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Graph powers and k-ordered Hamiltonicity

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Abstract

It is known that if G is a connected simple graph, then G^3 is Hamiltonian (in fact, Hamilton-connected). A simple graph is k-ordered Hamiltonian if for any sequence $v_1, v_2, \ldots, v_k$ of k vertices there is a Hamiltonian cycle containing these vertices in the given order. In this paper, we prove that if $k \geqslant 4$, then $G^{\lfloor 3k/2 \rfloor - 2}$ is k-ordered Hamiltonian for every connected graph G on at least k vertices. By considering the case of the path graph P_n, we show that this result is sharp. We also give a lower bound on the power of the cycle C_n that guarantees k-ordered Hamiltonicity.

Keywords: Graph powers; k-Ordered Hamiltonian; Hamiltonian cycles

1. Introduction

The concept of k-ordered Hamiltonian graphs has been recently introduced by Ng and Schultz [6]. A simple graph G is *k-ordered* (resp. *k-ordered Hamiltonian*) if for any sequence $v_1, v_2, \ldots, v_k$ of k vertices of G there is a cycle (resp. a Hamiltonian cycle) in G containing these vertices in the given order. Note that being 3-ordered Hamiltonian is equivalent to being Hamiltonian.

A natural direction of research related to this new Hamiltonian property is to generalize existing results implying graph Hamiltonicity and obtain results implying k-ordered Hamiltonicity. In [6] Ng and Schultz generalize classical theorems of Dirac and Ore and give minimum vertex degree conditions that guarantee a graph is k-ordered Hamiltonian. These conditions were improved by Faudree [3]. Another series of results appearing in [2] describes various forbidden subgraphs that force a graph to be k-ordered or k-ordered Hamiltonian. There are many open questions about whether these properties can be ensured by sufficient connectivity in a graph (see [3]).

In this paper we extend a well-known result on Hamiltonicity of the third power of a simple graph, which is defined below.

Definition 1.1. Let G be a simple graph with vertex set V and edge set E. The *nth power* of G, denoted by G^n, is the simple graph with the same vertex set V and with the edge set

$$E(G^n) = \{(v, w) | d_G(v, w) \leqslant n\}.$$

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Our goal is to explore the k-ordered Hamiltonicity of graph powers. In Section 3 we give a lower bound on the smallest power of the path P_n that is k-ordered Hamiltonian. In Section 4 we prove the main theorem, which states that $G^{\lfloor 3k/2 \rfloor - 2}$ is k-ordered Hamiltonian for $k \geqslant 4$ and a connected graph G on at least k vertices. Finally, in Section 5 we discuss the k-ordered Hamiltonicity of powers of the cycle C_n. We conclude the paper by mentioning possible directions for further research on the topic.

2. Preliminaries

All graphs considered in this paper are finite simple graphs. The distance between vertices v, w of a graph G is denoted by $d_G(v, w)$. The number of vertices of G is denoted by $|G|$. If P is a path with endpoints a and b, then aPb denotes the path P starting at a and ending at b. Since the edges of our graphs are undirected, we use the notation (u, v) and (v, u) to denote the same edge whose endpoints are u and v.

For $k \geqslant 3$ and a graph G, let $p_k(G)$ be the smallest integer p such that G^p is k-ordered Hamiltonian. In this paper we give bounds on $p_k(G)$ for an arbitrary connected graph G and in the special cases of G being a path or a cycle.

A graph G is *Hamilton-connected* if for any pair v, w of vertices of G there is a path in G between v and w containing all vertices of G. Such a path is called a *Hamiltonian path*. The following theorem is often included as an exercise in graph theory textbooks.

Theorem 2.1. *If G is a connected graph on two or more vertices, then G^3 is Hamilton-connected.*

Proof. Let T be a spanning tree of G. Notice that it suffices to show that T^3 is Hamilton-connected since T^3 is a subgraph of G^3 with the same vertex set.

We will show by induction that if T is a tree, then T^3 is Hamilton-connected. This is obvious if T has only two vertices.

Suppose that the assertion is true for trees with fewer than $|T|$ vertices. Let v_1 and v_2 be distinct vertices of T. Choose an edge $e = (w_1, w_2)$ of T such that $T - e$ consists of two connected components T_1 and T_2 satisfying $v_1, w_1 \in T_1$ and $v_2, w_2 \in T_2$. For $i = 1, 2$ let $u_i = w_i$ if $w_i \neq v_i$ or if $|T_i| = 1$, otherwise let u_i be a vertex of T_i such that $d_T(u_i, w_i) = 1$. Let P_i be a Hamiltonian path in T_i^3 between v_i and u_i (if $u_i = v_i = w_i$, then P_i is the "empty" path starting and ending at v_i). Notice that $d_T(u_1, u_2) \leqslant 3$, so $v_1 P_1 u_1 u_2 P_2 v_2$ is a Hamiltonian path in T^3 between v_1 and v_2. □

Since Hamilton-connectedness implies Hamiltonicity, we have the following immediate corollary:

Corollary 2.2. *If G is a connected graph on three or more vertices, then G^3 is Hamiltonian.*

The proof of Theorem 2.1 has been included for two reasons. First, it uses the fact that it suffices to prove the statement only for trees instead of arbitrary graphs. We will use the same idea in the proof of Theorem 4.4. Second, it is based on a simple induction argument, which in contrast to the case where we need to keep track of the order in which certain vertices are contained in the cycle, cannot be applied.

The following theorem about 2-connected graphs was proved by Fleischner [4] in 1974. A simpler proof can be found in [1].

Theorem 2.3. *If G is a 2-connected graph on three or more vertices, then G^2 is Hamiltonian.*

Another result relevant to the discussion of this paper was proved by Lou et al. [5].

Theorem 2.4. *Let G be a connected graph on three or more vertices. For any two edges e_1 and e_2 of G, there is a Hamiltonian cycle in G^4 containing e_1 and e_2.*

3. Powers of paths

Let P_n be the path on n vertices. In this section we present a lower bound on $p_k(P_n)$.

Theorem 3.1. *For $k \geqslant 3$ and $n \geqslant 2k-1$, the inequality $p_k(P_n) \geqslant \lfloor 3k/2 \rfloor - 2$ holds.*

Proof. First, consider the case of even k, and let $k = 2m$. We show that $(P_n)^{3m-3}$ is not $2m$-ordered. Let $v_1, v_3, \ldots, v_{2m-1}, u_1, u_2, \ldots, u_{2m-1}, v_2, v_4, \ldots, v_{2m}$ be consecutive vertices of the path P_n. Suppose that C is a cycle in $(P_n)^{3m-3}$ containing the vertices $v_1, v_2, \ldots, v_{2m}$ in order. For $1 \leqslant i \leqslant 2m$, let R_i denote the part of C between, and including, v_i and v_{i+1} (indices taken modulo $2m$). Put $U = \{u_1, u_2, \ldots, u_{2m-1}\}$. Since $|U| < 2m$, there is an index i such that R_i does not contain a vertex of U. In P_n, the set U is located between v_i and v_{i+1}, so R_i must contain an edge e that "skips over" U. The edge e connects one of $v_1, v_3, \ldots, v_{2m-1}$ and one of $v_2, v_4, \ldots, v_{2m}$ because $d_{P_n}(v_1, v_2) = d_{P_n}(v_{2m-1}, v_{2m}) = 3m-1 > 3m-3$. Also, the edge e connects two non-consecutive vertices of $v_1, v_2, \ldots, v_{2m}$ since the distance in P_n between two consecutive vertices of this sequence is at least $3m-2$. Thus, the cycle C cannot contain e, which contradicts R_i containing e.

We treat the case of odd k in a similar way. Let $k = 2m+1$. We show that $(P_n)^{3m-2}$ is not $(2m+1)$-ordered. Let $v_1, v_3, \ldots, v_{2m+1}, u_1, u_2, \ldots, u_{2m-1}, v_2, v_4, \ldots, v_{2m}$ be consecutive vertices of the path P_n. Suppose that C is a cycle in $(P_n)^{3m-2}$ containing the vertices $v_1, v_2, \ldots, v_{2m+1}$ in order. As in the previous case, we find an edge e of C connecting one of $v_1, v_3, \ldots, v_{2m+1}$ and one of $v_2, v_4, \ldots, v_{2m}$. Again, this edge connects two non-consecutive vertices of $v_1, v_2, \ldots, v_{2m+1}$ since $d_{P_n}(v_i, v_{i+1}) \geqslant 3m-1$ for $1 \leqslant i \leqslant 2m$. We obtain a similar contradiction.

We conclude that $p_{2m}(P_n) \geqslant 3m-2$ and $p_{2m+1}(P_n) \geqslant 3m-1$. The theorem follows. □

4. Main theorem

We now prove our main result which gives an upper bound on $p_k(G)$ for a connected graph G on at least k vertices. We begin by proving three technical lemmas.

Lemma 4.1. *Let W be a tree on at least two vertices, and let w be a vertex of W. Then $W^3 - w$ has a Hamiltonian path whose endpoints w_1 and w_2 satisfy $d_W(w, w_1) = 1$ and $d_W(w, w_2) \leqslant 2$.*

Proof. Let $W_1, \ldots, W_m$ be the connected components of $W - w$. Let $w_1^{(i)}$ be the vertex of W_i adjacent to w in W, and let $w_2^{(i)}$ be a vertex of W_i adjacent to $w_1^{(i)}$, or equal to $w_1^{(i)}$ if $|W_i| = 1$. Let $R^{(i)}$ be a Hamiltonian path in W_i^3 starting at $w_1^{(i)}$ and ending at $w_2^{(i)}$; such a path exists by Theorem 2.1. Note that $d_W(w_2^{(i)}, w_1^{(i+1)}) \leqslant d_W(w_2^{(i)}, w) + d_W(w, w_1^{(i+1)}) \leqslant 3$ for $1 \leqslant i \leqslant m-1$, so the concatenation $R^{(1)} \ldots R^{(m)}$ yields the desired path with $w_1 = w_1^{(1)}$ and $w_2 = w_2^{(m)}$. □

Lemma 4.2. *Let k and $p \geqslant 3$ be positive integers, and let G be a connected graph on at least k vertices. Let $v_1, \ldots, v_k$ be a sequence of k vertices of G. Suppose that there exists a tree $U \subseteq G$ and a cycle C in U^p satisfying the following conditions:*

(i) *C contains $v_1, \ldots, v_k$ in order;*
(ii) *C contains every leaf of U;*
(iii) *if x is a leaf of U, then x is adjacent in C to a vertex y such that either $d_U(x, y) \leqslant p-2$, or $d_U(x, y) = p-1$ and y is not a leaf of U.*

Then there is a Hamiltonian cycle in G^p containing $v_1, \ldots, v_k$ in order.

Proof. We begin by extending C to a Hamiltonian cycle in U^p. Let y be a vertex of U such that $y \notin C$. By condition (ii), y is not a leaf. Therefore, $U - y$ has at least two connected components, and some part of C between a leaf in one component and a leaf in another component does not contain y. This part contains an edge (x, z) such that y lies on the unique path in U between x and z. Let C' be the cycle obtained from C by replacing the edge (x, z) with the path xyz. Clearly, C' satisfies conditions (i) and (ii). Since $d_U(x, y) + d_U(y, z) = d_U(x, z) \leqslant p$, it follows that $d_U(x, y) \leqslant p-1$ and $d_U(z, y) \leqslant p-1$, hence C' satisfies condition (iii). In this way, we can add all remaining vertices of U to obtain a Hamiltonian cycle $\tilde{C}$ of U^p satisfying conditions (i)–(iii) with C replaced by $\tilde{C}$.

Let T be a spanning tree of G containing U. Consider the graph $T - E(U)$ obtained by removing the edges of U from T. Let $U_1, \ldots, U_\ell$ be the connected components of $T - E(U)$ such that $|U_i| > 1$. Let u_i denote the unique vertex of $U \cap U_i$. Note that the vertices $u_1, \dots, u_\ell$ are distinct. Put $T_0 = U$ and $T_i = U \cup U_1 \cup \dots \cup U_i$ for $1 \leqslant i \leqslant \ell$, so that $T_\ell = T$. We construct a sequence $\tilde{C} = C_0, C_1, \dots, C_\ell$, where C_i is a Hamiltonian cycle of T_i^p satisfying the following conditions:

(I) C_i contains the vertices $v_1, \dots, v_k$ in order;
(II) if x is a common leaf of U and T_i, then x is adjacent in C_i to a vertex y such that either $d_T(x, y) \leqslant p - 2$, or $d_T(x, y) = p - 1$ and y is not a leaf of U.

The cycle C_0 satisfies the above conditions. We construct C_{i+1} by inserting the vertices of $U_{i+1} - u_{i+1}$ between two consecutive vertices of C_i. By Lemma 4.1, there is a Hamiltonian path $w_1 R_i w_2$ in $U_{i+1}^3 - u_{i+1}$ whose endpoints w_1 and w_2 satisfy $d_T(u_{i+1}, w_1) = 1$ and $d_T(u_{i+1}, w_2) \leqslant 2$. We consider two cases.

Case 1: u_{i+1} is not a leaf of T_i. Then $T_i - u_{i+1}$ has at least two connected components, and one of the parts of C_i between a vertex in one component and a vertex in another component does not contain u_{i+1}. This part contains an edge (x, z) such that u_{i+1} belongs to the unique path between x and z in T_i. Without loss of generality, suppose that $d_T(x, u_{i+1}) \geqslant d_T(z, u_{i+1})$. Let C_{i+1} be the cycle obtained by replacing the edge (x, z) with the path $x w_1 R_i w_2 z$ in C_i. Then C_{i+1} is a Hamiltonian cycle in T_{i+1}^p because $d_T(x, w_1) = d_T(x, u_{i+1}) + 1 \leqslant p$ and $d_T(z, w_2) \leqslant d_T(z, u_{i+1}) + d_T(u_{i+1}, w_2) \leqslant \lfloor p/2 \rfloor + 2 \leqslant p$.

Since C_{i+1} contains the vertices of C_i in the same order as C_i, it follows that C_{i+1} satisfies condition (I). To show that C_{i+1} satisfies condition (II), we need to consider the case when x or z is a common leaf of U and T_{i+1}. If x is a common leaf of U and T_{i+1}, then x is also a leaf of T_i, so either $d_T(x, z) \leqslant p - 1$, or the vertex $y \neq z$ adjacent to x in C_i has the property of condition (II). In the former case, we have $d_T(x, w_1) = d_T(x, u_{i+1}) + 1 \leqslant p - 1$, and w_1 is not a leaf of U. In the latter case, the vertex y, which is adjacent to x in C_{i+1}, has the required property. The case when z is a common leaf of U and T_{i+1} is treated similarly.

Case 2: u_{i+1} is a leaf of T_i. Then, by condition (II), u_{i+1} is adjacent in C_i to a vertex y such that either $d_T(u_{i+1}, y) \leqslant p-2$, or $d_T(u_{i+1}, y) = p - 1$ and y is not a leaf of U. Let C_{i+1} be the cycle obtained by replacing the edge (y, u_{i+1}) with the path $y w_1 R_i w_2 u_{i+1}$ in C_i. Then C_{i+1} is a Hamiltonian cycle in T_{i+1}^p because $d_T(y, w_1) = d_T(y, u_{i+1}) + 1 \leqslant p$ and $d_T(w_2, u_{i+1}) \leqslant 2$.

As in the previous case, it is easy to see that C_{i+1} satisfies condition (I). To show that C_{i+1} satisfies condition (II), we need to consider the case when y is a common leaf of U and T_{i+1}. In this case, y is also a leaf of T_i, so either $d_T(y, u_{i+1}) \leqslant p - 2$, or the vertex $y' \neq u_{i+1}$ adjacent to y in C_i has the property of condition (II). In the former case, we have $d_T(y, w_1) \leqslant p - 1$, and w_1 is not a leaf of U, and in the latter case the vertex y', which is adjacent to y in C_{i+1}, has the required property.

Since $T_\ell = T$, it follows that C_ℓ is a Hamiltonian cycle in T^p, and hence in G^p, containing $v_1, \dots, v_k$ in order. The lemma follows. □

Lemma 4.3. *For every integer* $t \geqslant 1$ *and every tree* U, *there exists a map* $\alpha : V(U) \to \{1, \dots, t\}$ *such that, for every two distinct vertices* x *and* z *of* U *and every integer* $c \in \{1, \dots, t\}$, *there exists a sequence* $x = y_0, y_1, \dots, y_{\ell-1}, y_\ell = z$ *of distinct vertices satisfying the following conditions*:

(A) $d_U(y_i, y_{i-1}) \leqslant t$ *for* $1 \leqslant i \leqslant \ell$;
(B) y_i *is not a leaf of* U *for* $1 \leqslant i \leqslant \ell - 1$;
(C) $\alpha(y_i) = c$ *for* $1 \leqslant i \leqslant \ell - 1$;
(D) *if* $\alpha(x) \neq c$ (*resp*., $\alpha(z) \neq c$), *then* $d_U(x, y_1) \leqslant t - 1$ (*resp*., $d_U(y_{\ell-1}, z) \leqslant t - 1$);
(E) *if* U *has at least* $t - 1$ *non-leaves*, x *and* z *are leaves of* U, *and* $\alpha(x)$ *and* $\alpha(z)$ *are not both equal to* c, *then* $\ell \geqslant 2$.

Proof. If U has at most $t - 2$ non-leaves, then the distance between any two vertices in U is at most $t - 1$, so the map $\alpha(x) = 1$ for all x and the sequence $x = y_0, y_1 = z$ satisfy conditions (A)–(D) (condition (E) is not applicable in this case).

Suppose that U has exactly $t - 1$ non-leaves $w_1, \dots, w_{t-1}$. Put $\alpha(w_i) = i$ for $1 \leqslant i \leqslant t - 1$, and set $\alpha(v) = t$ for every leaf v of U. If x or z is a non-leaf, then $d_U(x, z) \leqslant t - 1$, so setting $x = y_0, y_1 = z$ satisfies the conditions of the lemma. If both x and z are leaves and $c = t$, then setting $x = y_0, y_1 = z$ satisfies the conditions of the lemma as well. Finally, if both x and z are leaves and $c \neq t$, then setting $x = y_0, y_1 = w_c, y_2 = z$ satisfies the conditions of the lemma.

It remains to consider the case when U has at least t non-leaves. Let $|U|=m$, and let q be the number of leaves of U. Construct the sequence $U_t \subset U_{t+1} \subset \ldots \subset U_m$ of subtrees of U as follows. Let $U_m = U$. For $m-q \leqslant i \leqslant m-1$, choose a leaf v of U, and put $U_i = U_{i+1} - v$. Then U_{m-q} is the subtree consisting of all non-leaves of U. For $t \leqslant i \leqslant m-q-1$, choose a leaf v of U_{i+1}, and put $U_i = U_{i+1} - v$.

Let $x_1, \ldots, x_t$ be the vertices of U_t. For $t+1 \leqslant i \leqslant m$, let x_i be the only vertex of $U_i - U_{i-1}$. Then $V(U_j)=\{x_1, \ldots, x_j\}$ for $t \leqslant j \leqslant m$, and $x_{m-q+1}, \ldots, x_m$ are the leaves of U.

For each vertex $x \in U$, we define $\alpha(x)$ and construct a sequence $b_1(x), \ldots, b_{t-1}(x)$ of non-leaves of U as follows. Begin by setting $\alpha(x_j) = j$ for $1 \leqslant j \leqslant t$. Also, set $b_i(x_j)$ be the ith closest vertex of U_t to x_j for $1 \leqslant i \leqslant t-1$. (That is, rank the vertices of U_t other than x_j by distance to x_j from closest to furthest, breaking ties arbitrarily, and set $b_i(x_j)$ to be the ith vertex in this ranking.) Now, let $s \geqslant t+1$, and suppose that $\alpha(x_j)$ and $b_i(x_j)$ have been defined for $j < s$. Let x_r be the vertex adjacent to x_s in U_s; then $r < s$, and x_r is not a leaf of U. Put $b_1(x_s) = x_r$, $b_i(x_s) = b_{i-1}(x_r)$ for $2 \leqslant i \leqslant t-1$, and $\alpha(x_s) = \alpha(b_{t-1}(x_r))$.

Observe that $\alpha(x_j)$ and $b_i(x_j)$ have the following properties:

(a) the sequence $\alpha(x_j), \alpha(b_1(x_j)), \alpha(b_2(x_j)), \ldots, \alpha(b_{t-1}(x_j))$ contains all elements of $\{1, \ldots, t\}$;
(b) $d_U(x_j, b_i(x_j)) \leqslant i$;
(c) $b_i(x_j) \in U_j$ for $j \geqslant t$.

We prove these properties by induction on j. Clearly, conditions (a)–(c) hold for $j = t$. Assume the above conditions for $j < s$, and let x_r be the vertex adjacent to x_s in U_s. The sequence $\alpha(x_s), \alpha(b_1(x_s)), \ldots, \alpha(b_{t-1}(x_s))$ is the same as $\alpha(b_{t-1}(x_r)), \alpha(x_r), \alpha(b_1(x_r)), \ldots, \alpha(b_{t-2}(x_r))$, which contains all elements of $\{1, \ldots, t\}$. Also, $d_U(x_s, b_1(x_s)) = d_U(x_s, x_r) = 1$ and $d_U(x_s, b_i(x_s)) = 1 + d_U(x_r, b_{i-1}(x_r)) \leqslant i$ for $2 \leqslant i \leqslant t-1$. Finally, we have $b_1(x_s) = x_r \in U_r \subset U_s$, and $b_i(x_s) = b_{i-1}(x_r) \in U_r \subset U_s$ for $2 \leqslant i \leqslant t-1$.

We now show that for $t \leqslant j \leqslant m$, two vertices x and z of U_j, and an integer $c \in \{1, \ldots, t\}$, there exists a sequence $x = y_0, y_1, \ldots, y_\ell = z$ satisfying conditions (A)–(E) of the lemma. We proceed by induction on j. For $j = t$, the vertices x and z are not leaves of U, and $d_U(x, z) = d_{U_t}(x, z) \leqslant t-1$, so setting $x = y_0$ and $y_1 = z$ satisfies the desired conditions.

Suppose that the assertion is true for $j < s$, where $s \geqslant t+1$. To prove the assertion for $j = s$, it suffices to consider the case when x or z is equal to x_s. Without loss of generality, suppose that $z = x_s$ and $x \in U_{s-1}$. Let x_r be the vertex adjacent to x_s in U_s, and let $x_{r'}$ be the element of the sequence $x_r, b_1(x_r), \ldots, b_{t-1}(x_r)$ such that $\alpha(x_{r'}) = c$. By property (c), we have $x_{r'} \in U_r$, and since x_r is not a leaf of U, it follows that $x_{r'}$ is not a leaf of U either. By the inductive hypothesis, there exists a sequence $x = y'_0, y'_1, \ldots, y'_{\ell'} = x_{r'}$ satisfying conditions (A)–(E). Set $\ell = \ell' + 1$ and $y_0 = y'_0$, $y_1 = y'_1, \ldots, y_{\ell-1} = y'_{\ell-1}$, $y_\ell = x_s$. It is easy to check that the new sequence $y_0, \ldots, y_\ell$ satisfies conditions (A)–(E). The only non-trivial verification needed is for condition (D): we have $d_U(y_{\ell-1}, z) = 1 + d_U(x_{r'}, x_r) \leqslant t$, and equality holds only if $x_{r'} = b_{t-1}(x_r)$, in which case $\alpha(z) = \alpha(x_s) = \alpha(b_{t-1}(x_r)) = \alpha(x_{r'}) = c$. □

We now prove the main theorem.

Theorem 4.4. *For $k \geqslant 4$, let G be a connected graph on at least k vertices. Then $G^{\lfloor 3k/2 \rfloor - 2}$ is k-ordered Hamiltonian.*

Proof. Put $t = \lfloor 3k/2 \rfloor - 2$. Let $v_1, \ldots, v_k$ be a sequence of k vertices of G. Our goal is to find a Hamiltonian cycle in G^t containing the vertices $v_1, \ldots, v_k$ in order. Let T be a spanning tree of G, and let U be the smallest subtree of T containing all the v_i's. Then all leaves of U are among the v_i's. Let q be the number of leaves of U.

Let $\alpha : V(U) \to \{1, \ldots, t\}$ be a map satisfying the conditions of Lemma 4.3. For convenience, let us refer to the value of $\alpha(v)$ as the *color* of the vertex v.

Perhaps the best way to present the construction is in the form of an algorithm. For the rest of the proof, all indices of the elements of the sequence $v_1, \ldots, v_k$, as well as the sequences $h_1, \ldots, h_k$ and $R_1, \ldots, R_k$ introduced later, are taken modulo k. Let $V = \{v_1, \ldots, v_k\}$, and let $\tilde{V}$ denote the set of all non-leaves of U contained in V.

Step 1. We begin with a procedure in which we mark certain elements of V. First, mark every element $v_i \in \tilde{V}$ such that no other element of $\tilde{V}$ has the same color as v_i. Next, for each color $c \in \alpha(V)$ such that no element of $\tilde{V}$ has color c, mark one of the elements of $V - \tilde{V}$ of color c.

Observe that at most one element of each color has been marked. We continue with a lemma.

Lemma 4.5. *Let r be the number of elements of V that were not marked in the above procedure. Then*

$$|\alpha(V)| + r \leqslant t + 1. \qquad (*)$$

Proof. Let s be the number of elements $v_i \in \tilde{V}$ such that no other element of $\tilde{V}$ has the same color as v_i, and let m be the number of colors encountered more than once among the elements of $\tilde{V}$. Then

$$m \leqslant \left\lfloor \frac{|\tilde{V}|}{2} \right\rfloor = \left\lfloor \frac{k-q}{2} \right\rfloor,$$

because each of the m colors is encountered at least twice. Let $p = k - q - s$ be the number of elements of $\tilde{V}$ that were not marked. Then

$$|\alpha(\tilde{V})| + p = s + m + p = k - q + m \leqslant \left\lfloor \frac{3(k-q)}{2} \right\rfloor.$$

The number of marked elements of $V - \tilde{V}$ is the number of colors encountered among the elements of $V - \tilde{V}$ but not encountered among the elements of $\tilde{V}$, or $|\alpha(V)| - |\alpha(\tilde{V})|$. The number of elements of $V - \tilde{V}$ that were not marked is $r - p$. Thus

$$|\alpha(V)| - |\alpha(\tilde{V})| + r - p = |V - \tilde{V}| = q$$

since $V - \tilde{V}$ is the set of all leaves of U. Adding this equality to the previous inequality yields

$$|\alpha(V)| + r \leqslant \left\lfloor \frac{3(k-q)}{2} \right\rfloor + q = \left\lfloor \frac{3k-q}{2} \right\rfloor \leqslant \left\lfloor \frac{3k}{2} \right\rfloor - 1 = t + 1$$

because $q \geqslant 2$. □

Step 2. In this step we consider several cases, in some of which we mark one additional vertex.

Case 2.1: $|\alpha(V)| + r \leqslant t$. In this case, proceed directly to Step 3.

Case 2.2: $|\alpha(V)| + r = t + 1$. From the proof of Lemma 4.5 it follows that equality in $(*)$ holds only if

(1) $m = \lfloor |\tilde{V}|/2 \rfloor$, and
(2) $q = 2$, or else $q = 3$ and k is odd.

We consider three further subcases.

Case 2.2.1: $q = 2$ and k is even. In this case, U is a path. Condition (1) implies that each color in $\alpha(\tilde{V})$ is represented by exactly two elements of $\tilde{V}$. Let v_ℓ be one of the endpoints of U. Let v_i be the element of $\tilde{V}$ closest to v_ℓ, and let v_j be the other representative of the color $c = \alpha(v_i)$ in $\tilde{V}$. Note that neither v_i nor v_j have been marked in Step 1. One of the vertices v_{j-1} and v_{j+1} is different from v_ℓ. Without loss of generality, we can assume that $v_{j+1} \neq v_\ell$, because finding a Hamiltonian cycle containing the sequence v_k, ..., v_2, v_1 in the given order is clearly equivalent to our task. Note that the path between v_j and v_{j+1} in U does not contain v_i in its interior, for then v_{j+1} would be closer to v_ℓ than v_i. It follows that no element of V of color c lies between v_j and v_{j+1}. Mark v_j, and proceed to Step 3.

Case 2.2.2: $q = 2$ and k is odd. Again, U is a path. Condition (1) now implies that a unique color $c_0 \in \alpha(\tilde{V})$ is represented by exactly three elements of $\tilde{V}$, and the other colors in $\alpha(\tilde{V})$ are each represented by exactly two elements of $\tilde{V}$. If one of the two elements v_{i_1} and v_{i_2} of $\tilde{V}$ closest to the two endpoints of U has color different from c_0, denote this element by v_i and act as in the previous case. Otherwise, both v_{i_1} and v_{i_2} have color c_0. Let v_j be the remaining element of $\tilde{V}$ of color c_0. Suppose there is no element of V of color c_0 between v_j and v_{j+1}, or between v_j and v_{j-1}. Then, as in the previous case, assume the former without loss of generality, mark v_j, and proceed to Step 3. We are left with the situation in which there is an element of $\tilde{V}$ of color c_0 between v_j and v_{j+1} and between v_j and v_{j-1}. Besides v_j, the only elements of $\tilde{V}$ of color c_0 are v_{i_1} and v_{i_2}. We conclude that v_{j-1} and v_{j+1} are the endpoints of U. At this point, we designate v_j as a “special” vertex, and we deal with it in Step 3.

Case 2.2.3: $q=3$ and k is odd. Here U consists of three paths with a common endpoint u_0. Since $|\tilde{V}|$ is even, condition (1) implies that each color in $\alpha(\tilde{V})$ is represented by exactly two elements of $\tilde{V}$. For each of the three branches of U, find the element of $\tilde{V}$ in this branch, if any, closest to the leaf belonging to this branch. Of these elements, let v_i be the furthest from u_0. Note that $v_i \neq u_0$ because $|\tilde{V}| \geqslant 2$. Let v_j be the other representative of the color $c = \alpha(v_i)$ in $\tilde{V}$. Neither v_i nor v_j have been marked in Step 1. As before, without loss of generality, assume that v_{j+1} is different from the leaf of U contained in the same branch as v_i. Then the unique path in U from v_j to v_{j+1} does not contain v_i in its interior, hence it contains no element of V of color c in its interior. Mark v_j and proceed to Step 3.

Step 3. Observe that in Cases 2.2.1–2.2.3, colors c and c_0 are not represented by the vertices marked in Step 1, so after Step 2 there is still at most one marked vertex of each color.

In this step, we construct paths in U^t between consecutive elements of the sequence $v_1, \dots, v_k$ and concatenate them to form a cycle containing $v_1, \dots, v_k$ in order. We analyze cases and subcases corresponding to those of Step 2.

Case 3.1: $|\alpha(V)| + r \leqslant t$. We define a sequence $h_1, \dots, h_k$ of elements of $\{1, \dots, t\}$ as follows. For each ℓ such that v_ℓ is marked, set $h_\ell = \alpha(v_\ell)$. Let $v_{a_1}, \dots, v_{a_r}$ be the unmarked elements of V. Set $h_{a_1}, \dots, h_{a_r}$ to be distinct elements of $\{1, \dots, t\}$ not contained in $\alpha(V)$. Such an assignment is possible because $|\alpha(V)| + r \leqslant t$. Thus the elements of the sequence $h_1, \dots, h_k$ are all distinct.

For $1 \leqslant s \leqslant k$, let R_s be the path in U^t from $x = v_s$ to $z = v_{s+1}$ satisfying the conditions of Lemma 4.3 with $c = h_s$. Since the h_s's are all distinct, the paths $R_1, \dots, R_k$ are interior vertex disjoint. Also, the path R_s contains no element of $\tilde{V}$, and hence no element of V, in its interior (by condition (B) of Lemma 4.3, interior vertices of R_s are non-leaves). Indeed, every interior vertex of R_s has color h_s, which is either not represented by any vertex of V if v_s is unmarked, or else represented by no vertex of $\tilde{V}$ except v_s if v_s is marked. Hence the concatenation $R_1 \cdots R_k$ is a cycle in U^t containing $v_1, \dots, v_k$ in order.

Case 3.2: $|\alpha(V)| + r = t + 1$. As in Case 2.2, we consider three subcases.

Case 3.2.1: $q = 2$ and k is even (see Case 2.2.1). Define $h_1, \dots, h_k$ and $R_1, \dots, R_k$ as in the previous case. The assignment of values of $h_1, \dots, h_k$ is possible because in this case there are $r-1$ unmarked vertices and $|\alpha(V)|+r-1=t$. Repeating the argument of Case 3.1, we conclude that the paths $R_1, \dots, R_k$ in U^t are interior vertex disjoint and that for $s \neq j$, the path R_s contains no element of V in its interior, where j is the index of the vertex v_j marked in Case 2.2.1. It remains to prove that R_j has the same property. Recall that there are no elements of V of color $c = \alpha(v_j)$ between v_j and v_{j+1} on the path U. Since v_j is marked, all interior vertices of R_j have color c, so it suffices to ensure that the interior vertices of R_j lie between v_j and v_{j+1}. If this is not the case, we modify R_j as follows. Assume without loss of generality that v_j lies to the left of v_{j+1}. Let $v_j = y_0, y_1, \dots, y_\ell = v_{j+1}$ be the sequence of vertices of R_j. Let y_p be the last element of this sequence, if any, lying to the left of v_j. Then v_j lies between y_p and y_{p+1}, so $d_U(v_j, y_{p+1}) < d_U(y_p, y_{p+1})$, hence the sequence $v_j = y_0, y_{p+1}, \dots, y_\ell = v_{j+1}$ satisfies the conditions of Lemma 4.3 (observe that $d_U(v_j, y_{p+1}) < t$ and that v_j is a non-leaf, so condition (E) is not applicable). Similarly, we can eliminate all interior vertices of R_j lying to the right of v_{j+1}. Thus the modified path R_j has the desired properties, and $R_1 \dots R_k$ is a cycle in U^t containing $v_1, \dots, v_k$ in order.

Case 3.2.2: $q=2$ and k is odd (see Case 2.2.2). This case can be treated in exactly the same way as Case 3.2.1 unless a "special" vertex v_j was introduced in Case 2.2.2. If so, define $h_1, \dots, \widehat{h_j}, \dots, h_k$ and $R_1, \dots, \widehat{R_j}, \dots, R_k$ as in Case 3.1 (the hat symbol denotes omission of a sequence element), which is possible because there are $r-1$ unmarked vertices besides v_j, and $|\alpha(V)| + r - 1 = t$. Then, as before, the paths $R_1, \dots, \widehat{R_j}, \dots, R_k$ in U^t are interior vertex disjoint and do not contain elements of V in their interior. Set $h_j = h_{j-1}$, and let R_j be the path in U^t from $x = v_j$ to $z = v_{j+1}$ satisfying the conditions of Lemma 4.3 with $c = h_j$. Using the modification described in Case 3.2.1, we can assume that the interior vertices of R_{j-1} and R_j lie between v_{j-1} and v_j and between v_j and v_{j+1}, respectively, and hence R_{j-1} and R_j are interior vertex disjoint. Also, since h_j is different from all other h_s's except h_{j-1}, the path R_j is interior vertex disjoint from all other R_s's. Finally, note that the entire path $v_{j-1}R_{j-1}v_jR_jv_{j+1}$in U^t does not contain any vertices of V except v_j in its interior because all interior vertices of this path except v_j have color h_{j-1}, which is not represented among the vertices of $\tilde{V}$ because either v_{j-1} is a marked leaf of U and h_{j-1} is the color of v_{j-1}, or else $h_{j-1} \notin \alpha(V)$. Thus we obtain the desired cycle $R_1 \cdots R_k$.

Case 3.2.3: $q = 3$ and k is odd (see Case 2.2.3). This case is almost identical to Case 3.2.1 except that we need to make sure that the path R_j in U^t contains no element of V in its interior, where j is the index of the vertex v_j marked in Case 2.2.3. Since $h_j = \alpha(v_j)$ and R_j does not contain leaves of U in its interior, we only need to show that the interior vertices of R_j do not include the only other element of $\tilde{V}$ of color h_j, namely, v_i. Suppose that v_i is an interior vertex of R_j. Let $v_j = y_0, y_1, \dots, y_\ell = v_{j+1}$ be the sequence of vertices of R_j. Let y_p be the last vertex of R_j belonging to the path in U between v_i and the leaf of the branch containing v_i. By the assumption at the end of Case 2.2.3, $y_p \neq v_{j+1}$ and hence $p < \ell$. First, suppose that y_{p+1} is not on the same branch of U as v_i. It is easy to see that $d_U(v_j, y_{p+1}) \leqslant d_U(v_i, y_{p+1}) \leqslant d_U(y_p, y_{p+1})$, the former inequality implied by the fact that v_i is the furthest from u_0 of all elements of $\tilde{V}$. Thus we can replace R_j with the shorter path $v_j = y_0, y_{p+1}, \ldots, y_\ell = v_{j+1}$. It remains to show that the modified path satisfies the conditions of Lemma 4.3. Conditions (A)–(C) hold trivially. We have $d_U(v_j, y_{p+1}) \leqslant d_U(y_p, y_{p+1}) \leqslant t$. Condition (D) for $x = v_j$ is not applicable because $\alpha(v_j) = h_j = c$, and if $y_{p+1} = v_{j+1} = z$ and $\alpha(z) \neq c$, then $d_U(v_j, y_{p+1}) \leqslant d_U(y_p, y_{p+1}) \leqslant t - 1$ by condition (D) for the old path. Finally, condition (E) is not applicable because $x = v_j$ is not a leaf of U. Now suppose that y_{p+1} is on the same branch of U as v_i. Let y_m be the first vertex of R_j belonging to the path in U between v_i and the leaf of the branch containing v_i. Then y_{m-1}, y_{p+1}, and v_i lie on the same path in U with v_i being one of the endpoints of the path. Also, $d_U(y_{m-1}, v_i) \leqslant d_U(y_{m-1}, y_m) \leqslant t$ and $d_U(y_{p+1}, v_i) \leqslant d_U(y_{p+1}, y_p) \leqslant t$, so $d_U(y_{m-1}, y_{p+1}) \leqslant t - 1$. It follows that the shorter path $v_j = y_0, \ldots, y_{m-1}, y_{p+1}, \ldots, y_\ell = v_{j+1}$ satisfies the conditions of Lemma 4.3.

Step 4. In this step we complete the construction of a Hamiltonian cycle in G^t containing the vertices $v_1, \ldots, v_k$ in order. We deal with two easy cases first.

Case 4.1: U has at most $t - 3$ non-leaves. In this case $d_U(v_i, v_{i+1}) \leqslant t - 2$, so applying Lemma 4.2 with $p = t$ to the cycle $C = v_1 \ldots v_k$ in U^t yields a Hamiltonian cycle in T^t, which is also a Hamiltonian cycle in G^t, containing $v_1, \ldots, v_k$ in order.

Case 4.2: U has exactly $t - 2$ non-leaves; then the distance in U between any two vertices is at most $t - 1$. Observe that $\lceil k/2 \rceil \leqslant t - 2$, and let $w_1, \ldots, w_{\lceil k/2 \rceil}$ be distinct non-leaves of U. Let C be the cycle $v_1 w_1 v_2 v_3 w_2 v_4 v_5 w_3 \ldots v_k$ in U^t if k is even, or the cycle $v_1 w_1 v_2 v_3 w_2 v_4 v_5 w_3 \ldots v_k w_{\lceil k/2 \rceil}$ in U^t if k is odd. Then every vertex of V, and in particular every leaf of U, is adjacent in C to a non-leaf, which is within distance $t - 2$ in U. Hence applying Lemma 4.2 with $p = t$ to the cycle C yields a desired Hamiltonian cycle in T^t and hence in G^t.

Finally, we come to the most general case.

Case 4.3: U has at least $t - 1$ non-leaves. In this case, we show that Lemma 4.2 can be applied to the cycle $C = R_1 \ldots R_k$ constructed in Step 3. All we need to verify is that condition (iii) of Lemma 4.2 holds for every leaf of U. Let v_s be a leaf of U. Then one of h_{s-1} and h_s is different from $\alpha(v_s)$ (the only case in which $h_{s-1} = h_s$ is when v_s is the "special" vertex introduced in Case 2.2.2, but then v_s is not a leaf of U). Suppose that $h_{s-1} \neq \alpha(v_s)$. Note that v_s is adjacent in R_{s-1} to a non-leaf of U, because if v_{s-1} is a leaf, then by condition (E) R_{s-1} has at least one interior vertex, and it is a non-leaf by condition (B). By condition (D), the vertex adjacent in R_{s-1} to v_s is within distance $t - 1$ of v_s in U. Hence condition (iii) of Lemma 4.2 is satisfied, and there is a Hamiltonian cycle in T^t, and hence in G^t, containing $v_1, \ldots, v_k$ in order. The case $h_s \neq \alpha(v_s)$ is treated in the same way.

The theorem is now proved. □

Corollary 4.6. *For $k \geqslant 4$ and a connected graph G on at least k vertices, the inequality $p_k(G) \leqslant \lfloor 3k/2 \rfloor - 2$ holds.*

Corollary 4.7. $p_k(P_n) = \lfloor 3k/2 \rfloor - 2$ *for $k \geqslant 4$ and $n \geqslant 2k - 1$.*

Proof. Immediate consequence of Theorem 3.1 and Corollary 4.6. □

5. Powers of cycles

In this section, we compute $p_5(C_n)$ and give a lower bound on $p_k(C_n)$.

Theorem 5.1. *Let C_n denote the cycle on n vertices. If $n \geqslant 5$, then $(C_n)^3$ is five-ordered Hamiltonian.*

Proof. Let $v_1, v_2, \ldots, v_5$ be a sequence of five vertices of C_n, and let $w_1, w_2, \ldots, w_5$ be the same five vertices in the order in which they appear in the cycle C_n. For $1 \leqslant i \leqslant 5$, let P_i denote the portion of C_n between w_i and w_{i+1} containing no other vertices of the sequence $w_1, \ldots, w_5$ (indices taken modulo 5). We will construct 10 internally disjoint paths R_{ij} in $(C_n)^3$, where R_{ij} is a path between w_i and w_j for $1 \leqslant i < j \leqslant 5$. We adopt the convention that $R_{ji} = R_{ij}$.

First, we construct $R_{i,i+2}$ for $1 \leqslant i \leqslant 5$. Let $a_1, a_2, \ldots, a_5$ be integers such that $a_i \in \{1, 2\}$ and $a_i + 1 \not\equiv |P_i| - 1 \pmod 3$ for all i (note that $|P_i| - 1$ is the length of P_i).

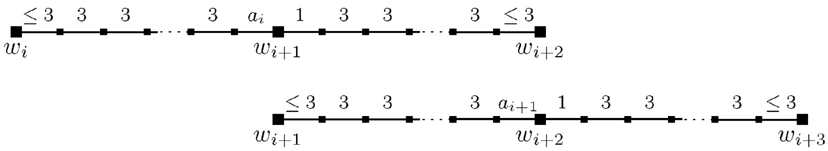

Fig. 1. The paths $R_{i,i+2}$ and $R_{i+1,i+3}$.

For $1 \leqslant i \leqslant 5$, let $R_{i,i+2}$ be the path $w_i t_1^i t_2^i \cdots t_{l_i}^i u_1^i u_2^i \cdots u_{m_i}^i w_{i+2}$, where $t_j^i \in P_i$, $u_j^i \in P_{i+1}$, and

$$d_{P_i}(w_i, t_1^i) \leqslant 3, \quad d_{P_i}(t_j^i, t_{j+1}^i) = 3 \text{ for } 1 \leqslant j \leqslant l_i - 1,$$
$$d_{P_i}(t_{l_i}^i, w_{i+1}) = a_i, \quad d_{P_{i+1}}(w_{i+1}, u_1^1) = 1,$$
$$d_{P_{i+1}}(u_j^i, u_{j+1}^i) = 3 \text{ for } 1 \leqslant j \leqslant m_i - 1 \quad \text{and} \quad d_{P_{i+1}}(u_{m_i}^i, w_{i+2}) \leqslant 3.$$

We allow $l_i = 0$ if $d_{P_i}(w_i, w_{i+1}) \leqslant a_i$ and $m_i = 0$ if $d_{P_{i+1}}(w_{i+1}, w_{i+2}) = 1$. Fig. 1 shows the paths $R_{i,i+2}$ and $R_{i+1,i+3}$. Since $a_{i+1} + 1 \not\equiv |P_{i+1}| - 1 \pmod 3$, these paths are internally disjoint.

For $1 \leqslant i \leqslant 5$, let $R_{i,i+1}$ be the path in $(C_n)^3$ between w_i and w_{i+1} containing all vertices of P_i that are not contained in $R_{i,i+2}$ and $R_{i-1,i+1}$.

Define i_j by $v_j = w_{i_j}$ and form a cycle C in $(C_n)^3$ containing $v_1, \ldots, v_5$ in order by linking the paths $R_{i_1 i_2}$, $R_{i_2 i_3}, \ldots, R_{i_5 i_1}$. Finally, we extend C to a Hamiltonian cycle using the following procedure.

(1) If C is not Hamiltonian, choose a vertex $z \notin C$ of C_n adjacent in C_n to a vertex $u \in C$.
(2) At most one vertex adjacent to u in $(C_n)^3$ is more than distance 3 away from z. Since u is adjacent to two vertices in C, we can choose a vertex t adjacent to u in C such that $d_{C_n}(t, z) \leqslant 3$.
(3) Replace the edge (t, u) of C with the path tzu.
(4) If C is not Hamiltonian, return to Step (1).

Since during the procedure we insert the remaining vertices into C without changing the order of the vertices already in C, the order in which $v_1, \ldots, v_5$ are contained in C is preserved, so $(C_n)^3$ is five-ordered Hamiltonian. This completes the proof of the theorem. □

Corollary 5.2. *If G is a Hamiltonian graph on five or more vertices, then G^3 is five-ordered Hamiltonian.*

Proof. Let C be a Hamiltonian cycle in G. Then C^3 is five-ordered Hamiltonian. Since C^3 is a subgraph of G^3 with the same vertex set, it follows that G^3 is five-ordered Hamiltonian, too. □

The next proposition yields a lower bound on $p_k(C_n)$.

Proposition 5.3. *If $m \geqslant 3$ and n is sufficiently large, then $(C_n)^m$ is not $2m$-ordered.*

Proof. Let $v_1, v_3, v_5, \ldots, v_{2m-1}, w_1, w_2, \ldots, w_{n-3m+1}, v_2, v_4, \ldots, v_{2m-4}, v_{2m}, v_{2m-2}, u_1, u_2, \ldots, u_{m-1}$ be the vertices of C_n (in this order; see Fig. 2). Suppose $(C_n)^m$ is $2m$-ordered; then there is a cycle C in $(C_n)^m$ that contains $v_1, v_2, \ldots, v_{2m}$ in order. For $1 \leqslant i \leqslant 2m$, let R_i be the portion of C between v_i and v_{i+1} that does not contain any other vertices of the sequence $v_1, \ldots, v_{2m}$ (indices taken modulo $2m$). Denote $U = \{u_1, u_2, \ldots, u_{m-1}\}$ and $W = \{w_1, w_2, \ldots, w_{n-3m+1}\}$.

If n is sufficiently large, then we have $d_{C_n}(v_i, v_{i+1}) > m$ for all i. Therefore, each R_i must contain a vertex of U or a vertex of W. Since $|U| = m - 1$, there are at least $m + 1$ paths R_i that do not contain vertices of U and hence consist only of vertices in $\{v_i, v_{i+1}\} \cup W$. Each such R_i must contain at least $\lfloor |W|/m \rfloor$ vertices of W, otherwise there would be m consecutive vertices of W not contained in R_i, and hence R_i would have two adjacent vertices that are more than distance m apart in C_n. We conclude that at least $m + 1$ of the paths R_i have at least $\lfloor |W|/m \rfloor$ interior vertices, thus

$$|C_n| > (m+1) \left\lfloor \frac{|W|}{m} \right\rfloor = (m+1) \left\lfloor \frac{n - 3m + 1}{m} \right\rfloor > n$$

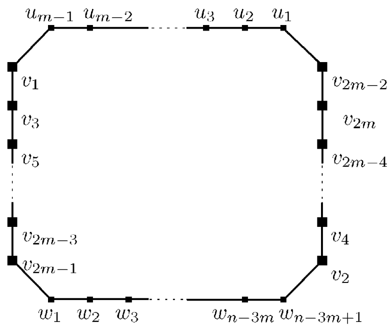

Fig. 2. The cycle C_n.

if n is sufficiently large—a contradiction. It follows that $(C_n)^m$ is not $2m$-ordered. $\square$

Corollary 5.4. $p_k(C_n) \geqslant \lfloor k/2 \rfloor + 1$ *for sufficiently large* n.

Corollary 5.5. $p_5(C_n) = 3$ *for sufficiently large* n.

Proof. Immediate consequence of Theorem 5.1 and Corollary 5.4 $\square$.

6. Conclusion and open problems

Our main result, Theorem 4.4, is a generalization of the classic theorem stating that the third power of a connected graph is Hamiltonian. In view of Theorem 3.1, the result is sharp. Having established a universal lower bound on $p_k(G)$ for all connected graphs G, it is natural to look for such a bound valid for a more restricted family of graphs. The case $G = P_n$ appears to represent the worst case scenario in that it requires the largest power to achieve k-ordered Hamiltonicity. Perhaps the next simplest case is $G = C_n$, some results on which are given in Section 5. It would be nice to find the actual value of $p_k(C_n)$. A more challenging task is to obtain a version of Theorem 4.4 generalizing Theorem 2.3 about two-connected graphs.

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