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Temporal aspects of the modal logic of subset spaces

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Abstract

Starting out from the 'topological' modal logic of Moss and Parikh (Ann. Pure Appl. Logic 78 (1996) 73–110) we present a *logic of knowledge and time* in which *subset trees* appear as the intended semantical structures. Our main results concern *completeness* and *decidability* of a corresponding system as well as the complexity of the satisfiability problem for certain subclasses of formulas. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

In recent years, the *logic of knowledge* has become an important part of computer science and AI. For instance, it represents a very appropriate formalism for analyzing distributed systems and specifying properties of them. Especially powerful tools result if this logic is combined with others like, e.g., *temporal logic* [5].

In the paper we give an extension of both the usual logic of knowledge of an agent and linear time temporal logic. But complementary to the standard systems of knowledge and time (see [5]) we start out from the so-called *topological* modal logic of Moss and Parikh [4]. Roughly speaking, the latter system describes situations in which the knowledge of an agent increases if computational *effort* is spent. As knowledge is usually represented by the set of states which seem alternative to the agent at his actual point of view of the world, the increasing of knowledge amounts to a successive shrinking of the set of states the agent considers possible in the course of time. Thus, describing knowledge acquisition in the formal model leads to a logical treatment of approximations in a system of sets. In this way a topological component comes into play, explaining the above naming. There is a corresponding axiom present in the logical system which controls the just indicated interaction between knowledge and

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effort in time. It says that an agent knows a formula at all future states whenever he knows that it is valid there. Axioms of this kind are typical of the systems formalizing topological reasoning.

For convenience, let us give some more details of topological modal logic. Actually it is a bimodal system with two operators, K representing *knowledge* and $\Box$ representing *effort*. The structures interpreting these operators are *two-dimensional* in a sense (cf. [11]). To be more precise, underlying frames consist of a non-empty set X and a distinguished set $\mathcal{O}$ of subsets of X called the *opens* (although they need not be open sets in the sense of topology); K then varies over the elements of an open set, whereas $\Box$ captures the shrinking of an open. Consequently, K is an **S5**-like and $\Box$ is an **S4**-like modality, and they are connected by the above mentioned axiom.

In the meantime various systems of topological modal logic have been studied. Sound and complete axiomatizations were proposed for the basic *subset space logic* (where $\mathcal{O}$ may be an arbitrary set of subsets of X) [4], and for *topologic* (where $\mathcal{O}$ is in fact a topology on X) [6]. Moreover, the topological modal theory of *treelike spaces* was investigated by Georgatos [7]. (A subset frame (as introduced above) is treelike by definition, iff $U \subseteq V$ or $V \subseteq U$ or $U \cap V = \emptyset$ holds for all opens U, V.) As to *finite-height trees* of opens and a "topological" variant of the modal system **G**, which is the basic **K** augmented by the "Löb rule" $\Box(\Box\alpha \to \alpha) \to \Box\alpha$, we refer the reader to [9].

In the present paper we make more explicit the temporal content of the effort operator by adding a *nextstep* operator to the Moss–Parikh formalism. Accordingly, a generalization of propositional linear time temporal logic with *nexttime* and *always in the future* to the topological context is achieved (*always* is the manner of reading "$\Box$" presently). The adequate semantical structures are, however, not linear in the sense that the opens form a chain w.r.t. set inclusion, as one could suspect at first glance. For the latter structures only the *nexttime*-fragment of linear time temporal logic could be generalized correspondingly [10]. Instead we have linearity only from the viewpoint of states, but a tree structure on the set of opens.

The paper requires some basic notions and techniques from propositional modal and temporal logic like *canonical models* and *filtrations*. In any case [8], Sections 1–5, or [2], Sections 1–4, provide sufficient background. In particular, we proceed as follows: In Section 2 the underlying logical language is defined. Then, giving a list of axioms and rules an appropriate logical system **LK** is introduced. The proof of the semantical completeness of **LK** follows in Section 3. A special filtration is involved therein, allowing the subsequent model construction for a non-deducible formula. Afterwards the decidability of the logic is shown. In Section 4 we deal with complexity. Turning to the multi-agent version of our logic we examine special classes of formulas having a *feasible* satisfiability problem. We utilize methods of [12] in order to extend and generalize Scherer's results on the ordinary logic of knowledge, $\mathbf{S5}_m$, correspondingly.

2. The logic

First, we introduce the syntax and semantics of a language, LK, which can express certain linear-time properties of knowledge. Let PV be a recursively enumerable set of strings, called *propositional variables* (denoted by upper case Roman letters). Based on PV, the set $\mathscr{F}$ of LK-formulas (denoted by lower case Greek letters) is defined by the following clauses:
- $\mathrm{PV} \cup \{\top\} \subseteq \mathscr{F}$;
- $\alpha, \beta \in \mathscr{F} \Rightarrow \neg\alpha, K\alpha, \circ\alpha, \Box\alpha, (\alpha \wedge \beta) \in \mathscr{F}$;
- no other strings belong to $\mathscr{F}$.

We use common conventions denoting formulas and, especially, the following abbreviations: $L\alpha$ for $\neg K \neg \alpha$ and $\Diamond \alpha$ for $\neg\Box\neg\alpha$.

The semantical structures are triples (X, d, σ) specified by the subsequent definition.

Definition 1. Let X be a non-empty set, and let $d = (E_j)_{j \in \mathbb{N}}$ be a sequence of equivalence relations on X such that every class of E_j splits up into some classes of E_{j+1}, for all $j \in \mathbb{N}$. Then the pair $\mathscr{F} = (X, d)$ is called a *subset tree frame.*

Let a mapping $\sigma : \mathrm{PV} \times \mathrm{X} \to \{0, 1\}$ be given additionally. Then σ is called a valuation, and the triple $\mathscr{M} = (X, d, \sigma)$ is called a subset tree model (based on $\mathscr{F}$).

The set of all equivalence classes w.r.t. the relations E_j ($j \in \mathbb{N}$) is called the set of *opens* of $\mathscr{F}$. Note that in every subset tree model the set of opens contained in any equivalence class of the relation E_0 forms in fact a tree w.r.t. (reverse) set inclusion such that no two opens on the same level intersect. Thus subset tree models are treelike in particular (see Section 1).

Let U_j^x denote the equivalence class of x w.r.t. the relation E_j. A pair x, U_j^x (designated without brackets mostly) is called a *situation* of $\mathscr{F}$. Using this notation we introduce the validity relation for LK-formulas. [1]

Definition 2 (*Semantics of* LK). Let $\mathscr{M} = (X, d, \sigma)$ be a subset tree model and x, U_j^x a situation. Then we define

$$x, U_j^x \models_{\mathscr{M}} A :\Leftrightarrow \sigma(A, x) = 1,$$

$$x, U_j^x \models_{\mathscr{M}} \neg\alpha :\Leftrightarrow x, U_j^x \not\models_{\mathscr{M}} \alpha,$$

$$x, U_j^x \models_{\mathscr{M}} \alpha \wedge \beta :\Leftrightarrow x, U_j^x \models_{\mathscr{M}} \alpha \text{ and } x, U_j^x \models_{\mathscr{M}} \beta,$$

$$x, U_j^x \models_{\mathscr{M}} K\alpha :\Leftrightarrow y, U_j^y \models_{\mathscr{M}} \alpha \text{ for all } y \in U_j^x,$$

$$x, U_j^x \models_{\mathscr{M}} \circ\alpha :\Leftrightarrow x, U_{j+1}^x \models_{\mathscr{M}} \alpha,$$

$$x, U_j^x \models_{\mathscr{M}} \Box\alpha :\Leftrightarrow (\forall k > j)\ x, U_k^x \models_{\mathscr{M}} \alpha,$$

for all $A \in \mathrm{PV}$ and $\alpha, \beta \in \mathscr{F}$.

[1] It is also possible to designate situations simply as pairs (x, j); we prefer the above notation to indicate the connection with the semantics of topological modal logic.

In case $x, U_j^x \models_{\mathcal{M}} \alpha$, we say that α *holds in* $\mathcal{M}$ *at the situation* x, U_j^x; moreover, the formula $\alpha \in \mathcal{F}$ *holds in* $\mathcal{M}$ (denoted by $\models_{\mathcal{M}} \alpha$), iff it holds in $\mathcal{M}$ at every situation. If there is no ambiguity, we omit the index $\mathcal{M}$ subsequently.

Note that the semantics of the operator K is the intended one because the sets U_j^x all are equivalence classes. The operator $\Box$ quantifies over all future time points, i.e., the actual state is excluded; one can also develop a version of the logic where $\Box$ models the reflexive and transitive closure of $\circ$.

The following schemes of formulas are suitable for the axiomatization of the set of LK-validities.

(1) All $\mathcal{F}$-instances of propositional tautologies
(2) $K(\alpha \to \beta) \to (K\alpha \to K\beta)$
(3) $K\alpha \to \alpha$
(4) $K\alpha \to KK\alpha$
(5) $L\alpha \to KL\alpha$
(6) $(A \to \circ A) \wedge (\neg A \to \circ \neg A)$
(7) $\circ(\alpha \to \beta) \to (\circ\alpha \to \circ\beta)$
(8) $\circ \neg \alpha \leftrightarrow \neg \circ \alpha$
(9) $\circ L \alpha \to L \circ \alpha$
(10) $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$
(11) $\Box\alpha \to \circ(\alpha \wedge \Box\alpha)$
(12) $\Box(\alpha \to \circ\alpha) \to (\circ\alpha \to \Box\alpha)$
for all $A \in \mathrm{PV}$ and $\alpha, \beta \in \mathcal{F}$.

Adding rules, we get a logical system designated **LK**. In fact, modus ponens as well as necessitation w.r.t. each modality are present:

$$(1)\ \frac{\alpha, \alpha \to \beta}{\beta}, \quad (2)\ \frac{\alpha}{K\alpha}, \quad (3)\ \frac{\alpha}{\circ\alpha}, \quad (4)\ \frac{\alpha}{\Box\alpha},$$

for all $\alpha, \beta \in \mathcal{F}$. For convenience, we comment on some of the axioms. Schemes (3)–(5) represent the standard axioms of knowledge. They characterize reflexivity, transitivity, and the euclidean property, respectively, of the accessibility relation in Kripke frames, which are the common semantical domains of modal logic. Scheme (8) corresponds in this sense with functionality. (9) relates the *nexttime* operator to the dual of K, saying that $\circ\alpha$ is possible at the actual situation whenever α is possible at the succeeding one. (The converse is not always true.) This axiom determines the interaction between knowledge and time. Schemes of this form are typical of the systems considered in topological modal logic; see also Lemma 4(c) below. Axioms (2), (7) and (10) first of all have a proof-theoretical meaning. Axioms (11) and (12) are suitable variants of the schemes *Mix* and *Ind*, respectively, from linear time temporal logic (compare with [8], Section 9). They express that the $\Box$-operator represents the transitive closure of the *nextstep* relation in a sense to be made precise later on. Finally, we should say a few words about scheme (6). It has to be added because we want the valuation to be independent of the time component of a situation. This requirement simplifies the definition of the semantics, but it clearly implies that the system is not closed under substitution.

Soundness of the axioms w.r.t. the intended structures can easily be established; hence, we omit the proof of the following proposition.

Proposition 3. *Axioms* (1)–(12) *hold in every subset tree model.*

To get to know the interplay between the operators $\circ$, K and $\Box$, respectively, we present some **LK**-deducible formulas.

Lemma 4. *For all* $\alpha \in \mathscr{F}$, *the following formulas are derivable in the system* **LK**:
(a) $\Box\alpha \rightarrow \Box\Box\alpha$,
(b) $\circ(\alpha \wedge \Box\alpha) \rightarrow \Box\alpha$,
(c) $K\Box\alpha \rightarrow \Box K\alpha$.

Proof. The formula $(\alpha \rightarrow \circ\,\alpha) \rightarrow (\Box(\alpha \rightarrow \circ\,\alpha) \rightarrow (\alpha \rightarrow \Box\alpha))$ can be derived from the induction axiom (12) by propositional reasoning. With the aid of this formula and $\Box$-necessitation one can easily show that

$$\vdash_{\mathbf{LK}} \alpha \rightarrow \circ\beta \;\Rightarrow\; \vdash_{\mathbf{LK}} \alpha \rightarrow \Box\beta$$

is valid (as usual, $\vdash_{\mathbf{LK}}$ designates derivability in the system **LK**). Since $\vdash_{\mathbf{LK}} \Box\alpha \rightarrow \circ\Box\alpha$ holds because of axiom (11), the derivability of $\Box\alpha \rightarrow \Box\Box\alpha$ follows.

The formula $\Box(\Box\alpha \rightarrow \circ\Box\alpha) \rightarrow (\circ\Box\alpha \rightarrow \Box\Box\alpha)$ is an instance of (12). As the premise can be obtained with the aid of (11) and rule (4), we get $\vdash_{\mathbf{LK}} \circ\Box\alpha \rightarrow \Box\Box\alpha$. Since we also get $\vdash_{\mathbf{LK}} \Box\Box\alpha \rightarrow \Box\circ\alpha$ from axiom (11) by means of $\Box$-necessitation, (10), and modus ponens, $\vdash_{\mathbf{LK}} \circ\Box\alpha \rightarrow \Box\circ\alpha$ follows. This implies $\vdash_{\mathbf{LK}} \circ\Box\alpha \rightarrow \Box(\alpha \rightarrow \circ\,\alpha)$. Now the induction axiom (12) and propositional reasoning (among other things, e.g., (7)) yield (b).

The details of the derivation of (c) are omitted. It can be done with the aid of (a), (b), the fact that also $\vdash_{\mathbf{LK}} \Box\circ\alpha \rightarrow \circ\Box\alpha$ holds, and, clearly, axioms (8) and (9). $\Box$

Note that (a) expresses transitivity of the accessibility relation belonging to the operator $\Box$. The scheme (b) is the reversal of axiom (11); it is applied in the completeness proof conclusively. Finally, (c) represents the so-called *cross axiom* which is typical of topological modal logic (see [4]; (c) is precisely the formula connecting knowledge and time which was mentioned in the introduction).

3. Completeness and decidability

The completeness proof for the system **LK** runs via a special filtration of a generated submodel C of the canonical model, which is essentially the Moss–Parikh filtration

[4, Section 2.3] suited to the presence of time. Let $\alpha \in \mathscr{F}$ be given. Then we define the following sets of formulas based on the set $sf(\alpha)$ of subformulas of α:

$$\Gamma_{\mathrm{PV}} := \{\circ A \mid A \in \mathrm{PV} \cap \mathrm{sf}(\alpha)\} \cup \{\circ\neg A \mid A \in \mathrm{PV} \cap \mathrm{sf}(\alpha)\},$$
$$\Gamma_{\circ} := \{\circ\beta \mid \Box\beta \in sf(\alpha)\} \cup \{\circ\Box\beta \mid \Box\beta \in sf(\alpha)\},$$
$$\Gamma := sf(\alpha) \cup \Gamma_{\circ} \cup \Gamma_{\mathrm{PV}},$$
$$\Gamma^{\neg} := \Gamma \cup \{\neg\beta \mid \beta \in \Gamma\},$$
$$\Gamma^{\wedge} := \Gamma^{\neg} \text{ joined with the set of all finite conjunctions of distinct elements of } \Gamma^{\neg},$$
$$\Gamma^{L} := \{L\beta \mid \beta \in \Gamma^{\wedge}\},$$
$$\widetilde{\Gamma} := \Gamma^{\wedge} \cup \Gamma^{L}.$$

Note that $\widetilde{\Gamma}$ is finite and closed under subformulas. Now, for all $s, t \in C$, we let

$$s \sim t :\Leftrightarrow s \cap \Gamma = t \cap \Gamma.$$

Moreover, let $\bar{s}$ designate the $\sim$-equivalence class of s, and let $\bar{C} := \{\bar{s} \mid s \in C\}$. Since Γ is finite, $\bar{C}$ is a finite set as well.

So far we have formed a filtration of C. Next, we introduce filtrations of the accessibility relations of C. For convenience, we repeat the definition. Let $\triangle$ be a modal operator (i.e., $\triangle = K$, $\circ$, or $\Box$) and ∇ its dual (note that $\circ$ is self-dual because of axiom (8)). Then, designating the accessibility relation on the canonical model which belongs to the operator $\triangle$ by $\xrightarrow{\nabla}$, as it is usual, a binary relation $\xmapsto{\nabla}$ on $\bar{C}$ is called a *$\widetilde{\Gamma}$-filtration* of $\xrightarrow{\nabla}$, iff the following two conditions are satisfied for all $s, t \in C$:
- $s \xrightarrow{\nabla} t$ implies $\bar{s} \xmapsto{\nabla} \bar{t}$,
- $\bar{s} \xmapsto{\nabla} \bar{t}$ implies $\{\beta \mid \triangle\beta \in s \cap \Gamma\} \subseteq t$.

Let the relation $\xmapsto{\circ}$ on $\bar{C}$ be a filtration of $\xrightarrow{\circ}$. Let $\xmapsto{\circ^+}$ designate the transitive closure of $\xmapsto{\circ}$. Then the following lemma holds, which gives a precise meaning to our earlier remark that $\Box$ represents the transitive closure of $\circ$.

Lemma 5. *The relation $\xmapsto{\circ^+}$ on $\bar{C}$ is a filtration of the relation $\xrightarrow{\Diamond}$ on the set C.*

Proof. First we remind of a well-known fact referred to as the *definability lemma* sometimes (see, e.g., [8], 9.7):

> Let $X \subseteq \bar{C}$. Then there exists a formula β_X such that for all $s \in C$ it holds that $\beta_X \in s \Leftrightarrow \bar{s} \in X$.

The proof of the lemma uses this fact. It is similar to that of the *ancestral lemma* 9.8 in [8]. The differences occur because the modal scheme **T** is not present for the operator $\Box$ in our system. We point out these differences only. For $s \in C$, let β_s be the formula characterizing the set $\{x \in \bar{C} \mid \bar{s} \xmapsto{\circ^+} x\}$ according to the definability lemma. Then we get

$\Box(\beta_s \rightarrow \circ\beta_s) \in s$ as in the proof of [8], 9.8. Axiom scheme (12) implies $\circ\beta_s \rightarrow \Box\beta_s \in s$. Now choose $t \in C$ such that $\bar{s} \overset{\circ}{\longmapsto} \bar{t}$ (t clearly exists). Then $\bar{s} \overset{\circ^+}{\longmapsto} \bar{t}$, hence $\beta_s \in t$. It follows that $\circ\beta_s \in s$ holds. Thus $\Box\beta_s \in s$ holds as well. This implies the first filtration condition.

As to the second, we prove

$$\bar{s} \overset{\circ^n}{\longmapsto} \bar{t} \Rightarrow \{\beta \mid \Box\beta \in s \cap \Gamma\} \subseteq t$$

by induction on $n \geqslant 1$ (where $\overset{\circ^n}{\longmapsto}$ denotes the n-fold composition of the relation $\overset{\circ}{\longmapsto}$). This suffices evidently.

Case $n = 1$: Let $\Box\beta \in s \cap \Gamma$. Then $\circ\beta \in s \cap \Gamma$. As $\overset{\circ}{\longmapsto}$ is a filtration of $\overset{\circ}{\longrightarrow}$, and by the definition of Γ, we obtain $\beta \in t$.

Case $n > 1$: Again assume $\Box\beta \in s \cap \Gamma$. Then, by axiom scheme (11) and the definition of Γ, $\circ\Box\beta \in s \cap \Gamma$. Let $u \in C$ be such that $\bar{s} \overset{\circ}{\longmapsto} \bar{u}$ and $\bar{u} \overset{\circ^{n-1}}{\longmapsto} \bar{t}$. It follows that $\Box\beta \in u \cap \Gamma$. Now the induction hypothesis applies. □

The *minimal* filtration of the relation $\overset{\nabla}{\longrightarrow}$ plays a central role in our investigations. It is defined by

$$\bar{s} \overset{\nabla}{\longmapsto} \bar{t} :\Leftrightarrow (\exists s' \in \bar{s})(\exists t' \in \bar{t})\, s \overset{\nabla}{\longrightarrow} t$$

for all $s, t \in C$, and it is in fact a filtration of $\overset{\nabla}{\longrightarrow}$. The following lemma is valid for the minimal filtration of $\overset{\circ}{\longrightarrow}$. It is a consequence of the scheme (8) (as to its proof, see [8], 9.9).

Lemma 6. *Let $\overset{\circ}{\longmapsto}$ be the minimal filtration of $\overset{\circ}{\longrightarrow}$. Let $s \in C$ be given and assume that $\circ\beta \in \Gamma$ holds. Then $\circ\beta \in s$ iff $(\exists t \in C)\, \bar{s} \overset{\circ}{\longmapsto} \bar{t}$ and $\beta \in t$.*

Passing to a filtration the functionality of $\overset{\circ}{\longrightarrow}$ is lost generally. The above lemma represents its substitute.

Due to the definition of the set $\widetilde{\Gamma}$ the minimal filtration $\overset{L}{\longmapsto}$ of the relation $\overset{L}{\longrightarrow}$ satisfies the conditions stated in the subsequent proposition.

Proposition 7. *The relation $\overset{L}{\longmapsto}$ is an equivalence relation on $\bar{C}$. Moreover, for all $s, t, u \in C$ such that $\bar{s} \overset{\circ}{\longmapsto} \bar{t} \overset{L}{\longmapsto} \bar{u}$ there exists $v \in C$ satisfying $\bar{s} \overset{L}{\longmapsto} \bar{v} \overset{\circ}{\longmapsto} \bar{u}$.*

A proof is given in the appendix. The second statement above asserts the so-called *cross property* to the accessibility relations $\overset{\circ}{\longmapsto}$ and $\overset{L}{\longmapsto}$ of the filtration. This property is a consequence of axiom (9) on the canonical model, and it is passed on the filtration according to Proposition 7.

The following considerations lead to a further property of the relation $\overset{\circ^+}{\longmapsto}$ which will be applied later on. Since $\overset{\circ^+}{\longmapsto}$ is transitive, we may analyse its clusters. These are built as follows: The relation

$$\bar{s} \approx \bar{t} :\Leftrightarrow \bar{s} = \bar{t} \text{ or } (\bar{s} \stackrel{\circ^+}{\longmapsto} \bar{t} \text{ and } \bar{t} \stackrel{\circ^+}{\longmapsto} \bar{s})$$

(for all $s, t \in C$) is easily seen to be an equivalence relation on $\bar{C}$. The $\approx$-class of an element $\bar{s} \in \bar{C}$ is commonly called the *cluster* of $\bar{s}$, and it is designated $cl_{\bar{s}}$. Letting

$$cl_{\bar{s}} \leqslant cl_{\bar{t}} :\Leftrightarrow \bar{s} \stackrel{\circ^+}{\longmapsto} \bar{t}$$

gives a partial order $\leqslant$ on the set of all clusters. As usual, we write $cl_{\bar{s}} < cl_{\bar{t}}$ iff $cl_{\bar{s}} \leqslant cl_{\bar{t}}$ and $cl_{\bar{s}} \neq cl_{\bar{t}}$. The following *cluster lemma* can be proved with the aid of Lemma 4(b).

Lemma 8. *Let $s \in C$, and let $cl_{\bar{s}}$ be a cluster which is not a last element w.r.t. "$<$". Assume that for $\Box\beta \in \Gamma$ it holds that $\Box\beta \notin s$. Then for all $v \in C$ such that $cl_{\bar{v}}$ is an immediate $<$-successor of $cl_{\bar{s}}$ there exists $t \in C$ such that $cl_{\bar{v}} = cl_{\bar{t}}$ and ($\beta \notin t$ or $\Box\beta \notin t$). Moreover, if $cl_{\bar{v}}$ is a $<$-last cluster, then the first alternative is valid.*

Proof. By assumption, $cl_{\bar{s}}$ is not a terminal $<$-element. Take any immediate $<$-successor cl of $cl_{\bar{s}}$. Then, by the definition of the relation $\stackrel{\circ^+}{\longmapsto}$, there exist $u, v \in C$ such that $cl_{\bar{s}} = cl_{\bar{u}} < cl_{\bar{v}} = cl$ and $u \stackrel{\circ}{\longrightarrow} v$. First we argue that $\Box\beta \notin u$. For otherwise $\circ\Box\beta \in u$ by axiom (10). Thus, as $\stackrel{\circ}{\longmapsto}$ is a filtration of $\stackrel{\circ}{\longrightarrow}$ and $\circ\Box\beta \in \Gamma$, $\Box\beta \in z$ for every $z \in C$ such that $\bar{u} \stackrel{\circ}{\longmapsto} \bar{z}$. Since there exists such a $\bar{z}$ in the cluster of $\bar{s}$, $\Box\beta \in s$ follows inductively, a contradiction. Consequently, $\Box\beta \notin u$. From Lemma 4(b) we get $\circ\beta \notin u$ or $\circ\Box\beta \notin u$. In the first case we conclude that $\circ\neg\beta \in u$ with the aid of axiom (8). We have $\neg\beta \in v$ then. In the second case we get $\circ\neg\Box\beta \in u$, thus $\neg\Box\beta \in v$, or, equivalently, $\Box\beta \notin v$. So we achieve our goal putting $t := v$.

Now assume that $cl_{\bar{v}}$ is a $<$-last cluster. Then β is "falsified" in any case: either $\neg\beta \in v$ already holds and we may put $t := v$ anew, or $\neg\Box\beta \in v$; in the second case it follows that there exists a $w \in C$ such that $v \stackrel{\Diamond}{\longrightarrow} w$ and $\neg\beta \in w$. As $\stackrel{\circ^+}{\longmapsto}$ is a filtration of $\stackrel{\Diamond}{\longrightarrow}$, we get $\bar{v} \stackrel{\circ^+}{\longmapsto} \bar{w}$, and, as $cl_{\bar{v}}$ is a last cluster, $cl_{\bar{w}} = cl_{\bar{v}}$. So, letting $t := w$ the proof of the lemma is complete. $\Box$

Let $\gamma \in \mathscr{F}$ be given and assume that $\not\vdash_{\mathbf{LK}} \gamma$. Then there exists an element s in the canonical model such that $\alpha := \neg\gamma \in s$. Form $\widetilde{\Gamma}$ dependent on α as above, and consider the filtration $(\bar{C}, \{\stackrel{L}{\longmapsto}, \stackrel{\circ}{\longmapsto}, \stackrel{\circ^+}{\longmapsto}\})$ of $(C, \{\stackrel{L}{\longrightarrow}, \stackrel{\circ}{\longrightarrow}, \stackrel{\Diamond}{\longrightarrow}\})$, where the latter structure is the submodel of the canonical model generated by s. For every $\bar{v} \in \bar{C}$ choose an ordering of the set of $\stackrel{\circ}{\longmapsto}$-successors of $\bar{v}$ such that this set may be represented as a list $(t_0, \ldots, t_{v_{\bar{v}}})$. Define for all $\bar{u}$ from the $\stackrel{L}{\longmapsto}$-equivalence class of $\bar{s}$ a finitely ramified tree $tr_{\bar{u}}$ of height ω by

- $root(tr_{\bar{u}}) := \bar{u}$.
- if $\bar{v}$ is a node of $tr_{\bar{u}}$ on level n, then $(t_0, \ldots, t_{v_{\bar{v}}})$ are its sons (from left to right),

for all $n \in \mathbb{N}$. Let $\hat{C}$ be the set of all paths through these trees. Note that $\hat{C}$ can be identified with a certain set of functions $f : \mathbb{N} \to \bar{C}$. The carrier set of the desired model will be a subset $\tilde{C}$ of $\hat{C}$, which is obtained in the following way:

- For each cluster cl, fix an element $v \in C$ such that $cl_{\bar{v}} = cl$; for each *first* cluster choose v such that $\bar{s} \stackrel{L}{\longmapsto} \bar{v}$ holds additionally (this is possible because we started with a generated submodel).
- For each cluster cl forget the $\stackrel{\circ}{\longmapsto}$-transitions leading to the same point.
- For each cluster cl having more than one element choose a shortest $\stackrel{\circ}{\longmapsto}$-circle ζ_{cl} containing $\bar{v}$, i.e., a shortest path *through* the cluster starting and ending in $\bar{v}$.
- Put all the circles and the single points one behind the other according to the ordering of the clusters.
- Let C' be the set of all functions $f : \mathbb{N} \to \bar{C}$ which pass faithfully through these circles; i.e., f starts out of a first and leads to a last $<$-element which is looped ad infinitum afterwards.

Now $\tilde{C}$ is defined inductively along this finite set of infinite paths: in step $n+1$ of the construction the already existing paths have to be lengthened as it is prescribed by the elements of C', if need is, and those partial paths have to be supplied, which are given by the *cross property*, for all elements of $\bar{C}$ equivalent to some point on level $n+1$ just obtained by prolongation.

Define the sequence d of equivalence classes on $\tilde{C}$ by

$$g \in U_j^f : \Leftrightarrow (\forall k \leqslant j)\, g(k) \stackrel{L}{\longmapsto} f(k)$$

for all $f, g \in \tilde{C}$ and $j \in \mathbb{N}$. Furthermore, let a valuation σ be given which satisfies

$$f(0) = \bar{t} \Rightarrow [\sigma(A, f) = 1 \Leftrightarrow A \in t]$$

for all $A \in \mathrm{PV} \cap \Gamma$, $t \in C$ and $f \in \tilde{C}$. Then we get the following theorem.

Theorem 9. *The structure* $\mathscr{M} := (\tilde{C}, d, \sigma)$ *is a subset tree model. Moreover, for all* $\beta \in \widetilde{\Gamma}$, $t \in C$, $f \in \tilde{C}$ *and* $j \in \mathbb{N}$ *it holds that*

$$f(j) = \bar{t} \Rightarrow [\beta \in t \Leftrightarrow f, U_j^f \models_{\mathscr{M}} \beta].$$

Proof. The first assertion of the theorem is obvious by the above construction. The proof of the second proceeds by induction on β. The details of the induction are not carried out here. In case β a propositional variable one uses axiom (6) as well as the fact that $\widetilde{\Gamma}$ contains Γ_{PV}. The truth-functional cases are evident. Lemma 6 is needed in case $\beta = \circ\gamma$, Lemma 8 in case $\beta = \Box\gamma$, and Proposition 7 in case $\beta = K\gamma$. $\square$

Applying the theorem to $\alpha \in s$ defined above and any $f \in \tilde{C}$ satisfying $f(0) = \bar{s}$, the desired completeness result follows as an immediate consequence. Combining this result and Proposition 3 we get:

Corollary 10. *The system* **LK** *is sound and complete w.r.t. the class of subset tree models.*

The set of LK-formulas holding in all subset tree models is decidable although the above completeness proof does not yield the finite model property for the system. Instead, we may proceed via suitable Kripke models having the same modal theory as subset tree models.

Definition 11. Let $\mathcal{M} := (W, \{R, S, T\}, \sigma)$ be a trimodal model (i.e., W is a non-empty set, R, S and T are binary relations on W, and σ is a valuation). Then $\mathcal{M}$ is called an LK-model, iff
- R is an equivalence relation and S is a function on W,
- T is the transitive closure of S,
- for all $s, t, u \in W$: if $(s,t) \in S$ and $(t,u) \in R$, then there exists a $v \in W$ such that $(s,v) \in R$ and $(v,u) \in S$ (this is the cross property),
- for all $s, t \in W$ such that $(s,t) \in T$ it holds that $\mathcal{M} \models A[s]$ iff $\mathcal{M} \models A[t]$ $(A \in \mathrm{PV})$.

Note that the relation R corresponds with the modality K; accordingly, S and $\circ$ as well as T and $\Box$ are related. In fact, we have the following theorem.

Theorem 12. *A formula* $\alpha \in \mathcal{F}$ *is* **LK**-*derivable, iff it holds in every* LK-*model.*

Proof. Soundness can easily be established. For the completeness part of the proof one can use Corollary 10. In fact, by means of
- $W := \{(x, U_j^x) \mid x \in X,\ j \in \mathbb{N}\}$,
- $((x, U_j^x), (y, U_k^y)) \in R :\Leftrightarrow j = k$ and $U_j^x = U_j^y$,
- $((x, U_j^x), (y, U_k^y)) \in S :\Leftrightarrow x = y$ and $k = j + 1$,
- $((x, U_j^x), (y, U_k^y)) \in T :\Leftrightarrow x = y$ and $k > j$,
- $\tilde{\sigma}(A, (x, U_j^x)) = 1 :\Leftrightarrow \sigma(A, x) = 1$,

every subset tree model $\mathcal{M} = (X, d, \sigma)$ gives rise to an LK-model $\widetilde{\mathcal{M}} := (W, \{R, S, T\}, \tilde{\sigma})$, and an easy induction shows that for all $\alpha \in \mathcal{F}$

$$(\forall x, U_j^x \in W)\,(x, U_j^x \models_{\mathcal{M}} \alpha \;\Leftrightarrow\; \widetilde{\mathcal{M}} \models \alpha[x, U_j^x])$$

holds. (On the right-hand side usual multimodal satisfaction is designated.) □

Apart from the functionality of the relation $\overset{\circ}{\longmapsto}$ the finite structures obtained by filtration in the course of the completeness proof for the subset tree logic are LK-models. We want to circumvent this deficiency which was left over next. For this purpose we mark every path $g \in \tilde{C}$ at the node where the corresponding $<$-terminal cluster is covered for the first time. Because of König's Lemma a finite set of finite trees results. Now we "spread" these trees by introducing new points equivalent to the original ones w.r.t. the valuation so that the paths become "parallel" lines, i.e., $\overset{\circ}{\longmapsto}$-sequences having no element in common. In this way we get a *finite* LK-model falsifying a given non-derivable formula. (In fact, the original model is a p-morphic image of the new one.). It should be mentioned that the proceeding here is similar to that in standard temporal logic; see [8], Section 9. Therefore, we do not work out the indicated ideas further, but state the corresponding result only.

Theorem 13. *The set of* LK-*formulas holding in every subset tree model is decidable.*

The complexity of the LK-satisfiability problem S is unfeasibly high. In fact, the corresponding problem for linear time temporal logic is PSPACE-complete already (see [3], 4.1). We do not want to determine the sharp bound for S here. Instead, we subsequently prove that there are interesting subclasses of formulas for which the complexity of the satisfiability problem is low even in the multi-agent case.

4. Prefix formulas

The content of this section is rather technical first. The main results of the technical part (Theorems 24 and 27 below) will be used in Section 4.3 to obtain "low" complexity bounds for the satisfiability problem of certain classes of *prefix formulas*. Such formulas consist of a block of modal operators followed by a literal. [2]

In Section 4.1 we study the *consistency problem* for prefix formulas. We establish consistency of every at most two-element subset of a set of formulas, X, as a sufficient criterion for the consistency of X. Afterwards, in Section 4.2, we analyse the complexity of deciding the word problem for a special Semi–Thue system originating from the axioms of the system **LK**. The corresponding result will be combined with the just mentioned consistency criterion then. This gives the desired complexity assertions.

As we use canonical model techniques later on, we have to confine ourselves to the *nexttime fragment* of our logic; i.e., axioms (10)–(12) and rule (4) have to be dropped. Furthermore, we release axiom (6) for the sake of technical simplicity. (Accordingly, time has to be made explicit in situations of subset tree models then, but we need not deal with those structures any more.) However, we consider the *multi-agent version* of this logic. Thus, for a fixed $m \geqslant 1$, there are m knowledge operators $K_1, \ldots, K_m$ present in the language, LK_m, each of which induces a corresponding axiom scheme $(2)_i$, $(3)_i$, $(4)_i$, $(5)_i$ and $(9)_i$, respectively. We continue to designate the resulting logical system **LK**.

The proceeding here follows [12] to a great extent. Thus only the different arguments are given in detail.

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[2] The same questioning was also considered for the original logic of subset spaces due to Moss and Parikh [4]; the corresponding results were presented at the 4th International Symposium on Logical Foundations of Computer Science, Yaroslavl, Russia, July 1997.

4.1. Consistency

Let Σ be a finite set of symbols. Denote by Σ^* the set of words over Σ, and let Λ designate the empty word.

Definition 14. The set $\mathscr{P} \subseteq \{K_1, \ldots, K_m, \circ, \neg\}^*$ of prefixes (of LK_m) is defined by the following clauses:
- $\Lambda \in \mathscr{P}$,
- if $P \in \mathscr{P}$, then also K_iP, L_iP, $\circ P \in \mathscr{P}$,
- no other strings belong to $\mathscr{P}$.

Now we define several (deleting, generating, commuting) relations on prefixes, which originate from the axioms in Section 2 and their duals.

Definition 15. Let $P, P' \in \mathscr{P}$.
(1) $P \xrightarrow{-} P'$:$\Leftrightarrow$ there are $Q, R \in \mathscr{P}$ such that
   - $P = QK_iR$ and $P' = QR$ or
   - $P = QL_iOR$ and $P' = QOR$ $(O \in \{K_i, L_i\})$.

(2) $P \xrightarrow{+} P'$:$\Leftrightarrow$ there are $Q, R \in \mathscr{P}$ such that
   - $P = QR$ and $P' = QL_iR$ or
   - $P = QOR$ and $P' = QK_iOR$ $(O \in \{K_i, L_i\})$.

(3) $P \xrightarrow{\mathrm{c}} P'$:$\Leftrightarrow$ there are $Q, R \in \mathscr{P}$ such that
   - $P = QK_i \circ R$ and $P' = Q \circ K_iR$ or
   - $P = Q \circ L_iR$ and $P' = QL_i \circ R$.

(4) $\xrightarrow{\mathrm{p}} := \xrightarrow{-} \cup \xrightarrow{+} \cup \xrightarrow{\mathrm{c}}$.

Let $\xrightarrow{\ldots *}$ denote the respective reflexive and transitive closure.

With the aid of some modal proof theory the following proposition can be shown.

Proposition 16. *Let $P, P' \in \mathscr{P}$ be prefixes such that $P \xrightarrow{\mathrm{p}*} P'$. Then, for all $\alpha \in \mathscr{F}$, $\vdash P\alpha \rightarrow P'\alpha$.*

Here (and in the remaining part of this section) derivability is understood w.r.t. the system **LK**. The above mentioned prefix formulas are introduced next. Afterwards we connect prefix formulas with the just defined relation $\xrightarrow{\mathrm{p}*}$.

Definition 17. Let $\mathscr{L}$ be the set of literals, i.e., the set PV joined with the set $\{\neg A \mid A \in \mathrm{PV}\}$. A formula $\alpha \in \mathscr{F}$ is called a prefix formula, iff $\alpha = P\lambda$ for some prefix $P \in \mathscr{P}$ and some $\lambda \in \mathscr{L}$.

Let $\mathscr{PF}$ designate the set of all prefix formulas. $\mathscr{PF}$ is closed under negation in the following sense: if $\alpha = P\lambda \in \mathscr{PF}$, then $\neg\alpha$ is equivalent to $\bar{P}\bar{\lambda}$, where $\bar{P}$ is obtained from P by substituting each modal operator $O \in \{K_i, L_i\}$ occurring in P by its dual, and

$$\bar{\lambda} = \begin{cases} \neg\lambda & \text{if } \lambda \in \mathrm{PV} \\ A & \text{if} \lambda = \neg A \text{ for some } A \in \mathrm{PV} \end{cases}$$

now, $\bar{P}\,\bar{\lambda}$ is a prefix formula which will be written as $\neg\,\alpha$ (by abuse of notation).

As it was announced above we connect the relation $\xrightarrow{\text{p}*}$ with prefix formulas.

Definition 18. A relation $\xrightarrow{\text{s}}$ on $\mathscr{PF}$ is defined by

$$P\lambda \xrightarrow{\text{s}} P'\lambda' : \Leftrightarrow P \xrightarrow{\text{p}*} P' \text{ and } \lambda = \lambda'$$

for all $P\lambda, P'\lambda' \in \mathscr{PF}$.

In view of Proposition 16 the relation $\xrightarrow{\text{s}}$ can be viewed as a "strong" implication between prefix formulas. Furthermore, it can easily be seen that $\xrightarrow{\text{s}}$ is transitive and respects duals in the following sense:

$$\text{if } \alpha \xrightarrow{\text{s}} \beta, \text{ then } \neg\beta \xrightarrow{\text{s}} \neg\,\alpha \quad (\alpha, \beta \in \mathscr{PF}).$$

Subsequently we define a certain notion of consistency based on $\xrightarrow{\text{s}}$.

Definition 19. Let $X \subseteq \mathscr{PF}$ be a set of prefix formulas.

(1) If $X = \{\alpha, \beta\}$, then X is called pseudo-consistent, iff not $\alpha \xrightarrow{\text{s}} \neg\beta$.

(2) X is called pseudo-consistent, iff every subset of X consisting of at most two elements is pseudo-consistent.

We get a lemma on pseudo-consistent sets, Lemma 20 below. So-called LK_m-*models* are used in the subsequent proof of this lemma, which are nothing but modified LK-models (see Definition 11): The relation T as well as the condition on the propositional variables have to be skipped now, and there are m equivalence relations $R_1, \ldots, R_m$ instead of a single one, R. Note that the completeness results stated in Corollary 10 and Theorem 12 hold correspondingly for the system considered here.

Lemma 20. *Let $X \subseteq \mathscr{PF}$ be pseudo-consistent, and let $\alpha \in \mathscr{PF}$ be an arbitrary prefix formula. Then the following conditions are satisfied*:

(a) *if $X \cup \{\neg\,\alpha\}$ is not pseudo-consistent, then there is some $\beta \in X$ such that $\beta \xrightarrow{\text{s}} \alpha$*;

(b) *if $\beta \xrightarrow{\text{s}} \alpha$ for some $\beta \in X$, then $X \cup \{\alpha\}$ is pseudo-consistent.*

Proof. In order to prove (a) we use that a singleton set $\{\beta\} \subseteq \mathscr{PF}$ is pseudo-consistent. To this end we first show that a single prefix formula $\beta = P\lambda$ is satisfiable. (This will also be applied in the proof of (b).) We take an LK_m-model having a one-element carrier $\{x\}$ and $\{(x,x)\}$ as the interpretation of all the relations R_i, $1 \leqslant i \leqslant m$, and S; moreover, we let $\sigma(A,x) = 1$ if $\lambda = A \in \mathrm{PV}$; otherwise, i.e., if $\lambda = \neg A$, then we let $\sigma(A,x) = 0$. In this way we obtain a model of β.

As a consequence of the satisfiability of β we get that $\neg\beta$ is not derivable. Thus also $\beta \rightarrow \neg\beta$ is not derivable. By Proposition 16, $\beta \xrightarrow{s} \neg\beta$ does not hold. Consequently, $\{\beta\}$ is pseudo-consistent. Now we prove (a) and (b).

(a) Since we assume that $X \cup \{\neg\alpha\}$ is not pseudo-consistent there exists a subset $\{\beta,\gamma\} \subseteq X \cup \{\neg\alpha\}$ which is not pseudo-consistent. As both X and $\{\neg\alpha\}$ are pseudo-consistent we conclude that neither $\{\beta,\gamma\} \subseteq X$ nor $\{\beta,\gamma\} \subseteq \{\neg\alpha\}$ holds. Hence $\beta \in X$ and $\gamma = \neg\alpha$ (w.l.o.g.). It follows that $\beta \xrightarrow{s} \alpha$.

(b) By the assumption on X and the first part of the proof we still have to show that $\{\gamma,\alpha\}$ is pseudo-consistent for all $\gamma \in X$. Suppose towards a contradiction that for some $\gamma \in X$ the set $\{\gamma,\alpha\}$ is not pseudo-consistent, i.e., $\gamma \xrightarrow{s} \neg\alpha$. Then, because of $\neg\alpha \xrightarrow{s} \neg\beta$, we get that $\gamma \xrightarrow{s} \neg\beta$ (see the remark following Definition 18). Consequently, $\{\gamma,\beta\}$ is not pseudo-consistent. This contradicts the assumption on X.

□

As in ordinary modal proof theory we can introduce the concept of *maximal* pseudo-consistent sets of formulas.

Definition 21. Let $X \subseteq \mathscr{PF}$ be a set of prefix formulas. Then X is called maximal pseudo-consistent, iff
- X is pseudo-consistent and
- $X \cup \{\alpha\}$ is not pseudo-consistent for all $\alpha \in \mathscr{PF} \setminus X$.

It turns out that maximal pseudo-consistent sets and maximal consistent sets share an important property. In fact, the following result follows easily from Lemma 20.

Lemma 22. *Let $X \subseteq \mathscr{PF}$ be maximal pseudo-consistent. Then for all $\alpha \in \mathscr{PF}$ either $\alpha \in X$ or $\neg\alpha \in X$ holds.*

Utilizing the notions and results obtained so far we are going to show that a "canonical" model exists for prefix formulas. The following notational convention will be used subsequently: for $i \in \{1,\ldots,m\}$ and $O \in \{K_i, L_i\}$ let $O\mathscr{PF} := \{O\alpha \mid \alpha \in \mathscr{PF}\}$. Define a structure

$$\mathscr{M}^{\mathrm{pr}} = (C^{\mathrm{pr}}, \{R_1^{\mathrm{pr}},\ldots,R_m^{\mathrm{pr}}, S^{\mathrm{pr}}\}, \sigma^{\mathrm{pr}}),$$

by
- $C^{\mathrm{pr}} := \{t \subseteq \mathscr{PF} \mid t \text{ maximal pseudo-consistent}\}$,

and, for all $t,u \in C^{\mathrm{pr}}$, $i \in \{1,\ldots,m\}$, and $A \in \mathrm{PV}$,
- $(t,u) \in R_i^{\mathrm{pr}} :\Leftrightarrow t \cap K_i\mathscr{PF} = u \cap K_i\mathscr{PF}$,
- $(t,u) \in S^{\mathrm{pr}} :\Leftrightarrow \{\alpha \in \mathscr{PF} \mid \circ\alpha \in t\} \subseteq u$, and
- $\sigma^{\mathrm{pr}}(A,t) = 1 :\Leftrightarrow A \in t$.

Then the following crucial *truth lemma* is valid.

Lemma 23. *The just defined structure $\mathscr{M}^{\mathrm{pr}}$ is an LK_m-model. Moreover, for all prefix formulas $\alpha \in \mathscr{PF}$ and every $t \in C^{\mathrm{pr}}$ it holds that*

$$\mathscr{M}^{\mathrm{pr}} \models \alpha[t] \Leftrightarrow \alpha \in t.$$

Proof. First we prove that $\mathcal{M}^{\mathrm{pr}}$ is an LK_m-model. Clearly, each R_i^{pr} is an equivalence relation on the set C^{pr}. Next we show that S^{pr} is a function. So let $s,t,u \in C^{\mathrm{pr}}$ be given such that (s,t), $(s,u) \in S^{\mathrm{pr}}$. Let $\alpha \in \mathscr{PF}$ be any element of t. Then $\circ\alpha \in s$. For otherwise $\neg \circ \alpha \in s$ by Lemma 22. Actually, $\neg \circ \alpha = \circ \neg \alpha$. So $\circ\neg\alpha \in s$ follows. This implies $\neg\alpha \in t$, a contradiction. Consequently, we get that $\alpha \in u$. This shows $t \subseteq u$. In the same way we obtain $u \subseteq t$. Thus S^{pr} is in fact a functional relation. It remains to prove that for all $s \in C^{\mathrm{pr}}$ there is at least one S^{pr}-successor $t \in C^{\mathrm{pr}}$. To this end we consider the set $X_\circ := \{\alpha \in \mathscr{PF} \mid x\circ\alpha \in s\}$. It suffices to show the pseudo-consistency of $X_\circ$. Suppose towards a contradiction that there are $\alpha_1, \alpha_2 \in \mathscr{PF}$ such that $\circ\alpha_1, \circ\alpha_2 \in s$ and $\alpha_1 \xrightarrow{\mathrm{s}} \neg\alpha_2$. It follows that $\circ\alpha_1 \xrightarrow{\mathrm{s}} \circ\neg\alpha_2$, hence $\circ\alpha_1 \xrightarrow{\mathrm{s}} \neg\circ\alpha_2$. Since s is maximal pseudo-consistent we get $\neg\circ\alpha_2 \in s$ because of Lemma 20(b). But this contradicts $\circ\alpha_2 \in s$. So the assumption is wrong. Therefore, $X_\circ$ is pseudo-consistent, as desired.

In order to prove the *cross property*, let $t,u,v \in C^{\mathrm{pr}}$ be given such that $(t,u) \in S^{\mathrm{pr}}$ and $(u,v) \in R^{\mathrm{pr}}$. Define $X := [t \cap (K_i\mathscr{PF} \cup L_i\mathscr{PF})] \cup \{\circ\beta | \beta \in v\}$. We claim that X is pseudo-consistent. Assume towards a contradiction that this is not the case. Then there is a two-element subset $Y \subseteq X$ which is not pseudo-consistent. We first convince ourselves that $Y \not\subseteq \{\circ\beta \mid \beta \in v\}$. Otherwise there would exist $\beta_1, \beta_2 \in v$ such that $\circ\beta_1 \xrightarrow{\mathrm{s}} \neg\circ\beta_2 = \circ\neg\beta_2$. According to the definition of the relation $\xrightarrow{\mathrm{s}}$ we get $\beta_1 \xrightarrow{\mathrm{s}} \neg\beta_2$. (For the precise argument the reader convinces himself first that the leftmost "$\circ$" of "$\circ\beta_1$" cannot be removed by an application of a rule from Definition 15; otherwise $\circ\neg\beta_2$ is no more reachable. Then

$$\circ P_1 \xrightarrow{\mathrm{p}^*} \circ P_2 \Rightarrow P_1 \xrightarrow{\mathrm{p}^*} P_2,$$

where P_1 is the prefix of β_1 and P_2 is the prefix of $\neg\beta_2$, is proved by an induction on the number of applications of $\xrightarrow{\mathrm{p}}$; some case distinctions are required therein.) The maximality of v and Lemma 20(b) imply that $\neg\beta_2 \in v$. This contradicts the pseudo-consistency of v. Since t is likewise pseudo-consistent Y can be written as $Y = \{\alpha, \circ\beta\}$, where $\alpha \in t \cap (K_i\mathscr{PF} \cup L_i\mathscr{PF})$ and $\beta \in v$.

Now $O\gamma \xrightarrow{\mathrm{s}} \circ\neg\beta$ holds because of the pseudo-inconsistency of Y, where $O \in \{K_i, L_i\}$ and $\gamma \in \mathscr{PF}$ are such that $\alpha = O\gamma$. We get $K_iO\gamma \xrightarrow{\mathrm{s}} K_i\circ\neg\beta$ as well. With the aid of Definition 15(2) and (3) as well as the transitivity of $\xrightarrow{\mathrm{s}}$ we obtain $\alpha = O\gamma \xrightarrow{\mathrm{s}} \circ K_i\neg\beta$. Using the same arguments as above it follows that $\circ K_i\neg\beta \in t$. Thus $K_i\neg\beta \in u$. We conclude that $K_i\neg\beta \in v$. So $\neg\beta \in v$ follows, a contradiction. Consequently, X is pseudo-consistent.

Let w be a maximal pseudo-consistent set of prefix formulas containing X. We want to show that $(t,w) \in R_i^{\mathrm{pr}}$ and $(w,v) \in S^{\mathrm{pr}}$. The first conjunct follows easily from the definition of the first component of X (applying Lemma 22). As to the second assume that there exists a prefix formula γ such that $\circ\gamma \in w$ and $\gamma \notin v$. Then $\neg\gamma \in v$ by Lemma 22, whence $\circ\neg\gamma = \neg\circ\gamma \in w$ by the definition of X. This contradicts the pseudo-consistency of w. It follows that $(w,v) \in S^{\mathrm{pr}}$. Altogether the *cross property* is proved.

The second assertion of the lemma is proved by an induction on the length n of the prefixes of α. Concerning the case $n=0$ (i.e., α is a literal) and the cases $\alpha = K_i\beta$ and $\alpha = L_i\beta$ of the induction step we refer the reader to [12], Section IV, proof of Lemma 2.5. Actually that proof can be transferred to the present case. Now let $\alpha = \circ\beta \in \mathscr{PF}$. Assume first that $\alpha \in t$. Take any $u \in C^{\mathrm{pr}}$ such that $(t,u) \in S^{\mathrm{pr}}$. Then $\beta \in u$ holds by the definition of S^{pr}. We get $\mathscr{M}^{\mathrm{pr}} \models \beta[u]$ by the induction hypothesis. Thus $\mathscr{M}^{\mathrm{pr}} \models \circ\beta[t]$ follows. On the other hand, assume that $\alpha \notin t$. Then the set $X' := \{\gamma \mid \circ\gamma \in t\} \cup \{\neg\beta\}$ is pseudo-consistent. For otherwise there would exist a two-element subset $Y' = \{\gamma_1, \gamma_2\}$ of X' which is pseudo-inconsistent as well. As in the proof of the functionality of S^{pr} above we conclude from $\gamma_1 \xrightarrow{\mathrm{s}} \neg\gamma_2$ that $\circ\gamma_1 \xrightarrow{\mathrm{s}} \neg\circ\gamma_2$ holds in case $Y' \subseteq \{\gamma \mid \circ\gamma \in t\}$. The maximality of t implies $\neg\circ\gamma_2 \in t$ then, a contradiction. Thus we are led to $\gamma_1 \in \{\gamma \mid \circ\gamma \in t\}$ and $\gamma_2 = \neg\beta$ (w.l.o.g.). But now from $\gamma_1 \xrightarrow{\mathrm{s}} \beta$ it follows that $\circ\gamma_1 \xrightarrow{\mathrm{s}} \circ\beta$ holds, which causes $\circ\beta \in t$, a contradiction as well. Therefore, the above set X' is pseudo-consistent. Finally, a maximal pseudo-consistent set u containing X' is S^{pr}-reachable from t and fulfils $\mathscr{M}^{\mathrm{pr}} \not\models \beta[u]$ because of the induction hypothesis. Consequently, $\mathscr{M}^{\mathrm{pr}} \not\models \circ\beta[t]$. □

The first main result of the technical part reads as follows:

Theorem 24. *Let $X \subseteq \mathscr{PF}$ be a set of prefix formulas. Then X is consistent, iff it is pseudo-consistent.*

Proof. First note that consistency coincides with satisfiability according to the completeness results of Section 2. If X is not pseudo-consistent, then there are elements $\alpha, \beta \in X$ such that $\alpha \xrightarrow{\mathrm{s}} \neg\beta$. By Proposition 16 we get $\vdash \alpha \rightarrow \neg\beta$. Thus $\{\alpha, \beta\}$ is not consistent. If on the other hand X is pseudo-consistent, then X is contained in a maximal pseudo-consistent set t by Zorn's Lemma. According to Lemma 23 the set t is satisfiable. Therefore, X is satisfiable as well. □

4.2. Analysing a special semi-thue system

In this subsection we analyse the relation $\xrightarrow{\mathrm{p}^*}$ in more detail. In particular, we give a polynomial upper bound for the computational complexity of deciding its *word problem*. First we introduce further designations based on Definition 15: We write $\xrightarrow{(\dots)_i}$ to indicate that only rule i of the relation $\xrightarrow{(\dots)}$ has been applied, and we consider the following union of relations, too:

$$\xrightarrow{\mathrm{q}} := \xrightarrow{-} \cup \xrightarrow{+}.$$

The word problem of the relation $\xrightarrow{\mathrm{q}^*}$ was studied in detail in [12, Section V, 2]

The corresponding algorithm of Scherer is based on the following facts:
Let P, Q be prefixes satisfying the presupposition of the theorem. Then

$$P \xrightarrow{\mathrm{q}^*} Q \Leftrightarrow P \xrightarrow{-^*} \Lambda \xrightarrow{+^*} Q \text{ or}$$
$$\text{there are decompositions } P = P'OP'',\ Q = Q'OQ''$$
$$\text{such that } P'O \xrightarrow{\mathrm{q}^*} Q'O \text{ and } P'' \xrightarrow{\mathrm{q}^*} Q'', \text{ for suitable}$$
$$\text{prefixes } P', P'', Q', Q'' \text{ and a modal operator } O.$$

Moreover, $P \xrightarrow{-^*} \Lambda$ (and, dually, $\Lambda \xrightarrow{+^*} Q$) can be decided in polynomial (even linear) time by deleting all possible operators in P according to the rules of Definition 15(1). This can be done while parsing P from left to right, thereby storing in the cache the sequence of the remaining modal operators L_i which is updated step by step $(i \in \{1, \ldots, m\})$.

We are going to describe an algorithm which decides the word problem of the relation $\xrightarrow{\mathrm{p}^*}$ in polynomial time. It uses the decision procedure for the word problem of $\xrightarrow{\mathrm{q}^*}$, which results from the just mentioned facts in a rather straightforward manner, as a subroutine, and it is based on the following decomposition of the relation $\xrightarrow{\mathrm{p}^*}$.

Lemma 26. $\xrightarrow{\mathrm{p}^*} = \xrightarrow{(c)_2^*} \circ \xrightarrow{\mathrm{q}^*} \circ \xrightarrow{(c)_1^*}$.

The lemma can be proved by several case distinctions. We omit the details.

Subsequently, we use the following designations for special sets of prefixes:

$$\mathscr{K} := \{K_1, \ldots, K_m\}^*, \qquad \mathscr{L} := \{\neg K_1 \neg, \ldots, \neg K_m \neg\}^*$$

and

$$\mathscr{M} := \mathscr{P} \cap \{K_1, \ldots, K_m, \neg\}^*.$$

Let $P_1, \ldots, P_n, Q_1, \ldots, Q_m \in \mathscr{M}$ be given. Then

$$P := P_1 \circ \cdots \circ P_n \xrightarrow{\mathrm{p}^*} Q_1 \circ \cdots \circ Q_m =: Q$$

can only hold if $n = m$. In this case the algorithm tries to establish $P_i' \xrightarrow{\mathrm{q}^*} Q_i'$ *from right to left*, i.e., for $i = n, n-1, \ldots, 1$ successively; here the prefixes P_i', Q_i' result from the original P_i, Q_i by certain commutations (as it is explained below) $(i = 1, \ldots, n)$.

In fact, if $P_n \xrightarrow{\mathrm{q}^*} Q_n$ does not hold and if there are prefixes $P_{n_1} \in \mathscr{L}$, $P_{n_2} \in \mathscr{M}$ such that $P_n = P_{n_1} P_{n_2}$, then, according to the above lemma, it is tested whether $P_{n_2} \xrightarrow{\mathrm{q}^*} Q_n$ holds. Analogously, if $Q_n = Q_{n_1} Q_{n_2}$ for some $Q_{n_1} \in \mathscr{K}$ and $Q_{n_2} \in \mathscr{M}$, then it is tried to establish $P_n \xrightarrow{\mathrm{q}^*} Q_{n_2}$.

In order to jeopardize a desired derivation $P_i \xrightarrow{q^*} Q_i$ at a later step of the computation concerning an index $i < n$ as little as possible, the procedure keeps track of the shortest "successful" prefixes P_{n_1} and Q_{n_1}, respectively, if such prefixes exist, and goes on with

$$\text{either } P'_{n-1} := P_{n-1}P_{n_1}, \quad Q_{n-1} \quad \text{or} \quad P_{n-1}, \quad Q'_{n-1} := Q_{n-1}Q_{n_1}$$

then. Afterwards the lower indices are treated correspondingly. (Note that $P \xrightarrow{p^*} Q$ cannot be verified if both some L_i prefixing P_n and some K_j prefixing Q_n have to be commuted in order to establish $P'_n \xrightarrow{q^*} Q'_n$; this is also valid for each index $i > 1$ treated later on.)

As all operations going beyond the algorithm of Scherer respect the polynomial time bound, we have outlined the proof of the following theorem.

Theorem 27. *The word problem of the relation* $\xrightarrow{p^*}$ *is decidable in polynomial time.*

4.3. Complexity

We introduce the classes of LK_m-formulas next, for which we want to examine the complexity of the satisfiability problem.

Definition 28. Let $n \in \mathbb{N}$, $n \geqslant 1$. Define nSAT $:= \{\alpha \in \mathscr{F} \mid \alpha$ is a conjunction of disjunctions of prefix formulas such that each conjunct contains at most n prefix formulas, and α is satisfiable$\}$.

Note that a formula belonging to nSAT can be obtained from a corresponding propositional formula, substituting literals by prefix formulas. In fact, our results are generalizations of the well-known results for propositional nSAT.

The key to the determination of a non-trivial upper bound of the complexity of nSAT is contained in part (a) of the following theorem. Note that its propositional counterpart holds trivially.

Theorem 29. (a) 1SAT *is decidable in polynomial time.*

(b) 2SAT *is decidable in polynomial time.*

(c) *For all* $n \geqslant 3$, *the set* nSAT *is* NP-*complete.*

Proof. (a) Let $\alpha = \beta_1 \wedge \cdots \wedge \beta_n$ be a conjunction of prefix formulas, i.e., $\beta_i \in \mathscr{PF}$ for $i = 1, \ldots, n$. Then, by Theorem 24,

$$\alpha \notin \text{1SAT} \Leftrightarrow \text{there are } i, j \in \{1, \ldots, n\} \text{ such that the set } \{\beta_i, \beta_j\} \text{ is not pseudo-consistent.}$$

The latter condition holds by definition, iff there are indices $i, j \in \{1, \ldots, n\}$, prefixes $P, P' \in \mathscr{P}$ and a literal λ such that

$$\beta_i = P\lambda, \quad \neg\beta_j = P'\lambda \quad \text{and} \quad P \xrightarrow{p^*} P'.$$

According to Theorem 27 $P \xrightarrow{\mathrm{p}^*} P'$ can be decided in polynomial time. Moreover, only $O(n^2)$ pairs of prefixes have to be checked. This gives the assertion of (a).

(b) Let $\alpha = \{\{\beta_{11}, \beta_{12}\}, \ldots, \{\beta_{n1}, \beta_{n2}\}\}$ be given such that $\beta_{ij} \in \mathscr{PF}$ for all $i \in \{1, \ldots, n\}$ and $j \in \{1, 2\}$. If $\beta_{ij} \xrightarrow{\mathrm{s}} \beta_{ik}$ for some $i \in \{1, \ldots, n\}$ and $j, k \in \{1, 2\}$, $j \neq k$, then we replace the ith clause by $\{\beta_{ik}\}$. Note that $\beta_{ij} \xrightarrow{\mathrm{s}} \beta_{ik}$ can be tested in polynomial time because of Theorem 27; if α' designates the formula where all such substitutions are carried out, then we get

$$\alpha \text{ is satisfiable} \Leftrightarrow \alpha' \text{ is satisfiable.}$$

Now fix some prefix formula β from a clause $\{\beta, \beta'\}$ of α'. Determine all prefix formulas γ' of α' such that $\beta \xrightarrow{\mathrm{s}} \neg\gamma'$. Then replace the clause $\{\beta, \beta'\}$ by all clauses $\{\neg\gamma', \beta'\}$ originating from such a formula γ'. It is not hard to see that for the resulting formula α''

$$\alpha'' \text{ is satisfiable} \Leftrightarrow \alpha' \in \text{is satisfiable}$$

holds (use Theorem 24). Iterating these steps we eventually arrive at a formula $\alpha''' = \{\{\gamma_{11}, \gamma_{12}\}, \ldots, \{\gamma_{m1}, \gamma_{m2}\}\}$ ($\gamma_{ij} \in \mathscr{PF}$ for $i = 1, \ldots, m$ and $j = 1, 2$) such that $\gamma_{ij} \wedge \gamma_{kl}$ ($i \neq k$ or $j \neq l$) is contradictory if and only if one conjunct is the negation of the other; moreover,

$$\alpha''' \text{ is satisfiable} \Leftrightarrow \alpha \text{ is satisfiable,}$$

and α''' can be obtained within a polynomial time bound. So it suffices to decide the satisfiability problem for formulas of the type α''' in polynomial time. This can be performed by generalizing propositional resolution [1, Section 2.10] in an obvious way. Since the resolvent of two clauses having at most two elements likewise has at most two elements and since there are not too many two-element clauses over a given set of prefix formulas, the desired time bound follows.

(c) Since propositional logic is a part of LK_m it suffices to prove that nSAT belongs to NP. But this is fairly clear: A non-deterministic Turing machine *guesses* a prefix formula in each clause and checks satisfiability of the resulting conjunction like in the proof of (a). In this way the asserted upper bound is established. □

As it is well-known [5], the satisfiability problem for the common multi-agent logic of knowledge, $\mathbf{S5}_m$, is PSPACE-complete. So the general LK_m-satisfiability problem is PSPACE-hard. Thus we have got a small hierarchy of LK_m-formulas w.r.t. the complexity of the satisfiability problem.

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Appendix A

For convenience of the reader we give a proof of Proposition 7 in this appendix (see also [4, Section 2.3]). We introduce some notations. Let a finite set of formulas Y and an element $s \in C$ be given. Let $Y_s := s \cap Y$. Then we define

$$\psi_{Y_s} := \bigwedge_{\beta \in Y_s} \beta \wedge \bigwedge_{\gamma \in Y \setminus Y_s} \neg \gamma.$$

We have the following basic facts.

Lemma A.1. *The subsequent formulas are* **LK**-*derivable:*
(a) $\psi_{\widetilde{\Gamma}_s} \leftrightarrow (\psi_{\Gamma_s^\wedge} \wedge \psi_{\Gamma_s^L})$,
(b) $\psi_{\Gamma_s} \leftrightarrow \psi_{\Gamma_s^\wedge}$,
(c) $\psi_{\Gamma_s^L} \wedge L\psi_{\Gamma_s^\wedge} \rightarrow L(\psi_{\Gamma_s^\wedge} \wedge \psi_{\Gamma_s^L})$.

The easy proof of the lemma is omitted. Note that the **S5**-axioms have to be used in order to establish (c). We reformulate Proposition 2.7 of [4] next. Let $\vdash$ designate **LK**-deducibility.

Proposition A.2. *Let $s, t \in C$. The following conditions are equivalent*:
(a) $\psi_{\widetilde{\Gamma}_s} \wedge L\psi_{\widetilde{\Gamma}_t}$ *is consistent,*
(b) $s \cap \Gamma^L = t \cap \Gamma^L$,
(c) $\vdash \psi_{\widetilde{\Gamma}_s} \rightarrow L\psi_{\widetilde{\Gamma}_t}$.

Proof. (a)$\Rightarrow$(b): Suppose that (b) does not hold. Then there exists a formula $L\beta \in \Gamma^L$ such that either $L\beta \in s$ and $L\beta \notin t$ or $L\beta \notin s$ and $L\beta \in t$. In the first case we get that $\psi_{\widetilde{\Gamma}_t} \wedge L\beta$ is not consistent (note that $t \cap \widetilde{\Gamma}$ is a maximal consistent subset of $\widetilde{\Gamma}$). From basic modal proof theory it follows that $\vdash \psi_{\widetilde{\Gamma}_t} \rightarrow K \neg \beta$. Thus we get $\vdash L\psi_{\widetilde{\Gamma}_t} \rightarrow \mathrm{LK} \neg \beta$. Because of axiom (5) we obtain $\vdash L\psi_{\widetilde{\Gamma}_t} \rightarrow K \neg \beta$. As $L\beta \in s \cap \widetilde{\Gamma}$, this contradicts the consistency of $\psi_{\widetilde{\Gamma}_s}$ with $L\psi_{\widetilde{\Gamma}_t}$. In the second case one argues analogously (using $\psi_{\widetilde{\Gamma}_s}$ and axiom (4) instead of $L\psi_{\widetilde{\Gamma}_t}$ and axiom (5)).

(b)$\Rightarrow$(c): Part (a) of Lemma A.1 implies $\vdash \psi_{\widetilde{\Gamma}_s} \rightarrow \psi_{\Gamma_s^L}$. The assumption that (b) holds gives $\vdash \psi_{\Gamma_s^L} \rightarrow \psi_{\Gamma_t^L}$. Clearly, $\vdash \psi_{\Gamma_t^L} \rightarrow L\psi_{\Gamma_t}$ is valid. Using Lemma A.1.(b) we get $\vdash L\psi_{\Gamma_t} \rightarrow L\psi_{\Gamma_t^\wedge}$. From these deducibilities we obtain $\vdash \psi_{\widetilde{\Gamma}_s} \rightarrow (\psi_{\Gamma_t^L} \wedge L\psi_{\Gamma_t^\wedge})$. With the aid of Lemma A.1.(c) $\vdash \psi_{\widetilde{\Gamma}_s} \rightarrow L(\psi_{\Gamma_t^\wedge} \wedge \psi_{\Gamma_t^L})$ follows. Thus we get (c) by means of Lemma A.1.(a).

(c) $\Rightarrow$ (a): If $\psi_{\widetilde{\Gamma}_s} \wedge L\psi_{\widetilde{\Gamma}_t}$ would be inconsistent, then $\vdash \psi_{\widetilde{\Gamma}_s} \rightarrow \neg L\psi_{\widetilde{\Gamma}_t}$. As $\psi_{\widetilde{\Gamma}_s}$ is consistent this contradicts (c). Hence $\psi_{\widetilde{\Gamma}_s} \wedge L\psi_{\widetilde{\Gamma}_t}$ must be consistent. □

Returning to the filtration $\overset{L}{\longmapsto}$ of $\overset{L}{\longrightarrow}$ we have the following corollary.

Corollary A.3. *Let s, t be as in the preceding proposition. Then each of the conditions* (a)–(c) *is equivalent to* $\bar{s} \overset{L}{\longmapsto} \bar{t}$.

Proof. Clearly, $\bar{s} \stackrel{L}{\longmapsto} \bar{t}$ implies (a) of Proposition A.2. Moreover, using that we deal with the minimal filtration it is easy to see that (c) implies $\bar{s} \stackrel{L}{\longmapsto} \bar{t}$. □

So far we have prepared the proof of Proposition 7, which turns out to be easy now:

Proof. Reflexivity and symmetry of $\stackrel{L}{\longmapsto}$ hold because this relation is the minimal filtration of $\stackrel{L}{\longrightarrow}$. Transitivity follows from Corollary A.3. and condition (b) of Proposition A.2. Finally, the *cross property* is yielded with the aid of Corollary A.3, Proposition A.2(c) and the fact that this property is present on the canonical model. □

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