MATHEMATICS

THE EFFECTIVENESS OF THE BASIC SETS OF POLYNOMIALS AT A POINT

BY

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1. *Introduction.* The subject of basic sets of polynomials was introduced by J. M. WHITTAKER. For general terminology the reader should refer to [1] [1]) and [2]. The set $\{p_n(z)\}$ of polynomials is said to be basic if it represents any polynomial and in particular z^n in a unique finite linear combination of the form:

$$z^n = \sum_i w_{ni} p_i(z), \tag{1.1}$$

where

$$p_i(z) = \sum_j p_{ij} z^j$$

and (w_{ij}) and (p_{ij}) are the basic sets of operators and coefficients respectively associated with the set $\{p_n(z)\}$.

The set $\{p_n(z)\}$ is said to be effective on $|z| = R$, if any function regular on $|z| = R$, can be represented in terms of $\{p_n(z)\}$. Thus, the basic set $\{p_n(z)\}$ of polynomials is effective at the origin, if the basic series represents at the origin every function which is regular there.

Let:

$$G_n(R) = \sum_i |w_{ni}|\, M_i(R), \tag{1.2}$$

where

$$M_i(R) = \max_{|z|=R} |p_i(z)|.$$

Write

$$G(R) = \limsup_n \, (G_n(R))^{1/n}.$$

Also, let:

$$F_n(R) = \max_{i,j} \max_{z=R} |w_{ni} p_i(z) + w_{ni+1} p_{i+1}(z) + \ldots + w_{nj} p_j(z)| \tag{1.3}$$

and write

$$F(R) = \limsup_n \, (F_n(R))^{1/n}.$$

It was shown by WHITTAKER [2] that the set $\{p_n(z)\}$ satisfying the

[1]) The numbers in square brackets refer to the reference at the end of the paper.

condition $\limsup_n (N_n)^{1/n} = 1$, (i.e. Cannon set) is effective at the origin if: (N_n is the number of terms in (1.1))

$$G(0+) = \lim_{R \downarrow 0} G(R) = 0.$$

If the set is such that $\limsup_n (N_n)^{1/n} > 1$, then such a set is effective at the origin if:

$$F(0+) = \lim_{R \downarrow 0} F(R) = 0.$$

The author has used his own method [1] to determine the effectiveness and the domain of effectiveness, as well, of some class of basic sets of polynomials. For this, he introduced the constants C, b and B defined by:

$$C = \sup_{d(n)} (\limsup_n (C(n, d(n))^{1/n}, \tag{1.4}$$

where

$$C(n, d(n)) = \sup_i (|w_{ni}| \, |p_{id(n)}|),$$

and $(d(n))$ is a sequence of integers such that $\limsup_n d(n)/n = 1$,

$$b = \sup_{s(n)} \limsup_n (b(n, s(n))^{(n-s(n))^{-1}}), \tag{1.5}$$

where

$$b(n, s(n)) = \sup_i (|w_{ni}| \, |p_{is(n)}|),$$

and $(s(n))$ is a sequence of integers such that $\limsup_n s(n)/n = k < 1$,

$$B = \inf_{g(n)} (\limsup_n (B(n, g(n))^{(n-g(n))^{-1}}),$$

where

$$B(n, g(n)) = \sup_i (|w_{ni}| \, |p_{ig(n)}|),$$

and $(g(n))$ is a sequence of integers such that $\limsup_n g(n)/n = k' > 1$.

Here, we discuss the effectiveness of *any* basic set of polynomials, in a way, altogether different, from that, usually, used before. This effectiveness is determined, here, in terms of the above mentioned constants.

Also, we investigate the effectiveness at the origin of the inverse and the product sets of simple basic sets of polynomials in an easy procedure, which differs from that used by Eweida [3].

We prove, in this paper, the following theorems:

Theorem 1. *Let $\{p_n(z)\}$ be any basic set of polynomials, then the necessary conditions for such a set to be effective at the origin are:*

(i) *C has a finite value* (ii) $b = 0$.

Theorem 2. *Let $\{p_n(z)\}$ be a simple basic set of polynomials effective* *at the origin, then the necessary condition for the effectiveness of its inverse at the origin is:*

$$\lim_{n\to\infty} |p_{nn}|^{1/n}$$

tends always to a non-zero finite limit.

Theorem 3. *Let* $\{p_n(z)\}$ and $\{q_n(z)\}$ *be simple basic sets of polynomials effective at the origin, then the necessary condition for the product set* $\{p_n(z)\}\,\{q_n(z)\}$ *to be effective at the origin is:*

$$\lim_{n\to\infty} |q_{nn}|^{1/n}$$

tends to a non-zero finite limit.

Thus, by theorem 1, it became possible to determine the effectiveness of *any* basic set of polynomials at the origin, *by a common procedure.* By theorems 2 and 3, we find that the condition imposed, for the effectiveness of the inverse sets and the product sets, is less resteictive than that given by EWEIDA [3] in this connection.

2. *Notation and previous results.* Let (v_{ij}) and (q_{ij}) be the basic sets of operators and coefficients, respectively, associated with the set $\{q_n(z)\}$, and (y_{ij}) and (u_{ij}) the corresponding sets associated with the set $\{u_n(z)\}$.

Let $\{\overset{*}{p}_n(z)\}$ be the inverse set of $\{p_n(z)\}$, then the sets (p_{ij}) and (w_{ij}) are the sets of operators and coefficients, respectively, associated with $\{\overset{*}{p}_n(z)\}$.

Write $\{u_n(z)\}=\{p_n(z)\}\,\{q_n(z)\}$, then $\{u_n(z)\}$ is the product set of the sets $\{p_n(z)\}$ and $\{q_n(z)\}$ in this order, and it is defined by:

$$y_{ij} = \sum_h u_{ih} w_{hj} \;\; \& \;\; u_{ij} = \sum_h p_{ih} q_{hj}\,. \tag{2.1}$$

We write, throughout, a's to represent constants.

The author has proved, in [1], that the basic set of polynomials satisfying the condition $C\leqslant 1$, is effective on $|z|=R$, where R is such that $b\leqslant R\leqslant B$.

3. We give in this section the proofs of the theorems.

Proof of theorem 1. In view of (1.2) and (1.3), we have:

$$\begin{aligned} R^n \leqslant F_n(R) \leqslant G_n(R) \leqslant D_n^{\,2)}\, N_n(C(n,d(n))\, R^{d(n)} + b(n,s(n))\, R^{s(n)} + \\ + B(n,g(n))\, R^{g(n)}). \\ \leqslant R^n D_n N_n\, (C(n,d(n)))\, R^{d(n)-n} + (b/R)^{n-s(n)} + (B/R)^{g(n)-n}). \end{aligned}$$

Since $0<t'\leqslant C\leqslant t<\infty$ and $b=0$, then on taking the n-th. root we get:

$$R \leqslant F(R) \leqslant G(R) \leqslant aR, \quad \text{for any value of } R\leqslant B,$$

i.e.

$$F(R) \leqslant G(R) = a_1 R.$$

[2]) D_n is the degree of the polynomial of the highest degree in (1.1).

Hence

$$F(0+) = G(0+) = 0.$$

Thus the set is effective at the origin.

Proof of theorem 2. Since the set $\{p_n(z)\}$ is effective at the origin then, by theorem 1, we have (i) $C \leqslant t < \infty$ (ii) $b = 0$.

In view of (1.4), we have:

$$\overset{*}{C} = \limsup_n (|p_{n\,i(n)}|\,|w_{i(n)\,d(n_0)}|)^{1/n}.$$

Since $\{p_n(z)\}$ is a simple set, then

$$\overset{*}{C} = \limsup_n [(|w_{nn}|\,|p_{n\,i(n)}|\,|w_{i(n)\,d(n_0)}|\,|p_{d(n_0)\,d(n_0)}|)^{1/n} \times |p_{nn}/p_{d(n_0)\,d(n_0)}|^{1/n}].$$

But we know that $\lim\limits_{n\to\infty} |p_{nn}|^{1/n}$ is non-zero finite and $n \geqslant i(n) \geqslant d(n_0)$, then we get:

$$\overset{*}{C} \leqslant a_2 C^2 < \infty.$$

Thus, the condition (i) of theorem 1 is satisfied.

In view of (1.5), we have in a similar way:

$$\overset{*}{b} = \limsup_n [(|w_{nn}|\,|p_{n\,i(n)}|\,|w_{i(n)\,s(n_0)}|\,|p_{s(n_0)\,s(n_0)}|)^{(n-s(n_0))^{-1}} \times \\ \times |p_{nn}/p_{s(n_0)\,s(n_0)}|^{(n-s(n_0))^{-1}}].$$

Since $b = 0$, then whether $i(n)$ belongs to $(d(n))$ or $(s(n))$, we have:

$$\overset{*}{b} \leqslant 0, \quad \text{i.e.} \quad \overset{*}{b} = 0.$$

Thus, the condition (ii) of theorem 1, is satisfied.

Hence, the result of the theorem follows.

Proof of theorem 3. Let

$$\{u_n(z)\} = \{p_n(z)\}\,\{q_n(z)\}.$$

Then, we have to prove that (i) C_u is finite (ii) $b_u = 0$.

(i) In view of (1.4) and (2.1), we have:

$$C_u = \limsup_n (|v_{nm(n)}|\,|w_{m(n)\,i(n)}|\,|p_{i(n)\,x(n)}|\,|q_{x(n)\,d(n_0)}|)^{1/n} \\ = \limsup_n [(|v_{nm(n)}|\,|q_{m(n)\,m(n)}|\,|w_{m(n)\,i(n)}|\,|p_{i(n)\,m(n)}|\,|v_{x(n)\,x(n)}| \times \\ \times |q_{x(n)\,d(n_0)}|)^{1/n}\,|q_{x(n)\,x(n)}/q_{m(n)\,m(n)}|^{1/n}].$$

Since $n \geqslant m(n) \geqslant i(n) \geqslant x(n) \geqslant d(n_0)$, then we have:

$$C_u \leqslant a_3\, C_q^2\, C_p.$$

Since both $\{p_n(z)\}$ and $\{q_n(z)\}$ are effective at the origin, then

C_u is finite.

(ii) In view of (1.5) and (2.1), we have in a similar way that:

$$b_u \leqslant (a_4 b_u;\ a_5 b_q) = 0, \qquad \text{(because } b_u = b_q = 0\text{)}$$

Hence, the result of the theorem follows.

Example

This example is given to show that the condition of theorem 3 is necessary.

Consider the sets $\{p_n(z)\}$ and $\{q_n(z)\}$ of polynomials defined by:

$$\begin{aligned} p_n(z) &= a^n z^n + z^{n-1} && n \text{ odd.} \\ &= a^{-(n+1)} z^n && n \text{ even or } 0. \\ q_n(z) &= b^{n+1} z^n && n \text{ odd or } 0. \\ &= b^{3n^2} z^n + a^{3n^2} z^{n-1} && n \text{ even.} \end{aligned}$$

(a and b are positive and finite)

Although both the sets $\{p_n(z)\}$ and $\{q_n(z)\}$ are effective at the origin, yet the product set $\{p_n(z)\}\{q_n(z)\}$, in this order, is not effective at the origin because $\lim_{n\to\infty} |q_{nn}|^{1/n}$ is infinite.

Moreover, the product set $\{q_n(z)\}\{p_n(z)\}$, in this order, is effective at the origin because $\lim_{n\to\infty} |p_{nn}|^{1/n}$ is finite.

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