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Asymptotic modelling of the linear dynamics of laminated beams

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ABSTRACT

We study the linear dynamics of a layered elastic beam by means of the asymptotic expansion method. The beam consists of three linearly elastic isotropic layers: the middle layer is considered to be thinner and softer than the upper and lower ones. We characterize the limit models by distinguishing three cases of natural frequencies: the *low* frequencies associated with flexural vibrations, the *mean* frequencies associated with axial vibrations and the *high* frequencies, associated with transversal shear and pinching vibrations.

1. Introduction

The precise determination of the natural frequencies of an elastic thin structure has great importance in engineering applications. Especially in the case of structures subjected to dynamic loads, the knowledge of the natural frequencies can prevent resonance phenomena, and, hence, the amplification of the eigen-modes and, finally, the definitive collapse of the structure. This becomes crucial in the case of thin three-dimensional beams and plates, and in the case of thin layered assemblies. For instance, for elastic isotropic thin plates, it has been shown in Ciarlet and Kesavan (1981) and Dauge et al. (1999), using an asymptotic analysis, that the lowest natural frequencies are associated with bending vibrations and they are proportional to the thickness of the structure. A three-dimensional modal analysis is quite complex and sometimes numerical results are not easy to be obtained. Moreover, when we are dealing with layered thin structures with strong contrasts in the geometry (small thicknesses) or in the mechanical properties (presence of "soft" or "rigid" layers), numerical instabilities can occur due to those differences and to the complicated finite elements discretization. Therefore much attention has been directed towards the formal derivation of simplified models.

The objective of the present work is to derive a mathematical model of the dynamical behavior of a laminated beam and to characterize its natural frequencies. Although the results of this paper can be applied to different mechanical problems, the authors have been inspired by laminated glass beams that have been studied from a practical, experimental and partially theoretical point of view in Consolini (2011). In this work we want to lay down a rational background to classical dynamical models of multilayer beam usually used in literature.

We consider a two-dimensional beam consisting of three thin layers made of linear elastic isotropic materials: the upper and lower layers are called adherents, the middle layer is called adhesive. In order to perform an asymptotic analysis of the dynamical problem we choose a *small* real parameter ε which is used to scale respectively the thicknesses and the elastic moduli of the three layers. The thicknesses of the adherents are of order ε, while the thickness of the adhesive, being thinner than the upper and lower layers, is scaled with ε^2. The elastic moduli of the upper and lower layers remain unscaled, while the Lamé's constants of the middle layer are scaled with ε^2 since the adhesive is considered to be softer than the adherents. We characterize the limit kinematics, the limit eigenvalue problems and the associated limit natural frequencies corresponding to three different models of free vibrations for the laminated beam: the *low*, *mean* and *high* frequencies. This problem is studied in the framework of linear elasticity, but in principle it can be also extended to more general contexts, like viscoelasticity or elastoplasticity. The choice of an isotropic constitutive law allows to decouple the three vibration problems.

We use the asymptotic expansion method, which consists of approximating the solution of the actual elastic problem through the successive terms of a power series with respect, for instance, to the dimensions of the section or to the thickness of the layers, considered as a small parameter. This mathematical technique has been employed to formally justify classical theories of beams, as in Trabucho and Viaño (1996) and theories of plates, as in Ciarlet (1997).

The asymptotic expansion method has been applied both to statical and dynamical problems. For what concerns with the asymptotic methods applied to the deduction of models for the dynamics of thin structures we can cite, for example, Davet

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(1987), for an accurate computation of low, mean and high frequencies of a beam, Irago et al. (1998) and Irago and Viaño (1998), for a formal deduction of axial and torsional high frequencies of a beam, and for the derivation of the second corrector term of the bending frequency of a beam, and, finally, Ciarlet and Kesavan (1981) and Dauge et al. (1999), for the derivation of bending and membrane eigen-frequencies of a plate.

Different models for the statics of layered panels have been formally derived through an asymptotic analysis. For instance, we can cite Serpilli and Lenci (2008), in which a quite similar problem is widely studied in the statical case for three different layered elastic strip with soft and hard core, and Avila-Pozos et al. (1999), in which the authors study an orthotropic layered plate by using ε^3 to scale the elastic moduli of the adhesive layer, instead of ε^2, which is what we use in the present work, and Åslund (2005), for non linear models of multilayer plates.

Other models of adhesive joints and other complex gluing of structures have been already studied in the statical case by means of the asymptotic expansion method and variational convergence. For instance, important contributions have been given by the works by Klarbring (1991) and Klarbring and Movchan (1998), for adhesively bonded joints, Geymonat and Krasucki (1997) and Zaittouni et al. (2002), for glued plates, Geymonat et al. (1999); Lenci (2000) and Licht and Michaille (1997), for limit models of soft thin adhesive joints. The asymptotic behavior of elastic thin "soft" and "hard" interfaces between two three-dimensional bodies using functional convergence and Γ-convergence has been investigated in several works and in different mathematical and mechanical frameworks in Abdelmoula et al. (1998); Lebon and Rizzoni (2011); Lebon and Rizzoni (2010); Lebon and Zaittouni (2010) and Bessoud et al. (2011). Moreover, it is worth mentioning the paper by Licht et al. (2009) which deals with the dynamics of elastic bodies connected by a thin adhesive layer using the Trotter method of approximation of semi-groups, (see Trotter (1958)). In this work the elastodynamical problem is transformed into a first order evolution problem for the pair (displacement, velocity) in an Hilbert space of possible state with finite energy: this gives the form of a so-called semi-group depending on the set of the mechanical parameters of the problem. As the parameters tend to zero, this semi-group converges in the sense of Trotter theory towards a limit which gives the convenient model.

The layout of this paper is as follows. In Section 2, we introduce the mathematical problem associated with the dynamical behavior of a three-layers beam with soft adhesive; we define a small real parameter ε and the geometrical and mechanical quantities related to it; then we apply the asymptotic methods to obtain the simplified models. In Section 3, we present the main Ansatz (9) for the asymptotic expansions of the displacement field and of the natural frequencies of the beam. Then we study three different cases of vibration. In Section 4, we present the problem of low frequencies associated with bending vibration modes; in Section 5, we derive the expression of mean frequencies associated with axial vibration modes; in Section 6, we study the high frequencies associated with transversal shear and pinching vibration modes. In Section 7, we give a numerical comparison between the limit natural frequencies deduced by the asymptotic analysis and those obtained through some numerical tests carried out by using a commercial FEM code.

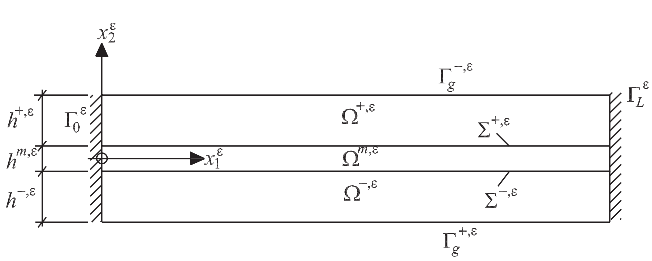

Fig. 1. Reference configuration.

2. Statement of the problem

This section is aimed at laying down an appropriate theoretical and formal ground for the rest of the article. In the sequel, Greek indices range in the set {1,2} and the Einstein's summation convention with respect to the repeated indices is adopted.

Let us consider a two-dimensional Euclidian space identified by $\mathbb{R}^2$. Let $0 < \varepsilon < 1$ be an dimensionless *small* real parameter which will tend to zero. Given a constant $L > 0$ we define the domain $\overline{\Omega}^\varepsilon = \overline{\Omega}^{+,\varepsilon} \cup \overline{\Omega}^{m,\varepsilon} \cup \overline{\Omega}^{-,\varepsilon} \subset \mathbb{R}^2$, as shown in Fig. 1:

$$\begin{aligned}
&\Omega^{m,\varepsilon} = \left(-\frac{h^{m,\varepsilon}}{2}, \frac{h^{m,\varepsilon}}{2}\right) \times (0, L),\\
&\Omega^{+,\varepsilon} = \left(\frac{h^{m,\varepsilon}}{2}, \frac{h^{m,\varepsilon}}{2} + h^{+,\varepsilon}\right) \times (0, L),\\
&\Omega^{-,\varepsilon} = \left(-\frac{h^{m,\varepsilon}}{2} - h^{-,\varepsilon}, -\frac{h^{m,\varepsilon}}{2}\right) \times (0, L),\\
&\Gamma_a^\varepsilon = \left(-\frac{h^{m,\varepsilon}}{2} - h^{-,\varepsilon}, \frac{h^{m,\varepsilon}}{2} + h^{+,\varepsilon}\right) \times \{a\}, \quad a = 0, L,\\
&\Gamma_g^{\pm,\varepsilon} = \left\{\pm\frac{h^{m,\varepsilon}}{2} \pm h^{\pm,\varepsilon}\right\} \times (0, L),\\
&\Sigma^{\pm,\varepsilon} = \overline{\Omega}^{\pm,\varepsilon} \cap \overline{\Omega}^{m,\varepsilon} = \left\{\pm\frac{h^{m,\varepsilon}}{2}\right\} \times [0, L].
\end{aligned} \tag{1}$$

We consider a three-layer strip of length L occupying the reference configuration $\overline{\Omega}^\varepsilon$. We study the physical problem corresponding to the mechanical behavior of a two-dimensional three-layer beam subjected to free vibrations. The structure is clamped at both ends on $\Gamma_a^\varepsilon, a = 0, L$, and the complementary part of the boundary $\Gamma_g^{\pm,\varepsilon}$ is traction free. The sets $\Omega^{+,\varepsilon}$, $\Omega^{-,\varepsilon}$, $\Omega^{m,\varepsilon}$ are respectively filled by three homogeneous, isotropic and linearly elastic materials characterized by the Lamé's coefficients $\lambda^{\pm,\varepsilon}$ and $\mu^{\pm,\varepsilon}$, and $\lambda^{m,\varepsilon}$ and $\mu^{m,\varepsilon}$ respectively. As usual we assume that $3\lambda^{\pm,\varepsilon} + 2\mu^{\pm,\varepsilon} > 0$, $\mu^{\pm,\varepsilon} > 0$, $3\lambda^{m,\varepsilon} + 2\mu^{m,\varepsilon} > 0$, $\mu^{m,\varepsilon} > 0$.

The physical eigenvalue value problem defined over the variable domain Ω^ε can be written as follows:

$$\begin{aligned}
&\text{Field equations in } \Omega^{\pm,\varepsilon} \text{ and } \Omega^{m,\varepsilon}:\\
&\mu^{\pm,\varepsilon}\Delta^\varepsilon \mathbf{u}^{\pm,\varepsilon} + (\lambda^{\pm,\varepsilon} + \mu^{\pm,\varepsilon})\nabla^\varepsilon \mathrm{div}^\varepsilon \mathbf{u}^{\pm,\varepsilon} = -\rho\omega_{k,\varepsilon}^2 \mathbf{u}^{\pm,\varepsilon},\\
&\mu^{m,\varepsilon}\Delta^\varepsilon \mathbf{u}^{m,\varepsilon} + (\lambda^{m,\varepsilon} + \mu^{m,\varepsilon})\nabla^\varepsilon \mathrm{div}^\varepsilon \mathbf{u}^{m,\varepsilon} = -\rho\omega_{k,\varepsilon}^2 \mathbf{u}^{m,\varepsilon},\\
&\text{Boundary conditions}:\\
&\mathbf{u}^\varepsilon := (\mathbf{u}^{+,\varepsilon}, \mathbf{u}^{m,\varepsilon}, \mathbf{u}^{-,\varepsilon}) = \mathbf{0} \text{ on } \Gamma_a^\varepsilon, \ a = 0, L,\\
&\sigma_{\alpha 2}^{\pm,\varepsilon}(\mathbf{u}^{\pm,\varepsilon}) = 0 \text{ on } \Gamma_g^{\pm,\varepsilon},\\
&\text{Interface conditions}:\\
&\mathbf{u}^{\pm,\varepsilon} = \mathbf{u}^{m,\varepsilon} \text{ on } \Sigma^{\pm,\varepsilon},\\
&\sigma_{\alpha 2}^{\pm,\varepsilon}(\mathbf{u}^{\pm,\varepsilon}) = \sigma_{\alpha 2}^{m,\varepsilon}(\mathbf{u}^{m,\varepsilon}) \text{ on } \Sigma^{\pm,\varepsilon},
\end{aligned} \tag{2}$$

where ρ is the surface mass density, $\omega_{k,\varepsilon}$ represent the natural frequencies of the beam, $\sigma_{\alpha\beta}^\varepsilon(\mathbf{u}^\varepsilon)$ are the components of the Cauchy stress tensor, associated with the displacement field $\mathbf{u}^\varepsilon = (u_\alpha^\varepsilon)$ by the Hooke's generalized law, $\sigma_{\alpha\beta}^\varepsilon(\mathbf{u}^\varepsilon) = \lambda^\varepsilon e_{\tau\tau}^\varepsilon(\mathbf{u}^\varepsilon)\delta_{\alpha\beta} + 2\mu^\varepsilon e_{\alpha\beta}^\varepsilon(\mathbf{u}^\varepsilon)$, and $e_{\alpha\beta}^\varepsilon(\mathbf{u}^\varepsilon) := \frac{1}{2}(\partial_\alpha^\varepsilon u_\beta^\varepsilon + \partial_\beta^\varepsilon u_\alpha^\varepsilon)$ are the components of the linearized strain tensor. We denote, respectively, with $\partial/\partial x_\alpha^\varepsilon := \partial_\alpha^\varepsilon$ and $\partial_{\alpha\alpha}^\varepsilon$, the first and the second partial derivatives with respect to x_α^ε, with $\Delta^\varepsilon f^\varepsilon := \partial_{\alpha\alpha}^\varepsilon f^\varepsilon$, the two-dimensional Laplacian operator of f^ε, with $\mathrm{div}^\varepsilon \mathbf{f}^\varepsilon := \partial_\alpha^\varepsilon f_\alpha^\varepsilon$, the two-dimensional divergence operator of $\mathbf{f}^\varepsilon = (f_\alpha^\varepsilon)$ and with $\nabla^\varepsilon f^\varepsilon := (\partial_\alpha^\varepsilon f^\varepsilon)$, the two-dimensional gradient operator of f^ε .

To proceed with the asymptotic analysis, we need to make some assumptions regarding the dependence on the small parameter ε of the thicknesses and on the elastic moduli of the three layers. The thicknesses $h^{\pm,\varepsilon}$ are linearly dependent of ε, i.e., $h^{\pm,\varepsilon} = \varepsilon h$, while the thickness of the middle layer is thinner and it is scaled with ε^2, i.e., $h^{m,\varepsilon} = \varepsilon^2 h$. The Lamé's constants of $\Omega^{+,\varepsilon}$ and $\Omega^{-,\varepsilon}$ are independent of ε and are equal between each other, so that $\lambda^{\pm,\varepsilon} = \lambda$ and $\mu^{\pm,\varepsilon} = \mu$, while the elastic moduli of $\Omega^{m,\varepsilon}$, which is assumed to be softer than the upper and lower layers, present the following dependence with respect to $\varepsilon : \lambda^m_\varepsilon = \varepsilon^2 \lambda^m$ and $\mu^m_\varepsilon = \varepsilon^2 \mu^m$.

In order to study the asymptotic behavior of $\mathbf{u}^\varepsilon$ when the thickness of the three layers goes to zero, by using the approach of Ciarlet (1997), we introduce the following change of variables π^ε: $x := (x_1, x_2) \in \overline{\Omega} \mapsto x^\varepsilon := (x^\varepsilon_1, x^\varepsilon_2) \in \overline{\Omega}^\varepsilon$, that allows to transform the problem posed on a variable domain (dependent of ε) onto a problem on a fixed domain (independent of ε):

$$
\begin{aligned}
&\pi^\varepsilon(x) = x^\varepsilon := \left(x_1, \varepsilon^2 \frac{h}{2} + \varepsilon\left(x_2 - \frac{h}{2}\right)\right), \ \forall x \in \overline{\Omega}^+,\\
&\pi^\varepsilon(x) = x^\varepsilon := (x_1, \varepsilon^2 x_2), \ \forall x \in \overline{\Omega}^m,\\
&\pi^\varepsilon(x) = x^\varepsilon := \left(x_1, -\varepsilon^2 \frac{h}{2} + \varepsilon\left(x_2 + \frac{h}{2}\right)\right), \ \forall x \in \overline{\Omega}^-,
\end{aligned} \tag{3}
$$

where

$$
\begin{aligned}
&\overline{\Omega} := \overline{\Omega}^+ \cup \overline{\Omega}^m \cup \overline{\Omega}^- \subset \mathbb{R}^2,\\
&\Omega^+ = \left(\frac{h}{2}, \frac{3h}{2}\right) \times (0, L), \ \Omega^- = \left(-\frac{3h}{2}, -\frac{h}{2}\right) \times (0, L),\\
&\Omega^m = \left(-\frac{h}{2}, \frac{h}{2}\right) \times (0, L)
\end{aligned} \tag{4}
$$

Likewise, one has

$$
\begin{aligned}
&\Gamma_a = \left(-\frac{3h}{2}, \frac{3h}{2}\right) \times \{a\}, \ a = 0, L, \ \Gamma^\pm_g = \left\{\pm\frac{3h}{2}\right\} \times (0, L),\\
&\Sigma^\pm = \overline{\Omega}^\pm \cap \overline{\Omega}^m = \left\{\pm\frac{h}{2}\right\} \times [0, L].
\end{aligned} \tag{5}
$$

Note that we have assumed that both adherents have the same thickness.

By using the bijection π^ε, we get that $\partial^\varepsilon_1 = \partial_1$ and $\partial^\varepsilon_2 = \frac{1}{\varepsilon}\partial_2$ in $\Omega^\pm$, and $\partial^\varepsilon_1 = \partial_1$ and $\partial^\varepsilon_2 = \frac{1}{\varepsilon^2}\partial_2$ in Ω^m.

With the unknowns $\mathbf{u}^\varepsilon := (\mathbf{u}^{+,\varepsilon}, \mathbf{u}^{m,\varepsilon}, \mathbf{u}^{-,\varepsilon})$ and natural frequencies $\omega_{k,\varepsilon}$, appearing in (2), we associate respectively the scaled unknowns $\mathbf{u}(\varepsilon) := (\mathbf{u}^+(\varepsilon), \mathbf{u}^m(\varepsilon), \mathbf{u}^-(\varepsilon))$ transformed by π^ε and the scaled natural frequencies $\omega_k(\varepsilon)$ by means of the following relations:

$$
\begin{aligned}
&\mathbf{u}(\varepsilon)(x) := \mathbf{u}^\varepsilon(x^\varepsilon), \ \forall x^\varepsilon = \pi^\varepsilon(x) \in \overline{\Omega}^\varepsilon,\\
&\omega^2_k(\varepsilon) := \omega^2_{k,\varepsilon}.
\end{aligned} \tag{6}
$$

According to the previous assumptions, problem (2) can be reformulated on the fixed domain Ω independent of ε. We obtain the following scaled eigenvalue problem:

$$
\begin{aligned}
&\text{Field equations in } \Omega^\pm \text{ and } \Omega^m:\\
&\frac{1}{\varepsilon^2}\mu\partial_{22}u^\pm_1(\varepsilon) + \frac{1}{\varepsilon}(\lambda+\mu)\partial_{12}u^\pm_2(\varepsilon) + (\lambda+2\mu)\partial_{11}u^\pm_1(\varepsilon)\\
&\quad = -\rho\omega^2_k(\varepsilon)u^\pm_1(\varepsilon),\\
&\frac{1}{\varepsilon^2}(\lambda+2\mu)\partial_{22}u^\pm_2(\varepsilon) + \frac{1}{\varepsilon}(\lambda+\mu)\partial_{12}u^\pm_1(\varepsilon) + \mu\partial_{11}u^\pm_2(\varepsilon)\\
&\quad = -\rho\omega^2_k(\varepsilon)u^\pm_2(\varepsilon),\\
&\frac{1}{\varepsilon^2}\mu^m\partial_{22}u^m_1(\varepsilon) + (\lambda^m+\mu^m)\partial_{12}u^m_2(\varepsilon) + \varepsilon^2(\lambda^m+2\mu^m)\partial_{11}u^m_1(\varepsilon)\\
&\quad = -\rho\omega^2_k(\varepsilon)u^m_1(\varepsilon),\\
&\frac{1}{\varepsilon^2}(\lambda^m+2\mu^m)\partial_{22}u^m_2(\varepsilon) + (\lambda^m+\mu^m)\partial_{12}u^m_1(\varepsilon) + \varepsilon^2\mu^m\partial_{11}u^m_2(\varepsilon)\\
&\quad = -\rho\omega^2_k(\varepsilon)u^m_2(\varepsilon),\\
&\text{Boundary conditions}:\\
&\mathbf{u}(\varepsilon) = \mathbf{0} \quad \text{on } \Gamma_{0,L},\\
&\sigma^\pm_{\alpha 2}(\mathbf{u}^\pm(\varepsilon)) = 0 \quad \text{on } \Gamma^\pm_g,\\
&\text{Interface conditions}:\\
&\mathbf{u}^\pm(\varepsilon) = \mathbf{u}^m(\varepsilon) \quad \text{on } \Sigma^\pm,\\
&\sigma^\pm_{\alpha 2}(\mathbf{u}^\pm(\varepsilon)) = \sigma^m_{\alpha 2}(\mathbf{u}^m(\varepsilon)) \quad \text{on } \Sigma^\pm,
\end{aligned} \tag{7}
$$

where the scaled stresses take the following form:

$$
\begin{aligned}
\sigma^\pm_{12}(\mathbf{u}^\pm(\varepsilon)) &:= \frac{1}{\varepsilon}\mu\partial_2 u^\pm_1(\varepsilon) + \mu\partial_1 u^\pm_2(\varepsilon),\\
\sigma^\pm_{22}(\mathbf{u}^\pm(\varepsilon)) &:= \frac{1}{\varepsilon}(\lambda+2\mu)\partial_2 u^\pm_2(\varepsilon) + \lambda\partial_1 u^\pm_1(\varepsilon),\\
\sigma^m_{12}(\mathbf{u}^m(\varepsilon)) &:= \mu^m\partial_2 u^m_1(\varepsilon) + \varepsilon^2\mu^m\partial_1 u^m_2(\varepsilon),\\
\sigma^m_{22}(\mathbf{u}^m(\varepsilon)) &:= (\lambda^m+2\mu^m)\partial_2 u^m_2(\varepsilon) + \varepsilon^2\lambda^m\partial_1 u^m_1(\varepsilon).
\end{aligned} \tag{8}
$$

The aim of the present work is to study the behavior of the problem when ε tends to zero and to characterize the limit solution. From a practical point of view, this means that we are looking for a simplified model of the original problem whose solution is easier to compute and still a good approximation of the actual solution $(\mathbf{u}(\varepsilon), \omega^2_k(\varepsilon))$: mathematically, this means to evaluate the $\lim_{\varepsilon\to 0}(\mathbf{u}(\varepsilon), \omega^2_k(\varepsilon))$.

3. Asymptotic analysis

We can now perform an asymptotic analysis of the rescaled problem (7). Since it has a polynomial structure with respect to the small parameter ε, we can look for the solution of the problem as a series of powers of ε:

$$
\begin{aligned}
&\mathbf{u}(\varepsilon) = \mathbf{u}^0 + \varepsilon\mathbf{u}^1 + \varepsilon^2\mathbf{u}^2 + \varepsilon^3\mathbf{u}^3 + \cdots,\\
&\omega^2_k(\varepsilon) = \varepsilon^p\omega^2_{k,p} + \varepsilon^{p+1}\Lambda_{k,p+1} + \varepsilon^{p+2}\Lambda_{k,p+2} + \cdots
\end{aligned} \tag{9}
$$

with $p \in \{2, 0, -2\}$.

The choice of the value of p has a fundamental influence on the limit model, as shown by Davet (1987) and Dauge et al. (1999). Indeed by choosing $p \in \{2, 0, -2\}$ we will cover part of the spectrum of the possible frequencies and modal forms of the considered layered beam. For instance by taking $p = 2$ we focus our attention to natural frequencies of order 1 (*low* frequencies) corresponding to wavelengths of the order of the unit. The vibration modes are independent of the transversal coordinate x_2 and from a mechanical point of view they represent bending waves propagating along the length of the beam. By choosing $p = 0$ we obtain natural frequencies of order 0 (*mean* frequencies). This waves propagates along the length of the beam, the modes are independent of x_2 and they represent axial or stretching waves. If we want to obtain a limit model in which the waves propagate in both directions, through the thickness and along the length of the beam, thus, we need to choose $p = -2$: this corresponds to *high* frequencies related to wavelengths of the order of ε.

By substituting (9) into the scaled eigenvalue value problem (7) and by identifying the terms with identical power of ε, we obtain the following sequence of equations for $p \in \{2, 0, -2\}$:

$$
\begin{aligned}
&\text{Field equations in } \Omega^\pm \text{ and } \Omega^m :=\\
&\mu\partial_{22}u^{\pm,i}_1 + (\lambda+\mu)\partial_{12}u^{\pm,i-1}_2 + (\lambda+2\mu)\partial_{11}u^{\pm,i-2}_1\\
&\quad = -\rho\left(\omega^2_{k,p}u^{\pm,q}_1 + \sum_{n=1}^{i}\Lambda_{k,p+n}u^{\pm,q-n}_1\right),\\
&(\lambda+2\mu)\partial_{22}u^{\pm,i}_2 + (\lambda+\mu)\partial_{12}u^{\pm,i-1}_1 + \mu\partial_{11}u^{\pm,i-2}_2\\
&\quad = -\rho\left(\omega^2_{k,p}u^{\pm,q}_2 + \sum_{n=1}^{i}\Lambda_{k,p+n}u^{\pm,q-n}_2\right),
\end{aligned}
$$

$$\mu^m\partial_{22}u_1^{m,i}+(\lambda^m+\mu^m)\partial_{12}u_2^{m,i-2}+(\lambda^m+2\mu^m)\partial_{11}u_1^{m,i-4}$$
$$=-\rho\left(\omega_{k,p}^2u_1^{m,q}+\sum_{n=1}^{i}\Lambda_{k,p+n}u_1^{m,q-n}\right),$$
$$(\lambda^m+2\mu^m)\partial_{22}u_2^{m,i}+(\lambda^m+\mu^m)\partial_{12}u_1^{m,i-2}+\mu^m\partial_{11}u_2^{m,i-4} \tag{10}$$
$$=-\rho\left(\omega_{k,p}^2u_2^{m,q}+\sum_{n=1}^{i}\Lambda_{k,p+n}u_2^{m,q-n}\right),$$

with $i,n\in\mathbb{N}$ and $q:=i-2-p\in\mathbb{N}$,
Boundary conditions:

$$\mathbf{u}^i=\mathbf{0}\quad\text{on } \Gamma_{0,L},$$

$$\sigma_{\alpha2}^{\pm,i}(\mathbf{u}^{\pm,i},\mathbf{u}^{\pm,i-1})=0\quad\text{on } \Gamma_g^{\pm},$$

Interface conditions:

$$\mathbf{u}^{\pm,i}=\mathbf{u}^{m,i}\quad\text{on } \Sigma^{\pm},$$

$$\sigma_{\alpha2}^{\pm,i}(\mathbf{u}^{\pm,i},\mathbf{u}^{\pm,i-1})\sigma_{\alpha2}^{m,i}(\mathbf{u}^{m,i-1},\mathbf{u}^{m,i-3})\quad\text{on } \Sigma^{\pm},$$

where

$$\begin{aligned}
\sigma_{12}^{\pm,i}(\mathbf{u}^{\pm,i},\mathbf{u}^{\pm,i-1})&:=\mu\partial_2u_1^{\pm,i}+\mu\partial_1u_2^{\pm,i-1},\\
\sigma_{22}^{\pm,i}(\mathbf{u}^{\pm,i},\mathbf{u}^{\pm,i-1})&:=(\lambda+2\mu)\partial_2u_2^{\pm,i}+\lambda\partial_1u_1^{\pm,i-1},\\
\sigma_{12}^{m,i}(\mathbf{u}^{m,i-1},\mathbf{u}^{m,i-3})&:=\mu^m\partial_2u_1^{m,i-1}+\mu^m\partial_1u_2^{m,i-3},\\
\sigma_{22}^{m,i}(\mathbf{u}^{m,i-1},\mathbf{u}^{m,i-3})&:=(\lambda^m+2\mu^m)\partial_2u_2^{m,i-1}+\lambda^m\partial_1u_1^{m,i-3}.
\end{aligned} \tag{11}$$

Now we can study each eigenvalue value problem (10) and characterize the displacement fields $\mathbf{u}^i$ for each $i\in\mathbb{N}$ and the natural frequencies $\Lambda_{k,p+n}$ for each $n\in\mathbb{N}$. We call respectively *limit displacement field* and *limit natural frequency* the leading terms $(\mathbf{u}^0,\omega_{k,p})$, with $p\in\{2,0,-2\}$, of the asymptotic expansions (9) and *limit problem* the associated eigenvalue problem.

4. Low frequencies: limit problem for $p=2$

In this section we characterize the limit eigenvalue problem associated with $(\mathbf{u}^0,\omega_{k,2})$, which mechanically represents free bending vibrations of a three-layers beam. The limit problem when $p=2$ is presented in the following theorem:

Theorem 1. *The leading term* $\mathbf{u}^0:=(\mathbf{u}^{+,0},\mathbf{u}^{m,0},\mathbf{u}^{-,0})$ *of the asymptotic expansion (9) is such that*

$$\begin{aligned}
&u_2^{+,0}(x_1,x_2)=u_2^{m,0}(x_1,x_2)=u_2^{-,0}(x_1,x_2):=U_2^0(x_1),\\
&u_1^0(x_1,x_2)=0,
\end{aligned} \tag{12}$$

and it satisfies the following eigenvalue value problem ($\phi''''(x_1):=\frac{d^4\phi}{dx_1^4}(x_1)$):

$$\begin{cases}
\frac{4\mu(\lambda+\mu)h^3}{3(\lambda+2\mu)}U_2^{0''''}(x_1)-\rho\omega_{k,2}^2hU_2^0(x_1)=0,\\
U_2^0(0)=U_2^0(L)=0,\\
U_2^{0'}(0)=U_2^{0'}(L)=0.
\end{cases} \tag{13}$$

The limit natural frequency of the beam $\omega_{k,2}$ *is the solution of the following equation:*

$$\cos(KL)\cosh(KL)=1, \tag{14}$$

where

$$K:=\sqrt[4]{\frac{3\rho\omega_{k,2}^2}{4h^2}\,\frac{\lambda+2\mu}{\mu(\lambda+\mu)}}. \tag{15}$$

Proof. For the sake of clarity the proof is divided into five parts.

(i) The boundary value problem corresponding to $i=0$ in (10) takes the following form:

$$\begin{aligned}
&\mu\partial_{22}u_1^{\pm,0}=0,\quad(\lambda+2\mu)\partial_{22}u_2^{\pm,0}=0\quad\text{in } \Omega^{\pm},\\
&\mu^m\partial_{22}u_1^{m,0}=0,\quad(\lambda^m+2\mu^m)\partial_{22}u_2^{m,0}=0\quad\text{in } \Omega^{m},\\
&\mu\partial_2u_1^{\pm,0}\left(x_1,\pm\frac{h}{2}\right)=0,\quad(\lambda+2\mu)\partial_2u_2^{\pm,0}\left(x_1,\pm\frac{h}{2}\right)=0,\\
&\mu\partial_2u_1^{\pm,0}\left(x_1,\pm\frac{3h}{2}\right)=0,\quad(\lambda+2\mu)\partial_2u_2^{\pm,0}\left(x_1,\pm\frac{3h}{2}\right)=0,\\
&\mathbf{u}^{\pm,0}\left(x_1,\pm\frac{h}{2}\right)=\mathbf{u}^{m,0}\left(x_1,\pm\frac{h}{2}\right).
\end{aligned} \tag{16}$$

This problem leads to the following characterization of $\mathbf{u}^0$:

$$\begin{cases}
u_\alpha^{\pm,0}(x_1,x_2):=A_\alpha^{\pm,0}(x_1),\\
u_\alpha^{m,0}(x_1,x_2)=\frac{1}{2}\left(A_\alpha^{+,0}+A_\alpha^{-,0}\right)+\frac{x_2}{h}\left(A_\alpha^{+,0}-A_\alpha^{-,0}\right).
\end{cases} \tag{17}$$

(ii) The next boundary value problem for $i=1$ is:

$$\begin{aligned}
&\mu\partial_{22}u_1^{\pm,1}=-(\lambda+\mu)\partial_{12}u_2^{\pm,0}\quad\text{in } \Omega^{\pm},\\
&(\lambda+2\mu)\partial_{22}u_2^{\pm,1}=-(\lambda+\mu)\partial_{12}u_1^{\pm,0}\quad\text{in } \Omega^{\pm},\\
&\mu^m\partial_{22}u_1^{m,1}=0\quad\text{in } \Omega^{m},\\
&(\lambda^m+2\mu^m)\partial_{22}u_2^{m,1}=0\quad\text{in } \Omega^{m},\\
&\mu\partial_2u_1^{\pm,1}\left(x_1,\pm\frac{h}{2}\right)=-\left(\mu\partial_1u_2^{\pm,0}-\mu^m\partial_2u_1^{m,0}\right)\left(x_1,\pm\frac{h}{2}\right),\\
&(\lambda+2\mu)\partial_2u_2^{\pm,1}\left(x_1,\pm\frac{h}{2}\right)=-\left(\lambda\partial_1u_1^{\pm,0}-(\lambda^m+2\mu^m)\partial_2u_2^{m,0}\right)\left(x_1,\pm\frac{h}{2}\right),\\
&\mu\partial_2u_1^{\pm,1}\left(x_1,\pm\frac{3h}{2}\right)=-\mu\partial_1u_2^{\pm,0}\left(x_1,\pm\frac{3h}{2}\right),\\
&(\lambda+2\mu)\partial_2u_2^{\pm,1}\left(x_1,\pm\frac{3h}{2}\right)=-\lambda\partial_1u_1^{\pm,0}\left(x_1,\pm\frac{3h}{2}\right),\\
&\mathbf{u}^{\pm,1}\left(x_1,\pm\frac{h}{2}\right)=\mathbf{u}^{m,1}\left(x_1,\pm\frac{h}{2}\right).
\end{aligned} \tag{18}$$

From Eq. (18), we derive that $A_\alpha^{+,0}(x_1)=A_\alpha^{-,0}(x_1)$, and, thus,

$$u_\alpha^{+,0}(x_1)=u_\alpha^{-,0}(x_1)=u_\alpha^{m,0}(x_1):=U_\alpha^0(x_1). \tag{19}$$

Thus, at order zero, the three layers beam is reduced to a monolithic one, consisting of just one layer. Moreover, we get an expression of the displacement field $\mathbf{u}^1$:

$$\begin{cases}
u_1^{\pm,1}(x_1,x_2)=B_1^{\pm,1}(x_1)-x_2U_2^{0'}(x_1),\\
u_2^{\pm,1}(x_1,x_2)=B_2^{\pm,1}(x_1)-x_2\frac{\lambda}{\lambda+2\mu}U_1^{0'}(x_1),\\
u_\alpha^{m,1}(x_1,x_2)=B_\alpha^{m,1}(x_1)+x_2A_\alpha^{m,1}(x_1),
\end{cases} \tag{20}$$

with

$$\begin{aligned}
&B_1^{m,1}:=\frac{1}{2}(B_1^{+,1}+B_1^{-,1}),\quad B_2^{m,1}:=\frac{1}{2}(B_2^{+,1}+B_2^{-,1}),\\
&A_1^{m,1}:=-U_2^{0'}+\frac{1}{h}(B_1^{+,1}-B_1^{-,1}),\\
&A_2^{m,1}:=-\frac{\lambda}{\lambda+2\mu}U_1^{0'}+\frac{1}{h}\left(B_2^{+,1}-B_2^{-,1}\right).
\end{aligned} \tag{21}$$

(iii) The boundary value problem for $i=2$ reads as follows:

$$\begin{aligned}
&\mu\partial_{22}u_1^{\pm,2}=-(\lambda+\mu)\partial_{12}u_2^{\pm,1}-(\lambda+2\mu)\partial_{11}u_1^{\pm,0}\ \text{ in } \Omega^{\pm},\\
&(\lambda+2\mu)\partial_{22}u_2^{\pm,2}=-(\lambda+\mu)\partial_{12}u_1^{\pm,1}-\mu\partial_{11}u_2^{\pm,0}\ \text{ in } \Omega^{\pm},\\
&\mu^m\partial_{22}u_1^{m,2}=-(\lambda^m+\mu^m)\partial_{12}u_2^{m,0}\ \in\text{ in } \Omega^{m},\\
&(\lambda^m+2\mu^m)\partial_{22}u_2^{m,2}=-(\lambda^m+\mu^m)\partial_{12}u_1^{m,0}\ \text{ in } \Omega^{m},\\
&\mu\partial_2u_1^{\pm,2}\left(x_1,\pm\frac{h}{2}\right)=\left(-\mu\partial_1u_2^{\pm,1}+\mu^m\partial_2u_1^{m,1}\right)\left(x_1,\pm\frac{h}{2}\right),
\end{aligned}$$

$$(\lambda+2\mu)\partial_2 u_2^{\pm,2}\left(x_1,\pm\frac{h}{2}\right)=\left(-\lambda\partial_1 u_1^{\pm,1}+(\lambda^m+2\mu^m)\partial_2 u_2^{m,1}\right)\left(x_1,\pm\frac{h}{2}\right),$$
$$\mu\partial_2 u_1^{\pm,2}\left(x_1,\pm\frac{3h}{2}\right)=-\mu\partial_1 u_2^{\pm,1}\left(x_1,\pm\frac{3h}{2}\right),$$
$$(\lambda+2\mu)\partial_2 u_2^{\pm,2}\left(x_1,\pm\frac{3h}{2}\right)=-\lambda\partial_1 u_1^{\pm,1}\left(x_1,\pm\frac{3h}{2}\right),$$
$$\mathbf{u}^{\pm,2}\left(x_1,\pm\frac{h}{2}\right)=\mathbf{u}^{m,2}\left(x_1,\pm\frac{h}{2}\right). \tag{22}$$

The interface conditions in (22) yield immediately to $U_1^0=0$ and $A_\alpha^{m,1}=0$. This implies that $B_1^{\pm,1}=B_1^{m,1}\pm\frac{h}{2}U_2^{0'}$ and $B_2^{\pm,1}=B_2^{m,1}$. The displacement $\mathbf{u}^1$ becomes:

$$\begin{cases} u_1^{\pm,1}(x_1,x_2)=B_1^{m,1}(x_1)-\left(x_2\mp\frac{h}{2}\right)U_2^{0'}(x_1),\\ u_2^{\pm,1}(x_1,x_2)=B_2^{m,1}(x_1),\\ u_\alpha^{m,1}(x_1,x_2)=B_\alpha^{m,1}(x_1). \end{cases} \tag{23}$$

At order one, we obtain a Navier–Bernoulli-type displacement field with no jump of the displacement field at the adhesive interface. From the equations of problem (22), we obtain the following form of displacement field $\mathbf{u}^2$:

$$\begin{cases} u_1^{\pm,2}(x_1,x_2)=B_1^{\pm,2}(x_1)-x_2B_2^{m,1'}(x_1),\\ u_2^{\pm,2}(x_1,x_2)=B_2^{\pm,2}(x_1)-x_2\frac{\lambda}{\lambda+2\mu}B_1^{m,1'}(x_1)+\frac{\lambda}{\lambda+2\mu}\left(\frac{x_2^2}{2}\mp\frac{hx_2}{2}\right)U_2^{0''}(x_1),\\ u_\alpha^{m,2}(x_1,x_2)=B_\alpha^{m,2}(x_1)+x_2A_\alpha^{m,2}(x_1), \end{cases} \tag{24}$$

where

$$B_1^{\pm,2}=B_1^{m,2}\pm\frac{h}{2}\left(A_1^{m,2}+B_2^{m,1'}\right),$$
$$B_2^{\pm,2}=B_2^{m,2}\pm\frac{h}{2}\left(A_2^{m,2}+\frac{\lambda}{\lambda+2\mu}B_1^{\pm,1'}\right). \tag{25}$$

At order two, we start seeing the presence of a shearable adhesive middle layer, which allows the adherents to slide on one another, creating a jump of the displacement field between the upper and lower layers at the adhesive interface.

(iv) By using the expressions of displacement fields $\mathbf{u}^0$, $\mathbf{u}^1$ and $\mathbf{u}^2$ obtained in (i), (ii) and (iii), and the following equations of the boundary value problem for $i=3$:

$$\mu\partial_{22}u_1^{\pm,3}=-(\lambda+\mu)\partial_{12}u_2^{\pm,2}-(\lambda+2\mu)\partial_{11}u_1^{\pm,1}\quad \text{in } \Omega^\pm,$$
$$(\lambda+2\mu)\partial_{22}u_2^{\pm,3}=-(\lambda+\mu)\partial_{12}u_1^{\pm,2}-\mu\partial_{11}u_2^{\pm,1}\quad \text{in } \Omega^\pm,$$
$$\mu^m\partial_{22}u_1^{m,3}=-(\lambda^m+\mu^m)\partial_{12}u_2^{m,1}\quad \text{in } \Omega^m,$$
$$(\lambda^m+2\mu^m)\partial_{22}u_2^{m,3}=-(\lambda^m+\mu^m)\partial_{12}u_1^{m,1}\quad \text{in } \Omega^m, \tag{26}$$

we can characterize the displacement field $\mathbf{u}^3$ as follows:

$$\begin{cases} u_1^{\pm,3}(x_1,x_2)=B_1^{\pm,3}(x_1)+x_2A_1^{\pm,3}(x_1)\\ \qquad -\frac{3\lambda+4\mu}{\lambda+2\mu}\left\{\frac{x_2^2}{2}B_1^{m,1''}(x_1)-\frac{x_2^2}{6}\left(x_2\mp\frac{3h}{2}\right)U_2^{0'''}\right\},\\ u_2^{\pm,3}(x_1,x_2)=B_2^{\pm,3}(x_1)+x_2A_2^{\pm,3}(x_1)+\frac{\lambda}{\lambda+2\mu}\frac{x_2^2}{2}B_2^{m,1''}(x_1),\\ u_\alpha^{m,3}(x_1,x_2)=B_\alpha^{m,3}(x_1)+x_2A_\alpha^{m,3}(x_1). \end{cases} \tag{27}$$

Functions $B_2^{\pm,2}$ and $A_1^{\pm,3}$ satisfy the following relation:

$$\left(B_2^{+,2}+B_2^{-,2}\right)''+\left(A_1^{+,3}+A_1^{-,3}\right)'=-\frac{3h^2(\lambda+\mu)}{\lambda+2\mu}U_2^{0''''}. \tag{28}$$

(v) Let us consider the following equations deriving from problem when $i=4$:

$$(\lambda+2\mu)\partial_{22}u_2^{\pm,4}+(\lambda+\mu)\partial_{12}u_1^{\pm,3}+\mu\partial_{11}u_2^{\pm,2}=-\rho\omega_{k,2}^2u_2^{\pm,0}\ \text{ in } \Omega^\pm,$$
$$(\lambda+2\mu)\partial_2 u_2^{\pm,4}\left(x_1,\pm\frac{3h}{2}\right)=-\lambda\partial_1 u_1^{\pm,3}\left(x_1,\pm\frac{3h}{2}\right),$$
$$(\lambda+2\mu)\partial_2 u_2^{\pm,4}\left(x_1,\pm\frac{h}{2}\right)$$
$$=\left(-\lambda\partial_1 u_1^{\pm,3}+(\lambda^m+2\mu^m)\partial_2 u_2^{m,3}+\lambda^m\partial_1 u_1^{m,1}\right)\left(x_1,\pm\frac{h}{2}\right), \tag{29}$$

By integrating, respectively, the first equations of (29) along the thickness of each layer and by applying the boundary and interface conditions, we obtain:

$$\mu\left[\left(B_2^{+,2''}+B_2^{-,2''}\right)+\left(A_1^{+,3'}+A_1^{-,3'}\right)\right]+\frac{h^2\mu(\lambda+\mu)}{3(\lambda+2\mu)}U_2^{0''''}=-2\rho\omega_{k,2}^2U_2^0. \tag{30}$$

By combining relations (28) and (30), we finally get the limit problem satisfied by the limit displacement field U_2^0:

$$\begin{cases} \frac{4\mu(\lambda+\mu)h^3}{3(\lambda+2\mu)}U_2^{0''''}(x_1)-\rho\omega_{k,2}^2hU_2^0(x_1)=0,\\ U_2^0(0)=U_2^0(L)=0,\\ U_2^{0'}(0)=U_2^{0'}(L)=0. \end{cases} \tag{31}$$

where the second boundary condition is obtained by virtue of the form of $\mathbf{u}^1$ in (23). The limit natural frequency of the beam $\omega_{k,2}$ can be characterized explicitly by the following relation (see, for instance, Timoshenko and Goodier (1951)):

$$\cos(KL)\cosh(KL)=1, \tag{32}$$

where $K:=\sqrt[4]{\frac{3\rho\omega_{k,2}^2}{4h^2}\frac{\lambda+2\mu}{\mu(\lambda+\mu)}}$. □

Remark 1. As already claimed at the beginning of Section 3, since the axial component U_1^0 of the limit displacement field is identically equal to zero, the limit model for $p=2$ corresponds to flexural vibrations propagating along the x_1-coordinate, i.e. along the length of the beam. Moreover, since $u_2^{+,0}(x_1)=u_2^{-,0}(x_1)=u_2^{m,0}(x_1):=U_2^0(x_1)$, the simplified model homogenizes the three-layer beam into one consisting of just one layer. Thus the influence of the middle soft layer is negligible at least at the zeroth and at the first order of the asymptotic expansion. This model is similar to the one obtained by Davet (1987) for the free vibrations of a single layer elastic beam and it is also coherent with the results obtained in Dauge et al. (1999) in the case of plates.

Rewriting Eq. (32) in the form $\cos(KL)=1/\cosh(KL)$ we see that for large values of KL one has $\cos(KL)\simeq 0$, namely, $KL\simeq\frac{\pi}{2}(2k+1)$, with $k\in\mathbb{N}$, and then, by definition of K,

$$\omega_{k,2}\simeq(2k+1)^2\frac{\pi^2h}{2L^2}\sqrt{\frac{\mu(\lambda+\mu)}{3\rho(\lambda+2\mu)}}=(2k+1)^2\frac{\pi^2h}{4L^2}\sqrt{\frac{E}{3\rho(1-\nu^2)}}, \tag{33}$$

where E and ν are, respectively, the Young's modulus and the Poisson's coefficient of the upper and lower layers. The coefficient $\frac{E}{1-\nu^2}$ is an equivalent elastic modulus of $\Omega^\pm$ which appears commonly in the two-dimensional theory of elasticity.

Remark 2. It is well-known that problem (31) is usually associated with a Navier–Bernoulli flexural beam model. Even though we cannot find a Navier–Bernoulli-type kinematics at order zero, however we can recover it at the first order of the asymptotic expansion. In fact, since $\mathbf{u}(\varepsilon)(x)=\mathbf{u}^0(x)+\varepsilon\mathbf{u}^1(x)+\cdots$, we have:

$$\begin{cases} u_1^{\pm}(x) = \varepsilon\Big[B_1^{m,1}(x_1) - \big(x_2 \mp \tfrac{h}{2}\big)U_2^{0\prime}(x_1)\Big] + \cdots, \\ u_2^{\pm}(x) = U_2^0(x_1) + \varepsilon B_2^{m,1}(x_1) + \cdots, \\ u_1^m(x) = \varepsilon B_1^{m,1}(x_1) + \cdots, \\ u_2^m(x) = U_2^0(x_1) + \varepsilon B_2^{m,1}(x_1) + \cdots \end{cases} \tag{34}$$

Going backward to the physical unscaled variables we have, up to the order zero,

$$\begin{cases} u_1^{\pm,\varepsilon}(x^\varepsilon) = -\varepsilon\big(x_2 \mp \tfrac{h}{2}\big)U_2^{0\prime}(x_1^\varepsilon) + \cdots = -x_2^\varepsilon U_2^{0\prime}(x_1^\varepsilon) + \cdots, \\ u_2^{\pm,\varepsilon}(x^\varepsilon) = U_2^0(x_1^\varepsilon) + \cdots, \\ u_1^{m,\varepsilon}(x^\varepsilon) = 0 + \cdots, \\ u_2^{m,\varepsilon}(x^\varepsilon) = U_2^0(x_1^\varepsilon) + \cdots, \end{cases} \tag{35}$$

which is the Navier–Bernoulli displacement field.

5. *Mean* frequencies: limit problem for $p = 0$

In this section we characterize the limit eigenvalue problem associated with $(\mathbf{u}^0, \omega_{k,2})$. From a mechanical point of view, it corresponds to free axial vibrations of the laminated beam. The limit problem when $p = 0$ is characterized in the following theorem:

Theorem 2. *The leading term* $\mathbf{u}^0 := (\mathbf{u}^{+,0}, \mathbf{u}^{m,0}, \mathbf{u}^{-,0})$ *of the asymptotic expansion (9) is such that*

$$u_1^{+,0}(x_1,x_2) = u_1^{m,0}(x_1,x_2) = u_1^{-,0}(x_1,x_2) := U_1^0(x_1), \quad u_2^0(x_1,x_2) = 0, \tag{36}$$

and it satisfies the following boundary value problem ($\phi''(x_1) := \frac{d^2\phi}{dx_1^2}(x_1)$):

$$\begin{cases} \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} U_1^{0''}(x_1) + \rho\omega_{k,0}^2 U_1^0(x_1) = 0, \\ U_1^0(0) = U_1^0(L) = 0, \end{cases} \tag{37}$$

where

$$\omega_{k,0} = \frac{k\pi}{L}\sqrt{\frac{4\mu(\lambda+\mu)}{\rho(\lambda+2\mu)}}, \tag{38}$$

is the limit natural frequency of the beam.

Proof. Since the boundary value problems for $i = 0, 1$ are identical to those presented in the Proof of Theorem 1 in Section 4, in the sequel, we use the results previously obtained, i.e., the characterization of the displacement fields $\mathbf{u}^0$ and $\mathbf{u}^1$.

We directly start by posing $p = 0$ and $i = 2$ in (10) and we get:

$$\begin{aligned} &(\lambda+2\mu)\partial_{22}u_2^{\pm,2} + (\lambda+\mu)\partial_{12}u_1^{\pm,1} + \mu\partial_{11}u_2^{\pm,0} = -\rho\omega_{k,0}^2 u_2^{\pm,0} \quad \text{in } \Omega^\pm, \\ &\left((\lambda+2\mu)\partial_2 u_2^{\pm,2} + \lambda\partial_1 u_1^{\pm,1}\right)\left(x_1, \pm\frac{h}{2}\right) = (\lambda^m + 2\mu^m)\partial_2 u_2^{m,1}\left(x_1, \pm\frac{h}{2}\right), \\ &\left((\lambda+2\mu)\partial_2 u_2^{\pm,2} + \lambda\partial_1 u_1^{\pm,1}\right)\left(x_1, \pm\frac{3h}{2}\right) = 0, \end{aligned} \tag{39}$$

Afterwards by integrating the first equations of (39) along the thickness of each layer, by applying the boundary conditions and the interface conditions, we obtain that $U_2^0 = 0$, i.e., the transversal displacement at order zero vanishes.

By choosing $i = 2$, we also obtain the following set of equations:

$$\begin{aligned} &\mu\partial_{22}u_1^{\pm,2} + (\lambda+\mu)\partial_{12}u_2^{\pm,1} + (\lambda+2\mu)\partial_{11}u_1^{\pm,0} = -\rho\omega_{k,0}^2 u_1^{\pm,0} \quad \text{in } \Omega^\pm, \\ &\left(\mu\partial_2 u_1^{\pm,2} + \mu\partial_1 u_2^{\pm,1}\right)\left(x_1, \pm\frac{h}{2}\right) = \mu^m\partial_2 u_1^{m,1}\left(x_1, \pm\frac{h}{2}\right), \\ &\left(\mu\partial_2 u_1^{\pm,2} + \mu\partial_1 u_2^{\pm,1}\right)\left(x_1, \pm\frac{3h}{2}\right) = 0. \end{aligned} \tag{40}$$

By using Eq. (40) we can finally characterize the limit problem for the axial displacement U_1^0. Indeed by integrating the first Eq. (40) through the thickness, we get:

$$\begin{cases} \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} U_1^{0''}(x_1) + \rho\omega_{k,0}^2 U_1^0(x_1) = 0, \\ U_1^0(0) = U_1^0(L) = 0. \end{cases} \tag{41}$$

We can now characterize the limit natural frequency $\omega_{k,0}$ (see, for instance, Timoshenko and Goodier, 1951), which takes the following form:

$$\omega_{k,0} = \frac{k\pi}{L}\sqrt{\frac{4\mu(\lambda+\mu)}{\rho(\lambda+2\mu)}}, \tag{42}$$

where $C \in \mathbb{R}$ is an arbitrary constant and $k \in \mathbb{N}$. □

Remark 3. As already announced at the beginning of Section 3, since the transversal component U_2^0 of the limit displacement field vanishes, the limit model for $p = 0$ corresponds to axial or stretching vibrations propagating along the x_1-coordinate, i.e. along the length of the beam. Besides, since $u_1^{+,0}(x_1) = u_1^{-,0}(x_1) = u_1^{m,0}(x_1) := U_1^0(x_1)$, the simplified model reduces the three layer beam to one consisting of just one layer. Thus in the limit model we cannot perceive the presence of the soft thin adhesive layer and its effects on the dynamics of the beam. This model is formally equivalent to the one obtained by Davet (1987) for a single layer elastic beam and Dauge et al. (1999) for membranes.

The limit natural frequency expressed in terms of the Young's modulus E and the Poisson's coefficient ν of the upper and lower layers takes the following form:

$$\omega_{k,0} = \frac{k\pi}{L}\sqrt{\frac{E}{\rho(1-\nu^2)}}. \tag{43}$$

6. *High* frequencies: limit problems for $p = -2$

In this Section we are looking for a limit model when $p = -2$. In this case we obtain two kinds of different waves: the transversal shear waves, in which the main displacement is the one along the x_1-coordinate, and the pinching waves, in which the main displacement is the one along the x_2-coordinate. In both cases, the waves propagate on both directions x_1 and x_2 with a wavelength along the x_2-coordinate of the order of magnitude of the thickness of the adherents.

6.1. Transversal shear frequencies

Theorem 3. *The leading term* $\mathbf{u}^0 := (\mathbf{u}^{+,0}, \mathbf{u}^{m,0}, \mathbf{u}^{-,0})$ *of the asymptotic expansion (9) is such that*

- *Even* $k \in \mathbb{N}$

$$\begin{cases} u_1^{\pm,0}(x_1,x_2) = A_1^{\pm,0}(x_1)\cos\left(\frac{k\pi}{h}x_2\right), \\ u_1^{m,0}(x_1,x_2) = \frac{(-1)^{\frac{k}{2}}}{2}\left[\frac{(A_1^{+,0}+A_1^{-,0})(x_1)}{\cos\left(\alpha\frac{k\pi}{2}\right)}\cos\left(\alpha\frac{k\pi}{h}x_2\right) \right. \\ \qquad\qquad \left. + \frac{(A_1^{+,0}-A_1^{-,0})(x_1)}{\sin\left(\alpha\frac{k\pi}{2}\right)}\sin\left(\alpha\frac{k\pi}{h}x_2\right)\right], \end{cases} \tag{44}$$

- *Odd* $k \in \mathbb{N}$

$$\mathbf{u}^0(x_1,x_2) = 0, \tag{45}$$

with $\alpha := \sqrt{\frac{\mu}{\mu^m}}$, *and the limit natural frequency takes the following form:*

$$\omega_{k,-2} = \frac{k\pi}{h}\sqrt{\frac{\mu}{\rho}}. \tag{46}$$

Moreover, functions $A_1^{\pm,0}$ satisfy the following eigenvalue problem:

$$\begin{cases} A_1^{+,0''}x(x_1) - \mathcal{F}_k(\Lambda_{k,0})A_1^{+,0}(x_1) = \mathcal{G}_k A_1^{-,0}(x_1), \\ A_1^{-,0''}(x_1) - \mathcal{F}_k(\Lambda_{k,0})A_1^{-,0}(x_1) = \mathcal{G}_k A_1^{+,0}(x_1), \\ A_1^{+,0}(0) = A_1^{+,0}(L) = 0, \quad A_1^{-,0}(0) = A_1^{-,0}(L) = 0, \end{cases} \tag{47}$$

where coefficients $\mathcal{F}_k(\Lambda_{k,0})$ and $\mathcal{G}_k$ are defined in (A.1).

Proof. For simplicity the proof is divided into four parts.

(i) Since $\omega_{k,-2}$ has to be unique, in the sequel we assume that $u_2^{\pm,0} = u_2^{m,0} = 0$. Thus the eigenvalue value problem corresponding to $i = 0$ in (10), when $p = -2$, reduces to:

$$\begin{aligned} &\mu\partial_{22}u_1^{\pm,0} + \rho\omega_{k,-2}^2 u_1^{\pm,0} = 0 \quad \text{in } \Omega^\pm, \\ &\mu^m\partial_{22}u_1^{m,0} + \rho\omega_{k,-2}^2 u_1^{m,0} = 0 \quad \text{in } \Omega^m, \\ &\mu\partial_2 u_1^{\pm,0}\left(x_1, \pm\frac{h}{2}\right) = 0, \\ &\mu\partial_2 u_1^{\pm,0}\left(x_1, \pm\frac{3h}{2}\right) = 0, \\ &u_1^{\pm,0}\left(x_1, \pm\frac{h}{2}\right) = u_1^{m,0}\left(x_1, \pm\frac{h}{2}\right). \end{aligned} \tag{48}$$

By solving this differential system one can find, respectively, the form of the natural frequency $\omega_{k,-2}$ and the form of the limit displacement field u_1^0 in this way:

$$\omega_{k,-2} := \frac{k\pi}{h}\sqrt{\frac{\mu}{\rho}}, \tag{49}$$

and,

- *Even k*

$$\begin{cases} u_1^{\pm,0}(x_1,x_2) = A_1^{\pm,0}(x_1)\cos\left(\frac{k\pi}{h}x_2\right), \\ u_1^{m,0}(x_1,x_2) = \frac{(-1)^{\frac{k}{2}}}{2}\left[\frac{(A_1^{+,0}+A_1^{-,0})(x_1)}{\cos\left(\alpha\frac{k\pi}{2}\right)}\cos\left(\alpha\frac{k\pi}{h}x_2\right)\right. \\ \qquad\qquad\left. + \frac{(A_1^{+,0}-A_1^{-,0})(x_1)}{\sin\left(\alpha\frac{k\pi}{2}\right)}\sin\left(\alpha\frac{k\pi}{h}x_2\right)\right], \end{cases} \tag{50}$$

- *Odd k*

$$\begin{cases} u_1^{\pm,0}(x_1,x_2) = B_1^{\pm,0}(x_1)\sin\left(\frac{k\pi}{h}x_2\right), \\ u_1^{m,0}(x_1,x_2) = \frac{(-1)^{\frac{k-1}{2}}}{2}\left[\frac{(B_1^{+,0}-B_1^{-,0})(x_1)}{\cos\left(\alpha\frac{k\pi}{2}\right)}\cos\left(\alpha\frac{k\pi}{h}x_2\right)\right. \\ \qquad\qquad\left. + \frac{(B_1^{+,0}+B_1^{-,0})(x_1)}{\sin\left(\alpha\frac{k\pi}{2}\right)}\sin\left(\alpha\frac{k\pi}{h}x_2\right)\right], \end{cases} \tag{51}$$

with $\alpha := \sqrt{\frac{\mu}{\mu^m}}$ and, for both, $\sin(\alpha k\pi) \neq 0$.

Functions $A_1^{\pm,0}$ and $B_1^{\pm,0}$ are unknown for the moment. Our main objective is to determine the form of these unknown functions and completely characterize the limit displacement field $\mathbf{u}^0$.

(ii) Let us characterize the frequency $\Lambda_{k,-1}$ by using the following equations for $i = 1$:

$$\begin{aligned} &\mu\partial_{22}u_1^{\pm,1} + \rho\omega_{k,-2}^2 u_1^{\pm,1} = -\rho\Lambda_{k,-1}u_1^{\pm,0} \quad \text{in } \Omega^\pm, \\ &\mu\partial_2 u_1^{\pm,1}\left(x_1, \pm\frac{h}{2}\right) = \mu^m\partial_2 u_1^{m,0}\left(x_1, \pm\frac{h}{2}\right), \\ &\mu\partial_2 u_1^{\pm,1}\left(x_1, \pm\frac{3h}{2}\right) = 0. \end{aligned} \tag{52}$$

Let us apply the solvability condition to problem (52) for, respectively, even and odd frequencies, defined over Ω^+ and Ω^-.

- *Even k*

The solvability conditions defined over Ω^+ and Ω^- imply that $A_1^{\pm,0}(x_1) \neq 0$ and give the following explicit expression of $\Lambda_{k,-1}$:

$$\begin{aligned} \Lambda_{k,-1} &= \frac{2k\pi\sqrt{\mu^m\mu}}{\rho h^2}\cot\left(\frac{k\pi}{2\alpha}\right), \\ \Lambda_{k,-1} &= -\frac{2k\pi\sqrt{\mu^m\mu}}{\rho h^2}\tan\left(\frac{k\pi}{2\alpha}\right). \end{aligned} \tag{53}$$

- *Odd k*

The solvability conditions for odd frequencies gives

$$B_1^{\pm,0}(x_1) = 0, \tag{54}$$

which implies that $u_1^{\pm,0} = u_1^{m,0} = 0$

Let us update the form of the displacement field $\mathbf{u}^0$ using the above results:

- *Even k*

$$\begin{cases} u_1^{\pm,0}(x_1,x_2) = A_1^{\pm,0}(x_1)\cos\left(\frac{k\pi}{h}x_2\right), \\ u_1^{m,0}(x_1,x_2) = \frac{(-1)^{\frac{k}{2}}}{2}\left[\frac{(A_1^{+,0}+A_1^{-,0})(x_1)}{\cos\left(\alpha\frac{k\pi}{2}\right)}\cos\left(\alpha\frac{k\pi}{h}x_2\right)\right. \\ \qquad\qquad\left. + \frac{(A_1^{+,0}-A_1^{-,0})(x_1)}{\sin\left(\alpha\frac{k\pi}{2}\right)}\sin\left(\alpha\frac{k\pi}{h}x_2\right)\right], \end{cases} \tag{55}$$

- *Odd k*

$$u_1^{\pm,0}(x_1,x_2) = u_1^{m,0}(x_1,x_2) = 0. \tag{56}$$

(iii) The knowledge of the natural frequency $\Lambda_{k,-1}$ allows to fully obtain the form of displacement field $\mathbf{u}^1$, as follows:

- *Even k*

$$\begin{cases} u_1^{\pm,1}(x_1,x_2) = A_1^{\pm,1}(x_1)\cos\left(\frac{k\pi}{h}x_2\right) + B_1^{\pm,1}(x_1)\sin\left(\frac{k\pi}{h}x_2\right) \\ \qquad - \frac{\rho h\Lambda_{k,-1}}{2k\pi\mu}A_1^{\pm,0}(x_1)x_2\sin\left(\frac{k\pi}{h}x_2\right), \\ u_1^{m,1}(x_1,x_2) = A_1^{m,1}(x_1)\cos\left(\alpha\frac{k\pi}{h}x_2\right) + B_1^{m,1}(x_1)\sin\left(\alpha\frac{k\pi}{h}x_2\right) \\ \qquad - \frac{(-1)^{\frac{k}{2}}\rho h\Lambda_{k,-1}}{4k\pi\sqrt{\mu^m\mu}}x_2\left[\frac{(A_1^{+,0}+A_1^{-,0})(x_1)}{\cos\left(\alpha\frac{k\pi}{2}\right)}\sin\left(\alpha\frac{k\pi}{h}x_2\right)\right. \\ \qquad\left. - \frac{(A_1^{+,0}-A_1^{-,0})(x_1)}{\sin\left(\alpha\frac{k\pi}{2}\right)}\cos\left(\alpha\frac{k\pi}{h}x_2\right)\right], \end{cases} \tag{57}$$

where $B_1^{\pm,1}$, $A_1^{m,1}$ and $B_1^{m,1}$ are known functions of $A_1^{\pm,0}$ and $A_1^{\pm,1}$.

- *Odd k*

$$\begin{cases} u_1^{\pm,1}(x_1,x_2) = C_1^{\pm,1}(x_1)\sin\left(\frac{k\pi}{h}x_2\right), \\ u_1^{m,1}(x_1,x_2) = C_1^{m,1}(x_1)\cos\left(\alpha\frac{k\pi}{h}x_2\right) + D_1^{m,1}(x_1)\sin\left(\alpha\frac{k\pi}{h}x_2\right), \end{cases} \tag{58}$$

where $C_1^{m,1}$ and $D_1^{m,1}$ are known functions of $B_1^{\pm,1}$.

The x_2-component of the displacement field $\mathbf{u}^1$ is obtained by integrating the following boundary value problem:

$$\begin{aligned} &\partial_{22}u_2^{\pm,1} + \frac{\rho\omega_{k,-2}^2}{\lambda+2\mu}u_2^{\pm,1} = -\frac{\lambda+\mu}{\lambda+2\mu}\partial_{12}u_1^{\pm,0} \quad \text{in } \Omega^\pm, \\ &(\lambda+2\mu)\partial_2 u_2^{\pm,1}\left(x_1, \pm\frac{h}{2}\right) = -\lambda\partial_1 u_1^{\pm,0}\left(x_1, \pm\frac{h}{2}\right), \\ &(\lambda+2\mu)\partial_2 u_2^{\pm,1}\left(x_1, \pm\frac{3h}{2}\right) = -\lambda\partial_1 u_1^{\pm,0}\left(x_1, \pm\frac{3h}{2}\right). \end{aligned} \tag{59}$$

The natural frequencies associated with problem (59) are $\omega_n^* = \frac{n\pi}{2h}$, with $n \in \mathbb{N}$. We assume that $\omega := \sqrt{\frac{\rho}{\lambda+2\mu}}\omega_{k,-2} = \sqrt{\frac{\mu}{\lambda+2\mu}}\frac{k\pi}{h} \neq \omega_n^*$, which implies that $k \in \mathbb{N}$ is such that $\sqrt{\frac{\mu}{\lambda+2\mu}}k \neq \frac{n}{2}$. Therefore ω is not a natural frequency for this problem.

As customary, by defining $\beta := \sqrt{\frac{\mu}{\lambda+2\mu}}$, we get:

- *Even k*

$$u_2^{\pm,1}(x_1,x_2) = A_2^{\pm,1}(x_1)\cos\left(\beta\frac{k\pi}{h}x_2\right) + B_2^{\pm,1}(x_1)\sin\left(\beta\frac{k\pi}{h}x_2\right) - \frac{h}{k\pi}A_1^{\pm,0\prime}(x_1)\sin\left(\frac{k\pi}{h}x_2\right), \tag{60}$$

with $A_2^{\pm,1}(x_1) = \mp c_k A_1^{\pm,0'}(x_1)$ and $B_2^{\pm,1}(x_1) = d_k A_1^{\pm,0'}(x_1)$. Coefficients c_k and d_k are shown in (A.1) in Appendix A.

- *Odd k*

$$u_2^{\pm,1}(x_1,x_2) = 0. \tag{61}$$

(iv) Let us consider the eigenvalue value problem for $i = 2$:

$$\begin{aligned} &\mu\partial_{22}u_1^{\pm,2} + \rho\omega_{k,-2}^2 u_1^{\pm,2} = -(\lambda+\mu)\partial_{12}u_2^{\pm,1} - (\lambda+2\mu)\partial_{11}u_1^{\pm,0} - \rho\Lambda_{k,-1}u_1^{\pm,1} - \rho\Lambda_{k,0}u_1^{\pm,0} \quad \text{in } \Omega^{\pm},\\ &\mu\partial_2 u_1^{\pm,2}\left(x_1,\pm\frac{h}{2}\right) = \left(-\mu\partial_1 u_2^{\pm,1} + \mu^m\partial_2 u_1^{m,1}\right)\left(x_1,\pm\frac{h}{2}\right),\\ &\mu\partial_2 u_1^{\pm,2}\left(x_1,\pm\frac{3h}{2}\right) = -\mu\partial_1 u_2^{\pm,1}\left(x_1,\pm\frac{h}{2}\right). \end{aligned} \tag{62}$$

Since $\omega_{k,-2}$ is a natural frequency, we need to apply the solvability condition, which allows us to determine functions $A_1^{\pm,0}$, and, hence, fully characterize the zeroth order displacement field. So we get:

$$\begin{cases} A_1^{+,0''}(x_1) - \mathcal{F}_k(\Lambda_{k,0})A_1^{+,0}(x_1) = \mathcal{G}_k A_1^{-,0}(x_1),\\ A_1^{-,0''}(x_1) - \mathcal{F}_k(\Lambda_{k,0})A_1^{-,0}(x_1) = \mathcal{G}_k A_1^{+,0}(x_1),\\ A_1^{+,0}(0) = A_1^{+,0}(L) = 0, \quad A_1^{-,0}(0) = A_1^{-,0}(L) = 0, \end{cases} \tag{63}$$

where coefficients $\mathcal{F}_k(\Lambda_{k,0})$ and $\mathcal{G}_k$ are listed in (A.1), in Appendix A. By solving system (63), we can characterize the second corrector term $\Lambda_{k,0}$, as follows:

$$\sinh\left(L\sqrt{\mathcal{F}_k(\Lambda_{k,0}) + \mathcal{G}_k}\right)\sinh\left(L\sqrt{\mathcal{F}_k(\Lambda_{k,0}) - \mathcal{G}_k}\right) = 0. \qquad \square \tag{64}$$

Remark 4. Coefficients $\sqrt{\mathcal{F}_k(\Lambda_{k,0}) \pm \mathcal{G}_k}$ could be either complex or real numbers, this changing significantly the behavior of functions $A_1^{\pm,0}$ and the values of natural frequencies $\Lambda_{k,0}$. As we already mentioned in the Introduction to this paper, this work is conceived to give a theoretical background and justification to classical mathematical models of the dynamics of laminated glass beams. If we take into account practical values of the adherents Lamé's coefficients λ and μ, and of the adhesive Lamé's coefficients λ^m and μ^m, deriving, for instance, from materials used in laminated glass beams (see, for instance, Aşik and Tezacan (2005) and De Belder et al. (2010)) we realize that $\sqrt{\mathcal{F}_k(\Lambda_{k,0}) \pm \mathcal{G}_k} \in \mathbb{C}$. Hence, one has:

$$\begin{aligned} A_1^{-,0}(x_1) &= \mathcal{C}_1\sin(\sqrt{\mathcal{F}_k(\Lambda_{k,0}) + \mathcal{G}_k}x_1) + \mathcal{C}_2\sin(\sqrt{\mathcal{F}_k(\Lambda_{k,0}) - \mathcal{G}_k}x_1),\\ A_1^{+,0}(x_1) &= \mathcal{C}_1\sin(\sqrt{\mathcal{F}_k(\Lambda_{k,0}) + \mathcal{G}_k}x_1) - \mathcal{C}_2\sin(\sqrt{\mathcal{F}_k(\Lambda_{k,0}) - \mathcal{G}_k}x_1). \end{aligned} \tag{65}$$

Besides,

$$\sin\left(L\sqrt{\mathcal{F}_k(\Lambda_{k,0}) + \mathcal{G}_k}\right)\sin(L\sqrt{\mathcal{F}_k(\Lambda_{k,0}) - \mathcal{G}_k}) = 0, \tag{66}$$

which permits to determine two categories of natural frequencies $\Lambda_{k,\ell,0}$ (with self-explanatory notation):

$$\begin{aligned} \Lambda_{k,\ell,0} &= \sqrt{\frac{2\mu}{\rho h}\left(\frac{\ell^2\pi^2\mathcal{M}_k}{L^2} - (\mathcal{K}_k + \mathcal{D}_k)\right)},\\ \Lambda_{k,\ell,0} &= \sqrt{\frac{2\mu}{\rho h}\left(\frac{\ell^2\pi^2\mathcal{M}_k}{L^2} - (\mathcal{K}_k - \mathcal{D}_k)\right)}. \end{aligned} \tag{67}$$

Therefore the zeroth order limit displacement field $\mathbf{u}^0$ for even k becomes:

$$\begin{cases} u_1^{\pm,0}(x_1,x_2) = \left[\mathcal{C}_1\sin(\sqrt{\mathcal{F}_k(\Lambda_{k,0}) + \mathcal{G}_k}x_1) \mp\mathcal{C}_2\sin(\sqrt{\mathcal{F}_k(\Lambda_{k,0}) - \mathcal{G}_k}x_1)\right]\cos\left(\frac{k\pi}{h}x_2\right),\\ u_1^{m,0}(x_1,x_2) = (-1)^{\frac{k}{2}}\left[\frac{\mathcal{C}_1\sin(\mathcal{A}_{k,\ell}x_1)}{\cos\left(\alpha\frac{k\pi}{2}\right)}\cos\left(\alpha\frac{k\pi}{h}x_2\right) - \frac{\mathcal{C}_2\sin(\mathcal{B}_{k,\ell}x_1)}{\sin\left(\alpha\frac{k\pi}{2}\right)}\sin\left(\alpha\frac{k\pi}{h}x_2\right)\right],\\ u_2^{\pm,0}(x_1,x_2) = u_2^{m,0}(x_1,x_2) = 0. \end{cases} \tag{68}$$

Remark 5. It is remarkable that, by virtue of this *high* frequencies asymptotic approach, we can formally derive a limit model for wave propagation in a generalized waveguide consisting of three layers with different rigidities without any *a priori* assumption, (see, for instance, Achenbach, 1973).

Besides, in the case of transversal shear frequencies, we can characterize not only the leading term $\omega_{k,-2}$, but also, respectively, the second and third corrector terms of the asymptotic expansion of natural frequencies, $\Lambda_{k,-1}$ and $\Lambda_{k,0}$. This implies a much better approximation of the natural frequencies $\omega_{k,\varepsilon}$ of the initial dynamical problem for a layered beam.

6.2. Pinching frequencies

The choice of $p = -2$ in the asymptotic expansion of the frequencies (9) induces a decoupling of the possible vibrations of the layered beam, respectively, the transversal shear and the pinching frequencies. Since the calculations to obtain the expressions of the limit displacement field $\mathbf{u}^0$ and the limit natural frequencies $\omega_{k,-2}$ for pinching vibrations are analogous to those developed in Section 6.1, in what follows we just summarize the main results.

Hence, the leading term $\mathbf{u}^0 := (\mathbf{u}^{+,0}, \mathbf{u}^{m,0}, \mathbf{u}^{-,0})$ of the asymptotic expansion (9) is such that.

- Even $k \in \mathbb{N}$

$$\begin{cases} u_1^{\pm,0}(x_1,x_2) = u_1^{m,0}(x_1,x_2) = 0,\\ u_2^{\pm,0}(x_1,x_2) = A_2^{\pm,0}(x_1)\cos\left(\frac{k\pi}{h}x_2\right),\\ u_2^{m,0}(x_1,x_2) = \frac{(-1)^{\frac{k}{2}}}{2}\left[\frac{(A_2^{+,0}+A_2^{-,0})(x_1)}{\cos\left(\widetilde{\alpha}\frac{k\pi}{2}\right)}\cos\left(\widetilde{\alpha}\frac{k\pi}{h}x_2\right) + \frac{(A_2^{+,0}-A_2^{-,0})(x_1)}{\sin\left(\widetilde{\alpha}\frac{k\pi}{2}\right)}\sin\left(\widetilde{\alpha}\frac{k\pi}{h}x_2\right)\right], \end{cases} \tag{69}$$

- Odd $k \in \mathbb{N}$

$$\mathbf{u}^{\pm,0}(x_1,x_2) = \mathbf{u}^{m,0}(x_1,x_2) = 0, \tag{70}$$

with $\widetilde{\alpha} := \sqrt{\frac{\lambda+2\mu}{\lambda^m+2\mu^m}}$, and the limit natural frequency takes the following form:

$$\omega_{k,-2} = \frac{k\pi}{h}\sqrt{\frac{\lambda+2\mu}{\rho}}. \tag{71}$$

Moreover, functions $A_2^{\pm,0}$ satisfy the following eigenvalue problem:

$$\begin{cases} A_2^{+,0''}(x_1) - \widetilde{\mathcal{F}}_k(\Lambda_{k,0})A_2^{+,0}(x_1) = \widetilde{\mathcal{G}}_k A_2^{-,0}(x_1),\\ A_2^{-,0''}(x_1) - \widetilde{\mathcal{F}}_k(\Lambda_{k,0})A_2^{-,0}(x_1) = \widetilde{\mathcal{G}}_k A_2^{+,0}(x_1),\\ A_2^{+,0}(0) = A_2^{+,0}(L) = 0, \quad A_2^{-,0}(0) = A_2^{-,0}(L) = 0, \end{cases} \tag{72}$$

where coefficients $\widetilde{\mathcal{F}}_k(\Lambda_{k,0})$ and $\widetilde{\mathcal{G}}_k$ are shown in Appendix A.

Even for pinching vibrations, we find a form of the first corrector term $\Lambda_{k,-1}$ of the natural frequency,

$$\Lambda_{k,-1} = \frac{2k\pi\sqrt{(\lambda^m + 2\mu^m)(\lambda + 2\mu)}}{\rho h^2}\cot\left(\frac{k\pi}{2\widetilde{\alpha}}\right),$$
$$\Lambda_{k,-1} = -\frac{2k\pi\sqrt{(\lambda^m + 2\mu^m)(\lambda + 2\mu)}}{\rho h^2}\tan\left(\frac{k\pi}{2\widetilde{\alpha}}\right). \tag{73}$$

and, the equation satisfied by the second corrector term $\Lambda_{k,0}$:

$$\sinh\left(L\sqrt{\widetilde{\mathcal{F}}_k(\Lambda_{k,0}) + \widetilde{\mathcal{G}}_k}\right)\sinh\left(L\sqrt{\widetilde{\mathcal{F}}_k(\Lambda_{k,0}) - \widetilde{\mathcal{G}}_k}\right) = 0. \tag{74}$$

7. A numerical example

This section is devoted to a numerical comparison between the theoretical limit natural frequencies obtained by means of the asymptotic expansions approach and the natural frequencies of a two-dimensional layered thin beam with soft adhesive. The numerical analysis is carried out by using a commercial finite element code.

The real problem is solved numerically by considering a two-dimensional three-layers beam, clamped at both ends. The implemented layered beam model preserves the same thickness and stiffness ratios between the adherents and the adhesive used in the asymptotic model, i.e., by defining a small parameter ε, we have:

$$h^{\pm,\varepsilon} = \varepsilon h, \quad h^{m,\varepsilon} = \varepsilon^2 h, \quad E^{\pm,\varepsilon} = E, \quad E^{m,\varepsilon} = \varepsilon^2 E^m. \tag{75}$$

The problem is numerically solved by decreasing values of ε. As starting point we choose the typical Young's moduli of a laminated structural glass beam. For the upper and lower layers, we set $E^{\pm,\varepsilon} = 7 \times 10^{10}$ N/m^2 and $\nu^{\pm} = 0.2$. For the middle layer we consider $E^{m,\varepsilon} = 9.5 \times 10^8$ N/m^2 and $\nu^m = 0.49$.

The comparison with the asymptotic analysis is performed in the three cases presented in the previous Sections, namely, the *low* bending frequencies, the *mean* axial frequencies, and, finally, the *high* transversal shear frequencies.

7.1. Low frequencies

Let us recall the value of the first bending eigen-frequency solution of (32):

$$f_{0,2} := \frac{\omega_{0,2}}{2\pi} = 22.373\frac{h}{2\pi L^2}\sqrt{\frac{E}{3\rho(1-\nu^2)}}. \tag{76}$$

At first we set $\varepsilon = 0.1$ and we let it decreases. We plot the first natural frequency $f_{0,2}$ over ε vs the thickness of the adherents ε in Fig. 2.

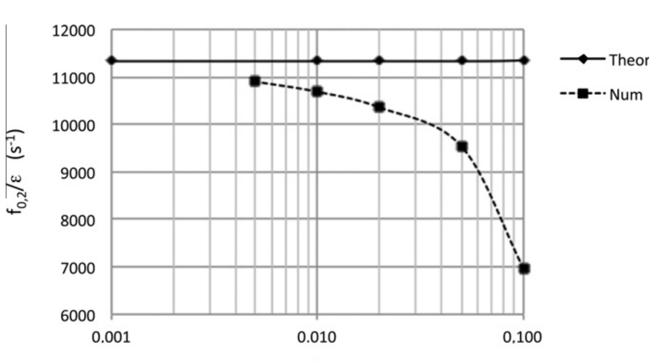

Fig. 2. First bending natural frequency.

The asymptotic development of the frequencies for $p = 2$,

$$\left(\frac{\omega_k(\varepsilon)}{\varepsilon}\right)^2 = \omega_{k,2}^2 + \varepsilon\Lambda_{k,3} + \varepsilon^2\Lambda_{k,4} + \cdots, \tag{77}$$

suggests that the ratio $\frac{\omega_k(\varepsilon)}{\varepsilon}$ converges to $\omega_{k,2}$, when ε tends to zero. The numerical results confirm this guess: indeed, the ratio $\frac{f_{0,2}}{\varepsilon}$ tends to the theoretical value as ε decreases.

Fig. 2 shows that the limit natural frequency we found is a good approximation of the natural frequency of the two-dimensional layered beam, when ε tends to zero. Indeed, in the last numerical simulation that we performed, with $\varepsilon = 0.005$, the relative error for the first natural bending frequency is of 3.8%. Below that value of ε, numerical instabilities and lack of convergence appeared due to thinness of the middle layer which entails a very difficult discretization. This fact confirms the importance of adopting a reduced model, which overcomes these numerical instabilities with a reliable approximation.

7.2. Mean frequencies

Let us recall the value of the first axial eigen-frequency:

$$f_{1,0} := \frac{\omega_{1,0}}{2\pi} = \frac{1}{2L}\sqrt{\frac{E}{\rho(1-\nu^2)}}. \tag{78}$$

We plot the first natural frequency $f_{1,0}$ vs the thickness of the adherents ε in Fig. 3.

From the asymptotic analysis, when $p = 0$, we have that

$$\omega_k^2(\varepsilon) = \omega_{k,0}^2 + \varepsilon\Lambda_{k,1} + \varepsilon^2\Lambda_{k,2} + \cdots \tag{79}$$

Hence, we expect that $\omega_k(\varepsilon)$ will tend to $\omega_{k,0}$, as ε tends to zero. As we can notice from Fig. 3, the numerical results confirm that the first axial natural frequency $f_{1,0}$ converges to the theoretical value when ε decreases. The relative error related to the last numerical test is less than 0.9%, for thin layered beam with slender ratio less

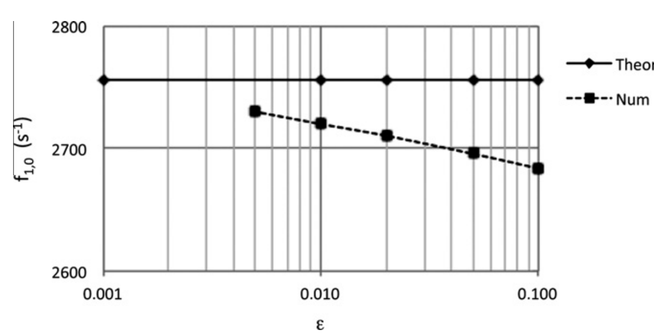

Fig. 3. First axial natural frequency.

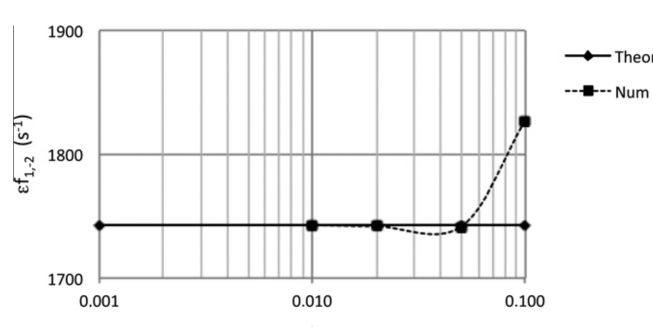

Fig. 4. First transversal shear natural frequency.

than 0.5%. Thus the first axial frequency predicted by the asymptotic analysis is a very good approximation of the "real" axial frequency.

7.3. High frequencies

Let us recall the value of the first transversal shear eigenfrequency:

$$f_{1,-2} := \frac{\omega_{1,-2}}{2\pi} = \frac{1}{2h}\sqrt{\frac{E}{2\rho(1+\nu)}}. \tag{80}$$

We plot the first natural frequency $f_{1,-2}$ multiplied by ε vs the thickness of the adherents ε in Fig. 4.

Since in the case $p=-2$ we have

$$(\varepsilon\omega_k(\varepsilon))^2 = \omega_{k,-2}^2 + \varepsilon\Lambda_{k,-1} + \varepsilon^2\Lambda_{k,0} + \cdots, \tag{81}$$

the quantity $\varepsilon\omega_k(\varepsilon)$ is expected to converge toward $\omega_{k,-2}$ when ε tends to zero. The comparison between the numerical and the theoretical values shows that $\varepsilon f_{1,-2}$ actually converges toward limit asymptotic value. In this particular case, with a thickness ratio of 0.5% the relative error becomes negligible (less than 0.01%).

8. Concluding remarks and further developments

A rational deduction of three different vibration models for a laminated beam has been carried out by means of the asymptotic expansion method. By varying $p \in \{-2, 0, 2\}$ in (9) we manage two study three different types of frequencies: the *low*, *mean* and *high* frequencies.

For what concerns *low* and *mean* frequencies, we characterize, respectively, the limit displacement field $\mathbf{u}^0$ and the limit natural frequencies $\omega_{k,2}$ and $\omega_{k,0}$. The *low* frequencies model corresponds to a beam subjected to free bending vibrations. The *mean* frequencies model corresponds to a beam subjected to free axial vibrations. For both cases we find that the presence of the adhesive is negligible, and, thus, the limit model corresponds to a Navier–Bernoulli-type monolithic beam. For further developments it will be interesting to find, at least, two corrector terms of the asymptotic expansion of the frequencies to better understand the behavior of the frequencies as a function of ε, i.e., the thickness of the layers.

Concerning the *high* frequencies models, we find two types of possible vibration modes: the transversal shear vibrations and the pinching vibrations. In this case the adhesive is perceived and we do not obtain a classical one-dimensional vibration model for a beam, but a vibration model in a layered two-dimensional waveguide in which the waves propagate in both directions. It is also interesting that we naturally deduced the expression of the natural frequency $\omega_{k,-2}$ and two corrector terms, namely, $\Lambda_{k,-1}$ and $\Lambda_{k,0}$.

The analytical deduction is supported by some numerical simulations obtained by the finite element method. It is shown that, by decreasing ε, i.e., the thicknesses of the layers and the stiffness of the adhesive, the numerical values of the first natural frequencies, in the three different cases, tend to the theoretical values, thus entailing a better approximation. The speed of convergence is larger with the *high* frequencies models and smaller with the *low* frequencies model.

As far as further developments are concerned, we believe that obtaining some 'formal' convergence results of the two-dimensional solution towards the solution of the simplified models for the three cases of study is worthy. This would justify even better the proposed limit models.

It would be also interesting to derive a simplified model for the dynamical behavior of a three layer beam with strong adhesive or with an adhesive with similar rigidity with respect to the adherents.

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Appendix A

The expressions of $\mathcal{F}_k(\Lambda_{k,0}), \mathcal{G}_k, \widetilde{\mathcal{F}}_k(\Lambda_{k,0})$ and $\widetilde{\mathcal{G}}_k$ take the following forms:

$$\begin{aligned}
\mathcal{F}_k(\Lambda_{k,0}) &:= \frac{\mathcal{K}_k + \mathcal{W}_{k,0}(\Lambda_{k,0})}{\mathcal{M}_k},\\
\mathcal{G}_k &:= \frac{\mathcal{D}_k}{\mathcal{M}_k},\\
\mathcal{W}_{k,0}(\Lambda_{k,0}) &:= \frac{\rho h \Lambda_{k,0}}{2\mu},\\
\mathcal{M}_k &:= 4(-1)^{\frac{k}{2}} d_k \sin\left(\beta\frac{k\pi}{2}\right)\cos(\beta k\pi) - \frac{h}{2},\\
\mathcal{K}_k &:= \frac{\rho h \Lambda_{k,-1}}{8\mu}\left[\tan^2\left(\alpha\frac{k\pi}{2}\right) + \cot^2\left(\alpha\frac{k\pi}{2}\right) + 2\right]\\
&\quad + \frac{\rho h \Lambda_{k,-1}}{4k\pi\mu\alpha}\left[\tan\left(\alpha\frac{k\pi}{2}\right) - \cot\left(\alpha\frac{k\pi}{2}\right)\right],\\
\mathcal{D}_k &:= \frac{\rho h \Lambda_{k,-1}}{8\mu}\left[\tan^2\left(\alpha\frac{k\pi}{2}\right) - \cot^2\left(\alpha\frac{k\pi}{2}\right)\right]\\
&\quad + \frac{\rho h \Lambda_{k,-1}}{4k\pi\mu\alpha}\left[\tan\left(\alpha\frac{k\pi}{2}\right) + \cot\left(\alpha\frac{k\pi}{2}\right)\right],\\
c_k &:= \frac{4(-1)^{\frac{k}{2}} h\beta}{k\pi}\sin\left(\beta\frac{k\pi}{2}\right),\\
d_k &:= \frac{2(-1)^{\frac{k}{2}} h\beta}{k\pi}\,\frac{1 - 2\sin\left(\beta\frac{k\pi}{2}\right)\sin\left(\beta\frac{3k\pi}{2}\right)}{\cos\left(\beta\frac{3k\pi}{2}\right)},
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
\widetilde{\mathcal{F}}_k(\Lambda_{k,0}) &:= \frac{\widetilde{\mathcal{K}}_k + \widetilde{\mathcal{W}}_{k,0}(\Lambda_{k,0})}{\widetilde{\mathcal{M}}_k},\\
\widetilde{\mathcal{G}}_k &:= \frac{\widetilde{\mathcal{D}}_k}{\widetilde{\mathcal{M}}_k},\\
\widetilde{\mathcal{W}}_{k,0}(\Lambda_{k,0}) &:= \frac{\rho h \Lambda_{k,0}}{2(\lambda+2\mu)},\\
\widetilde{\mathcal{M}}_k &:= \frac{4(-1)^{\frac{k}{2}}\mu}{\lambda+2\mu}\left[\tilde{c}_k \sin\left(\frac{k\pi}{2\beta}\right)\sin\left(\frac{k\pi}{\beta}\right) + \tilde{d}_k \sin\left(\frac{k\pi}{2\beta}\right)\cos\left(\frac{k\pi}{\beta}\right)\right]\\
&\quad - \frac{h}{2},\\
\widetilde{\mathcal{K}}_k &:= \frac{\rho h \Lambda_{k,-1}}{8(\lambda+2\mu)}\left[\tan^2\left(\tilde{\alpha}\frac{k\pi}{2}\right) + \cot^2\left(\tilde{\alpha}\frac{k\pi}{2}\right) + 2\right]\\
&\quad + \frac{\rho h \Lambda_{k,-1}}{4k\pi(\lambda+2\mu)\tilde{\alpha}}\left[\tan\left(\tilde{\alpha}\frac{k\pi}{2}\right) - \cot\left(\tilde{\alpha}\frac{k\pi}{2}\right)\right],\\
\widetilde{\mathcal{D}}_k &:= \frac{\rho h \Lambda_{k,-1}}{8(\lambda+2\mu)}\left[\tan^2\left(\tilde{\alpha}\frac{k\pi}{2}\right) - \cot^2\left(\tilde{\alpha}\frac{k\pi}{2}\right)\right]\\
&\quad + \frac{\rho h \Lambda_{k,-1}}{4k\pi(\lambda+2\mu)\tilde{\alpha}}\left[\tan\left(\tilde{\alpha}\frac{k\pi}{2}\right) + \cot\left(\tilde{\alpha}\frac{k\pi}{2}\right)\right],\\
\tilde{c}_k &:= \frac{4(-1)^{\frac{k}{2}} h\beta}{k\pi}\sin\left(\frac{k\pi}{2\beta}\right),\\
\tilde{d}_k &:= \frac{2(-1)^{\frac{k}{2}} h\beta}{k\pi}\,\frac{1 - 2\sin\left(\frac{k\pi}{2\beta}\right)\sin\left(\frac{3k\pi}{2\beta}\right)}{\cos\left(\frac{3k\pi}{2\beta}\right)}.
\end{aligned} \tag{A.2}$$

References

Åslund, J., 2005. Asymptotic analysis of adhesively bonded nonlinearly elastic plates. ZAMM – Z. Angew. Math. Mech. 85, 23–35.

Abdelmoula, R., Coutris, M., Marigo, J.-J., 1998. Comportement asymptotique d'une interphase élastique mince. C. R. Acad. Sci. Paris Sér. IIb 326, 237–242.

Achenbach, J.D., 1973. Wave propagation in elastic solids. North-Holland series in Applied Mathematics and Mechanics, Amsterdam.

Aşik, M.Z., Tezacan, S., 2005. A mathematical model for the behavior of laminated glass beams. Comp. Struct. 83, 1742–1753.

Avila-Pozos, O., Klarbring, A., Movchan, A.B., 1999. Asymptotic model of orthotropic highly inhomogeneous layered structures. Mechanics of Materials 31, 101–115.

Bessoud, A.-L., Krasucki, F., Serpilli, M., 2011. Asymptotic analysis of shell like inclusion with high rigidity. J. Elast. 103 (2), 153–172.

Ciarlet, P.G., 1997. Mathematical elasticity – Vol.II: Theory of plates. North-Holland, Amsterdam.

Ciarlet, P.G., Kesavan, S., 1981. Two-dimensional approximations of three-dimensional eigenvalue problems in plate theory. Comp. Meth. Appl. Mech. Eng. 26, 145–172.

Consolini, L., 2011. Structural glass between design, tests and models, Ph.D Thesis, Polytechnic University of Marche, Ancona, Italy.

Dauge, M., Djurdjevic, I., Faou, E., Rossle, A., 1999. Eigenmode asymptotics in thin elastic plates. J. Math. Pures. Appl. 78, 925–964.

Davet, J.L., 1987. Modélisation asymptotique en dynamique des structures élancées et en mécanique des matériaux composites – Chapitre 2: Un essai de classification des ondes dans le structures élancées. Thèse de Doctorat de l'Université Pierre et Marie Curie, Paris.

De Belder, K., Pintelon, R., Demol, C., Roose, P., 2010. Estimation of the equivalent complex modulus of laminated glass beams and its application to sound transmission loss prediction. Mech. Sys. Sign. Pro. 24, 809–822.

Geymonat, G., Krasucki, F., 1997. Analyse asymptotique du comportement en flexion de deux plaques collées. C.R. Acad. Sci. Paris Sér. IIb 325, 307–314.

Geymonat, G., Krasucki, F., Lenci, S., 1999. Mathematical Analysis of a bonded joint with a soft thin adhesive. Math. Mech. Solids 4, 201–225.

Irago, H., Viaño, J.M., 1998. Second-order asymptotic approximation of flexural vibrations in elastic rods. Math. Mod. Meth. Appl. Sci. 8, 1343–1362.

Irago, H., Kerdid, N., Viaño, J.M., 1998. Analyse asymptotique des modes de hautes fréquences dans les poutres minces. C.R. Acad. Sci. Paris, Série I 326, 1255–1260.

Klarbring, A., 1991. Derivation of a model of adhesively bonded joints by asymptotic expansion method. Int. J. Engrg. Sci. 29 (4), 493–512.

Klarbring, A., Movchan, A.B., 1998. Asymptotic modeling of adhesive joints. Mech. Mat. 28, 137–145.

Lebon, F., Rizzoni, R., 2010. Asymptotic analysis of a thin interface: the case of similar rigidity. Int. J. Eng. Sci. 48, 473–486.

Lebon, F., Rizzoni, R., 2011. Asymptotic behavior of a hard thin linear elastic interphase: an energy approach. Int. J. Solids Struct. 48, 441–449.

Lebon, F., Zaittouni, F., 2010. Asymptotic modelling of interfaces taking contact conditions into account: asymptotic expansions and numerical implementation. Int. J. Eng. Sci. 48, 111–127.

Lenci, S., 2000. A limit model in the analysis of steel-concrete beams. Math. Mod. Meth. Appl. Sci. 10, 1233–1250.

Licht, C., Michaille, G., 1997. A modeling of elastic adhesive bonded joints. Adv. Math. Sci. Appl. 7, 711–740.

Licht, C., Lebon, F., Léger, A., 2009. Dynamics of elastic bodies connected by a thin adhesive layer. Ultrasonic wave propagation in non homogeneous media. Springer Proc. Phys. 128, 99–110.

Serpilli, M., Lenci, S., 2008. Limit models in the analysis of three different layered elastic strips. Eur. J. Mech. A Solids 27, 247–268.

Timoshenko, S., Goodier, J.N., 1951. Theory of Elasticity. McGraw-Hill, New York.

Trabucho, L., Viaño, J.M., 1996. Mathematical modeling of rods. In: Ciarlet, P.G., Lions, J.L. (Eds.), Handbook of Numerical Analysis. Vol.IV. North-Holland, Amsterdam.

Trotter, H.F., 1958. Approximation of semi-groups operators. Pacific J. Math. 8, 887–919.

Zaittouni, F., Lebon, F., Licht, C., 2002. Étude théorique et numérique du comportement d'un assemblage de plaques. C.R. Acad. Sci. Paris 330 (5), 359–364.