ELSEVIER

Journal of Computational and Applied Mathematics 82 (1997) 199–212

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Best bounds for expected financial payoffs I Algorithmic evaluation

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Received 19 August 1996

Abstract

A systematic approach to the evaluation of best bounds for expected financial payoffs, in case the mean, variance and range of the distribution are known, is presented. It is based on the majorant/minorant mathematical technique, which consists to bound a payoff by some quadratic polynomial. For the class of piecewise linear payoff functions, a classification of the global triatomic extrema is given, and a general algorithm for evaluation is formulated.

Keywords: Best bounds; Triatomic risks; Piecewise linear; Algorithm; Reinsurance; Derivatives

AMS classification: 65; 90

1. Introduction

In Applied Risk Theory some of the most important mathematical objects of interest are *risks* represented by real random variables X taking values in some interval $I=[a,b]$, $-\infty \leqslant a < b \leqslant \infty$, and which have a fixed mean $\mu=E[X]$ and variance $\sigma^2=\mathrm{Var}[X]$. The space of all risks with the characteristics I,μ,σ, is denoted by $D:=D(I,\mu,\sigma)$. Given a risk $X\in D$, as well as a transformed risk $f(X)$, which represents any contingent financial payoff (e.g. a reinsurance payment or the payoff of a derivative financial instrument), it is of considerable practical interest to know the solutions to the extremal problems

$$\text{(Pmax)} \qquad f^* := \max_{X\in D}\{E[f(X)]\},$$

$$\text{(Pmin)} \qquad f_* := \min_{X\in D}\{E[f(X)]\}.$$

The notation X^* (resp. X_*) is used for a maximizing (resp. minimizing) risk such that the maximum is $f^*=E[f(X^*)]$ (resp. the minimum is $f_*=E[f(X_*)]$).

In practice most of the encountered optimization problems of the types (Pmax), (Pmin) have triatomic risks as extremal solutions [1, Theorem 13 of Section 5.3]. At the price of obtaining

PII S0377-0427(97)00053-8

perhaps only approximate extrema in some cases, it is of primordial interest to study systematically the related optimization problems

$$\text{(P3max)} \qquad f_3^* := \max_{X \in D^3}\{E[f(X)]\},$$

$$\text{(P3min)} \qquad f_{3*} := \min_{X \in D^3}\{E[f(X)]\},$$

where $D^3 := D^3(I, \mu, \sigma)$ is the subspace of D of all triatomic risks with the characteristics I, μ, σ.

The present paper offers a systematic approach to the well-known *majorant/minorant mathematical technique*, which consists to bound $f(x)$ by some quadratic polynomial $q(x)$ such that all mass points of $f(X)$ are simultaneously mass points of $q(X)$. Suppose $q(x)$ and $X \in D^3$ have been found such that $\Pr(q(X)=f(X))=1$ and $q(x) \geqslant f(x)$ on I (resp. $q(x) \leqslant f(x)$). Then the expected payoff $E[q(X)]=E[f(X)]$ depends only on μ, σ, and thus necessarily $X=X^*$ (resp. $X=X_*$) and clearly $f^*=f_3^*=E[f(X^*)]$ (resp. $f_*=f_{3*}=E[f(X_*)]$), which means in particular that the extremal problems (Pmax) and (Pmin) have been solved. In the last decade this general technique has been applied successfully in Actuarial Science by Jansen et al. [6], Goovaerts et al. [2] and Kaas et al. [7] among others. However its origin goes back to Chebyshev, Markov, Possé and has been first formulated as general principle by Isii [5] and Karlin as mentioned by Karlin and Studden [8]. In the last monograph the described method consists in the main Theorem 2.1, Chap. XII. More theoretical views of the majorant/minorant method include Whittle [9, Section 12.4].

Important examples for which the above technique works include reinsurance contracts such as stop-loss, limited stop-loss, franchise and disappearing deductible, two-layers stop-loss, etc. (see Part II: applications). All these choices belong to the class of *piecewise linear* contingent payoff functions, which besides Reinsurance play also a fundamental role as derivative pricing instruments in Finance. For this class of financial payoffs the problems (P3max), (P3min) can be treated in an unified manner and, for numerical evaluation, a *(finite) algorithm* can be formulated (Section 3).

It seems that our method is general enough for practical purposes. Indeed it suffices in principle to define the payoff function $f(x)$ for integers only (in terms of a unit of money payment), and any such $f(x)$ can be assumed to be a piecewise linear function. One can also argue that any $f(x)$ can be bounded by piecewise linear functions $g(x)$, $h(x)$ such that $g(x) \leqslant f(x) \leqslant h(x)$, which leads after optimization to practical upper and lower bounds $g_* := \min E[g(X)] \leqslant E[f(X)] \leqslant \max E[h(X)] = h^*$. Another economic justification in favor of piecewise linear functions is the fact that piecewise linear sharing rules can be solutions of equilibrium models of risk exchange (e.g. [3]).

2. Some preliminaries

The present section introduces some definitions, notations and conventions, which will be used throughout the paper.

Suppose that μ, σ are finite. Applying the location-scale transformation $Z=(X-\mu)/\sigma$, it suffices to work in a *standardized risk scale*, for which all risks have zero mean and one unit of variance. Once results have been obtained in this scale, it is not difficult to transform the formulas back in the *original risk scale* by replacing atoms z of Z by $\mu + \sigma z$ and amounts of payment m by $\mu + \sigma m$.

In the standardized risk scale the space $D^3 := D^3(I, \mu = 0, \sigma = 1)$ of all standardized triatomic risks identifies with the following convex subspace of the euclidean 3-dimensional space:

$$D^3 = \{(x, y, z) \in \mathbb{R}^3 \colon a \leqslant x < y < z \leqslant b,\ 1 + xy \geqslant 0,\ 1 + xz \leqslant 0,\ 1 + yz \geqslant 0\}. \tag{2.1}$$

This representation follows since an element $X \in D^3$ is defined by its ordered support, a fact one denotes here by $X = \{x, y, z\}$, $a \leqslant x < y < z \leqslant b$, where the mass points x, y, z take the following probabilities:

$$p_x = \frac{1 + yz}{(y - x)(z - x)}, \qquad p_y = \frac{-(1 + xz)}{(y - x)(z - y)}, \qquad p_z = \frac{1 + xy}{(z - x)(z - y)}. \tag{2.2}$$

Each $X \in D^3$ in this representation will be called a *feasible* triatomic risk. If the support of a triatomic risk is not necessarily an ordered triple, the attribute "feasible" will be omitted, but the risk itself can always be viewed as element of D^3 by permuting its atoms if necessary. In order that the set D^3 is nonempty, the rand points a, b of the interval I must satisfy the inequalities (constraints on the mean and variance):

$$a < 0, \quad b > 0, \quad ab \leqslant -1. \tag{2.3}$$

Each of the boundary conditions $1 + xy = 0$ (z arbitrary), $1 + xz = 0$ (y arbitrary), $1 + yz = 0$ (x arbitrary), identifies the subspace $D^2 := D^2(I, \mu = 0, \sigma = 1)$ of all *standardized diatomic risks* as subspace of the triatomic space D^3. By convention one sets

$$D^2 := \{(x, y) \in \mathbb{R}^2 \colon a \leqslant x < y \leqslant b,\ xy = -1\}. \tag{2.4}$$

A *feasible* diatomic risk is defined by its ordered support, written as $X = \{x, y\}$, $a \leqslant x < y \leqslant b$, where the mass points x, y take the probabilities:

$$p_x = \frac{y}{y - x}, \qquad p_y = \frac{-x}{y - x}. \tag{2.5}$$

For mathematical convenience one introduces an *involution*, abbreviated by the superscript $*$, which maps x to $x^* = -1/x$. By definition an involution is a function whose square is the identity, that is $x^{**} = x$. With this notation each $X \in D^2$ is uniquely of the form $X = \{x, x^*\}$, where $x \in [a, b^*]$, $x^* \in [a^*, b]$. The following equivalent representation of the space D^3 will be used throughout as equivalent alternative.

Lemma 2.1. *The space of all feasible triatomic risks admits the following set representation*:

$$D^3 = \{(x, y, z) \in \mathbb{R}^3 \colon a \leqslant x < y < z \leqslant b,\ x \leqslant z^* < 0,\ z^* \leqslant y \leqslant x^*\}. \tag{2.6}$$

Proof. A feasible triatomic risk is by definition an ordered triple $X = \{x, y, z\}$ such that $1 + yz \geqslant 0$, $1 + xz \leqslant 0$, $1 + xy \geqslant 0$. Since $xz \leqslant -1$ one must have $x < 0 < z$, hence also $z^* < 0 < x^*$. It follows that

$$z^* \cdot (1+yz) = z^* - y \leqslant 0 \Leftrightarrow z^* \leqslant y,$$
$$z^* \cdot (1+xz) = z^* - y \geqslant 0 \Leftrightarrow x \leqslant z^*,$$
$$x^* \cdot (1+xy) = x^* - y \geqslant 0 \Leftrightarrow y \leqslant x^*.$$

These formulas show that (2.1) and (2.6) are equivalent representations. □

3. Majorants/minorants for piecewise linear payoff functions

Suppose that the financial payoff function $f(x)$ is *piecewise linear* on $I=[a,b]$. Thus there exists a decomposition in subintervals

$$I = \bigcup_{i=m}^{n} I_i \quad (-\infty \leqslant m \leqslant n \leqslant \infty) \tag{3.1}$$

such that $I_i=[a_i,b_i]$, $a_m=a$, $a_{i+1}=b_i$, $i=m,\ldots,n$, $b_n=b$, and

$$f(x)=\ell_i(x), \quad x\in I_i \quad \text{with } \ell_i(x)=\alpha_i+\beta_i x, \quad x\in\mathbb{R}. \tag{3.2}$$

If there are only finitely many subintervals in $(-\infty,0]$, one can start with $m=0$. Otherwise one starts with $m=-\infty$. The abscissa of the point of intersection of two nonparallel lines $\ell_i(x)\neq\ell_j(x)$ is denoted by

$$d_{ij}=d_{ji}=\frac{\alpha_i-\alpha_j}{\beta_j-\beta_i}.$$

A triatomic risk $X\in D^3$ is determined by its support, a fact denoted by $X=\{u,v,w\}$, where $(u,v,w)\in I_i\times I_j\times I_k$ for some indices $i,j,k\in\{m,\ldots,n\}$. The piecewise quadratic function $q(x)-f(x)$ is denoted by $Q(x)$. Note that $Q(x)$ coincides on I_i with the quadratic polynomial $Q_i(x):=q(x)-\ell_i(x)$. Use is made of the backward functional operator defined by $\nabla_{ij}\ell(x):=\ell_j(x)-\ell_i(x)$.

To apply the majorant (minorant) method it is necessary to determine the set of risks X such that all mass points of the transformed risk $f(X)$ are mass points of some quadratic risk $q(X)$, where $q(x)$ is some quadratic polynomial, and such that $q(x)\geqslant f(x)$ on I for a maximum ($q(x)\leqslant f(x)$ on I for a minimum). In a first step we restrict our attention to quadratic polynomials $q(x)$ with *non-zero* quadratic term such that $\Pr(q(X)=f(X))=1$. One observes that the piecewise quadratic function $Q(x)=q(x)-f(x)$ can have at most two zeros on each subinterval I_i (double zeros being counted twice). If an atom of X, say u, is an interior point of some I_i, then it must be a double zero of $Q_i(x)$. Indeed $q(x)\geqslant\ell_i(x)$ (resp. $q(x)\leqslant\ell_i(x)$) for $x\in I_i$ can only be fulfilled if the line $\ell_i(x)$ is tangent to $q(x)$ at u, that is $q'(u)=\ell_i'(u)=f'(u)$, hence u is a double zero. Therefore in a first step, one has to describe the following set of triatomic risks

$$D^3_{f,q}=\{X=\{u,v,w\}\in D^3: \text{there exists a quadratic polynomial } q(x) \text{ with } \textit{non-zero} \text{ quadratic term such that } \Pr(q(X)=f(X))=1 \text{ and } q'(x)=f(x) \text{ if } x\in\{u,v,w\} \text{ is an interior point of some subinterval } I_i\}. \tag{3.3}$$

In case $f(x)$ is piecewise linear, this set may be described completely.

Theorem 3.1. *Let $X=\{u,v,w\}$ be a triatomic risk such that $(u,v,w)\in I_i\times I_j\times I_k$. An element $X\in D^3_{f,q}$ belongs necessarily to one of the following six different types, where permutations of the atoms are allowed*:

(D1) *$X=\{u,v\}$ is diatomic with $u,v=u^*$ double zeros of some $Q(x)$ such that $(u,v)=(d_{ij}\mp\sqrt{1+d_{ij}^2}, d_{ij}\pm\sqrt{1+d_{ij}^2})$, $\beta_j\neq\beta_i$.*

(D2) *$X=\{u,v\}$ is diatomic with v a rand point of I_j and $u=v^*$ a double zero of some $Q(x)$, such that either* (i) *$\beta_j\neq\beta_i$, $v\neq d_{ij}$ or* (ii) *$\beta_j=\beta_i$, $\alpha_j\neq\alpha_i$.*

(T1) *$X=\{u,v,w\}$ with u,v,w double zeros of some $Q(x)$ such that β_i,β_j,β_k are pairwise different, $d_{jk}-d_{ik}\neq 0, d_{ik}-d_{ij}\neq 0, d_{ij}-d_{jk}\neq 0$, and*

$$u=d_{ij}+d_{ik}-d_{jk},$$

$$v=d_{jk}+d_{ij}-d_{ik},$$

$$w=d_{ik}+d_{jk}-d_{ij}.$$

(T2) *$X=\{u,v,w\}$ with w a rand point of I_k,u,v double zeros of some $Q(x)$ such that β_i,β_j,β_k are pairwise different, $w\neq d_{ik},d_{jk}$, and*

$$u=w-\frac{2}{(\beta_j-\beta_i)}\cdot\left\{\nabla_{ik}\ell(w)-\operatorname{sgn}\left(\frac{w-v}{w-u}\right)\sqrt{\nabla_{ik}\ell(w)\nabla_{jk}\ell(w)}\right\},$$

$$v=w+\frac{2}{(\beta_j-\beta_i)}\cdot\left\{\nabla_{jk}\ell(w)-\operatorname{sgn}\left(\frac{w-v}{w-u}\right)\sqrt{\nabla_{ik}\ell(w)\nabla_{jk}\ell(w)}\right\}.$$

(T3) *$X=\{u,v,w\}$ with v,w rand points of I_j,I_k,u a double zero of some $Q(x)$, such that either* (i) *$\beta_j\neq\beta_i$, $v\neq d_{ij}$, or* (ii) *$\beta_j=\beta_i$, $\alpha_j\neq\alpha_i$, and either* (iii) *$\beta_k\neq\beta_i$, $w\neq d_{ik}$, or* (iv) *$\beta_k=\beta_i,\alpha_k\neq\alpha_i$, and*

$$u=\tfrac{1}{2}(v+w)\quad \textit{if}\ \frac{\nabla_{ik}\ell(w)}{\nabla_{ij}\ell(v)}=1,$$

$$u=v+\frac{w-v}{\operatorname{sgn}\left(\frac{w-u}{v-u}\right)\cdot\sqrt{\frac{\nabla_{ik}\ell(w)}{\nabla_{ij}\ell(v)}}-1}\quad \textit{if}\ \frac{\nabla_{ik}\ell(w)}{\nabla_{ij}\ell(v)}\neq 1.$$

(T4) *$X=\{u,v,w\}$ with u,v,w rand points of I_i,I_j,I_k, and either* (i) *β_i,β_j,β_k not all equal, or* (ii) *$\alpha_i,\alpha_j,\alpha_k$ not all equal.*

Proof. The definition (3.3) implies that an element $X\in D^3_{f,q}$ has either an atom u, which is double zero of $Q(x)$ (types (D1), (D2), (T1)–(T3)), or all three atoms of X are rand points of subintervals I_k (type (T4)). The stated specific forms of the different types are now derived.

Repeated use of the fact that a quadratic polynomial is uniquely determined by three conditions is made. If u is a double zero of $Q_i(x)=q(x)-\ell_i(x)$, one has for a zero v of $Q_j(x)$:

$$q(x)=c_{ij}(v)(x-u)^2+\ell_i(x),$$

with

$$c_{ij}(v) = \frac{\nabla_{ij}\ell(v)}{(v-u)^2} = \begin{cases} \dfrac{(\beta_j-\beta_i)(v-d_{ij})}{(v-u)^2} & \text{if } \beta_j \neq \beta_i, \\ \dfrac{\alpha_j-\alpha_i}{(v-u)^2} & \text{if } \beta_j = \beta_i. \end{cases} \tag{3.4}$$

Type D1: Since v is a double zero of $Q_j(x)$, the tangent line to $q(x)$ at v coincides with $\ell_j(x)$, which implies the condition $q'(v)=\ell_j'(v)$. Using (3.4) one gets

$$2c_{ij}(v)(v-u) = \beta_j - \beta_i.$$

If $\beta_j=\beta_i$ then $c_{ij}(v)=0$, hence $\alpha_j=\alpha_i$, and $q(x)=\ell_i(x)$ has a vanishing quadratic term. Therefore only $\beta_j \neq \beta_i$ must be considered, which implies that $u+v=2d_{ij}$. Since $v=u^*=u^{-1}$ one gets immediately the desired formulas for u, v.

Type D2: Formula (3.4) shows the existence of $q(x)$ and the conditions (i) and (ii) assure that the quadratic term of $q(x)$ is nonzero.

Type T1: Since u, v, w are double zeros of $Q_i(x)$, $Q_j(x)$, $Q_k(x)$, respectively, cyclic permutations of i, j, k and u, v, w in (3.4) yield 3 different expressions for $q(x)$:
(i) $q(x)=c_{ij}(v)(x-u)^2+\ell_i(x)$,
(ii) $q(x)=c_{jk}(w)(x-v)^2+\ell_j(x)$,
(iii) $q(x)=c_{ki}(u)(x-w)^2+\ell_k(x)$.
Inserting the three necessary conditions $q'(v)=\ell_j'(v)$, $q'(w)=\ell_k'(w)$, $q'(u)=\ell_i'(u)$, one gets the equations
(i) $2c_{ij}(v)(v-u)=\beta_j-\beta_i$,
(ii) $2c_{jk}(w)(w-v)=\beta_k-\beta_j$,
(iii) $2c_{ki}(u)(u-w)=\beta_i-\beta_k$.
One must have $\beta_i, \beta_j, \beta_k$ pairwise different. Otherwise $q(x)$ is a linear form (same argument as for type D1). One obtains the system of equations
(i) $u+v=2d_{ij}$,
(ii) $v+w=2d_{jk}$,
(iii) $w+u=2d_{ik}$
with the indicated solution. Moreover one has $c_{ij}(v)\neq 0$, $c_{jk}(w)\neq 0$, $c_{ki}(u)\neq 0$, hence $v-d_{ij}=d_{jk}-d_{ik}$, $w-d_{jk}=d_{ik}-d_{ij}$, $u-d_{ik}=d_{ij}-d_{jk}$ are all different from zero.

Type T2: In case u, v are double zeros of $Q_i(x), Q_j(x)$, respectively, one considers the following two different expressions:
(i) $q(x)=c_{ik}(w)(x-u)^2+\ell_i(x)$,
(ii) $q(x)=c_{jk}(w)(x-v)^2+\ell_j(x)$.
The additional conditions $q'(v)=\ell_j'(v)$, $q'(u)=\ell_i'(u)$ imply the equations
(i) $2c_{ik}(w)(v-u)=\beta_j-\beta_i$,
(ii) $2c_{jk}(w)(u-v)=\beta_i-\beta_j$.
If $\beta_j=\beta_i$ one has $c_{ik}(w)=c_{jk}(w)=0$, hence $q(x)$ is a linear form. Thus one has $\beta_j\neq\beta_i$. Since $c_{ik}(w)\neq 0$, $c_{jk}(w)\neq 0$ one has also $\beta_k\neq\beta_i$, $w\neq d_{ik}$, $\beta_k\neq\beta_j$, $w\neq d_{jk}$. Rearranging (i), (ii) one has

equivalently

(i) $\frac{1}{2}\left(\frac{\beta_j-\beta_i}{v-u}\right)=\frac{\nabla_{ik}\ell(w)}{(w-u)^2}$,

(ii) $\frac{1}{2}\left(\frac{\beta_j-\beta_i}{v-u}\right)=\frac{\nabla_{jk}\ell(w)}{(w-v)^2}$.

Through comparison one gets the relation

$$\frac{w-v}{w-u}=\operatorname{sgn}\left(\frac{w-v}{w-u}\right)\sqrt{\frac{\nabla_{jk}\ell(w)}{\nabla_{ik}\ell(w)}}.$$

Now rewrite (i) in the form

$$(u-w)^2=-\left(\frac{2}{\beta_j-\beta_i}\right)\nabla_{ik}\ell(w)\{(u-w)+(w-v)\}.$$

Divide by $(u-w)$ and use the obtained relation to get the desired formula for u. The expression for v is obtained similarly.

Type T3: Using (3.4) the condition $q(w)=\ell_k(w)$ can be written as

$$\nabla_{ij}\ell(v)(w-u)^2=\nabla_{ik}\ell(w)(v-u)^2.$$

In case the constraints (i)–(iv) are not fulfilled, $q(x)$ is linear. Otherwise one gets

$$\frac{w-v}{v-u}=\operatorname{sgn}\left(\frac{w-u}{v-u}\right)\sqrt{\frac{\nabla_{ik}\ell(w)}{\nabla_{ij}\ell(v)}},$$

which implies the formula for the mass point u.

Type T4: If the constraints are not fulfilled, then $q(x)$ is linear. Otherwise $\ell_i(u)$, $\ell_j(v)$, $\ell_k(w)$ do not lie on the same line and there exists always a $q(x)$ through these points. □

In the situation that $f(x)$ is composed of only *finitely* many piecewise linear segments, the formulas of Theorem 3.1 show that the set $D^3_{f,q}$, among which global extrema are expected to be found, is *finite*. An *algorithm* to determine the global extrema involves the following steps. For each $X\in D^3_{f,q}$ with corresponding $q(x)$ such that $\Pr(q(X)=f(X))=1$, test if $q(x)$ is *QP-admissible* (read quadratic polynomial admissible), which means that $q(x)$ is either a *QP-majorant* (read quadratic polynomial majorant) such that $q(x)\geqslant f(x)$ on I, or it is a *QP-minorant* (read quadratic polynomial minorant) such that $q(x)\leqslant f(x)$ on I. If $q(x)$ is a QP-majorant (resp. a QP-minorant) then the global maximum (resp. minimum) is attained at X, and X induces a so-called *QP-global maximum* (resp. *QP-global minimum*). If for all $X\in D^3_{f,q}$ the described test fails, and there exists global triatomic extrema, then there must exist a linear function $\ell(x)$ and triatomic risks X such that $\Pr(\ell(X)=f(X))=1$ and $\ell(x)\geqslant f(x)$ on I for a maximum (resp. $\ell(x)\leqslant f(x)$ on I for a minimum). This follows because the set $D^3_{f,\ell}$ of such risks has been excluded from $D^3_{f,q}$. Observe that these linear types of global extrema are usually not difficult to find (e.g. Proposition 5.1). To design an *efficient algorithm*, it remains to formulate simple conditions, which guarantee that a given $q(x)$ is *QP*-admissible. This is done in the next section.

4. Characterization of global triatomic extrema

The same notations as in Section 3 are used. The conditions under which a given quadratic polynomial is QP-admissible are determined. The general idea is as follows. If $X = \{u, v, w\}$ with $(u, v, w) \in I_i \times I_j \times I_k$, one determines first the *condition*, say (C1), under which $Q_i(x)$, $Q_j(x)$, $Q_k(x) \geqslant 0$ (resp. $\leqslant 0$). Then, given an index $s \neq i, j, k$, one imposes the condition that $q(x)$ does not intersect with the open line segment defined by $\ell_s(x) = \beta_s x + \alpha_s$, $x \in \mathring{I}_s$. Geometrically this last condition can be fulfilled in two logically distinct ways:

(C2) $Q_s(x) \geqslant 0$ (resp. $\leqslant 0$), that is $q(x)$ has at most one point of intersection with $\ell_s(x)$. This holds exactly when the discriminant of $Q_s(x)$ is nonpositive.

(C3) The quadratic polynomial $q(x)$ has two distinct points of intersection with $\ell_s(x)$, whose first coordinates lie necessarily outside the open interval $\mathring{I}_s$, that is $\{\xi, \eta : Q_s(\xi) = Q_s(\eta) = 0, \xi \neq \eta\} \not\subset \mathring{I}_s$. Two cases must be distinguished.

Case 1: *One of* $Q_s(x)$, $s = i, j, k$, *has a double zero.* Permuting the indices if necessary, one can assume that u is a double zero of $Q_i(x)$. One has $Q_i(x) = q(x) - \ell_i(x) = c_{ij}(v)(x-u)^2$ and for $s \neq i, j$ one has

$$Q_s(x) = q(x) - \ell_s(x) = Q_i(x) - \nabla_{is}\ell(x) = c_{ij}(v)(x-u)^2 + (\beta_i - \beta_s)(x-u) - \nabla_{is}\ell(u). \tag{4.1}$$

Its discriminant equals

$$\Delta_{ijs}(u, v) = (\beta_s - \beta_i)^2 + 4\frac{\nabla_{ij}\ell(v)\nabla_{is}\ell(u)}{(v-u)^2}. \tag{4.2}$$

Case 2: u, v, w *are simple zeros of* $Q_s(x)$, $s = i, j, k$. By assumption $Q_i(x)$ has besides u a second zero, say $z_i = z_{ijk}(u, v, w)$. One can set

$$Q_i(x) = q(x) - \ell_i(x) = c_{ijk}(u, v, w)(x-u)(x-z_i),$$

where the unknown constants $c := c_{ijk}(u, v, w)$, $z := z_i$ are determined by the conditions $q(v) = \ell_j(v)$, $q(w) = \ell_k(w)$, which yield the equations

$$c(v-u)(x-z) = \nabla_{ij}\ell(v), \tag{4.3}$$

$$c(w-u)(w-z) = \nabla_{ik}\ell(w). \tag{4.4}$$

Rewrite (4.4) as

$$c(w-z) = \frac{\nabla_{ik}\ell(w)}{w-u}. \tag{4.5}$$

From (4.3) one gets

$$c(v-u)(v-w) + c(v-u)(w-z) = \nabla_{ij}\ell(v).$$

Inserting (4.3) it follows that

$$c = \left(\frac{1}{w-v}\right)\left(\frac{\nabla_{ik}\ell(w)}{w-u} - \frac{\nabla_{ij}\ell(v)}{v-u}\right),$$

which can be transformed to the equivalent form

$$c = c_{ijk}(u,v,w) = \left(\frac{1}{w-u}\right)\left(\frac{\nabla_{jk}\ell(w)}{w-v} - \frac{\nabla_{ij}\ell(u)}{v-u}\right). \tag{4.6}$$

Insert (4.6) into (4.5) to obtain

$$z_i = z_{ijk}(u,v,w) = w - \frac{\nabla_{ik}\ell(w)}{\frac{\nabla_{jk}\ \ell(w)}{w-v} - \frac{\nabla_{ij}\ \ell(u)}{v-u}}. \tag{4.7}$$

For $s \neq i,j,k$ one considers now the quadratic polynomial

$$Q_s(x) = q(x) - \ell_s(x) = Q_i(x) + \nabla_{is}\ell(x),$$

that is written out

$$Q_s(x) = c_{ijk}(x-u)^2 + (\beta_i - \beta_s + c_{ijk}(u - z_{ijk}))(x-u) - \nabla_{is}\ell(u). \tag{4.8}$$

Its discriminant equals

$$\Delta_{ijks}(u,v,w) = (\beta_i - \beta_s + c_{ijk}(u-z_{ijk}))^2 + 4c_{ijk}\nabla_{is}\ell(u), \tag{4.9}$$

where one uses the expression

$$c_{ijk}(u - z_{ijk}) = c(w-z) - c(w-u) = \frac{\nabla_{ik}\ell(w)}{w-u} + \frac{\nabla_{ij}\ell(v)}{v-u} - \frac{\nabla_{jk}\ell(v)}{w-u}. \tag{4.10}$$

Making use of these preliminaries, the set of *QP-global extrema* for the expected piecewise linear financial payoff $E[f(X)]$, described as the subset of $D^3_{f,q}$ of those risks leading to a QP-admissible quadratic polynomial, is determined as follows.

Theorem 4.1 (Characterization of QP-global triatomic extrema). *The quadratic polynomial* $q(x)$ *associated to a triatomic distribution* $X = \{u,v,w\} \in D^3_{f,q}$, $(u,v,w) \in I_i \times I_j \times I_k$, *is a QP-majorant* (*resp. a QP-minorant*) *if and only if the following conditions hold*:

I. Diatomic types D1, D2
(C1) $Q_i(x), Q_j(x) \geqslant 0$ (*resp.* $\leqslant 0$), *type* D1: $\beta_j > \beta_i$ (resp. $\beta_j < \beta_i$)
(C1) $Q_i(x) \geqslant 0$ (resp. $\leqslant 0$), *type* D2:
(a1) $\beta_j > \beta_i$ (resp. $\beta_j < \beta_i$), if $\beta_j \neq \beta_i$,
(b1) $\alpha_j > \alpha_i$ (resp. $\alpha_j < \alpha_i$), if $\beta_j = \beta_i$.
(C1) $Q_j(x) \geqslant 0$ (resp. $\leqslant 0$), *type* D2:
(a2) $\beta_j > \beta_i$ (resp. $\beta_j < \beta_i$), and $\eta_j := d_{ij} + (d_{ij} - u)^2/(v - d_{ij}) \notin \mathring{I}_j$, if $\beta_j \neq \beta_i$,
(b2) $\alpha_j > \alpha_i$ (resp. $\alpha_j < \alpha_i$), and $\eta_j := 2u - v \notin \mathring{I}_j$ if $\beta_j = \beta_i$.
For all $s \neq i,j$ one has either
(C2) $\Delta := \Delta_{ijs}(u,v) \leqslant 0$, or
(C3) $\Delta > 0$ and $\xi_s, \eta_s := (\beta_s - \beta_i \pm \sqrt{\Delta})/2c_{ij}(v) \notin \mathring{I}_s$.
II. Triatomic types T1–T4

(C1) $Q_i(x), Q_j(x), Q_k(x) \geqslant 0$ (resp. $\leqslant 0$):

Type T1: $\operatorname{sgn}\left(\frac{\beta_j-\beta_i}{d_{jk}-d_{ik}}\right)=\operatorname{sgn}\left(\frac{\beta_k-\beta_i}{d_{ik}-d_{ij}}\right)=\operatorname{sgn}\left(\frac{\beta_i-\beta_k}{d_{ij}-d_{jk}}\right)=1$ (resp. $=-1$),

Type T2: $\operatorname{sgn}\left(\frac{\beta_k-\beta_i}{w-d_{ik}}\right)=\operatorname{sgn}\left(\frac{\beta_k-\beta_i}{w-d_{jk}}\right)=1$ (resp. $=-1$)

and

$$\eta_k := d_{jk} + \frac{(d_{jk}-v)^2}{w-d_{jk}} \notin \mathring{I}_k$$

Type T3:

(a1) $\operatorname{sgn}\left(\frac{\beta_j-\beta_i}{v-d_{ij}}\right)=1$ (resp. $=-1$), if $\beta_j \neq \beta_i$,

(b1) $\alpha_j > \alpha_i$ (resp. $\alpha_j < \alpha_i$), if $\beta_j = \beta_i$,

(a2) $\operatorname{sgn}\left(\frac{\beta_k-\beta_i}{w-d_{ik}}\right)=1$ (resp. $=-1$), if $\beta_k \neq \beta_i$,

(b2) $\alpha_k > \alpha_i$ (resp. $\alpha_k < \alpha_i$), if $\beta_k = \beta_i$,

and furthermore

(a3) $\eta_j := d_{ij} + \frac{(d_{ij}-u)^2}{v-d_{ij}} \notin \mathring{I}_j$, if $\beta_j \neq \beta_i$,

(b3) $\eta_j := 2u - v \notin \mathring{I}_j$, if $\beta_j = \beta_i$,

(a4) $\eta_k := d_{ik} + \frac{(d_{ik}-u)^2}{w-d_{ik}} \notin \mathring{I}_k$, if $\beta_k \neq \beta_i$,

(b4) $\eta_k := 2u - w \notin \mathring{I}_k$, if $\beta_k = \beta_i$.

Type T4: $\operatorname{sgn}\{c_{ijk}(u,v,w)\}=\operatorname{sgn}\{c_{jki}(v,w,u)\}=\operatorname{sgn}\{c_{kij}(w,u,v)\}=1$ (resp. $=-1$), and furthermore

$$\eta_i := z_{ijk}(u,v,w) \notin \mathring{I}_i, \qquad \eta_j := z_{jki}(v,w,u) \notin \mathring{I}_j, \qquad \eta_k := z_{kij}(w,u,v) \notin \mathring{I}_k.$$

(C2), (C3) for Types T1–T3:

For all $s \neq i,j,k$ one has either $\Delta := \Delta_{ijs}(u,v) \leqslant 0$, or

$\Delta > 0$ and $\xi_s, \eta_s := (\beta_s - \beta_i \pm \sqrt{\Delta})/2c_{ij}(v) \notin \mathring{I}_s$.

(C2), (C3) for Type T4:

For all $s \neq i,j,k$ one has either $\Delta := \Delta_{ijks}(u,v,w) \leqslant 0$, or

$\Delta > 0$ and $\xi_s, \eta_s := (\beta_s - \beta_i + c_{ijk}(z_{ijk}-u) \pm \sqrt{\Delta})/2c_{ijk}(v) \notin \mathring{I}_s$.

Proof. One proceeds case by case.

Case I: *Diatomic types*

(C1) Type D1:

Use (3.4) and its permuted version obtained by replacing u by v to get

$$Q_i(x) = (\beta_j - \beta_i)\frac{(v - d_{ij})}{(v-u)^2}(x-u)^2 = \frac{(\beta_j - \beta_i)}{4\sqrt{1+d_{ij}^2}}(x-u)^2,$$

$$Q_j(x) = (\beta_j - \beta_i)\frac{(d_{ij} - u)}{(u-v)^2}(x-v)^2 = \frac{(\beta_j - \beta_i)}{4\sqrt{1+d_{ij}^2}}(x-v)^2,$$

which implies the displayed condition.

(C1) $Q_i(x) \geqslant 0$ (resp. $\leqslant 0$), Type D2.

If $\beta_j \neq \beta_i$ one argues as for Type D1, hence (a1). Otherwise one has

$$Q_i(x) = \frac{(\alpha_j - \alpha_i)}{(v-u)^2}(x-u)^2,$$

which shows (b1).

(C1) $Q_j(x) \geqslant 0$ (resp. $\leqslant 0$), Type D2.

Besides $\xi = v$ the quadratic polynomial $Q_j(x)$ has a second zero η, which is solution of the equation $q(\eta) = \ell_j(\eta)$, and which must lie outside the open interval $\mathring{I}_j$. Using (3.4) one has to solve the equation

$$\nabla_{ij}\ell(v)(\eta - u)^2 = \nabla_{ij}\ell(\eta)(v-u)^2.$$

One finds

$$\eta = \begin{cases} d_{ij} + \dfrac{(d_{ij}-u)^2}{v - d_{ij}} & \text{if } \beta_j \neq \beta_i, \\ 2u - v & \text{if } \beta_j = \beta_i. \end{cases}$$

Furthermore one has $Q_j(x) = c_{ij}(v)(x-u)^2 + \nabla_{ji}\ell(x)$ and the sign of $Q_j(x)$ is determined by the sign of $c_{ij}(v)$, leading to the same conditions as for $Q_i(x)$.

Conditions (C2) and (C3) follow immediately using the formulas (4.1) and (4.2) described in the text under Case 1.

Case II: *Triatomic types*

(C1) Type T1: From the proof of Theorem 3.1 one borrows the formulas

$$Q_i(x) = c_{ij}(v)(x-u)^2, \qquad Q_j(x) = c_{jk}(w)(x-v)^2, \qquad Q_k(x) = c_{ki}(u)(x-w)^2,$$

which imply the desired condition.

(C1) Type T2:

The following formulas are found in the proof of Theorem 3.1:

$$Q_i(x) = c_{ik}(w)(x-u)^2,$$
$$Q_j(x) = c_{jk}(w)(x-v)^2,$$
$$Q_k(x) = Q_j(x) - \ell_k(x) = c_{jk}(w)(x-v)^2 - \nabla_{jk}\ell(x).$$

The sign of these quadratic polynomials is determined by the sign of its quadratic terms, which implies the first statement. On the other side, $Q_k(x)$ has besides $\xi = w$ a second zero η, which must lie outside the open interval $\mathring{I}_k$. The equation $Q_k(\eta) = 0$ implies the relation

$$\nabla_{jk}\ell(w)(\eta - v)^2 = \nabla_{jk}\ell(\eta)(w - v)^2,$$

which has the unique solution

$$\eta = d_{jk} + \frac{(d_{jk} - v)^2}{w - d_{jk}}.$$

This implies the second statement.

(C1) Type T3: One has the formulas

$$\begin{aligned} Q_i(x) &= c_{ij}(v)(x - u)^2 = c_{ik}(w)(x - u)^2, \\ Q_j(x) &= Q_i(x) - \ell_j(x) = c_{ij}(v)(x - u)^2 - \nabla_{ij}\ell(x), \\ Q_k(x) &= Q_j(x) - \ell_k(x) = c_{ik}(w)(x - u)^2 - \nabla_{ik}\ell(x). \end{aligned}$$

Looking at the sign of the quadratic terms implies the first statement. Besides $\xi_j = v$ the second zero η_j of $Q_j(x)$ must lie outside $\mathring{I}_j$. Similarly $Q_k(x)$ has two zeros $\xi_k = v$, η_k, of which the second one must lie outside $\mathring{I}_k$. The above formulas imply the following equivalent statements

$$\begin{aligned} & Q_j(\eta_j) = 0 \Leftrightarrow \nabla_{ij}\ell(v)(\eta_j - u)^2 = \nabla_{ij}\ell(\eta_j)(v - u)^2 \\ & \quad \Leftrightarrow \eta_j = d_{ij} + \frac{(d_{ij} - u)^2}{v - d_{ij}}, \quad \text{if } \beta_j \neq \beta_i, \ \eta_j = 2u - v, \ \text{if } \beta_j = \beta_i, \\ & Q_k(\eta_k) = 0 \Leftrightarrow \nabla_{ik}\ell(w)(\eta_k - u)^2 = \nabla_{ik}\ell(\eta_k)(w - u)^2 \\ & \quad \Leftrightarrow \eta_k = d_{ik} + \frac{(d_{ik} - u)^2}{w - d_{ik}}, \quad \text{if } \beta_k \neq \beta_i, \ \eta_k = 2u - w, \ \text{if } \beta_k = \beta_i \end{aligned}$$

from which the required conditions are shown.

(C1) Type T4: The formulas in the text under Case 2 show through permutation of indices that

$$Q_i(x) = c_{ijk}(u, v, w)(x - \xi_i)(x - \eta_i), \quad \xi_i = u, \ \eta_i = z_{ijk}\ (u, v, w),$$

$$Q_j(x) = c_{jki}(v, w, u)(x - \xi_j)(x - \eta_j), \quad \xi_j = u, \ \eta_j = z_{jki}\ (v, w, u),$$

$$Q_k(x) = c_{kij}(w, u, v)(x - \xi_k)(x - \eta_k), \quad \xi_k = u, \ \eta_k = z_{kij}\ (w, u, v).$$

The signs of the quadratic terms imply the first statement. The second affirmation is the fact that the corresponding zeros must lie outside the displayed open intervals.

Finally, the conditions (C2) and (C3) are clear from the distinction in the text between Cases 1 and 2. □

5. The minimum problem for piecewise linear convex payoff functions

For specific choices of payoff functions and/or triatomic distributions, it is sometimes possible to derive general rules, which are useful in the optimization process. To illustrate let us derive some minimizing decision criteria. In the present work these have been applied to handle the minimum problem for the "two-layers stop-loss contract" (see Part II: applications).

Proposition 5.1. *Assume the payoff function $f(x)$ is piecewise linear convex on I. Suppose there exists a triatomic risk $X_* = \{x,y,z\} \in D^3_{f,f_i}$ such that* $\Pr(f_i(X_*) = f(X_*)) = 1$ *and that $0 \in I_i$. Then X_* is a minimizing solution of the extremal problem*:

$$\min_{X\in D^3}\{E[f(X)]\} = E[f(X_*)] = f_i(0).$$

Proof. Since $f(x)$ is convex on I, one has from Jensen's inequality and using the fact $\mu = 0 \in I_i$ that $E[f(X)] \geqslant f(\mu) = f_i(0)$ for all $X \in D$. By assumption all the mass points of X_* belong to I_i and since $f(x) = f_i(x)$ on I_i, one gets $E[f(X_*)] = f_i(0)$. Therefore the lower bound is attained. □

Proposition 5.2. *Assume the payoff function $f(x)$ is piecewise linear convex on I. Suppose $X \in D^3_{f,q}$ is not a type* T4. *Then X cannot minimize* $E[f(X)]$.

Proof. Without loss of generality let us assume that $X = \{u,v\}$ or $X = \{u,v,w\}$ with $u \in I_i$ a double zero of $Q_j(x) = q(x) - \ell_i(x)$, $v \in I_j$. A straightforward calculation shows that $q(x) = c_{ij}(v)(x-u)^2 + \ell_i(x)$, where

$$c_{ij}(v) = \frac{\nabla_{ij}\ell(v)}{(v-u)^2} = \frac{f'(u) - h(v,u)}{u-v} = \frac{h(u,v) - f'(u)}{v-u}$$

with

$$h(u,v) = h(v,u) = \frac{f(v) - f(u)}{v-u}.$$

Let us distinguish between two subcases.

Case 1: $v < u$. Since $f(x)$ is convex on I, one has for all x such that $v < u < x$ the inequality

$$h(v,u) \leqslant \frac{f(u) - f(x)}{u-x}.$$

Taking limits as $x \to u$ one has also $h(v,u) \leqslant f'(u)$, hence $c_{ij}(v) \geqslant 0$.

Case 2: $v > u$. Similarly for all x such that $x < u < v$ one has the inequality

$$h(u,v) \leqslant \frac{f(u) - f(x)}{u-x},$$

and in the limit as $x \to u$ one has also $h(u,v) \geqslant f'(u)$, hence $c_{ij}(v) \geqslant 0$.

In both cases one has $q(x) \geqslant \ell_i(x)$. This implies that $q(x) \leqslant f(x)$ cannot hold, which means that X cannot minimize $E[f(X)]$. □

Combining both results, it is possible to restrict considerably the set of triatomic risks, which can minimize the expected payoff.

Corollary 5.3. *Suppose the payoff function $f(x)$ is piecewise linear convex. Then the minimum expected payoff* $\min_{X\in D^3}\{E[f(X)]\}=E[f(X_*)]$ *is attained either for* $X_*\in D^3_{f,q}$ *of type* T4 *or for* $X_*\in D^3_{f,f_i}$, $0\in I_i$, $m\leqslant i\leqslant n$.

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