On the oscillation of certain mixed neutral equations

T. Candan\textsuperscript{a}, R.S. Dahiya\textsuperscript{b,∗}

\textsuperscript{a} Department of Mathematics, Faculty of Art and Science, Niğde University, Niğde, 51200, Turkey
\textsuperscript{b} Department of Mathematics, Iowa State University, Ames, IA 50011, USA

Received 13 February 2007; accepted 19 February 2007

Abstract

Sufficient conditions are obtained for oscillatory behavior of solutions for certain \(n\)th-order neutral functional differential equations with distributed deviating arguments. The approach proposed in this work makes it possible to get different results. The results obtained are illustrated with examples.

© 2007 Elsevier Ltd. All rights reserved.

Keywords: Neutral equations; Oscillations; Distributed delay

1. Introduction

In this work, we study the oscillatory behavior of the solutions of the \(n\)th-order mixed neutral functional differential equations with distributed deviating arguments of the form

\[
\left[ x(t) + h \int_a^b x(t - \xi)d\xi + g \int_a^b x(t + \xi)d\xi \right]^{(n)} = p \int_c^d x(t - \mu)d\mu + q \int_c^d x(t + \mu)d\mu
\]

and

\[
\left[ x(t) + h \int_a^b x(t - \xi)d\xi + g \int_a^b x(t + \xi)d\xi \right]^{(n)} = px(t - \tau) + qx(t + \nu),
\]

where \(h\) and \(g\) are nonnegative real constants, \(p, q, \tau\) and \(\nu\) are positive real numbers, \([a, b]\) and \([c, d]\) are positive intervals.

Similar kinds of equations were studied with discrete delay; for example, see ([2,5–9]). Some of these results are extended to the distributed deviating argument case in [3,4] but our results are different since distributed deviating arguments involve under \(n\)th-order derivatives. The case of Eq. (1) with discrete delay and \(n = 2\) is encountered in the study of vibrating masses attached to an elastic bar (see [10]). For related books we refer the reader to [1,10,13].

∗ Corresponding author. Tel.: +1 515 294 8159; fax: +1 515 294 5454.
E-mail addresses: tcandan@nigde.edu.tr (T. Candan), rdahiya@iastate.edu (R.S. Dahiya).
A solution of (1) and (2), which is nontrivial for all large \( t \), is called oscillatory if it has no last zero. Otherwise, a solution is called nonoscillatory.

The purpose of this work is to provide sufficient conditions for Eqs. (1) and (2), involving the coefficients, limits of integration and deviating arguments only.

2. Main results

The following lemma will be used in our proofs, from [11,12].

**Lemma 1.** Suppose that \( a \) and \( h \) are positive constants and let

\[
a^{1/n}(h/n)e > 1.
\]

Then,

(i) the inequality

\[
x^{(n)}(t) - ax(t - h) \geq 0
\]

has no eventually positive bounded solutions when \( n \) is even;

(ii) the inequality

\[
x^{(n)}(t) - ax(t + h) \geq 0
\]

has no eventually positive unbounded solutions when \( n \) is even, i.e., the last inequality in the above has no solution \( x \) with \( x^{(i)}(t) > 0 \) for \( i = 0, 1, \ldots, n \) and all large \( t \);

(iii) the inequality

\[
x^{(n)}(t) - ax(t + h) \geq 0
\]

has no eventually positive solutions when \( n \) is odd;

(iv) the inequality

\[
x^{(n)}(t) + ax(t - h) \leq 0
\]

has no eventually positive solutions when \( n \) is odd.

**Theorem 1.** Suppose \( c > b \), \( n \) is odd and

\[
\left( \frac{q(d - c)}{1 + (h + g)(b - a)} \right)^{1/n} \left( \frac{c - b}{n} \right)e > 1.
\]

Then Eq. (1) is oscillatory.

**Proof.** Suppose that \( x(t) \) is a nonoscillatory solution of (1). We may assume that \( x(t) \) is eventually positive, that is, there exists a \( t_0 \) such that \( x(t) > 0 \) for \( t \geq t_0 \). If \( x(t) \) is an eventually negative solution, this can be proved by the same arguments. Set

\[
z(t) = x(t) + h \int_a^b x(t - \xi)d\xi + g \int_a^b x(t + \xi)d\xi.
\]

Then from (1)

\[
z^{(n)}(t) = p \int_a^d x(t - \mu)d\mu + q \int_a^d x(t + \mu)d\mu
\]

for \( t \geq t_1 \) for some \( t_1 \geq t_0 \). Therefore, it is clear from (4) that \( z^{(i)}(t) \), \( i = 0, 1, \ldots, n \), are of constant sign on \([t_2, \infty)\) and \( z^{(i)}(t) > 0 \) for \( t \geq t_2 \). Set

\[
w(t) = z(t) + h \int_a^b z(t - \xi)d\xi + g \int_a^b z(t + \xi)d\xi.
\]
Then
\[ w^{(n)}(t) = z^{(n)}(t) + h \int_a^b z^{(n)}(t - \xi)d\xi + g \int_a^b z^{(n)}(t + \xi)d\xi \]
\[ = p \int_c^d x(t - \mu)d\mu + q \int_c^d x(t + \mu)d\mu \]
\[ + h \left[ p \int_a^b \int_c^d x(t - \xi - \mu)d\xi d\mu + q \int_a^b \int_c^d x(t - \xi + \mu)d\xi d\mu \right] \]
\[ + g \left[ p \int_a^b \int_c^d x(t + \xi - \mu)d\xi d\mu + q \int_a^b \int_c^d x(t + \xi + \mu)d\xi d\mu \right]. \]

By Fubini’s Theorem we can change the order of integration, i.e.,
\[ w^{(n)}(t) = p \int_c^d \left[ x(t - \mu) + h \int_a^b x(t - \xi - \mu)d\xi + g \int_a^b x(t + \xi - \mu)d\xi \right] d\mu \]
\[ + q \int_c^d \left[ x(t + \mu) + h \int_a^b x(t - \xi + \mu)d\xi + g \int_a^b x(t + \xi + \mu)d\xi \right] d\mu \]
\[ = p \int_c^d z(t - \mu)d\mu + q \int_c^d z(t + \mu)d\mu. \quad (5) \]

We can show by using (5) that \( w \) satisfies Eq. (1) and therefore we have
\[ \left[ w(t) + h \int_a^b w(t - \xi)d\xi + g \int_a^b w(t + \xi)d\xi \right]^{(n)} = p \int_c^d w(t - \mu)d\mu + q \int_c^d w(t + \mu)d\mu. \quad (6) \]

Since \( z(t) > 0 \) for \( t \geq t_2 \), it follows from (5) that \( w^{(n+1)}(t) > 0 \) for \( t \geq t_3 \geq t_2 \) and therefore we have
\[ w^{(i)}(t) > 0 \quad \text{for} \quad i = 0, 1, \ldots, n + 1 \quad \text{and} \quad t \geq t_3. \quad (7) \]

Now making use of (6) and (7), we can see that
\[ w^{(n)}(t) + h(b - a)w^{(n)}(t - a) + g(b - a)w^{(n)}(t + b) \geq q \int_c^d w(t + \mu)d\mu \geq q(d - c)w(t + c) \]
and hence
\[ [1 + (h + g)(b - a)]w^{(n)}(t + b) \geq q(d - c)w(t + c) \]
or
\[ w^{(n)}(t) - \frac{q(d - c)}{1 + (h + g)(b - a)}w(t + (c - b)) \geq 0. \quad (8) \]

Thus, in view of Lemma 1(iii) and condition (3) and (8) has no solution satisfying (7). Thus we obtain the desired contradiction which completes the proof. \( \Box \)

**Theorem 2.** Suppose \( c > b \), \( n \) is even and
\[ \left( \frac{q(d - c)}{1 + (h + g)(b - a)} \right) \frac{1}{n} \left( \frac{c - b}{n} \right) e > 1 \quad (9) \]
and
\[ \left( \frac{p(d - c)}{1 + (h + g)(b - a)} \right) \frac{1}{n} \left( \frac{c - b}{n} \right) e > 1. \quad (10) \]

Then Eq. (1) is oscillatory.
**Proof.** Suppose that \( x(t) \) is a nonoscillatory solution of (1). We may assume that \( x(t) > 0 \) for \( t \geq t_0 \). If \( x(t) \) is an eventually negative solution, this can be proved by the same arguments. Set

\[
z(t) = x(t) + \int_{a}^{b} x(t - \xi) d\xi + g \int_{a}^{b} x(t + \xi) d\xi.
\]

It is clear from (1) that

\[
z^{(n)}(t) = p \int_{c}^{d} x(t - \mu) d\mu + q \int_{c}^{d} x(t + \mu) d\mu
\]

for \( t \geq t_1 \) for some \( t_1 \geq t_0 \). Therefore, \( z^{(i)}(t) \), \( i = 0, 1, \ldots, n \), are of constant sign on \([t_2, \infty)\). Let

\[
w(t) = z(t) + \int_{a}^{b} z(t - \xi) d\xi + g \int_{a}^{b} z(t + \xi) d\xi.
\]

Then as in the proof of Theorem 1 one can show that

\[
w^{(n)}(t) = p \int_{c}^{d} z(t - \mu) d\mu + q \int_{c}^{d} z(t + \mu) d\mu.
\]

On the other hand it can be shown that \( w \) satisfies Eq. (1) and therefore we have

\[
\left[ w(t) + h \int_{a}^{b} w(t - \xi) d\xi + g \int_{a}^{b} w(t + \xi) d\xi \right]^{(n)} = p \int_{c}^{d} w(t - \mu) d\mu + q \int_{c}^{d} w(t + \mu) d\mu.
\]

Now we have two cases to consider: \( z'(t) > 0 \) and \( z'(t) < 0 \) for \( t \geq t_2 \). If \( z'(t) > 0 \), as in the proof of Theorem 1 we can find

\[
w^{(n)}(t) - \frac{q(d - c)}{1 + (h + g)(b - a)} w(t + (c - b)) \geq 0.
\]

Thus, in view of Lemma 1(ii) and condition (9) and (12) has no solution satisfying (7). This is a contradiction.

If \( z'(t) < 0 \) for \( t \geq t_2 \), then we have \( w'(t) < 0 \), \( w^{(n)}(t) > 0 \) and \( w^{(n+1)}(t) < 0 \) for \( t \geq t_3 \geq t_2 \). Using the decreasing nature of \( w^{(n)}(t) \) on \([t_3, \infty] \) in (11), we obtain

\[
[1 + (h + g)(b - a)] w^{(n)}(t - b) \geq p(d - c) w(t - c)
\]

or

\[
w^{(n)}(t) - \frac{p(d - c)}{1 + (h + g)(b - a)} w(t - (c - b)) \geq 0.
\]

This result contradicts Lemma 1(i) and condition (10) and therefore the proof is complete. □

The proofs of the next two theorems are similar to those of Theorem 1 and Theorem 2, respectively; therefore they are omitted.

**Theorem 3.** Suppose \( v > b \), \( n \) is odd and

\[
\left( \frac{q}{1 + (h + g)(b - a)} \right)^{1/n} \left( \frac{v - b}{n} \right) e > 1.
\]

Then Eq. (2) is oscillatory.

**Theorem 4.** Suppose \( \min[\tau, v] > b \), \( n \) is even and

\[
\left( \frac{q}{1 + (h + g)(b - a)} \right)^{1/n} \left( \frac{v - b}{n} \right) e > 1
\]

and

\[
\left( \frac{p}{1 + (h + g)(b - a)} \right)^{1/n} \left( \frac{\tau - b}{n} \right) e > 1.
\]

Then Eq. (2) is oscillatory.
Example 1. Consider the following neutral differential equation:
\[
\left[ x(t) + \frac{3\pi}{2} \int_{\pi/2}^{3\pi/2} x(t-\xi) d\xi + \frac{3\pi}{2} \int_{\pi/2}^{3\pi/2} x(t+\xi) d\xi \right]^{(3)} = \frac{3}{4} \int_{5\pi}^{6\pi} x(t-\mu) d\mu + \frac{9}{4} \int_{5\pi}^{6\pi} x(t+\mu) d\mu,
\]
and note that \( n = 3, h = g = 1, a = \frac{3\pi}{2}, b = 3\pi, c = 5\pi, d = 6\pi, p = \frac{3}{4} \) and \( q = \frac{9}{4} \).

One can easily check that the conditions of Theorem 1 are satisfied. It is easy to verify that \( x(t) = \cos t + \sin t \) is a solution of this problem.

Example 2. Consider the following neutral differential equation:
\[
\left[ x(t) + \frac{1}{4} \int_{\pi/2}^{3\pi/2} x(t-\xi) d\xi + \frac{1}{4} \int_{\pi/2}^{3\pi/2} x(t+\xi) d\xi \right]^{(4)} = \frac{1}{3\pi} \int_{4\pi}^{6\pi} x(t-\mu) d\mu + \frac{1}{6\pi} \int_{4\pi}^{6\pi} x(t+\mu) d\mu,
\]
and note that \( n = 4, h = \frac{1}{4}, g = \frac{1}{4}, a = \frac{\pi}{2}, b = \frac{3\pi}{2}, c = 4\pi, d = 6\pi, p = \frac{1}{3\pi} \) and \( q = \frac{1}{6\pi} \).

One can easily check that the conditions of Theorem 2 are satisfied. It is easy to verify that \( x(t) = t \cos t \) is a solution of this problem.

References