof. For any triple of indecomposable modules A_i, B_j, C_l appearing in the direct sum decompositions of A, B, and C above, we have $soc(A_i \otimes B_j \otimes C_l) = soc(A_i) \otimes soc(B_j) \otimes soc(C_l)$. Thus, $soc(A \otimes B \otimes C) = soc(A) \otimes soc(B) \otimes soc(C) = k[d_4] \otimes k[d_2] \otimes k[d_1]$. This shows that the map above is an isomorphism on the socle, hence it is injective. The Poincaré series of the two sides are equal, by a calculation, so we have an isomorphism. □

Remark 4.8. No analogous factorization exists for larger fields or for more variables.

We can rewrite this factorization as a direct sum decomposition

Proposition 4.9. *There is an isomorphism of* U_3*-modules*:

$$\begin{aligned} k[x,y,z] = {} & k[d_4] \otimes (k \oplus M) \\ & \oplus k[d_1, d_4] \otimes N \\ & \oplus k[d_2, d_4] \otimes (K \oplus W) \\ & \oplus k[d_1, d_2, d_4] \otimes F. \end{aligned}$$

Here $F = N \otimes K$ *is a free* U_3*-module, appearing in degree four, while* $k, M, N, W = M \otimes K$, *and* K *appear in degrees* 0, 1, 2, 3, *and* 2, *respectively.*

5. The Sylow 2-subgroup of the Higman–Sims group: modules and cohomology

As we mentioned, the Sylow 2-subgroup can be written as the semidirect product $(\mathbb{Z}_4)^3 \rtimes U_3$, where the action of U_3 on the vector space $H^1((\mathbb{Z}_4)^3)$ gives this cohomology group the structure of the natural U_3-module M. It follows that as a U_3-module, $H^*((\mathbb{Z}_4)^3) = \Lambda^*(M) \otimes S^*(M)$.

In this section, we compute the Poincaré series of $H^*(U_3; \Lambda^*(M) \otimes S^*(M))$. We will also try to find a bound on the degrees of generating elements. To do this, we must compute the Poincaré series of the cohomologies of all the U_3-modules appearing in the decomposition of $\Lambda^*(M) \otimes S^*(M)$. Since $\Lambda^*(M) = k \oplus M \oplus M^* \oplus k$, it is enough to compute the cohomologies of the modules appearing in decomposition (4.9) of $S^*(M)$, plus the cohomologies of the tensor products of these modules with M and M^*.

The first thing to understand is how the tensor products of the modules k, M, N, K, W, and F with the modules M and M^* break up into direct sums of indecomposables. The answers are displayed in Table 4. A couple of new modules appear; they are described below the table.

The module X_{10} fits in an exact sequence $X_{10} \hookrightarrow F \oplus F \twoheadrightarrow W$. The module Y_9 is by definition $M \otimes M$, and there is an exact sequence $M \hookrightarrow T \oplus F \twoheadrightarrow Y_9$. This second exact sequence is constructed by taking the exact sequence $k \hookrightarrow T \twoheadrightarrow M$ and applying $- \otimes_k M$. Here we note that T is the permutation module $U_3/\langle b \rangle$.

The next step is to compute cohomologies for each of the modules listed in the table; we will need Poincaré series and degrees of generators for the cohomologies $H^*(U_3, X)$

Table 4

$\otimes$	M	M^*
k	M	M^*
M	Y_9	$F \oplus k$
N	$F \oplus N$	$F \oplus N$
K	W	W^*
W	$F \oplus X_{10}$	$F^{\oplus 2} \oplus K$
F	$F^{\oplus 3}$	$F^{\oplus 3}$

Table 5

Module	Poincaré series	Bound on generator degrees
k	$\frac{1}{(1-x)^2}$	0
M	$\frac{1}{(1-x)^2}$	1
M^*	$\frac{1}{(1-x)^2}$	1
N	$\frac{1}{(1-x)}$	0
K	$\frac{1}{(1-x)^2}$	1
W	$1+\frac{x}{(1-x)^2}$	2
W^*	$\frac{2-x}{(1-x)^2}$	1
F	1	0
Y_9	$\frac{2-2x-x^2}{(1-x)^2}$	2
X_{10}	$2+x+\frac{x^2}{(1-x)^2}$	3

as modules over the cohomology $H^*(U_3, k)$, for each indecomposable module X. We display the results in the form of Table 5 and give the methods of proof afterwards.

Now, we present brief arguments for these cohomology computations.

In the case of a permutation module, i.e. the cases k, N, K, and F, the cohomology is just the cohomology of the appropriate subgroup, regarded as a module over the cohomology of U_3 via the restriction map. The degrees of the module generators can be worked out from a knowledge of the restriction image.

Now we turn to the more subtle cases. First, let us note that there is an exact sequence $k \hookrightarrow T \twoheadrightarrow M$. In fact, more is true, there is also an exact sequence $M \hookrightarrow N \twoheadrightarrow k$, and by dualizing we can get exact sequences for M^*. We will just work with the first exact sequence.

Lemma 5.1. *The cohomology of M, i.e.* $H^*(U_3, M)$, *has Poincaré series* $(1-x)^{-2}$ *and is generated by elements in degrees* 0 *and* 1.

Proof. Consider the long exact sequence in cohomology arising from the short exact sequence $k \hookrightarrow T \twoheadrightarrow M$. The maps $H^i(U_3, k) \to H^i(U_3, T)$ are just the restriction homomorphisms for $\langle b \rangle = \mathbb{Z}_2 \subset U_3$, which can be shown to be surjective in all degrees i. This implies that $H^i(U_3, M)$ is isomorphic to the kernel of the restriction map $H^{i+1}(U_3, k) \to H^{i+1}(U_3, T)$ for all $i \geqslant 0$. This means that we can write the Poincaré series of the cohomology of M as

$$x^{-1}\left(\frac{1}{(1-x)^2}-\frac{1}{(1-x)}\right)=\frac{1}{(1-x)^2}.$$

The module generators may be taken to be in degrees 0, 1. □

Lemma 5.2. *The cohomology of* M^*, *i.e.* $H^*(U_3,M^*)$, *has Poincaré series* $(1-x)^{-2}$ *and is generated by elements in degrees* 0 *and* 1.

Proof. M and M^* are interchanged by an outer automorphism of U_3. □

Lemma 5.3. $H^*(U_3,W)$, *has Poincaré series* $1+x(1-x)^{-2}$ *and is generated by elements in degrees* ≤ 2.

Proof. There is an exact sequence $W \hookrightarrow F \twoheadrightarrow K$. □

Lemma 5.4. $H^*(U_3,W^*)$, *has Poincaré series* $(2-x)(1-x)^{-2}$ *and is generated by elements in degrees* ≤ 1.

Proof. There is an exact sequence $K \hookrightarrow F \twoheadrightarrow W^*$. □

Lemma 5.5. $H^*(U_3,Y_9)$, *has Poincaré series* $(2-2x+x^2)(1-x)^{-2}$ *and is generated by elements in degrees* ≤ 2.

Proof. There is an exact sequence $M \hookrightarrow T\otimes M^* \twoheadrightarrow Y_9$, and we consider the associated long exact sequence in cohomology. Since $T\otimes M^* \approx T\oplus F$, we see that $H^i(U_3,T\oplus F)= H^i(U_3,T)$ is one dimensional if $i>0$. Furthermore, the induced maps $H^i(U_3,M)\to H^i(U_3,T)$ are surjective for all $i>0$. This, plus the fact that the socle of Y_9 is two dimensional, determines the Poincaré series. For the information on the degrees of the generators, we must study the kernel of the map $H^*(U_3,M)\to H^*(U_3,T)$. This kernel is generated by elements in degrees ≤ 2. □

Lemma 5.6. $H^*(U_3,X_{10})$, *has Poincaré series* $2+x+x^2(1-x)^{-2}$ *and is generated by elements in degrees* ≤ 3.

Proof. Use the exact sequence $X_{10} \hookrightarrow F\oplus F \twoheadrightarrow W$. □

6. The E_2-term for the Sylow 2-subgroup of the Higman–Sims group

The algebraic computations in the preceding section will now be assembled to describe the E_2-term of the Lyndon–Hochschild–Serre spectral sequence associated to the group extension $S=4^3:D_8$. It can of course be described precisely as $H^*(U_3,\Lambda^*(M)\otimes S^*(M))$.

Table 6

Module	Decomposition	Poincaré series	Degree bound
k	k	$\frac{1}{(1-x)^2}$	0
M	M	$\frac{1}{(1-x)^2}$	1
M^*	M^*	$\frac{1}{(1-x)^2}$	1
$M \otimes M$	Y_9	$\frac{2-2x+x^2}{(1-x)^2}$	2
$M \otimes M^*$	$F \oplus k$	$1 + \frac{1}{(1-x)^2}$	0
N	N	$\frac{1}{(1-x)}$	0
$N \otimes M$	$F \oplus N$	$1 + \frac{1}{(1-x)}$	0
$N \otimes M^*$	$F \oplus N$	$1 + \frac{1}{(1-x)}$	0
K	K	$\frac{1}{(1-x)^2}$	1
$K \otimes M$	W	$1 + \frac{x}{(1-x)^2}$	2
$K \otimes M^*$	W^*	$\frac{2-x}{(1-x)^2}$	1
W	W	$1 + \frac{x}{(1-x)^2}$	2
$W \otimes M$	$F \oplus X_{10}$	$3 + x + \frac{x^2}{(1-x)^2}$	3
$W \otimes M^*$	$F \oplus F \oplus K$	$2 + \frac{1}{(1-x)^2}$	1
F	F	1	0
$F \otimes M$	$F^{\oplus 3}$	3	0
$F \otimes M^*$	$F^{\oplus 3}$	3	0

Table 7

Module	Propagators	Degree	Poincaré series	C-Deg.
k	d_8	0	$\frac{1}{1-x^8}\frac{1}{(1-x)^2}$	$0+0=0$
M	d_8	2	$\frac{x^2}{1-x^8}\frac{1}{(1-x)^2}$	$2+1=3$
N	d_8, d_2	4	$\frac{x^4}{(1-x^8)(1-x^2)}\frac{1}{1-x}$	$4+0=4$
K	d_8, d_4	4	$\frac{x^4}{(1-x^8)(1-x^4)}\frac{1}{(1-x)^2}$	$4+1=5$
W	d_8, d_4	6	$\frac{x^6}{(1-x^8)(1-x^4)}(1+\frac{x}{(1-x)^2})$	$6+2=8$
F	d_8, d_4, d_2	8	$\frac{x^8}{(1-x^8)(1-x^4)(1-x^2)}$	$8+0=8$

To start, we make a table of the modules appearing in the E_2-term, which we regard as $\Lambda^*(M) \otimes S^*(M)$, and the Poincaré series of their cohomologies (Table 6).

Now, we must describe the propagation of these modules in the cohomology of $(\mathbb{Z}_4)^3$. This follows from the description of $S^*(M)$ in Section 4. It is important to note, however, that we have doubled all degrees in the symmetric algebra. For this reason, we will refer to the invariants as d_2, d_4, and d_8, by abuse of notation. We want to compute the Poincaré series of the E_2-term, and what we shall do is compute the Poincaré series of the "tensored with a trivial" part first, then compute the Poincaré series of the "tensored with M" part, and then compute the Poincaré series of the "tensored with M^*" part. Finally, we combine these pieces.

Table 8

Module	Propagators	Degree	Poincaré series	C-Deg.
M	d_8	1	$\frac{x}{1-x^8}\frac{1}{(1-x)^2}$	$1+1=2$
$M \otimes M$	d_8	3	$\frac{x^3}{1-x^8}\frac{2-2x+x^2}{(1-x)^2}$	$3+2=5$
$N \otimes M$	d_8, d_2	5	$\frac{x^5}{(1-x^8)(1-x^2)}(1+\frac{1}{1-x})$	$5+0=5$
$K \otimes M$	d_8, d_4	5	$\frac{x^5}{(1-x^8)(1-x^4)}(1+\frac{x}{(1-x)^2})$	$5+2=7$
$W \otimes M$	d_8, d_4	7	$\frac{x^7}{(1-x^8)(1-x^4)}(3+x+\frac{x^2}{(1-x)^2})$	$7+3=10$
$F \otimes M$	d_8, d_4, d_2	9	$\frac{3x^9}{(1-x^8)(1-x^4)(1-x^2)}$	$9+0=9$

We make a table for the "tensored with a trivial part" (Table 7). Since there are two trivial modules in $\Lambda^*(M)$, we just consider the one in degree 0 and then multiply our Poincaré series by $1+x^3$. In Table 7, "Degree" means the degree in which the propagated module first appears in the symmetric algebra, and "C-Deg." means the degree in the E_2-term after which we know no further generators in cohomology appear.

Thus the total Poincaré series for the "tensored with a trivial" part is

$$(1+x^3)\left[\frac{1}{1-x^8}\frac{1}{(1-x)^2}+\frac{x^2}{1-x^8}\frac{1}{(1-x)^2}+\frac{x^4}{(1-x^8)(1-x^2)}\frac{1}{1-x}\right.$$
$$+\frac{x^4}{(1-x^8)(1-x^4)}\frac{1}{(1-x)^2}+\frac{x^6}{(1-x^8)(1-x^4)}\left(1+\frac{x}{(1-x)^2}\right)$$
$$\left.+\frac{x^8}{(1-x^8)(1-x^4)(1-x^2)}\right].$$

Note that the maximum degree of an algebra generator for the E_2-term is eight. Although we are tensoring with a trivial module in degree three, this is really just multiplying by an element of the E_2-term and so no new algebra generators in degrees 11 or greater are produced.

Now, let us move on to the "tensored with M" part.

We produce the desired information in the form of Table 8 and a Poincaré series, as above.

Thus, the total Poincaré series for the "tensored with M" part is

$$\frac{x}{1-x^8}\frac{1}{(1-x)^2}+\frac{x^3}{1-x^8}\frac{2-2x+x^2}{(1-x)^2}+\frac{x^5}{(1-x^8)(1-x^2)}\left(1+\frac{1}{1-x}\right)$$
$$+\frac{x^5}{(1-x^8)(1-x^4)}\left(1+\frac{x}{(1-x)^2}\right)$$
$$+\frac{x^7}{(1-x^8)(1-x^4)}\left(3+x+\frac{x^2}{(1-x)^2}\right)+\frac{3x^9}{(1-x^8)(1-x^4)(1-x^2)}.$$

Now, the "tensored with M^*" part is shown in Table 9.

Table 9

Module	Propagators	Degree	Poincaré series	C-Deg.
$k \otimes M^*$	d_8	2	$\frac{x^2}{1-x^8}\frac{1}{(1-x)^2}$	$2+1=3$
$M \otimes M^*$	d_8	4	$\frac{x^4}{1-x^8}(1+\frac{1}{(1-x)^2})$	$4+0=4$
$N \otimes M^*$	$d_8,\ d_2$	6	$\frac{x^6}{(1-x^8)(1-x^2)}(1+\frac{1}{1-x})$	$6+0=6$
$K \otimes M^*$	$d_8,\ d_4$	6	$\frac{x^6}{(1-x^8)(1-x^4)}\frac{2-x}{(1-x)^2}$	$6+1=7$
$W \otimes M^*$	$d_8,\ d_4$	8	$\frac{x^8}{(1-x^8)(1-x^4)}(2+\frac{1}{(1-x)^2})$	$8+1=9$
$F \otimes M^*$	$d_8,\ d_4,\ d_2$	10	$\frac{3x^{10}}{(1-x^8)(1-x^4)(1-x^2)}$	$10+0=10$

Thus, the total Poincaré series for the "tensored with M^*" part is

$$\frac{x^2}{1-x^8}\frac{1}{(1-x)^2}+\frac{x^4}{1-x^8}\left(1+\frac{1}{(1-x)^2}\right)+\frac{x^6}{(1-x^8)(1-x^2)}\left(1+\frac{1}{1-x}\right)$$
$$+\frac{x^6}{(1-x^8)(1-x^4)}\frac{2-x}{(1-x)^2}+\frac{x^8}{(1-x^8)(1-x^4)}\left(2+\frac{1}{(1-x)^2}\right)$$
$$+\frac{3x^{10}}{(1-x^8)(1-x^4)(1-x^2)}.$$

Adding these series up, we obtain the Poincaré series for the E_2-term. We summarize our computation in the following

Theorem 6.1. *The Poincaré Series for the E_2 term of the Lyndon–Hochschild–Serre spectral sequence for the cohomology of the group extension $S = Syl_2(HS) = 4^3 : D_8$ is given by the rational function*

$$p(x)=\frac{(1+x)^2(1-x+x^2)(1+2x^2-x^5)}{(1-x)^2(1-x^4)(1-x^8)}.$$

Moreover, a complete set of algebra generators for the E_2 term of the spectral sequence occur in degrees eight and below.

Using Maple, this series can be expanded to yield

$$1+3x+7x^2+14x^3+23x^4+34x^5+48x^6+65x^7+84x^8+105x^9$$
$$+131x^{10}+163x^{11}+198x^{12}+236x^{13}+280x^{14}+330x^{15}+383x^{16}$$
$$+439x^{17}+503x^{18}+\cdots.$$

Our goal is to analyze the behavior of this spectral sequence. Our description provides us with very good control of the generators, especially given that they arise in low degree. To make effective use of this, we require an explicit computer-assisted calculation of $H^*(S)$ in low degrees. This can indeed be implemented; details are provided in a subsequent section. For now, we use this information to prove the main result in this paper.

Theorem 6.2. *Let $S=(\mathbb{Z}/4)^3 \rtimes U_3$ denote the Sylow 2-subgroup of HS, the Higman–Sims group. The Lyndon–Hochschild–Serre spectral sequence for this semidirect product collapses at E_2 and hence the polynomial $p(x)$ is the Poincaré series for $H^*(S,\mathbb{F}_2)$.*

Proof. A computer calculation using MAGMA for $H^*(S,\mathbb{F}_2)$ shows that through degree 10 the coefficients of the polynomial $p(x)$ agree with the ranks of the cohomology. Now according to the preceding tables, all algebra generators occur by this degree. Hence, using the multiplicative structure of the spectral sequence we infer that it must collapse at E_2, i.e. $E_2=E_\infty$. □

7. Computer calculations of the cohomology of $S=Syl_2(HS)$

The cohomology $H^*(S,\mathbb{F}_2)$ was calculated directly through degree 10 using a computer. The calculation included a determination of the Betti numbers, a minimal set of ring generators, and a complete collection of relations among the generators in the first ten degrees. We also obtained the images of the generators under various restrictions. Using the latter information and the analysis of the last section, we were able to construct the cohomology ring with all generators and relations using computer technology. We give a summary of the calculations in this section.

The calculation was begun by first obtaining a minimal projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to k,$$

of the trivial module $k=\mathbb{F}_2$. The process is mostly linear algebra. The free module over S is generated as a collection of matrices representing the actions of the generators of the group. At each stage in the construction we get a minimal set of generators for the kernel of the previous boundary map, create the free module with exactly that number of generators, make the matrix for the boundary map from the free module to the kernel of the previous boundary map – this is the new boundary map – and find its kernel. The programs are conservative with both time and memory in that they save only the minimal amount of information necessary to reconstruct the boundary homomorphisms and create module structure only when it is necessary. The method is described in detail in [5,4].

Because the resolution is minimal, any non-zero homomorphism $\gamma': P_n \to k$ is a cocycle representing a non-zero cohomology class γ. Any such cocycle can be lifted to a chain map $\hat{\gamma}: P_* \to P_*$ of degree n which also represents γ. Again this is a exercise in linear algebra on the computer and details of the implementation can be found in the references given above. The point of this operation is that the cup product of two cohomology elements is the class of the composition of the representing chain maps. Thus, we can compute the product structure of the cohomology ring. Note that the program computes the chain maps for only a minimal set of generators for the cohomology. That is, at each stage it constructs the subspace of $Hom_{kG}(P_n,k)$ that can be obtained by compositions with chain maps of lower degrees.

The same implementations have been used to compute the cohomology rings of all but a few of the groups of order dividing 64.[5] The programs are written in the MAGMA language and run in the MAGMA computer algebra system [3]. The computations of the cohomology of S were all run on an SUN ULTRA2200 (named the sloth) that has approximately 1 Gb of RAM and 14 Gb of hard disk. The computation of the projective resolution out to degree 10 for the trivial module k of S took slightly under 31 h. The computation of the chain maps of the minimal generators of the cohomology ring took more than 55 h. The attempt to compute an 11th step in the projective resolution failed for lack of memory. Some of the results of this calculation are given in the following. Note that this is precisely the information that is needed to complete the proof of Theorem 6.2.

Proposition 7.1. *The Betti numbers for the cohomology $H^*(S,k)$ through degree* 10 *are*

$$1, 3, 7, 14, 23, 34, 48, 65, 84, 105, 131,$$

and the degrees of the minimal generators of the cohomology ring through degree 10 *are*

$$1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 5, 6, 7, 8.$$

We also computed the restrictions of the cohomology to the elementary abelian subgroups and to the centralizers of the elementary abelian subgroups. The restriction maps are obtained by first converting the projective resolution of k for the group to a projective resolution of the subgroup and then constructing the chain map, lifing the identity on k, from a minimal projective resolution of k for the subgroup to the converted projective resolution for the group. The cocycles for the cohomology elements of the group are then pulled back along the chain map. The process is described in more detail in [6].

Of particular interest was the restrictions to the centralizers of the elementary abelian groups of order four. This calculation was made on the assumption (hope) that the depth of the cohomology ring $H^*(S,k)$ is at least two and hence that the cohomology of S is detected on these centralizers. An easy calculation shows the following.

Lemma 7.2. *If E is an elementary abelian subgroup of order four in S then the centralizer of E is contained in the subgroup K_β (defined in Section* 2) *or in one of the three subgroups*

$$M = \langle v_1, v_2, v_3, ts, s^2 \rangle,$$

$$D_1 = \langle (v_1v_2)^2, v_1v_3, t, s \rangle,$$

$$D_2 = \langle (v_1v_2)^2, v_1v_3, v_2^{-1}t, ts, v_1v_2^{-1}s^2 \rangle.$$

[5] See the second author's web page, http://www.math.uga.edu/˜jfc/groups2/cohomology2.html for the results and some further discussion of the methods.

Here M is the centralizer of $(v_1v_2)^2$, D_1 is the centralizer of s^2 and D_2 is the centralizer of $v_1v_2^{-1}s^2$. The order of M is 256 while the orders of D_1 and D_2 are 64.

The computer was able to calculate the cohomology ring of all four of the groups in the lemma. In the case of K_β it was known that the cohomology is detected on the centralizers of the maximal elementary abelian subgroups and this aided the computation. In all three of the other cases the minimal set of generators for the cohomology ring contained no element of degree greater than four and the computation was possible even though all three of the groups have two-rank four and the Betti numbers of the cohomology grow rapidly. In fact, the cohomology rings of D_1 and D_2 are included in the calculations of the second author (see the web page). These groups have Hall–Senior numbers 170 and 110, respectively.

We were able to compute the restriction maps and then get the kernels of the restriction to each of the subgroups. Then the intersection of the kernel of the restriction maps was computed. Actually, the problem of getting the intersection of the kernels seemed to be too difficult for the computer to attack directly using Gröbner basis machinery. Instead, we employed an indirect method of turning the restriction maps into linear transformations on the spaces of monomials in each degree and computing the intersection of the kernels out to degree 17. Thus, we had a complete set of generators for the intersection of the kernels out through degree 17.

There is one uncertainty that should be noted. The cohomology of M was only calculated out to degree 10 and that is not far enough to pass our test for completeness of the calculation [4]. Nonetheless, the generators of the cohomology of S are in degrees at most 8 and so their restrictions to M are expressed as polynomials in the calculated generators of the cohomology of M. Any polynomial in those generators which is computed to be in the kernel of restriction to M is, in fact, in that kernel. Thus, we get the following result, which is used in the next section.

Proposition 7.3. *The ideal $\mathscr{I}$ of Theorem 8.1 is generated by the elements of degree at most 17 that are in the intersection of the kernels of the restriction maps to K_β, M, D_1 and D_2. Hence the ideal is in the intersection of the kernels of the restrictions to the centralizers of the elementary abelian subgroups of order four.*

In the next section, we argue that this information together with some extra verification is sufficient.

8. Generators and relations for the cohomology of S

Our purpose in this section is to give a proof of the following.

Theorem 8.1. *The cohomology of S has the form*

$$H^*(S, \mathbb{F}_2) \cong \mathbb{F}_2[z, y, x, w, v, u, t, s, r, q, p, n, m, k, j, i, h]/\mathscr{I},$$

where the degree of z, y, x is 1, *the degree of w, v, u, t is* 2, *the degree of s, r, q, p is* 3, *the degree of n, m is* 4 *and the degrees of k, j, i, h are* 5, 6, 7 *and* 8, *respectively. The ideal $\mathscr{I}$ is generated by the relations in the appendix.*

Let $R = \mathbb{F}_2[z, y, x, w, v, u, t, s, r, q, p, n, m, k, j, i, h]$ be the polynomial ring in 17 variables in weighted degrees as in the theorem. For $n \geq 0$ let R^n be the space of homogeneous polynomials of degree n. From Proposition 7.1, we know that there is a natural homomorphism $\phi : R \to H^*(S, \mathbb{F}_2)$ taking each variable to the computed cohomology element. Let $\mathscr{J}$ be the kernel of ϕ. Our task is to establish that $\mathscr{I} \subseteq \mathscr{J}$.

Recall that the following is a complete list of representatives for the conjugacy classes of centralizers of maximal elementary abelian subgroups in S:

1. $C\mathrm{I} = 4^3$, a characteristic subgroup of S.
2. $C\mathrm{II} = 2^2 \times 4$.
3. $C\mathrm{III} = 2^2 \times 4$.
4. $C\mathrm{IV} = 2^2 \times 4$.
5. $C\mathrm{V} = 2^2 \times 4$.
6. $C\mathrm{VI} = 2^2 \times 4$.
7. 2^4_A.
8. 2^4_B.
9. 2^4_C.

The following basic detection result involving these groups was proved by computer calculation.

Lemma 8.2. *Any homogeneous polynomial of R^n in degree $n \leq 20$ is in $\mathscr{I}$ if and only if it is in the kernel of restriction to all of the centralizers listed above.*

Proof. First it was shown that the elements of $\mathscr{I}$ are in the kernels of restriction by computing the restriction maps on the elements. Then, for each degree $n \leq 20$ the restriction maps $R^n/\mathscr{I}^n \to H^*(C, \mathbb{F}_2)$ for C in the list was written as a linear transformation and the intersection of the null spaces was computed to be zero. □

Proof of Theorem 8.1. From the lemma we see that $\mathscr{J}^n \subseteq \mathscr{I}^n$ for $n \leq 20$. But by the Poincaré series (Theorem 6.2) we have that $R^n/\mathscr{I}^n$ has the same dimension as $R^n/\mathscr{J}^n \cong H^*(S, \mathbb{F}_2)$. On the other hand we know that $\mathscr{I}$ is generated by elements of degree at most 14. So $\mathscr{I} \subset \mathscr{J}$, since all generators are in $\mathscr{J}$. So ϕ induces a surjective homomorphism $R/\mathscr{I} \to H^*(S, \mathbb{F}_2)$. But again because the Poincaré series are the same this is an isomorphism. □

Appendix A. Relations in the cohomology of S

The following is a list of the relations in the cohomology ring $H^*(S, \mathbb{F}_2)$. These elements generate the ideal $\mathscr{I}$ in Theorem 8.1. Note that there are 79 relations generating $\mathscr{I}$ and that the relations are minimal in the sense that no collection of fewer than 79 elements will generate the ideal. This set is not a Gröbner basis for the ideal. The computed Gröbner basis using the grevlex ordering on the variables consisted of 884 elements in degrees up to 73. The elements in the list are ordered by degree. The largest degree of any element in the list is 14.

$$yx,\ zx,\ zy,\ xv,\ xw,\ yw+yv+yu,\ zt,\ zw,\ vt,\ xs,$$

$$yq+xq+w^2+wu,\ yq+wv,\ yr+xr+xq+xp+ut,\ yr+wt,\ ys+yq,$$

$$y^2u+yq+xq+wu,\ zp+yq+xq+v^2+vu,\ zr,\ zs+zq+xq,$$

$$tq,\ ts,\ vr,\ vs+vq,\ ws+wq,\ x^2r+xn+wp+vp+up,$$

$$x^2r+xn+wr+ur,\ yu^2+yn+wp,\ yvu+ws,$$

$$y^2r+wr,\ z^2p+zvu+zn+ws+vs+us,$$

$$z^3u+z^2p+zvu+zu^2+zn+zm+x^2r+xu^2+xn+xm,$$

$$rq,\ sp+qp,\ sq+q^2,\ sr,\ s^2+sq,\ xuq+vu^2+vn+sp,$$

$$ytr+x^3p+x^2n+xup+xtp+r^2,\ ytr+x^3p+xuq+xtr+xk+r^2,$$

$$yur+x^3p+xuq+xk+tn+rp,\ y^2n+yvp+yuq+wn,$$

$$z^3p+z^2n+zvq+zuq+y^2m+ytr+ytp+yk+x^3p+xk+vn \\ +vm+tn+tm+p^2,$$

$$z^3p+zvq+zup+yuq+xuq+vu^2+sq,\ z^3q+z^3p+zuq+zk,\ sn+qn,$$

$$ytn+xtm+xp^2+rn,\ yvn+yqp+vuq+vup+qn,$$

$$y^5v+y^4q+y^4p+y^3n+y^3m+y^2vq+y^2tp+y^2k+yum \\ +ytn+yr^2+yrp+yj+wk,$$

$$zvm+zq^2+zj+yvm+yum+yr^2+yrp+x^3u^2+x^3m+vup+vk \\ +u^2q+u^2p+uk+sm+rn+rm+pn,$$

$$z^2vq+zvm+zq^2+zqp+zj+yvn+yqp+vuq+u^2s,$$

$$z^3u^2+zu^3+zum+yvn+yqp+x^3u^2+x^3n+xu^3+xum+xtn+xj+vuq+u^2s,$$

$$z^5u+z^3m+zu^3+zum+x^3n+x^3m+xu^3+xum+xtn+xj+wk \\ +vk+u^2s+u^2q+uk,$$

$$y^4m + y^3k + y^2vn + y^2vm + y^2p^2 + yt^2p + ytk + yrm + x^4u^2 + x^3k + x^2un + x^2tn + x^2j + xu^2p + xtk + xrm + xpn + tr^2 + tj + rk,$$

$$y^5p + y^3k + y^2vn + y^2vm + yrm + ypm + yi + x^4u^2 + x^3k + x^2un + x^2tn + x^2j + xt^2p + xqm + wun + wum + wj + tr^2 + sk + qk,$$

$$zpn + ypn + xpn + wun + vun + vp^2 + uqp + n^2,$$

$$z^2q^2 + zuk + zqn + zqm + y^4m + y^3vq + y^3tp + y^3k + y^2vm + y^2tm + y^2p^2 + yqm + ypn + x^2un + xuk + xrm + wun + wj + vum + vp^2 + vj + up^2 + qk,$$

$$z^3k + z^2vm + z^2q^2 + zi + y^3tp + y^2tm + yuk + yt^2p + ytk + yqm + x^3k + x^2u^3 + x^2un + x^2tn + xt^2p + xi + wum + vun + vum + urp + up^2 + tr^2 + tj + sk + rk + qk,$$

$$z^3k + z^2vm + zqn + zpn + zi + y^5q + y^4n + y^2q^2 + yqm + x^2u^3 + xu^2p + xuk + xrm + wj + vq^2 + vp^2 + u^2n + uq^2 + uj + sk,$$

$$z^4u^2 + z^2u^3 + z^2un + z^2um + z^2q^2 + zqn + y^5q + y^4n + y^2q^2 + yqm + wun + vun + vq^2 + vp^2 + vj + uq^2 + sk,$$

$$z^5p + z^3k + z^2un + z^2j + zqn + zi + y^3vq + y^2vn + yuk + yrm + ypn + x^2u^3 + xuk + xqm + wum + vp^2 + u^2n + urp + uj + sk,$$

$$y^5n + y^3q^2 + y^2vk + y^2qm + yvqp + yup^2 + yuj + yt^2n + yrk + yqk + ynm + wuk + wqn + wpn + wi + trm + sj + r^2p + rp^2 + qj,$$

$$z^2pn + zuq^2 + zuj + zn^2 + znm + y^6p + y^5m + y^3vm + y^3tm + y^2qn + y^2qm + y^2i + yvqp + yvj + yt^2n + yt^2m + ytj + yrk + ypk + ynm + wqn + wpm + wi + vpm + u^3q + u^2k + urn + trn + q^3 + pj + nk,$$

$$z^2qn + zuj + zn^2 + znm + yup^2 + yt^2n + yrk + ynm + xpk + wpn + wpm + vuk + vqm + u^3q + u^2k + upn + trm + sj + rp^2 + rj + qj,$$

$$z^3un + z^2qn + z^2pn + zvj + zuq^2 + zuj + znm + sj + qj,$$

$$y^2tk + y^2qm + yup^2 + yt^2m + ytp^2 + ytj + yqk + ynm + x^3tn + xrk + xn^2 + wuk + wqn + wpm + t^3r + trn + tpm + ti + sj + r^2p + rp^2 + qj,$$

$$z^3vn + z^3un + z^2vk + z^2pn + zvj + y^5n + y^4vq + y^3vm + y^2qn + y^2pn + yvq^2 + yvqp + yup^2 + yn^2 + xnm + wuk + vqm + vpm + vi + u^3q + u^3p + u^2k + upn + q^3,$$

$$z^3vn + z^2qn + z^2pn + zvj + y^5n + y^2vk + y^2tk + y^2qn + yvqp + yup^2 + yt^2n + yt^2m + ytp^2 + ytj + yrk + yn^2 + x^3tn + wpn + wpm + wi + vuk + vqm + urn + t^3r + trm + tpm + ti + sj + r^2p,$$

$$y^3t^2p + y^2t^2m + yt^3p + yt^2k + yr^2p + yrp^2 + x^2u^2n + x^2t^2n + xrj + t^3n + t^2rp + t^2j + trk + r^2n + r^2m + rpm + ri,$$

$$z^2uq^2 + z^2qk + zpj + y^5vq + y^4j + y^3vk + y^3t^2p + y^3qn + y^3qm + y^3pm + y^3i + y^2vq^2 + y^2vj + y^2t^2m + y^2tp^2 + y^2tj + y^2pk + y^2h + yvpn + yupn + yt^3p + yt^2k + yr^2p + yqp^2 + yqj + x^2t^2n + xu^3p + xtpn + xrj + xpj + wqk + wpk + wnm + vup^2 + upk + t^3n + t^2p^2 + t^2j + trk + r^2n + r^2m + rpn + rpm + q^2n + p^2n + pi + k^2,$$

$$zuqn + y^7q + y^4q^2 + y^3vk + y^3qm + y^2vq^2 + y^2vj + yvqm + yvpn + yvi + yupn + yqj + x^2u^2n + x^2nm + xu^3p + xpj + wqk + wpk + vuj + upk + un^2 + tn^2 + r^2n + q^2n + qpn + qpm + nj,$$

$$z^3uk + zuqm + zupm + zmk + y^3vk + y^3qn + y^3pm + y^2tp^2 + y^2qk + yupn + yvi + yupn + yupm + yr^2p + yq^2p + yqj + ypj + x^2u^2n + xpj + wpk + vup^2 + upk + t^2rp + rpn + qpm + p^2n + p^2m + pi,$$

$$z^3uk + z^2u^2n + z^2uj + z^2qk + zupm + zui + zpj + y^3qn + y^3qm + y^3pn + y^2vq^2 + y^2nm + yq^2p + x^3uk + x^2u^4 + x^2nm + xu^3p + xui + xrj + wpk + wnm + vuj + usk + upk + un^2 + tn^2 + si + r^2n + qpn + nj,$$

$$z^4vm + z^4j + z^3vk + zuqn + zuqm + zupm + zpj + y^7q + y^4q^2 + y^3vk + y^3qn + y^3pn + y^2vj + y^2nm + yupn + yvi + yq^2p + yqj + x^2u^2n + x^2nm + vup^2 + u^3n + usk + un^2 + unm + qpn + qi,$$

$$z^2u^2k + z^2uqm + z^2q^3 + z^2qj + zvqk + zu^3n + zu^2j + zupk + zun^2 + y^6vq + y^4qm + y^4pn + y^3tp^2 + y^2qj + yvpk + yvn^2 + yvnm + yupk + yq^2n + yqpm + yqi + x^2u^2k + x^2nk + xunm + xtpk + xtnm + xri + xnj + wupm + wpj + wnk + vq^2p + vpj + urp^2 + unk + t^2rn + t^2pm + trp^2 + tp^3 + tpj + rpk + qpk + qnm + pnm + ni,$$

$$z^2vqm + z^2u^2k + z^2uqm + z^2q^3 + zvqk + zu^3n + zu^2j + zun^2 + y^9v + y^6vq + y^6k + y^5vm + y^5tm + y^5q^2 + y^4qn + y^4qm + y^3tp^2 + y^3tj + y^3pk + y^3nm + y^3m^2 + y^3h + y^2vi + y^2qj + y^2nk + yvpk + yvnm + yvm^2 + yvh + yupk + yuh + ytm^2 + x^2mk + xunm + xtnm + xqi + wupm + wpj + wnk + wmk + vmk + urj + uqp^2 + upj + unk + t^2pm + tp^3 + r^2k + rn^2 + qpk + qn^2 + qnm + ni + kj,$$

$$z^2q^2n + zvqj + zvmk + zupj + zqn^2 + zni + y^8m + y^7vq + y^7tp + y^7k + y^6tm + y^5vk + y^5qn + y^4pk + y^4nm + y^3p^3 + y^2vnm + y^2vh + yvq^3 + yvqj + yvnk + yvmk + yup^3 + yumk + ytrj + ytnk + ytmk + yrpk + yqm^2 + yp^2k + ypnm + yni + ykj + x^2mj + xqm^2 + xkj + wum^2 + wq^2m + wpi + vqpn + vp^2m + u^2qk + u^2pk + u^2n^2 + urpm + uri + uq^2m + up^2n + t^2nm + t^2m^2 + tr^2n + tp^2m + rpj + q^4 + q^3p + q^2j + qp^3 + p^2j + pnk + j^2,$$

$$z^5pm + z^2q^2n + z^2qi + zvqj + zupj + zpm^2 + zkj + y^8m + y^7vq + y^7k + y^6vm + y^5tk + y^5pm + y^4pk + y^3vqn + y^3t^3p + y^3ti + y^2vnm + y^2vh + y^2th + y^2q^2m + y^2k^2 + yvqj + yvnk + yumk + yt^4p + yt^3k + ytnk + ytmk + yrpk + yrn^2 + yq^2k + yqn^2 + yp^2k + ypnm + yni + ykj + x^8u^2 + x^8m + x^7k + x^6u^3 + x^5tk + x^5i + x^4tj + x^2t^2j + x^2mj + xt^2pn + xni + wun^2 + wum^2 + wq^2m + wmj + vum^2 + vqpn + vnj + vmj + u^2qk + uri + uqi + up^2n + upi + uk^2 + t^4n + t^3rp + t^3p^2 + t^3j + t^2nm + t^2m^2 + trpm + tk^2 + r^3p + r^2p^2 + rpj + q^4 + q^3p + q^2j + qp^3 + p^2j + pmk + ki + j^2,$$

$$z^{11}u + z^9u^2 + z^9m + z^7vn + z^6vk + z^6pm + z^3vnm + z^3un^2 + z^3nj + z^2unk + z^2pm^2 + z^2mi + z^2kj + zumj + y^7p^2 + y^6vk + y^4pj + y^4nk + y^3qi + y^3pi + y^2vnk + y^2tmk + y^2q^2k + y^2qnm + y^2qh + y^2p^2k + yvq^2m + yvqpm + yvqi + yvk^2 + yup^2n + yup^2m + yumj + yuk^2 + ytri + yr^2p^2 + yr^2j + yq^4 + yq^2j + yqpj + yn^2m + x^6uk + x^5u^2m + x^4ui + x^4mk + x^3nj + xu^4n + xu^2nm + xupi + xumj + xtnj + xpnk + wunk + wq^2k + wmi + vupj + vumk + vqnm + u^5q + u^4k + u^2p^3 + u^2nk + usm^2 + uq^2k + uqn^2 + t^3rn + trnm + tpnm + tpm^2 + q^3n + qp^2n + qnj + p^3m + pnj + nmk + ji,$$

$$z^{11}p + z^5vi + z^5u^2k + z^4pi + z^3umk + z^2qmk + z^2j^2 + zvqn^2 + zuqnm + zumi + zukj + zq^3n + zm^2k + y^{10}n + y^9tp + y^8q^2 + y^8j + y^7tk + y^7qn + y^7i + y^6vj + y^6tj + y^6nm + y^6h + y^5vi + y^5ti + y^4vn^2 + y^4vm^2 + y^4mj + y^4k^2 + y^3vqj + y^3tmk + y^3p^2k + y^3mi + y^2vqi + y^2vnj + y^2t^2m^2 + y^2q^2j + y^2n^2m + y^2mh + y^2ki + yvq^2k + yvqm^2 + yvqh + yupnm + yvmi + yupn^2 + yuph + yuni + yt^3pm + yt^2pj + ytp^2k + ytmi + yr^2pm + yq^3n + yp^2i + yn^2k + x^8u^3 + x^7uk + x^6u^2m + x^6uj + x^4um^2 + x^3mi + x^3kj + x^2u^4n + x^2umj + x^2j^2 + xu^2mk + xrk^2 + xji + wunj + wqnk + wnm^2 + wnh + vumj + vq^2p^2 + vqmk + vn^2m + vj^2 + u^2qi + u^2p^2m + u^2mj + urpj + urmk + uqnk + upmk + un^3 + unm^2 + uki + t^3nm + t^2r^2m + t^2rpm + t^2ri + t^2p^2n + tr^2j + trp^3 + trnk + tp^4 + skj + r^3k + r^2pk + rpn^2 + rni + qmi + pni + mk^2 + i^2.$$

Appendix B. Steenrod operations

In this appendix we describe the Steenrod operations on the cohomology generators listed previously. As before, this was done on a computer using MAGMA. Note that the program used puts all of the polynomials in "'normal form" relative to the Groebner basis of the ideal of relations.

$$Sq^1 w = yv, \quad Sq^1 v = zv + yv, \quad Sq^1 u = zu + xu, \quad Sq^1 t = yt + xt,$$

$$Sq^1 s = vu + yq + xq, \quad Sq^2 s = z^2 q + us + zn,$$

$$Sq^1 r = xr + xp, \quad Sq^2 r = ur + tr + xn,$$

$$Sq^1 q = vu + yq + xq, \quad Sq^2 q = xu^2 + z^2 q + uq + wp + vp + up + zn + xm,$$

$$Sq^1 p = zp, \quad Sq^2 p = z^2 q + z^2 p + y^2 p + tr + wq + uq + vp + ym + k,$$

$$\begin{aligned} Sq^1 n = x^2 p + wp + vp, \quad Sq^2 n &= y^4 v + y^3 q + y^3 p + zvq + yvq + xuq + yvp \\ &+ zup + yup + ytp + z^2 n + y^2 n + x^2 n + y^2 m + x^2 m + q^2 + rp + qp + tn \\ &+ vm + yk + xk + j, \end{aligned}$$

$$Sq^1 m = zu^2 + xu^2 + y^2 p + us + vq + up + zm + xm,$$

$$\begin{aligned} Sq^2 m &= y^4 v + y^3 q + y^3 p + u^3 + zvq + xuq + yvp + xup + xtp + z^2 m + y^2 m \\ &+ r^2 + q^2 + rp + vn + un + wm + vm + um + tm + xk + j, \end{aligned}$$

$$Sq^1 k = y^3 p + ytr + zvq + yvq + yvp + yup + xtp + z^2 n + y^2 m + qp + p^2 + vn,$$

$$\begin{aligned} Sq^2 k &= y^4 p + xu^3 + x^3 m + u^2 s + u^2 p + yrp + yp^2 + zun + yun + zvm + yvm \\ &+ xum + xtm + y^2 k + x^2 k + rn + qn + pn + sm + rm + vk + xj, \end{aligned}$$

$$\begin{aligned} Sq^4 k &= xu^4 + y^4 k + yvq^2 + xt^2 n + z^2 pn + xu^2 m + yt^2 m + y^2 vk + y^3 j + r^2 p \\ &+ qp^2 + p^3 + urn + uqn + tpn + zn^2 + uqm + wpm + upm + znm + xnm \\ &+ ym^2 + wuk + vuk + u^2 k + t^2 k + yrk + xrk + zqk + yqk + xpk + xuj + ytj \\ &+ xtj + z^2 i + x^2 i + mk + pj + vi + yh, \end{aligned}$$

$$\begin{aligned} Sq^1 j &= y^4 q + y^2 vq + y^3 n + y^3 m + x^3 m + zq^2 + vup + zqp + yvn + ytn + zvm \\ &+ ytm + y^2 k + qn + pn + vk + zj + yj + xj, \end{aligned}$$

$$\begin{aligned} Sq^2 j &= y^2 p^2 + x^2 tn + z^2 vm + ur^2 + vq^2 + tp^2 + t^2 n + zqn + vum + t^2 m + zqm \\ &+ xqm + zpm + zvk + yvk + zuk + xtk + z^2 j + n^2 + sk + vj, \end{aligned}$$

$$\begin{aligned} Sq^4 j &= y^2 t^2 m + y^3 pm + z^4 j + x^4 j + yq^3 + t^2 rp + yqp^2 + yp^3 + yvqn + yupn \\ &+ y^2 n^2 + yvqm + yupm + z^2 nm + xu^2 k + z^2 qk + z^2 vj + y^2 tj + x^2 tj + y^3 i \\ &+ qpn + un^2 + q^2 m + qpm + wnm + tnm + wm^2 + vm^2 + tm^2 + urk + trk \\ &+ wqk + uqk + wpk + upk + znk + xnk + ymk + vuj + yrj + zqj + zpj \\ &+ xpj + xti + y^2 h + nj + mj + qi + pi + wh, \end{aligned}$$

$$Sq^1 i = y^2q^2 + y^2p^2 + z^2vm + y^2tm + ur^2 + tr^2 + wq^2 + urp + trp + vqp + vp^2 + tp^2 + vun + xrn + yqn + zpn + xpn + vum + t^2m + yqm + xqm + zpm + ypm + zuk + xuk + z^2j + n^2 + sk + rk + wj + vj + tj + yi,$$

$$Sq^2 i = y^3tm + yvq^2 + xt^2n + y^2pn + y^2tk + q^2p + rp^2 + qp^2 + p^3 + vqn + uqn + zn^2 + xn^2 + urm + trm + wqm + uqm + wpm + vpm + upm + tpm + ynm + xnm + ym^2 + xm^2 + vuk + xrk + zqk + xpk + zvj + xuj + z^2i + x^2i + nk + mk + sj + rj + qj + vi + ti,$$

$$Sq^4 i = y^4tk + xu^3m + yt^3m + y^3vj + y^3tj + z^4i + y^4i + x^4i + vq^3 + vupn + yqpn + yvn^2 + yun^2 + t^2pm + yqpm + xunm + yum^2 + xum^2 + ytm^2 + u^3k + xupk + z^2nk + y^2nk + z^2mk + y^2mk + yt^2j + xt^2j + z^2qj + y^2qj + z^2pj + y^2vi + y^2ti + rn^2 + pn^2 + rnm + rm^2 + qm^2 + r^2k + vnk + tnk + vmk + umk + tmk + yk^2 + urj + wqj + vqj + uqj + wpj + ynj + xnj + zmj + ymj + xmj + t^2i + yqi + ypi + xpi + yvh + kj + ni + mi + sh + qh,$$

$$Sq^1 h = z^7v + x^7u + y^2qn + z^2pn + y^2pn + yt^2m + y^2pm + y^2vk + y^2tk + x^2tk + y^3j + rp^2 + p^3 + vpn + upn + tpn + xn^2 + usm + trm + wqm + vqm + wpm + vpm + xnm + wuk + vuk + yrk + zpk + ypk + xpk + zvj + yvj + zuj + yuj + xuj + x^2i + nk + pj,$$

$$Sq^2 h = z^8v + x^8u + u^5 + y^3pm + y^3tk + t^2rp + yrp^2 + yqp^2 + yupn + y^2n^2 + u^3m + yvqm + ytpm + z^2nm + y^2nm + x^2nm + z^2m^2 + y^2m^2 + x^2m^2 + xu^2k + xt^2k + z^2qk + y^2vj + x^2uj + y^2tj + x^2tj + x^3i + r^2n + q^2n + p^2n + wn^2 + vn^2 + un^2 + q^2m + rpm + qpm + wnm + vnm + trk + wqk + uqk + wpk + upk + ynk + xnk + zmk + vuj + zqj + ypj + yvi + xui + yti + xti + y^2h + pi,$$

$$Sq^4 h = z^6j + x^6j + u^6 + y^2t^3m + y^3t^2k + x^5i + t^3p^2 + t^4n + t^4m + z^2um^2 + xt^3k + y^3nk + y^3pj + y^3vi + x^3ti + z^4h + y^4h + x^4h + q^4 + p^4 + vq^2n + tp^2n + u^2n^2 + yrn^2 + yqn^2 + ypn^2 + tr^2m + vq^2m + vqpm + vp^2m + tp^2m + vunm + u^2nm + t^2nm + yrnm + ypnm + wum^2 + vum^2 + u^2m^2 + yqm^2 + xqm^2 + ypm^2 + zq^2k + yrpk + yqpk + zvmk + zumk + yumk + u^3j + t^3j + yvpj + xupj + ytpj + xtpj + y^2nj + x^2nj + z^2mj + xu^2i + xt^2i + z^2qi + y^2qi + y^2pi + x^2pi + y^2vh + z^2uh + nm^2 + m^3 + pmk + wk^2 + vk^2 + r^2j + q^2j + rpj + vnj + unj + tnj + wmj + tmj + zkj + ykj + xkj + usi + uri + vqi + uqi + wpi + upi + zni + yni + ymi + vuh + t^2h + yrh + xrh + zph + ki + nh + mh.$$

References

[1] A. Adem, R.J. Milgram, Cohomology of Finite Groups, Grundlehren der mathematische Wissenschaften, Vol. 309, Springer, Berlin, 1994.

[2] D.J. Benson, Conway's group Co_3 and the Dickson invariants, Manuscripta Math. 85 (1994) 177–193.
[3] W. Bosma, J. Cannon, Handbook of Magma Functions, Magma Computer Algebra, Sydney, 1996.
[4] J.F. Carlson, Calculating group cohomology: Tests for completion, J. Symbolic Comput., in preparation.
[5] J.F. Carlson, E. Green, G.J.A. Schneider, Computing the ext algebras for the group algebras of finite groups, J. Symbolic Comput. 24 (1997) 317–325.
[6] J.F. Carlson, J. Maginnis, R.J. Milgram, The cohomology of the sporadic groups J_2 and J_3, J. Algebra 214 (1999) 143–173.
[7] D. Karagueuzian, P. Symonds, The module structure of a group action on a polynomial ring, J. Algebra 218 (1999) 672–692.
[8] I. Madsen, R.J. Milgram, The Classifying Spaces for Surgery and Cobordism of Manifolds, Annals of Mathematics Studies, Vol. 92, Princeton University Press, Princeton, NJ, 1979.
[9] C.B. Thomas, Characteristic Classes and the Cohomology of Finite Groups, Cambridge Studies in Advanced Mathematics, Vol. 9, Cambridge University Press, Cambridge, 1986.