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Injectivity, Annihilators and Orders*

ROBERT C. SHOCK

Department of Mathematics, Southern Illinois University
Carbondale, Illinois 62901

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1. INTRODUCTION

Throughout this paper let R denote a ring with unity and M a unital right module over R. Defined in the literature are two important extensions for the module M. Echmann and Schopf [1] has given us the injective hull of M which we will denote by $I(M)$; Lambek [6], the rational completion of M which we will denote by $\bar{M}$. There is always a containment relation $M \subseteq \bar{M} \subseteq I(M)$. One problem is to determine whether or not the rational completion of a module is injective without computing it, that is, when does $\bar{M} = I(M)$? A partial solution has been given by R. E. Johnson [5]. He has shown that if the right singular ideal of a ring is zero, then its rational completion is injective. One purpose of this paper is to give necessary and sufficient conditions for the rational completion of a module to be injective; these conditions involve only the module M and the ring R.

It is well-known that the existence of the injective hull of a module involves some form of Zorn's lemma. There is no known "computational procedure" to construct the injective hull of a module. However, we are able to give necessary and sufficient conditions which depend only on R and M for a right ideal of R to be an annihilator of $I(M)$.

For a module M, a right ideal K of R is said to be M-dense if for each $x \in R - K$ the draw back $x^{-1}K = \{y \in R : xy \in K\}$ lies above the annihilators of M, that is, $mx^{-1}K \neq (0)$ for all $m \in M - (0)$ (Theorem 2.5). A right ideal L of R is an annihilator of the injective hull of M if and only if for each $x \in R - L$, the draw back $x^{-1}L$ is not M-dense; equivalently, for each $x \in R - L$ there does exist $y \in R$ and $m \in M - (0)$ such that $m(xy)^{-1} L = (0)$. If each R-homomorphism from a right ideal of R into M has an extension whose domain is M-dense, then we say that the module M has the *dense*

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extension property (Theorem 2.2). The rational completion of a module is injective if and only if the module has the dense extension property.

In Section 4, we apply our results to rings. Following Jans [3] we define a solid Goldie ring to be a ring R with the maximum condition on annihilators of the injective hull of R; recall a Goldie ring is a ring which has the maximum condition on right annihilators of R and contains no infinite direct sum of right ideals. Solid Goldie rings are Goldie rings. Jans states [3, p. 38] that it is not known if all Goldie rings are solid Goldie rings. An "intrinsic" description of a solid Goldie ring is given which enables us to give an example of a Goldie ring which is not a solid Goldie ring.

If R is a subring of $\underset{\sim}{Q}$ and the unity of R is the unity of $\underset{\sim}{Q}$, then R is a right order in $\underset{\sim}{Q}$ if

(a) every nonzero divisor of R is a unit in $\underset{\sim}{Q}$ and

(b) every element $q \in \underset{\sim}{Q}$ can be written $q = ab^{-1}$, where $a, b \in R$, and b is a nonzero divisor of R.

Jans [3] followed by Mewborn and Winton [7] have given necessary and sufficient conditions for a ring R to be a right order in a quasi-Frobenius ring. We also state similar conditions. Our conditions are entirely "intrinsic" in that they are stated in terms of R, not $I(R)$, and also nothing is mentioned about the existence of nonzero divisors; however we do use Goldie's theorem [2] on orders in semiprime Artinian rings. We prove that R is a right order in a quasi-Frobenius ring if and only if R is a solid Goldie ring with the dense extension property and the prime radical of R is the right singular ideal of R.

2. Rational Completion and Injectivity

Throughout this paper R will always denote a ring with unity. All modules are right unital modules over R. Let A, B denote modules. For $b \in B - A$ we define $b^{-1}A = \{x \in R : bx \in A\}$. A is essential in B means that each nonzero submodule of B has nonzero intersection with A. Also, $I(A)$ denotes the injective hull of the module A.

Definition 2.1. Let M be an R-module. A right ideal K of R is said to be M-dense provided that $m \in I(M) - (0)$ implies $mK \neq (0)$. That is, K lies above the annihilators of $I(M)$.

Lemma 2.2. *Assume that K is a right ideal of R and that for some $x \in R - K$, $x^{-1}K$ is not M-dense. Then there does exist $p \in I(M)$ such that $pK = (0)$ and $px \neq 0$. Thus, K is not M-dense.*

Proof. By the hypothesis $m(x^{-1}K) = (0)$ for some $m \in I(M) - (0)$. Let $f(xr + k) = mr$ for all $k \in K$, $r \in R$. This map from $xR + K$ into $I(M)$ is well defined since $m(x^{-1}K) = (0)$ and is a scalar map since $I(M)$ is injective. Thus, for some $p \in I(M)$ we have $f(y) = py$ for all y in the domain of f, $pK = (0)$ and $px \neq 0$.

PROPOSITION 2.3. (1) *Let M be an R module and K be a right ideal of R. The following are equivalent*:

(a) *K is M-dense.*

(b) *For all $x \in R$, $x^{-1}K$ is M-dense.*

(c) *For each $x \in R$ and each $m \in M - (0)$ there does exist $r \in R$ such that $xr \in K$ and $mr \neq 0$.*

(2) *The finite intersection of M-dense right ideals is again M-dense.*

Proof. From Lemma 2.2, (a) implies (b). Clearly, (b) implies (c). Now, (c) implies (a) for if $iK = (0)$, where $i \in I(M) - (0)$, then let $m = ix \in M - (0)$ for some $x \in R$. Therefore, $mx^{-1}K = (0)$, a contradiction. For part (2), suppose that K and N are M-dense. If $p \in I(M) - (0)$, then $pk \neq 0$ for some $k \in K$ and thus $pk(k^{-1}N) \neq (0)$ which implies that $p(N \cap K) \neq (0)$. Therefore, $N \cap K$ is M-dense.

For module M let H be the ring of R-homomorphism from $I(M)$ into $I(M)$. The rational completion of M, which we denote by $\bar{M}$, is defined to be the set $\{i \in I(M) : \text{if } f \in H \text{ and } f(M) = (0), \text{ then } f(i) = 0\}$, see Ref. [6]. The rational completion of M is precisely the set $\{i \in I(M) : i^{-1}M \text{ is } M\text{-dense}\}$. To see this we show that the complements of the above two sets are identical. If $f \in H$, $f(M) = (0)$ and $f(y) \neq 0$, where $y \in I(M)$, then $f(y)\, y^{-1}M = (0)$ and $y^{-1}M$ is not M-dense. If $p(y^{-1}M) = (0)$ for some $p, y \in I(M) - (0)$, then the mapping h from $yR + M$ into $I(M)$ where $h(yr + m) = pr$ for $r \in R$, $m \in M$ has an extension $h' \in H$ such that $h'(M) = (0)$ and $h'(y) \neq 0$.

DEFINITION 2.4. Let L be a right ideal of R and let M be an R-module. We say that an R-homomorphism g from L into M has a *dense extension* if it has an extension whose domain is M-dense, that is, there does exist an M-dense right ideal $L' \supseteq L$ and an R-homomorphism g' from L' into M such that g' and g agree on L. M has the *dense extension property* if each R-homomorphism from a right ideal of R into M has a dense extension.

THEOREM 2.5. *The rational completion of a module is injective if and only if the module has the dense extension property.*

Proof. If $\bar{M} = I(M)$, then for each R-homomorphism g from a right ideal L into M there does exist $m \in \bar{M}$ such that $g(x) = mx$ for all $x \in L$.

Let $L' = m^{-1}M$. $L' \supseteq L$ and L' is M-dense since $m \in \bar{M}$. The map g' where $g'(x) = mx$ for all $x \in L'$ extends g.

We now show that $\bar{M}$ is injective. Let f be an R-homomorphism from a right ideal L into M and $f(x) = qx$ for all $x \in L$, $q \in I(M)$. If $q \in I(M) - \bar{M}$, then apply the hypothesis to the map h from $q^{-1}M$ into M, where $h(x) = qx$ for all $x \in q^{-1}M$. Thus, h has a dense extension h' and for some $m \in I(M)$, $h'(y) = my \in M$. Since $m^{-1}M$ contains the domain of h' which is M-dense, $m^{-1}M$ is M-dense. Therefore, $(m - q)\, m^{-1}M \neq 0$ and $(m - q)z \neq 0$ for some $z \in m^{-1}M$. M is essential in $I(M)$ and we have for some $r \in R$, $(m - q)\, zr \in M - (0)$. However, $mzr \in M$ forces $qzr \in M$ and thus $zr \in q^{-1}M$. This implies $(h' - h)\, zr = (m - q)\, zr = 0$, a contradiction. Hence $q \in \bar{M}$ and $\bar{M}$ is injective.

3. Annihilators and Rationally Closed Ideals

Let A, B, and M be R-modules. We say that B is an M-rational extension of A provided that $A \subseteq B$ and if f is any R-homomorphism from a submodule of B into M and the kernel of $f \supseteq A$, then f must be the zero map. Also, a submodule A of B is said to be *M-rationally closed* if A has no proper M-rational extensions in B [6]. Let $A' = \{b \in B : b^{-1}A \text{ is } M\text{-dense}\}$. With the use of (2) of Proposition 2.3, we see that A' is an R-module and is called the *M-rational closure* of A in B.

Lemma 3.1. (1) *Assume that A is a submodule of B. The following are equivalent*:

(a) *A is M-rationally closed in B.*

(b) *If $b \in B - A$, then $b^{-1}A$ is not M-dense.*

(c) *$A = A'$.*

(2) *Assume $N \subseteq M \subseteq B$. Then $N' \subsetneq M'$ if and only if $b^{-1}N$ is not M-dense for some $b \in M - N$.*

Proof. (a) implies (b) for if A is M-rationally closed in B, then for each $b \in B - A$ there does exist an R-mapping g from $bR + A$ into M such that $g(b) \in M - (0)$ and $g(A) = (0)$. Thus, $g(b)(b^{-1}A) = (0)$ and the result follows. (b) implies (c) is clear. For (c) implying (a), we note that for each $b \in B - A$ there does exist a nonzero $i \in I(M)$ such that $i(b^{-1}A) = (0)$. Let $y \in R$ such that $iy \in M - (0)$. The map g from $byR + A$ into M, where $g(byr + a) = iyr$ for all $r \in R$, $a \in A$ has the property that $g(A) = (0)$ and $g(by) \neq 0$ which completes the proof. Part (2) is straight-forward and the details are omitted.

A right ideal K of R is said to be an annihilator of $I(M)$ if there does exist a subset S of $I(M)$ such that $K = \{x \in R : Sx = 0\}$.

THEOREM 3.2. *The annihilators of the injective hull of a module M are precisely the M-rationally closed right ideals of R. That is, a right ideal K is an annihilator of $I(M)$ if and only if $x^{-1}K$ is not M-dense for all $x \in R - K$.*

Proof. Let K be a right ideal of R. If K is an annihilator of $I(M)$, then for each $x \in R - K$ there does exist some $m \in I(M)$ such that $mK = (0)$ and $mx \neq 0$. Clearly $mx(x^{-1}K) = (0)$ and $x^{-1}K$ is not M-dense. It follows from part (1), (b) of Lemma 3.1 that K is M-rationally closed. Assume that K is an M-rationally closed in R and let $S = \{m \in I(M) : mK = (0)\}$. For $x \in R - K$ the hypothesis implies that $i(x^{-1}K) = (0)$ for some $i \in I(M) - (0)$. By Lemma 2.2 there does exist some $p \in I(M)$ such that $pK = (0)$ and $px \neq 0$. Thus, $p \in S$ and $\{z \in R : Sz = 0\} = K$.

In the following section we will use the corollary below.

COROLLARY 3.3. *Let M be an R-module. The following are equivalent:*

(1) *The collection of M-rationally closed right ideals of R satisfies the maximum condition.*

(2) *Each sequence $K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots$ of right ideals of R with the property that $K_i \subsetneqq K_{i+1}$ implies $k_i^{-1}K_i$ is not M-dense for some $k_i \in K_{i+1} - K_i$ becomes constant.*

(3) *Let $H = \{r \in R : mr = 0$ for all $m \in I(M)\}$. The factor ring R/H is a right finite-dimensional ring and each sequence $L_1 \subseteq L_2 \subseteq L_3 \subseteq \ldots$ of essential right ideals of R, where $L_i \subsetneqq L_{i+1}$ implies for some $x \in L_{i+1} - L_i$, $x^{-1}L_i$ is not M-dense becomes constant.*

Proof It follows from (2) of Lemma 3.1 that statements (1) and (2) are equivalent. We now show that (2) implies (3). Clearly H is an ideal of R. If right ideals A, B of R contain H and $A/H + B/H$ is direct, there does exist $m \in I(M)$ such that $mb \neq 0$ for some $b \in B - H$ and the map f from $A + bR$ into $I(M)$, where $f(a + br) = mr$, $a \in A$ is well defined since $mH = (0)$. There does exist $p \in I(M)$ such that $pA = (0)$, $pb \neq 0$, and $A' \subsetneqq (A + B)'$. It now follows that R/H is right finite dimensional. Finally, we prove (3) implies (1) by using an indirect argument. If $A_1' \subsetneqq A_2' \subsetneqq A_3' \subsetneqq \ldots$ is a sequence of M-rationally closed right ideals of R, then $\dim A_i/H = \dim A_{i+j}/H$ for some i and all $j \geqslant 1$. There does exist a right ideal $B \supseteq H$ such that the sum $A_i/H + B/H$ is direct and essential in R/H; thus, $A_i + B$ is essential in R since $A_i + B \supseteq H$. It is straightforwarded to check that $(A_i + B)' \subsetneqq (A_{i+1} + B)' \subsetneqq \ldots$, a contradiction.

4. Goldie Rings Are Not Solid Goldie Rings

In this section we give an example of a Goldie ring which is not a solid Goldie ring.

Example. Let R be the vector space over the two-element field with a set of base elements $\{1\} \cup \{y^k : k > 0\} \cup \{x_i : i > 0\}$ where i, k are positive integers. Define multiplication on R as follows: For all $y^i y^j = y^{i+j}, i, j, k$ we have $x_i x_j = 0$, $yx_i = 0$, $x_i y^k = x_{i-k}$ if $k < i$ and $x_i y^k = 0$ otherwise, $1b = b1 = b$ for all base elements b. R is right finite dimensional since $x_1 R + yR$ is essential. The prime radical P of R is generated by $\{x_1, x_2, x_3, ...\}$. If S is a subset of R and $r(S) \neq (0)$, then $r(S)$ is of the form $P + y^k R$; R is a Goldie ring. Consider the sequence $x_1 R \subset x_2 R \subset x_3 R \subset ...$, where $x_{i+1}^{-1}(x_i R) = yR + P$. Since $x_1(yR + P) = (0)$, it follows from (2) of Corollary 3.3 that R does not satisfy the maximum condition on annihilators of $I(R)$. One can also show that R is not left finite dimensional and does not have the maximum conditions on left annihilators of R.

We point out that if the right singular ideal of R is zero, then R is a solid Goldie ring if and only if it is a Goldie ring.

5. Orders in Quasi-Frobenius Rings

Throughout this last section we will write dense instead of R-dense. Thus, a right ideal K of R is dense if for each $x, y \in R - (0)$ there does exist $z \in R$ such that $xz \neq 0$ and $yz \in K$. Also, we will say that a right ideal K of R is rationally closed instead of saying K is $I(R)$-rationally closed (equivalently, K is an annihilator of $I(R)$). Q will always denote the complete ring of right quotients of R [5, p. 94] and $Z(Q)$ ($Z(R)$) the right singular ideal of Q (of R). Our first major goal is to establish a general setting for which R will be a right order in Q. If $Z(Q)$ is rationally closed and is the Jacobson radical of Q, $Q/Z(Q)$ is a semiprime Artinian ring and $R/Z(R)$ is semiprime, then R will be a right order in Q. We will need the proposition below.

Proposition 5.1. *Let K be a rationally closed ideal of R.*

(1) *If D is a dense right ideal of R and $D \supset K$, then D/K is dense in R/K.*

(2) *If B is a right ideal of R, $B \supset K$ and B/K is rationally closed in R/K, then B is rationally closed in R.*

(3) *If R is a solid Goldie ring, then R/K is also a solid Goldie ring.*

(4) *The complete lattice of rationally closed right ideals of R which contain K is order isomorphoric to the complete lattice of rationally closed right ideals*

of R/K if and only if for each right ideal $D \supset K$ we have D/K dense in R/K, implying D is dense in R.

Proof. (1) Let D be a dense right ideal of R and $D \supset K$. Let $a + K \neq 0 + K$, $b + K$ be given in R/K. Since $a^{-1}D \cap b^{-1}D$ is dense, $a(a^{-1}D \cap b^{-1}D) \not\subset K$ because K is rationally closed and $a^{-1}K$ is not dense. There does exist $r \in a^{-1}D \cap b^{-1}D$ such that $(a + K)(r + K) \neq 0 + K$ and $(b + K)(r + K) \in (D + K)$. (2) This follows from the fact that if $x^{-1}B$ were dense, where $x \in R - B$, then $(x + K)^{-1}B/K$ would be dense in R/K by part (1), this would be a contradiction. The remaining part of the proposition is now straightforward and the details are omitted.

If H is an ideal of Q, then the ring map f from $R/(H \cap R)$ into Q/H, where $f(r + (H \cap R)) = r + H$ is injective and $f(R/(H \cap R))$ is a subring of Q/H. We denote $f(R/(H \cap R))$ by R/H. If R is a subring of S, then S is called a *ring of right quotients* of R if for every $(0) \neq s \in S$, $s^{-1}R$ is a dense right ideal of R and $s(s^{-1}R) \neq (0)$ [5, p. 99].

LEMMA 5.2. *If H is a rationally closed ideal of Q, then Q/H is a ring of right quotients of R/H.*

Proof. Suppose $q \in Q - H$. We must show that $(q + H)^{-1} R/H$ is dense in R/H and $(q + H)[(q + H)^{-1} R/H] \neq (0) + H$. Since $q^{-1}R$ is dense in R, $q^{-1}R + H/H$ is dense in Q/H by (1) of Proposition 5.1. Since H is rationally closed in Q and $q \in R - H$ there does exist $i \in I(R)$ such that $iH = (0)$ and $iq \neq 0$. If $q^{-1}R \subset H$, then $iq(q^{-1}R) = (0)$, a contradiction. Therefore, there does exist $r \in R$ such that $qr \in R$ but $qr \notin H$ and we have our result.

THEOREM 5.3. *Suppose that $Z(Q)$ is the Jacobson radical of Q and is rationally closed. If $Q/Z(Q)$ is a semiprime Artinian ring and $R/Z(Q)$ is a semiprime ring, then R is a right order in Q.*

Proof. $R/Z(R)$ is a right order in $Q/Z(Q)$; this follows from Lemma 5.2, $Q/Z(Q)$ being a semiprime Artinian ring and from Goldie's theorem on right orders in semiprime Artinian rings. If D is a dense right ideal of R, then $D + Z(R)/Z(R)$ is dense in $R/Z(Q)$ and contains an invertible element $d + Z(Q)$ in $Q/Z(Q)$. Thus, $1 = m + xd = n + dx$ for some $m, n \in Z(Q)$. Since $1 - m$ and $1 - n$ are units in Q, d is invertible in Q. To complete the theorem it suffices to show that nonzero divisors of R are invertible in Q. If b is a nonzero divisor of R and $by \in Z(Q)$, where $y \in R - (0)$, then $r(by) = r(y)$ which implies that $y \in Z(R)$. Thus $b + Z(Q)$ is a right nonzero divisor in $R/Z(Q)$ and it follows as before that b is invertible in Q.

Remark. It is known that Q_R is an injective R-module if and only if Q is a self-injective ring [5, p. 95]. It follows from Theorem 2.3 that Q is a self-injective ring if and only if R has the dense extension property. If $Z(R) = (0)$, then each essential right ideal is dense and R has the dense extension property. Thus, if $Z(R) = (0)$, then $\underset{\sim}{Q}$ is a self-injective ring [4]. Also, it is not difficult to show that R is a solid Goldie ring if and only if $\underset{\sim}{Q}$ is a solid Goldie ring.

We define a ring R to be quasi-Frobenius if R is a (right) self-injective ring and is right Artinian. There are many characterizations of quasi-Frobenius rings and we will use the one stated by C. Faith [1]:

LEMMA 5.5. *A self-injective solid Goldie ring is quasi-Frobenius.*

Proof. See Ref. [1].

THEOREM 5.6. *R is a right order in a quasi-Frobenius ring if and only if R is a solid Goldie ring with the dense extension property and the prime radical of R is the right singular ideal of R.*

Proof. Suppose R is a right order in a quasi-Frobenius ring $\underset{\sim}{Q}$. Clearly, R_R is an essential submodule of $\underset{\sim}{Q}_R$. From the remark following Lemma 5.2, $\underset{\sim}{Q}_R$ is self-injective since $\underset{\sim}{Q}_Q$ is. $\underset{\sim}{Q}$ is the complete ring of right quotients of R and thus R is a solid Goldie ring with the dense extension property. $R/Z(R)$ is a right order in $\underset{\sim}{Q}/Z(\underset{\sim}{Q})$, since R is a right order in $\underset{\sim}{Q}$ and $\underset{\sim}{Q}/Z(\underset{\sim}{Q})$ is a semiprime Artinian ring. By an order theorem of A. W. Goldie [2], $R/Z(R)$ is semiprime. $Z(R)$ is nilpotent since $Z(\underset{\sim}{Q})$ is and we have $Z(R) =$ prime radical of R. We now prove the other implication. $\underset{\sim}{Q}$ is a self-injective solid Goldie ring and is a quasi-Frobenius by Lemma 5.5. The hypothesis of Theorem 5.3 is satisfied and R is a right order in $\underset{\sim}{Q}$.

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