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Continuity of magnetic Weyl calculus

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Abstract

We investigate continuity properties of the operators obtained by the magnetic Weyl calculus on nilpotent Lie groups, using modulation spaces associated with unitary representations of certain infinite-dimensional Lie groups.

Keywords: Weyl calculus; Magnetic field; Lie group; Modulation spaces

1. Introduction

There are three main themes that occur in the present paper:

- The pseudo-differential Weyl calculus that takes into account a magnetic field on $\mathbb{R}^n$, which has been recently developed by techniques of hard analysis, with motivation coming from quantum mechanics; some references in this connection include [24,22,25].
- The modulation spaces from the time-frequency analysis, which have become an increasingly useful tool in the classical pseudo-differential calculus on $\mathbb{R}^n$; see for instance the seminal papers [11] and [19].
- The theory of locally convex Lie groups and their representations, recently surveyed in [28]. See also [29].

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There is a huge literature devoted to various aspects of magnetic fields. From the point of view of the present paper, it is relevant to mention that some spectral properties of Schrödinger operators with magnetic fields were established by using representation theory for nilpotent Lie groups; see for instance [23,21,26]. However, the magnetic Weyl calculus has been rather recently developed. It gives a functional calculus for the operators of position and magnetic momentum in just the same way in which the classical Weyl calculus is an operator calculus for the positions and momenta, and its key feature is that it is gauge covariant. It follows by our earlier papers [1] and [3] that some of the very basic ideas of infinite-dimensional Lie theory prove to be very useful for understanding the aforementioned magnetic Weyl calculus as a Weyl quantization of a certain coadjoint orbit of a semi-direct product group $M = \mathcal{F} \rtimes \mathbb{R}^n$. Here $\mathcal{F}$ is a suitable translation-invariant space of smooth functions on $\mathbb{R}^n$ and the coadjoint orbit is associated with a natural unitary representation of M on $L^2(\mathbb{R}^n)$.

This representation theoretic approach to the magnetic Weyl calculus is further developed in the present paper by using the second of the themes mentioned above. Specifically, we introduce appropriate versions of modulation spaces and use them for describing the continuity properties of the magnetic pseudo-differential operators.

We recall from [1] that our approach to the magnetic Weyl calculus actually allows us to extend the constructions of [24] from the abelian group $(\mathbb{R}^n, +)$ to any simply connected nilpotent Lie group, and this will also be the setting of some of the main results of the present paper. However, the proofs are greatly helped by a more general framework that we develop, in the first sections of the paper, for the so-called localized Weyl calculus for representations of locally convex Lie groups that satisfy suitable smoothness conditions. In order to develop this abstract setting we provide infinite-dimensional extensions of some ideas and constructions related to irreducible representations of finite-dimensional nilpotent Lie groups, which we had developed in [2]. These extensions may also be interesting on their own, however their importance consists in pointing out that the magnetic Weyl calculus of [24] and the Weyl–Pedersen calculus initiated in [30] are merely different shapes of the same phenomenon.

We now briefly present the structure of the paper. The aim of Sections 2 and 3 is to give general conditions on representations of locally convex Lie groups that ensure good properties of a Weyl calculus and related objects, as Wigner distributions and modulation spaces. In fact, in this way we set up a rather general and systematic procedure for constructing spaces of symbols associated with a group representation and eventually proving continuity of the operators obtained by the Weyl calculus, and of the Weyl calculus itself. The main technical result of the paper could thus be considered the continuity property of the cross-Wigner distributions (Theorem 3.16). A special case of this procedure, that motivated the present paper, appeared in our earlier work [2] on Weyl–Pedersen calculus for irreducible representations of finite-dimensional nilpotent Lie groups. The developments in this paper allow us to treat the magnetic Weyl calculus as a particular case. In Section 4 we show that the conditions in Sections 2 and 3 are met in this case, and continuity/trace-class results are thus derived.

1.1. Notation

Throughout the paper we denote by $\mathcal{S}(\mathcal{V})$ the Schwartz space on a finite-dimensional real vector space $\mathcal{V}$. That is, $\mathcal{S}(\mathcal{V})$ is the set of all smooth functions that decay faster than any polynomial together with their partial derivatives of arbitrary order. Its topological dual—the space of tempered distributions on $\mathcal{V}$—is denoted by $\mathcal{S}'(\mathcal{V})$. We use the notation $\mathcal{C}^\infty_{\mathrm{pol}}(\mathcal{V})$ for the space of smooth functions that grow polynomially together with their partial derivatives of arbitrary order; the natural locally convex topology of this function space along with some of its special properties are discussed in [31].

For every complex vector space $\mathcal{Y}$ we denote by $\overline{\mathcal{Y}}$ the complex vector space defined by the conditions that $\mathcal{Y}$ and $\overline{\mathcal{Y}}$ have the same underlying real vector space, and the identity mapping $\mathcal{Y} \to \overline{\mathcal{Y}}$ is antilinear. If $\mathcal{Y}$ is a topological vector space, then $\mathcal{Y}'$ will always denote the weak topological dual of $\mathcal{Y}$, that is, the space of continuous linear functionals on $\mathcal{Y}$ endowed with the topology of uniform convergence on the compact subsets.

We shall always denote by $\cdot \widehat{\otimes} \cdot$ the completed projective tensor product of locally convex spaces and by $\cdot \bar{\otimes} \cdot$ the natural tensor product of Hilbert spaces. Our references for topological tensor products are [10,32,35].

We shall also use the convention that the Lie groups are denoted by upper case Latin letters and the Lie algebras are denoted by the corresponding lower case Gothic letters.

2. Smooth unitary representations of locally convex Lie groups

Let M be a locally convex Lie group with a smooth exponential mapping

$$\exp_M : \mathbf{L}(M) = \mathfrak{m} \to M$$

(see [28]). Assume that $\pi : M \to \mathcal{B}(\mathcal{H})$ is a unitary representation. We denote by $\mathcal{H}_\infty$ the space of *smooth vectors* for the representation π, that is,

$$\mathcal{H}_\infty := \{\phi \in \mathcal{H} \mid \pi(\cdot)\phi \in \mathcal{C}^\infty(M, \mathcal{H})\}.$$

We note that $\pi(M)\mathcal{H}_\infty = \mathcal{H}_\infty$ and, as proved in [27, Sect. IV], the derived representation $\mathrm{d}\pi : \mathfrak{m} \to \mathrm{End}(\mathcal{H}_\infty)$ is well defined and is given by

$$(\forall X \in \mathfrak{m})(\forall \phi \in \mathcal{H}_\infty) \quad \mathrm{d}\pi(X)\phi = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \pi\big(\exp_M(tX)\big)\phi.$$

Remark 2.1. If we denote by $\mathrm{U}(\mathfrak{m}_\mathbb{C})$ the universal enveloping algebra of the complexified Lie algebra $\mathfrak{m}_\mathbb{C}$, then the homomorphism of Lie algebras $\mathrm{d}\pi$ extends to a unique homomorphism of unital associative algebras $\mathrm{d}\pi : \mathrm{U}(\mathfrak{m}_\mathbb{C}) \to \mathrm{End}(\mathcal{H}_\infty)$. The space of smooth vectors $\mathcal{H}_\infty$ will always be considered endowed with the locally convex topology defined by the family of seminorms $\{p_u\}_{u \in \mathrm{U}(\mathfrak{m}_\mathbb{C})}$, where for every $u \in \mathrm{U}(\mathfrak{m}_\mathbb{C})$ we define

$$p_u : \mathcal{H}_\infty \to [0, \infty), \quad p_u(\phi) = \|\mathrm{d}\pi(u)\phi\|.$$

The inclusion mapping $\mathcal{H}_\infty \hookrightarrow \mathcal{H}$ is continuous and, for all $u \in \mathrm{U}(\mathfrak{m}_\mathbb{C})$ and $m \in M$, the linear operators $\mathrm{d}\pi(u) : \mathcal{H}_\infty \to \mathcal{H}_\infty$ and $\pi(m) : \mathcal{H}_\infty \to \mathcal{H}_\infty$ are continuous as well.

Definition 2.2. Assume the above setting.

If the linear subspace of smooth vectors $\mathcal{H}_\infty$ is dense in $\mathcal{H}$, then the unitary representation $\pi : M \to \mathcal{B}(\mathcal{H})$ is said to be *smooth*. If this is the case, then π is necessarily continuous, in the sense that the group action $M \times \mathcal{H} \to \mathcal{H}$, $(m, f) \mapsto \pi(m)f$, is continuous.

The representation π is said to be *nuclearly smooth* if the following conditions are satisfied:

(1) π is a smooth representation;
(2) $\mathcal{H}_\infty$ is a nuclear Fréchet space;
(3) both mappings $M \times \mathcal{H}_\infty \to \mathcal{H}_\infty$, $(m,\phi) \mapsto \pi(m)\phi$, and $\mathfrak{m} \times \mathcal{H}_\infty \to \mathcal{H}_\infty$, $(X,\phi) \mapsto \mathrm{d}\pi(X)\phi$ are continuous.

Let $\mathcal{B}(\mathcal{H})_\infty$ be the space of smooth vectors for the unitary representation

$$\pi \otimes \bar{\pi} : M \times M \to \mathcal{B}\big(\mathfrak{S}_2(\mathcal{H})\big), \quad (\pi \otimes \bar{\pi})(m_1, m_2)T = \pi(m_1)T\pi(m_2)^{-1}.$$

We shall say that the representation $\pi : M \to \mathcal{B}(\mathcal{H})$ is *twice nuclearly smooth* if it satisfies the following conditions:

(1) The representation π is nuclearly smooth.
(2) There exists the commutative diagram

$$\begin{array}{ccc} \mathcal{H}_\infty \widehat{\otimes} \overline{\mathcal{H}_\infty} & \hookrightarrow & \mathcal{H} \bar{\otimes} \overline{\mathcal{H}} \\ \downarrow & & \downarrow \\ \mathcal{B}(\mathcal{H})_\infty & \hookrightarrow & \mathfrak{S}_2(\mathcal{H}) \end{array} \tag{2.1}$$

where the vertical arrow on the left is a linear topological isomorphism, while the vertical arrow on the right is the natural unitary operator defined by $(\phi_1, \phi_2) \mapsto \phi_1 \bar{\otimes} \bar{\phi}_2 := (\cdot \mid \phi_2)\phi_1$.

Remark 2.3. Note that there can exist at most one Fréchet topology on $\mathcal{H}_\infty$ such that the inclusion $\mathcal{H}_\infty \hookrightarrow \mathcal{H}$ be continuous, as a direct consequence of the closed graph theorem.

Remark 2.4. Let π be a smooth representation and denote by $\mathcal{H}_{-\infty}$ the strong dual of $\overline{\mathcal{H}_\infty}$. Equivalently, $\mathcal{H}_{-\infty}$ can be described as the space of continuous antilinear functionals on $\mathcal{H}_\infty$ endowed with the topology of uniform convergence on the bounded subsets of $\mathcal{H}_\infty$. Then there exist the dense embeddings

$$\mathcal{H}_\infty \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{-\infty},$$

and the duality pairing $(\cdot \mid \cdot) : \mathcal{H}_{-\infty} \times \mathcal{H}_\infty \to \mathbb{C}$ extends the scalar product of $\mathcal{H}$.

Proposition 2.5. *If the unitary representation $\pi : M \to \mathcal{B}(\mathcal{H})$ is twice nuclearly smooth, then it also has the following properties*:

(1) *The representation $\pi \otimes \bar{\pi} : M \times M \to \mathcal{B}(\mathfrak{S}_2(\mathcal{H}))$ is nuclearly smooth.*
(2) *We have $\mathcal{L}(\mathcal{H}_{-\infty}, \mathcal{H}_\infty) \simeq \mathcal{B}(\mathcal{H})_\infty \hookrightarrow \mathfrak{S}_1(\mathcal{H})$ and there exists the commutative diagram*

$$\begin{array}{ccc} \mathcal{B}(\mathcal{H}) & \hookrightarrow & \mathcal{L}(\mathcal{H}_\infty, \mathcal{H}_{-\infty}) \\ \downarrow & & \downarrow \\ \mathfrak{S}_1(\mathcal{H})' & \hookrightarrow & \mathcal{L}(\mathcal{H}_{-\infty}, \mathcal{H}_\infty)' \end{array}$$

where the vertical arrow on the left is the natural linear topological isomorphism defined by the trace duality, and the vertical arrow on the right is also a linear topological isomorphism.

Proof. (1) The representation π is twice nuclearly smooth, hence $\mathcal{H}_\infty$ is a nuclear Fréchet space and $\mathcal{H}_\infty \widehat{\otimes} \overline{\mathcal{H}_\infty} \simeq \mathcal{B}(\mathcal{H})_\infty$. Then $\mathcal{B}(\mathcal{H})_\infty$ is in turn a nuclear Fréchet space (see for instance [35, Props. 50.1 and 50.6]). Moreover, since $\mathcal{H}_\infty$ is dense in $\mathcal{H}$, it follows that $\mathcal{B}(\mathcal{H})_\infty$ is dense in $\mathfrak{S}_2(\mathcal{H})$. To complete the proof of the fact that $\pi \otimes \bar{\pi}$ is twice nuclearly smooth, we still have to check that the mappings

$$M \times M \times \mathcal{B}(\mathcal{H})_\infty \to \mathcal{B}(\mathcal{H})_\infty, \quad (m_1, m_2, T) \mapsto \pi(m_1) T \pi(m_2)^{-1}$$

and

$$\mathfrak{m} \times \mathfrak{m} \times \mathcal{B}(\mathcal{H})_\infty \to \mathcal{B}(\mathcal{H})_\infty, \quad (X_1, X_2, T) \mapsto \mathrm{d}\pi(X_1) T - T\, \mathrm{d}\pi(X_2)$$

are continuous. To this end use again the fact that $\mathcal{H}_\infty \widehat{\otimes} \overline{\mathcal{H}_\infty} \simeq \mathcal{B}(\mathcal{H})_\infty$ and both mappings $M \times \mathcal{H}_\infty \to \mathcal{H}_\infty$, $(m, \phi) \mapsto \pi(m)\phi$, and $\mathfrak{m} \times \mathcal{H}_\infty \to \mathcal{H}_\infty$, $(X, \phi) \mapsto \mathrm{d}\pi(X)\phi$ are continuous.

(2) Since $\mathcal{H}_\infty$ is a nuclear Fréchet space, we get

$$\mathcal{L}(\mathcal{H}_{-\infty}, \mathcal{H}_\infty) = \mathcal{L}\big(\overline{\mathcal{H}_\infty}', \mathcal{H}_\infty\big) \simeq \mathcal{H}_\infty \widehat{\otimes} \overline{\mathcal{H}_\infty} \simeq \mathcal{B}(\mathcal{H})_\infty$$

(see [35, Eq. (50.17)]).

Moreover, for every $T \in \mathcal{B}(\mathcal{H})_\infty$ we have $T\mathcal{H} \subseteq \mathcal{H}_\infty$. Therefore one can prove (as in [4], for instance) that $\mathcal{B}(\mathcal{H})_\infty \subseteq \mathfrak{S}_1(\mathcal{H})$. Moreover, by considering the duals of the above topological linear isomorphisms, we get

$$\mathcal{L}(\mathcal{H}_{-\infty}, \mathcal{H}_\infty)' \simeq (\mathcal{H}_\infty \widehat{\otimes} \overline{\mathcal{H}_\infty})' \simeq \mathcal{L}\big(\mathcal{H}_\infty, \overline{\mathcal{H}_\infty}'\big) \simeq \mathcal{L}(\mathcal{H}_\infty, \mathcal{H}_{-\infty})$$

(see [35, Eqs. (50.19) and (50.16)]), and these isomorphisms agree with the isomorphism $\mathfrak{S}_1(\mathcal{H})' \simeq \mathcal{B}(\mathcal{H})$ in the sense of the commutative diagram in the statement. □

Remark 2.6. For every $f_1, f_2 \in \mathcal{H}$ we denote by $f_1 \otimes \bar{f_2} \in \mathcal{B}(\mathcal{H})$ the rank-one operator $f \mapsto (f \mid f_2) f_1$. If the representation $\pi \otimes \bar{\pi}$ is twice nuclearly smooth, then for any $f_1, f_2 \in \mathcal{H}_{-\infty}$ we can use Proposition 2.5 to define the continuous antilinear functional $f_1 \otimes \bar{f_2} : \mathcal{B}(\mathcal{H})_\infty \to \mathbb{C}$ by $(f_1 \otimes \bar{f_2})(T) = (f_1 \mid T f_2)$ for every $T \in \mathcal{B}(\mathcal{H})_\infty$.

2.1. Group square

Definition 2.7. The *group square* of M, denoted by $M \ltimes M$, is the semi-direct product defined by the action of M on itself by inner automorphisms. That is, $M \ltimes M$ is a locally convex Lie group whose underlying manifold is $M \times M$ and the group operation is

$$(m_1, m_2)(n_1, n_2) = \big(m_1 n_1, n_1^{-1} m_2 n_1 n_2\big)$$

for all $m_1, m_2, n_1, n_2 \in M$.

Lemma 2.8. *The following assertions hold*:

(1) *The mapping*

$$\mu : M \ltimes M \to M \times M, \quad (m_1, m_2) \mapsto (m_1 m_2, m_1)$$

is an isomorphism of Lie groups with tangent map

$$\mathbf{L}(\mu) : \mathfrak{m} \ltimes \mathfrak{m} \to \mathfrak{m} \times \mathfrak{m}, \quad (X, Y) \mapsto (X + Y, X).$$

(2) *The Lie group* $M \ltimes M$ *has a smooth exponential map*

$$\exp_{M \ltimes M} : \mathfrak{m} \ltimes \mathfrak{m} \to M \ltimes M, \quad (X, Y) \mapsto \big(\exp_M X, \exp_M(-X)\exp_M(X + Y)\big).$$

Proof. The arguments of Ex. 2.3 in [2] carry over to the present setting. □

Definition 2.9. We introduce the continuous unitary representation

$$\pi^{\ltimes} : M \ltimes M \to \mathcal{B}\big(\mathfrak{S}_2(\mathcal{H})\big), \quad \pi^{\ltimes}(m_1, m_2)T = \pi(m_1 m_2)T\pi(m_1)^{-1}.$$

To see that $\pi^{\ltimes}$ is a representation, one can use a direct computation or the fact that so is $\pi \otimes \bar{\pi}$ and we have

$$\pi^{\ltimes} = (\pi \otimes \bar{\pi}) \circ \mu, \tag{2.2}$$

where $\mu : M \ltimes M \to M \times M$ is the group isomorphism of Lemma 2.8.

3. Localized Weyl calculus and modulation spaces

The localized Weyl calculus (see Definition 3.10 below) was introduced in [1] as a tool for dealing with the magnetic Weyl calculus on nilpotent Lie groups. In the present section we further develop that circle of ideas by introducing the modulation spaces and extending some related techniques of [2] to the general framework provided by the localized Weyl calculus for representations of infinite-dimensional Lie groups.

Here we single out fairly general conditions that allow for a Weyl calculus to be defined, modulation spaces to be considered and continuity properties in these spaces to hold as in the classical time-frequency analysis; see [11,17,18] and the references therein. All of these conditions are satisfied in at least two important situations: the Weyl–Pedersen calculus for irreducible representations of finite-dimensional nilpotent Lie groups (see [2]) and the magnetic Weyl calculus of [1] to be treated in the last section.

3.1. Ambiguity functions and Wigner distributions

Setting 3.1. Throughout this section we keep the following notation:

(1) M is a locally convex Lie group (see [28]) with a smooth exponential mapping $\exp_M : \mathbf{L}(M) = \mathfrak{m} \to M$.

(2) $\pi : M \to \mathcal{B}(\mathcal{H})$ is a nuclearly smooth unitary representation.
(3) Ξ and Ξ^* are real finite-dimensional vector spaces with a duality pairing $\langle\cdot,\cdot\rangle : \Xi^* \times \Xi \to \mathbb{R}$ and with Lebesgue measures on Ξ and Ξ^* suitably normalized for the Fourier transform

$$\widehat{\cdot} : L^1(\Xi) \to L^\infty(\Xi^*), \quad b(\cdot) \mapsto \widehat{b}(\cdot) = \int_{\Xi} \mathrm{e}^{-\mathrm{i}\langle\cdot,x\rangle} b(x)\,\mathrm{d}x$$

to give a unitary operator $L^2(\Xi) \to L^2(\Xi^*)$. The inverse of this transform will be denoted by $a \mapsto \check{a}$.

Definition 3.2. Let $\theta : \Xi \to \mathfrak{m}$ be a linear mapping.

(a) *Orthogonality relations.* If either $\phi \in \mathcal{H}_\infty$ and $f \in \mathcal{H}_{-\infty}$, or $\phi, f \in \mathcal{H}$, then we define the *ambiguity function* along the mapping θ,

$$\mathcal{A}^{\pi,\theta}_\phi f : \Xi \to \mathbb{C}, \quad \big(\mathcal{A}^{\pi,\theta}_\phi f\big)(\cdot) = \big(f \mid \pi\big(\exp_M\big(\theta(\cdot)\big)\big)\phi\big).$$

Note that this is a continuous function on Ξ. We say that the representation π satisfies the *orthogonality relations* along the mapping θ if

$$\big(\mathcal{A}^{\pi,\theta}_{\phi_1} f_1 \mid \mathcal{A}^{\pi,\theta}_{\phi_2} f_2\big)_{L^2(\Xi)} = (f_1 \mid f_2)_{\mathcal{H}} \cdot (\phi_2 \mid \phi_1)_{\mathcal{H}} \tag{3.1}$$

for arbitrary $\phi_1, \phi_2, f_1, f_2 \in \mathcal{H}$. In particular, $\mathcal{A}^{\pi,\theta}_\phi f \in L^2(\Xi)$ for all $\phi, f \in \mathcal{H}$.

(b) *Modulation spaces.* Consider any direct sum decomposition $\Xi = \Xi_1 \dotplus \Xi_2$ and $r, s \in [1,\infty]$. For arbitrary $f \in \mathcal{H}_{-\infty}$ define

$$\|f\|_{M^{r,s}_\phi(\pi,\theta)} = \bigg(\int_{\Xi_2}\bigg(\int_{\Xi_1}\big|\big(\mathcal{A}^{\pi,\theta}_\phi f\big)(X_1,X_2)\big|^r\,\mathrm{d}X_1\bigg)^{s/r}\mathrm{d}X_2\bigg)^{1/s} \in [0,\infty]$$

with the usual conventions if r or s is infinite. The space

$$M^{r,s}_\phi(\pi,\theta) := \big\{f \in \mathcal{H}_{-\infty} \bigm| \|f\|_{M^{r,s}_\phi(\pi,\theta)} < \infty\big\}$$

is called a *modulation space* for the unitary representation $\pi : M \to \mathcal{B}(\mathcal{H})$ with respect to the linear mapping $\theta : \Xi \to \mathfrak{m}$, the decomposition $\Xi \simeq \Xi_1 \times \Xi_2$, and the *window vector* $\phi \in \mathcal{H}_\infty \setminus \{0\}$.

In connection with the above definition, we note that more general “co-orbit spaces” $\mathcal{X}_\phi(\pi,\theta)$ can be defined in $\mathcal{H}_{-\infty}$ by using any Banach space $\mathcal{X}$ of functions on Ξ instead of the mixed-norm Lebesgue spaces $L^{r,s}(\Xi_1 \times \Xi_2)$. More specifically, one can define for any window vector $\phi \in \mathcal{H}_\infty$,

$$\mathcal{X}_\phi(\pi,\theta) = \big\{f \in \mathcal{H}_{-\infty} \bigm| \mathcal{A}^{\pi,\theta}_\phi f \in \mathcal{X}\big\}.$$

A systematic investigation of these spaces can be done in a broader context (see [5]). However, the modulation spaces $M^{r,s}_\phi(\pi,\theta)$ introduced in Definition 3.2 above will suffice for the purposes of the present paper. See [12–14] for these constructions in the case of representations of locally compact groups.

There could be two sources for the intuition underlying the direct sum decomposition $\Xi = \Xi_1 + \Xi_2$: Firstly, the spaces of symbols are associated to a representation of the group $G \ltimes G$, which gives rise to such a decomposition with $\Xi_1 = \Xi_2$ is a linear subspace of the Lie algebra of G; this is the case in both examples in the paper. Secondly, there is the case of the modulation spaces of functions on which the operators act, and which are defined in terms of the representation of the group G. In this case the decomposition $\Xi = \Xi_1 + \Xi_2$ corresponds to canonical coordinates for the symplectic structure on the coadjoint orbit associated with the representation. This is the phase space decomposition, on which we did not focus in the present paper; see however [1] for some more details on the coadjoint orbits relevant for the magnetic case.

Another natural question concerns the independence of the modulation spaces on the choice of a window vector. We have discussed this issue in [2, Subsect. 3.1] for square-integrable representations of nilpotent Lie groups, which covers the case of time-frequency analysis. So far we have no such result in any other case.

Remark 3.3. If the representation π satisfies the orthogonality relations along the linear mapping $\theta : \Xi \to \mathfrak{m}$, then for any decomposition $\Xi = \Xi_1 \dot{+} \Xi_2$ and any choice of the window vector $\phi \in \mathcal{H}_\infty \setminus \{0\}$, we have $M^{2,2}_\phi(\pi, \theta) = \mathcal{H}$.

Remark 3.4. Let $V : \mathcal{H} \to \mathcal{H}_1$ be a unitary operator and consider the unitary representation $\pi_1 : M \to \mathcal{B}(\mathcal{H}_1)$ such that $V\pi(m) = \pi_1(m)V$ for every $m \in M$. Denote by $\mathcal{H}_{1,\infty}$ the space of smooth vectors for π_1 and let $\mathcal{H}_{1,-\infty}$ be the strong dual of $\overline{\mathcal{H}_{1,\infty}}$. Then there exist the linear topological isomorphisms $V|_{\mathcal{H}_\infty} : \mathcal{H}_\infty \to \mathcal{H}_{1,\infty}$ and $V_{-\infty} : \mathcal{H}_{-\infty} \to \mathcal{H}_{1,-\infty}$, where $V_{-\infty} f = f \circ V^*|_{\mathcal{H}_{1,\infty}}$ for every $f \in \mathcal{H}_{-\infty}$. It is easy to check that for every linear mapping $\theta : \Xi \to \mathfrak{m}$ and arbitrary $\phi \in \mathcal{H}_\infty$ and $f \in \mathcal{H}_{-\infty}$ we have $\mathcal{A}^{\pi,\theta}_\phi f = \mathcal{A}^{\pi_1,\theta}_{V\phi}(V_{-\infty} f)$. Therefore $V_{-\infty}$ naturally gives rise to isometric isomorphisms from the modulation spaces of the representation π onto the corresponding modulation spaces of the representation π_1.

Definition 3.5 *(Growth condition).* We say that the representation π satisfies the *growth condition* along the linear mapping $\theta : \Xi \to \mathfrak{m}$ if

$$\mathcal{A}^{\pi,\theta}_{\phi_2}\phi_1 \in \mathcal{S}(\Xi), \quad \text{for all } \phi_1, \phi_2 \in \mathcal{H}_\infty. \tag{3.2}$$

Note that (3.2) implies that the sesquilinear map

$$\mathcal{A}^{\pi,\theta} : \mathcal{H}_\infty \times \mathcal{H}_\infty \to \mathcal{S}(\Xi), \quad (\phi_1, \phi_2) \mapsto \mathcal{A}^{\pi,\theta}_{\phi_2}\phi_1$$

is separately continuous as a straightforward application of the closed graph theorem, and then it is jointly continuous by [32, Cor. 1 to Th. 5.1 in Ch. III].

If the representation π satisfies the orthogonality relations along the mapping θ, and $\phi, f \in \mathcal{H}$, then $\mathcal{A}^{\pi,\theta}_\phi f \in L^2(\Xi)$, hence we can define the *cross-Wigner distribution* $\mathcal{W}(f, \phi) \in L^2(\Xi^*)$ by the condition $\widehat{\mathcal{W}(f, \phi)} := \mathcal{A}^{\pi,\theta}_\phi f$.

Definition 3.6 *(Density condition).* The representation π is said to satisfy the *density condition* along the linear mapping $\theta : \Xi \to \mathfrak{m}$ if $\{\mathcal{A}^{\pi,\theta}_\phi f \mid \phi, f \in \mathcal{H}\}$ is a total subset of $L^2(\Xi)$, in the sense that it spans a dense linear subspace.

Remark 3.7. If the representation π satisfies the orthogonality relations along θ, then it follows in particular that $\{\mathcal{A}^{\pi,\theta}_{\phi} f \mid \phi, f \in \mathcal{H}\} \subseteq L^2(\Xi)$, however it is not clear in general that this subset of $L^2(\Xi)$ is total. Similarly, if π satisfies the growth condition along θ, then $\{\mathcal{A}^{\pi,\theta}_{\phi} f \mid \phi, f \in \mathcal{H}_\infty\} \subseteq \mathcal{S}(\Xi) \subseteq L^2(\Xi)$, however in this way we may not get a total subset of $L^2(\Xi)$.

Lemma 3.8. *If the representation π satisfies the orthogonality relations along the linear mapping $\theta : \Xi \to \mathfrak{m}$, then the following assertions hold*:

(1) *The representation $\pi \otimes \bar{\pi}$ satisfies the orthogonality relations along the linear mapping $\theta \times \theta : \Xi \times \Xi \to \mathfrak{m} \times \mathfrak{m}$.*
(2) *The representation $\pi^{\ltimes}$ satisfies the orthogonality relations along each of the linear mappings $\mathbf{L}(\mu)^{-1} \circ (\theta \times \theta) : \Xi \times \Xi \to \mathfrak{m} \ltimes \mathfrak{m}$ and $\theta \times \theta : \Xi \times \Xi \to \mathfrak{m} \ltimes \mathfrak{m}$.*

Proof. To see that assertion (1) holds, first prove the orthogonality relations for rank-one operators in $\mathfrak{S}_2(\mathcal{H})$, then extend them by sesquilinearity to the finite-rank operators, and eventually extend them by continuity to arbitrary Hilbert–Schmidt operators. Then assertion (2) on $\mathbf{L}(\mu) \circ (\theta \times \theta)$ follows by assertion (1) along with Eq. (2.2).

Then, to see that also the representation $\pi^{\ltimes}$ satisfies the orthogonality relations along the mapping $\theta \times \theta : \Xi \times \Xi \to \mathfrak{m} \ltimes \mathfrak{m}$, just note that

$$\big(\mathbf{L}(\mu)^{-1} \circ (\theta \times \theta)\big)(X, Y) = \big(\theta(Y), \theta(X) - \theta(Y)\big) = (\theta \times \theta)(Y, X - Y)$$

and the linear mapping $\Xi \times \Xi \to \Xi \times \Xi$, $(X, Y) \mapsto (Y, X - Y)$, has the Jacobian identically equal to 1. □

Lemma 3.9. *If the representation π satisfies the growth condition along the linear mapping $\theta : \Xi \to \mathfrak{m}$, then the following assertions hold*:

(1) *The representation $\pi \otimes \bar{\pi}$ satisfies the growth condition along the linear mapping $\theta \times \theta : \Xi \times \Xi \to \mathfrak{m} \times \mathfrak{m}$.*
(2) *The representation $\pi^{\ltimes}$ satisfies the growth condition along each of the linear mappings $\mathbf{L}(\mu)^{-1} \circ (\theta \times \theta) : \Xi \times \Xi \to \mathfrak{m} \ltimes \mathfrak{m}$ and $\theta \times \theta : \Xi \times \Xi \to \mathfrak{m} \ltimes \mathfrak{m}$.*

Proof. The growth condition for the representation π along θ implies that the bilinear map $\mathcal{A}^{\pi,\theta} : \mathcal{H}_\infty \times \overline{\mathcal{H}_\infty} \to \mathcal{S}(\Xi)$ is continuous, hence extends to a continuous linear map

$$\mathcal{A}^{\pi,\theta} : \mathcal{H}_\infty \mathbin{\widehat{\otimes}} \overline{\mathcal{H}_\infty} \to \mathcal{S}(\Xi).$$

By complex conjugation we also have

$$\overline{\mathcal{A}^{\pi,\theta}} : \overline{\mathcal{H}_\infty \mathbin{\widehat{\otimes}} \overline{\mathcal{H}_\infty}} = \overline{\mathcal{H}_\infty} \mathbin{\widehat{\otimes}} \mathcal{H}_\infty \to \mathcal{S}(\Xi).$$

Thus we get the continuous mapping

$$\mathcal{A}^{\pi,\theta} \mathbin{\widehat{\otimes}} \overline{\mathcal{A}^{\pi,\theta}} : \mathcal{H}_\infty \mathbin{\widehat{\otimes}} \overline{\mathcal{H}_\infty} \mathbin{\widehat{\otimes}} \overline{\mathcal{H}_\infty} \mathbin{\widehat{\otimes}} \mathcal{H}_\infty \to \mathcal{S}(\Xi) \mathbin{\widehat{\otimes}} \mathcal{S}(\Xi) = \mathcal{S}(\Xi \times \Xi).$$

By composing this with the permutation $(f_1, \phi_1, f_2, \phi_2) \mapsto (f_1, f_2, \phi_1, \phi_2)$ and using the isomorphism $\mathcal{H}_\infty \widehat{\otimes} \overline{\mathcal{H}_\infty} \simeq \mathcal{B}(\mathcal{H})_\infty$, we get a continuous operator $\mathcal{B}(\mathcal{H})_\infty \widehat{\otimes} \overline{\mathcal{B}(\mathcal{H})_\infty} \to \mathcal{S}(\Xi \times \Xi)$ which extends $\mathcal{A}^{\pi\otimes\bar{\pi},\theta\times\theta}$, since

$$\mathcal{A}^{\pi\otimes\bar{\pi},\theta\times\theta}_{\phi_1\otimes\bar{\phi}_2}(f_1 \otimes \bar{f}_2) = \mathcal{A}^{\pi,\theta}_{\phi_1} f_1 \otimes \overline{\mathcal{A}^{\pi,\theta}_{\phi_2} f_2}.$$

The second part in the growth condition can be checked similarly, by using that $\mathcal{H}_{-\infty}$ is nuclear, like $\mathcal{H}_\infty$ (see [32, Ch. IV, Th. 9.6]), and noting that $\mathcal{H}_{-\infty} \widehat{\otimes} \overline{\mathcal{H}_{-\infty}} \simeq (\overline{\mathcal{H}_\infty} \widehat{\otimes} \mathcal{H}_\infty)' \simeq \overline{\mathcal{B}(\mathcal{H})_\infty}'$.

Assertion (2) on $\mathbf{L}(\mu) \circ (\theta \times \theta)$ follows by assertion (1) along with Eq. (2.2).

Then, to see that also the representation $\pi^{\ltimes}$ satisfies the growth condition along $\theta \times \theta$, just note that

$$\big(\mathbf{L}(\mu)^{-1} \circ (\theta \times \theta)\big)(X, Y) = \big(\theta(Y), \theta(X) - \theta(Y)\big) = (\theta \times \theta)(Y, X - Y)$$

and the linear mapping $\Xi \times \Xi \to \Xi \times \Xi$, $(X, Y) \mapsto (Y, X - Y)$, is invertible. □

3.2. Localized Weyl calculus and its continuity properties

Definition 3.10. Let $\theta : \Xi \to \mathfrak{m}$ be a linear mapping.

The *localized Weyl calculus for π along θ* is the mapping $\mathrm{Op}^\theta : \widehat{L^1(\Xi)} \to \mathcal{B}(\mathcal{H})$ given by

$$\mathrm{Op}^\theta(a) = \int_\Xi \check{a}(X)\pi\big(\exp_M\big(\theta(X)\big)\big)\,\mathrm{d}X \tag{3.3}$$

for $a \in \widehat{L^1(\Xi)}$ where we use weakly convergent integrals.

The localized Weyl calculus for π along θ is said to be *regular* if

- π satisfies the growth condition along the mapping θ,
- π is twice nuclearly smooth, and
- $\mathrm{Op}^\theta(a) \in \mathcal{B}(\mathcal{H})_\infty$ whenever $a \in \mathcal{S}(\Xi^*)$.

Note that the closed graph theorem then implies that $\mathrm{Op}^\theta : \mathcal{S}(\Xi^*) \to \mathcal{B}(\mathcal{H})_\infty$ is a continuous linear mapping.

If the representation π satisfies the growth condition along the mapping θ, then one can think of (3.3) in the distributional sense in order to define the localized Weyl calculus $\mathrm{Op}^\theta : \mathcal{S}'(\Xi^*) \to \mathcal{L}(\mathcal{H}_\infty, \mathcal{H}_{-\infty})$. More specifically, for every $a \in \mathcal{S}'(\Xi^*)$ and $\phi, \psi \in \mathcal{H}_\infty$ we have

$$\big(\mathrm{Op}^\theta(a)\phi \mid \psi\big) = \big\langle \check{a}, \overline{\mathcal{A}^{\pi,\theta}_\phi \psi}\big\rangle \tag{3.4}$$

where $\langle\cdot,\cdot\rangle : \mathcal{S}'(\Xi) \times \mathcal{S}(\Xi) \to \mathbb{C}$ is the usual duality pairing.

Remark 3.11. If the localized Weyl calculus for π along θ is regular and moreover defines a linear topological isomorphism $\mathrm{Op}^\theta : \mathcal{S}(\Xi^*) \to \mathcal{B}(\mathcal{H})_\infty$ (see Proposition 3.12 for sufficient conditions), then we also have the linear topological isomorphism $\mathrm{Op}^\theta : \mathcal{S}'(\Xi^*) \to \mathcal{L}(\mathcal{H}_\infty, \mathcal{H}_{-\infty})$

by Proposition 2.5(2). Therefore, by using Remark 2.6, we see that there exist the sesquilinear mappings

$$\mathcal{A}^{\pi,\theta} : \mathcal{H}_{-\infty} \times \mathcal{H}_{-\infty} \to \mathcal{S}'(\Xi) \quad \text{and} \quad \mathcal{W} : \mathcal{H}_{-\infty} \times \mathcal{H}_{-\infty} \to \mathcal{S}'(\Xi^*) \tag{3.5}$$

such that

$$\mathrm{Op}^{\theta}\big(\mathcal{W}(f_1, f_2)\big) = f_1 \otimes \bar{f_2}$$

and $\widehat{\mathcal{W}(f_1, f_2)} = \mathcal{A}^{\pi,\theta}_{f_2} f_1$ for all $f_1, f_2 \in \mathcal{H}_{-\infty}$. In addition, it follows by (3.4) and the definition of the Fourier transform for tempered distributions that for every $a \in \mathcal{S}'(\Xi^*)$ and $\phi, \psi \in \mathcal{H}_\infty$ we have

$$\big(\mathrm{Op}^{\theta}(a)\phi \mid \psi\big) = \big(a \mid \mathcal{W}(\psi, \phi)\big). \tag{3.6}$$

If moreover the representation π satisfies the orthogonality relations along the linear mapping θ, then it follows by Proposition 3.12 below that the mappings (3.5) agree with the ambiguity functions and the cross-Wigner distributions (see Definition 3.5).

Proposition 3.12. *If π satisfies the orthogonality relations along the linear mapping $\theta : \Xi \to \mathfrak{m}$, then the following assertions are equivalent*:

(1) *The representation π satisfies the density condition along θ.*
(2) *There exists a unique unitary operator $\mathrm{Op}^{\theta} : L^2(\Xi^*) \to \mathfrak{S}_2(\mathcal{H})$ which agrees with the localized Weyl calculus for π along θ.*

If these assertions hold true, then we have

$$(\forall f, \phi \in \mathcal{H}) \quad \mathrm{Op}^{\theta}\big(\mathcal{W}(f, \phi)\big) = f \otimes \bar{\phi}. \tag{3.7}$$

If moreover the localized Weyl calculus for π along θ is regular, then the mapping $\mathrm{Op}^{\theta} : \mathcal{S}(\Xi^*) \to \mathcal{B}(\mathcal{H})_\infty$ is a linear topological isomorphism.

Proof. We begin with some general remarks. Since we have a unitary Fourier transform $L^2(\Xi) \to L^2(\Xi^*)$, it follows by the orthogonality relations along with (3.4) that for arbitrary $f, \phi \in \mathcal{H}$ we have

$$\mathrm{Op}^{\theta}\big(\mathcal{W}(f, \phi)\big) = f \otimes \bar{\phi} \quad \text{and} \quad \big\|\mathcal{W}(f, \phi)\big\|_{L^2(\Xi^*)} = \|f\| \cdot \|\phi\| = \|f \otimes \bar{\phi}\|_{\mathfrak{S}_2(\mathcal{H})}. \tag{3.8}$$

Moreover,

$$\mathrm{span}\big(\{f \otimes \bar{\phi} \mid f, \phi \in \mathcal{H}\}\big) \text{ is dense in } \mathfrak{S}_2(\mathcal{H}). \tag{3.9}$$

We now come back to the proof.

"(1) ⇒ (2)" Let π satisfy the density condition along θ. Since the Fourier transform $L^2(\Xi) \to L^2(\Xi^*)$ is unitary, it follows that $\operatorname{span}(\{\mathcal{W}(f,\phi) \mid f,\phi \in \mathcal{H}\})$ is a dense linear subspace of $L^2(\Xi^*)$. Therefore, by using (3.8) and (3.9), we see that Op^θ uniquely extends to a unitary operator $L^2(\Xi^*) \to \mathfrak{S}_2(\mathcal{H})$.

"(2) ⇒ (1)" If the operator $\operatorname{Op}^\theta : L^2(\Xi^*) \to \mathfrak{S}_2(\mathcal{H})$ is unitary, then it follows by (3.8) and (3.9) that $\operatorname{span}(\{\mathcal{W}(f,\phi) \mid f,\phi \in \mathcal{H}\})$ is a dense linear subspace of $L^2(\Xi^*)$. Then, by using again the fact that the Fourier transform $L^2(\Xi) \to L^2(\Xi^*)$ is unitary, we can see that $\operatorname{span}(\{\mathcal{A}^{\pi,\theta}_\phi f \mid f,\phi \in \mathcal{H}\})$ is a dense linear subspace of $L^2(\Xi)$, that is, π satisfies the density condition along θ.

Now assume that the assertions (1) and (2) in the statement are satisfied and the localized Weyl calculus for π along θ is regular. Then π satisfies the growth condition along θ, hence the ambiguity function defines a continuous sesquilinear mapping $\mathcal{A}^{\pi,\theta} : \mathcal{H}_\infty \times \mathcal{H}_\infty \to \mathcal{S}(\Xi)$ (see Definition 3.5). Since the Fourier transform is a linear topological isomorphism $\mathcal{S}(\Xi) \to \mathcal{S}(\Xi^*)$, the cross-Wigner distributions also define a continuous sesquilinear mapping $\mathcal{W} : \mathcal{H}_\infty \times \mathcal{H}_\infty \to \mathcal{S}(\Xi^*)$, which further induces a continuous linear mapping $\mathcal{W} : \mathcal{H}_\infty \mathbin{\widehat{\otimes}} \overline{\mathcal{H}_\infty} \to \mathcal{S}(\Xi^*)$. On the other hand, the condition that the localized Weyl calculus for π along θ is regular (see Definition 3.10) includes the assumption that the representation π is twice nuclearly smooth, hence we have a topological linear isomorphism $\mathcal{H}_\infty \mathbin{\widehat{\otimes}} \overline{\mathcal{H}_\infty} \simeq \mathcal{B}(\mathcal{H})_\infty$.

We thus eventually get a continuous linear mapping $\mathcal{W} : \mathcal{B}(\mathcal{H})_\infty \to \mathcal{S}(\Xi^*)$ which, by (3.8), has the property $\operatorname{Op}^\theta \circ \mathcal{W} = \mathrm{id}$ on $\mathcal{B}(\mathcal{H})_\infty$. In other words, $\mathcal{W} = (\operatorname{Op}^\theta)^{-1}|_{\mathcal{B}(\mathcal{H})_\infty}$. Thus the unitary operator $\operatorname{Op}^\theta : L^2(\Xi^*) \to \mathfrak{S}_2(\mathcal{H})$ restricts to a continuous linear map $\mathcal{S}(\Xi^*) \to \mathcal{B}(\mathcal{H})_\infty$ (since the localized Weyl calculus for π along θ is regular), while its inverse $(\operatorname{Op}^\theta)^{-1}$ restricts to a continuous linear map $\mathcal{W} : \mathcal{B}(\mathcal{H})_\infty \to \mathcal{S}(\Xi^*)$. It then follows that $\operatorname{Op}^\theta : \mathcal{S}(\Xi^*) \to \mathcal{B}(\mathcal{H})_\infty$ is a linear topological isomorphism (whose inverse is $\mathcal{W}$). □

Definition 3.13. Assume that the localized Weyl calculus for π along the linear mapping $\theta : \Xi \to \mathfrak{m}$ is regular and the representation π satisfies both the density condition and the orthogonality relations along θ. It follows by Proposition 3.12 that the localized Weyl calculus Op^θ defines a unitary operator $L^2(\Xi^*) \to \mathfrak{S}_2(\mathcal{H})$, and also linear topological isomorphisms $\mathcal{S}(\Xi^*) \to \mathcal{B}(\mathcal{H})_\infty \simeq \mathcal{L}(\mathcal{H}_{-\infty}, \mathcal{H}_\infty)$ and $\mathcal{S}'(\Xi^*) \to \mathcal{L}(\mathcal{H}_\infty, \mathcal{H}_{-\infty})$. Hence we can introduce the following notions:

(1) If $a, b \in \mathcal{S}'(\Xi^*)$ and the operator product $\operatorname{Op}^\theta(a)\operatorname{Op}^\theta(b) \in \mathcal{L}(\mathcal{H}_\infty, \mathcal{H}_{-\infty})$ is well defined, then Remark 3.11 shows that the *Moyal product* $a \#^\theta b \in \mathcal{S}'(\Xi^*)$ is uniquely determined by the condition

$$\operatorname{Op}^\theta\bigl(a \#^\theta b\bigr) = \operatorname{Op}^\theta(a)\operatorname{Op}^\theta(b).$$

Thus the Moyal product defines bilinear mappings $\mathcal{S}(\Xi^*) \times \mathcal{S}(\Xi^*) \to \mathcal{S}(\Xi^*)$ and $L^2(\Xi^*) \times L^2(\Xi^*) \to L^2(\Xi^*)$.

(2) We define the unitary representation $\pi^\# : M \ltimes M \to \mathcal{B}(L^2(\Xi^*))$ such that for every $m \in M \ltimes M$ there exists the commutative diagram

$$\begin{array}{ccc} L^2(\Xi^*) & \xrightarrow{\pi^\#(m)} & L^2(\Xi^*) \\ {\scriptstyle \operatorname{Op}^\theta}\downarrow & & \downarrow{\scriptstyle \operatorname{Op}^\theta} \\ \mathfrak{S}_2(\mathcal{H}) & \xrightarrow{\pi^{\ltimes}(m)} & \mathfrak{S}_2(\mathcal{H}) \end{array}$$

These constructions provide extensions of some notions introduced in [2].

Remark 3.14. In the setting of Definition 3.13 we note the following facts:

(1) For every $m_1, m_2 \in M$ and $f \in L^2(\Xi^*)$ we have

$$\pi^{\#}(m_1, m_2) f = \big(\mathrm{Op}^\theta\big)^{-1}\big(\pi(m_1 m_2)\big)\#^\theta f \#^\theta \big(\mathrm{Op}^\theta\big)^{-1}\big(\pi(m_1)\big)^{-1}.$$

(2) For every $X_1, X_2 \in \Xi$ we have $\mathrm{Op}^\theta(\mathrm{e}^{\mathrm{i}\langle\cdot, X_j\rangle}) = \pi(\exp_M(\theta(X_j)))$ for $j = 1, 2$, whence by Lemma 2.8(2)

$$\begin{aligned}\pi^{\#}\big(\exp_{M\ltimes M}\big(\theta(X_1), \theta(X_2)\big) f &= \pi^{\#}\big(\exp_M\big(\theta(X_1)\big), \exp_M\big(-\theta(X_1)\big)\exp_M\big(\theta(X_1 + X_2)\big)\big)\big) f \\ &= \mathrm{e}^{\mathrm{i}\langle\cdot, X_1+X_2\rangle}\#^\theta f \#^\theta \mathrm{e}^{-\mathrm{i}\langle\cdot, X_1\rangle}\end{aligned}$$

whenever $f \in L^2(\Xi^*)$.

Proposition 3.15. *Assume that the representation π is twice nuclearly smooth. If we have either $\phi_1, \phi_2, f_1, f_2 \in \mathcal{H}$, or $\phi_1, \phi_2 \in \mathcal{H}_\infty$ and $f_1, f_2 \in \mathcal{H}_{-\infty}$, then*

$$(\forall X, Y \in \Xi) \quad \big(\mathcal{A}^{\pi^{\ltimes}, \theta\times\theta}_{\phi_1 \otimes \bar{\phi}_2}(f_1 \otimes \bar{f}_2)\big)(X, Y) = \big(\mathcal{A}^{\pi,\theta}_{\phi_1} f_1\big)(X + Y) \cdot \overline{\big(\mathcal{A}^{\pi,\theta}_{\phi_2} f_2\big)(X)}.$$

If moreover the localized Weyl calculus for π along θ is regular and the representation π satisfies both the density condition and the orthogonality relations along θ, then

$$(\forall X, Y \in \Xi) \quad \big(\mathcal{A}^{\pi^{\#}, \theta\times\theta}_{\mathcal{W}(\phi_1, \phi_2)}\big(\mathcal{W}(f_1, f_2)\big)\big)(X, Y) = \big(\mathcal{A}^{\pi,\theta}_{\phi_1} f_1\big)(X + Y) \cdot \overline{\big(\mathcal{A}^{\pi,\theta}_{\phi_2} f_2\big)(X)}.$$

Proof. It follows at once by definition that

$$\mathcal{A}^{\pi\otimes\bar{\pi}, \theta\times\theta}_{\phi_1 \otimes \bar{\phi}_2}(f_1 \otimes \bar{f}_2) = \mathcal{A}^{\pi,\theta}_{\phi_1} f_1 \otimes \overline{\mathcal{A}^{\pi,\theta}_{\phi_2} f_2}.$$

On the other hand, we easily get by (2.2)

$$(\forall X, Y \in \Xi) \quad \big(\mathcal{A}^{\pi^{\ltimes}, \theta\times\theta}_{\phi_1 \otimes \bar{\phi}_2}(f_1 \otimes \bar{f}_2)\big)(X, Y) = \big(\mathcal{A}^{\pi\otimes\bar{\pi}, \theta\times\theta}_{\phi_1 \otimes \bar{\phi}_2}(f_1 \otimes \bar{f}_2)\big)(X + Y, X).$$

For the second part of the statement, just recall that $\mathrm{Op}^\theta(\mathcal{W}(f_1, f_2)) = f_1 \otimes \bar{f}_2$ and use Proposition 3.12 along with Remark 3.4. □

Now we are ready to give one of the main technical results of the present paper. It extends a result in [2] (which is recovered for representations of finite-dimensional nilpotent Lie groups). The general lines of the proof go back to [34] (which is recovered for Heisenberg groups); see also [7].

Theorem 3.16. *Let $\phi_1, \phi_2 \in \mathcal{H}_\infty \setminus \{0\}$, and assume the following hypotheses:*

(1) *The representation π satisfies both the density condition and the orthogonality relations along the linear mapping $\theta : \Xi \to \mathfrak{m}$.*
(2) *The localized Weyl calculus for the representation π along θ is regular.*

Let $\Xi = \Xi_1 \dotplus \Xi_2$ be any direct sum decomposition. If $1 \leqslant r \leqslant s \leqslant \infty$ and $r_1, r_2, s_1, s_2 \in [r, s]$ satisfy the equations $\frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{r} + \frac{1}{s}$, then the cross-Wigner distribution defines a continuous sesquilinear map

$$\mathcal{W}(\cdot,\cdot) : M^{r_1,s_1}_{\phi_1}(\pi,\theta) \times M^{r_2,s_2}_{\phi_2}(\pi,\theta) \to M^{r,s}_{\mathcal{W}(\phi_1,\phi_2)}\big(\pi^{\#}, \theta \times \theta\big).$$

Proof. The assertion follows from Proposition 3.15 along the same lines as in the proof of [2, Th. 2.22]. □

The next corollary records a standard consequence of the continuity of cross-Wigner distributions; see [19,34,18].

Corollary 3.17. *Let $\phi_1, \phi_2 \in \mathcal{H}_\infty \setminus \{0\}$, and assume the following hypotheses:*

(1) *The representation π satisfies both the density condition and the orthogonality relations along the linear mapping $\theta : \Xi \to \mathfrak{m}$.*
(2) *The localized Weyl calculus for the representation π along θ is regular.*

Now let $\Xi = \Xi_1 \dotplus \Xi_2$ be any direct sum decomposition. If $r, s, r_1, s_1, r_2, s_2 \in [1, \infty]$ satisfy the conditions

$$r \leqslant s, \qquad r_2, s_2 \in [r, s], \quad \textit{and} \quad \frac{1}{r_1} - \frac{1}{r_2} = \frac{1}{s_1} - \frac{1}{s_2} = 1 - \frac{1}{r} - \frac{1}{s},$$

then for every symbol $a \in M^{r,s}_{\mathcal{W}(\phi_1,\phi_2)}(\pi^{\#}, \theta \times \theta)$ we have a bounded linear operator

$$\mathrm{Op}^\theta(a) : M^{r_1,s_1}_{\phi_1}(\pi,\theta) \to M^{r_2,s_2}_{\phi_2}(\pi,\theta).$$

Moreover, the linear mapping

$$\mathrm{Op}^\theta : M^{r,s}_{\mathcal{W}(\phi_1,\phi_2)}\big(\pi^{\#}, \theta \times \theta\big) \to \mathcal{B}\big(M^{r_1,s_1}_{\phi_1}(\pi,\theta), M^{r_2,s_2}_{\phi_2}(\pi,\theta)\big)$$

is continuous.

Proof. The assertion follows from Theorem 3.16 along the same lines as in the proof of [2, Cor. 2.24]. We just recall that the conditions on the parameters come from Hölder's inequality and duality theory for the mixed-norm Lebesgue spaces; see [19,34,18] again. □

Corollary 3.18. *Let $\phi_1, \phi_2 \in \mathcal{H}_\infty \setminus \{0\}$, and assume the following hypotheses*:

(1) *The representation π satisfies both the density condition and the orthogonality relations along the linear mapping $\theta : \Xi \to \mathfrak{m}$.*
(2) *The localized Weyl calculus for the representation π along θ is regular.*

Then for every $a \in M^{\infty,1}_{\mathcal{W}(\phi_1,\phi_2)}(\pi^\#)$ we have $\mathrm{Op}^\theta(a) \in \mathcal{B}(\mathcal{H})$, and the linear mapping $\mathrm{Op}^\theta : M^{\infty,1}_{\mathcal{W}(\phi_1,\phi_2)}(\pi^\#, \theta \times \theta) \to \mathcal{B}(\mathcal{H})$ is continuous.

Proof. This is the special case of Corollary 3.17 with $r_1 = s_1 = r_2 = s_2 = 2$, $r = 1$, and $s = \infty$, since Remark 3.3 shows that $M^{2,2}_{\phi_j}(\pi, \theta) = \mathcal{H}$ for $j = 1, 2$. □

We note that $M^{\infty,1}_{\mathcal{W}(\phi_1,\phi_2)}(\pi^\#)$ is precisely Sjöstrand's algebra introduced in [33] in the case of the Heisenberg groups and their Schrödinger representations; see [2, Sect. 4].

3.3. Trace-class operators obtained by localized Weyl calculus

In this subsection we give a standard sufficient condition for a pseudo-differential operator to belong to the trace class. In the special case of the Schrödinger representation of a Heisenberg group, this result goes back to [33]. A proof for this result was also provided in [16], was extended to arbitrary nilpotent Lie groups in [2], and will be adapted below to the present setting.

Lemma 3.19. *Let the representation $\pi : M \to \mathcal{B}(\mathcal{H})$ satisfy the orthogonality relations along the linear mapping $\theta : \Xi \to \mathfrak{m}$, and pick $\phi_0 \in \mathcal{H}_\infty$ with $\|\phi_0\| = 1$. Then the following assertions hold*:

(1) *The operator $\mathcal{A}^{\pi,\theta}_{\phi_0} : \mathcal{H} \to L^2(\Xi)$, $f \mapsto \mathcal{A}^{\pi,\theta}_{\phi_0} f$, is an isometry whose image is the reproducing kernel Hilbert space associated with the reproducing kernel*

$$K : \Xi \times \Xi \to \mathbb{C}, \quad K(X_1, X_2) = \big(\pi\big(\exp_M\big(\theta(X_1)\big)\big)\phi_0 \,\big|\, \pi\big(\exp_M\big(\theta(X_2)\big)\big)\phi_0\big).$$

The orthogonal projection from $L^2(\Xi)$ onto $\mathrm{Ran}\,\mathcal{A}^{\pi,\theta}_{\phi_0}$ is just the integral operator defined by the integral kernel K.

(2) *For every $\phi, f \in \mathcal{H}$ we have*

$$\int_\Xi \big(\mathcal{A}^{\pi,\theta}_{\phi_0} f\big)(X) \cdot \pi\big(\exp_M\big(\theta(X)\big)\big)\phi \,\mathrm{d}X = (\phi \mid \phi_0) f.$$

In particular, for every $f \in \mathcal{H}$ we have

$$\int_\Xi \big(\mathcal{A}^{\pi,\theta}_{\phi_0} f\big)(X) \cdot \pi\big(\exp_M\big(\theta(X)\big)\big)\phi_0 \,\mathrm{d}X = f, \tag{3.10}$$

where the integral is weakly convergent in $\mathcal{H}$.

Assume that the representation π satisfies the growth condition along θ. Also, assume that for every $u \in \mathrm{U}(\mathfrak{m}_{\mathbb{C}})$ the function $\|\mathrm{d}\pi(u)\pi(\exp_M(\theta(\cdot)))\phi_0\|$ has polynomial growth, then moreover we have:

(3) *If $f \in \mathcal{H}_\infty$, then the integral in* (3.10) *is convergent with respect to the topology of $\mathcal{H}_\infty$.*
(4) *If $f \in \mathcal{H}_{-\infty}$, then* (3.10) *holds with the integral convergent in the w^*-topology.*
(5) *We have $\mathcal{H}_\infty = \{f \in \mathcal{H}_{-\infty} \mid \mathcal{A}^{\pi,\theta}_{\phi_0} f \in \mathcal{S}(\Xi)\}$.*

Proof. Assertion (1) follows at once by the orthogonality relations along with [15, Prop. 2.12]. Then assertion (2) follows by an application of [15, Prop. 2.11]. The proof for assertions (3)–(5) can be supplied by adapting the method of proof of [2, Cor. 2.9]. We omit the details. □

Remark 3.20. We note here that in the setting of Lemma 3.19, the condition that for all $u \in \mathrm{U}(\mathfrak{m}_{\mathbb{C}})$ and $\phi \in \mathcal{H}_\infty$ the function $\|\mathrm{d}\pi(\mathrm{Ad}_{\mathrm{U}(\mathfrak{m}_{\mathbb{C}})}(\exp_M(\theta(\cdot)))u)\phi\|$ has polynomial growth on Ξ implies that for all $f \in \mathcal{H}_{-\infty}$, $\phi \in \mathcal{H}_\infty$, the function $\mathcal{A}^{\pi,\theta}_{\phi} f$ has polynomial growth as well.

In fact, if $f \in \mathcal{H}_{-\infty}$, then there exists $u \in \mathrm{U}(\mathfrak{m}_{\mathbb{C}})$ such that for every $\psi \in \mathcal{H}_\infty$ we have $|(f \mid \psi)| \leqslant \|\mathrm{d}\pi(u)\psi\|$. (See Remark 2.1.) Then we have

$$\begin{aligned}\bigl|\bigl(\mathcal{A}^{\pi,\theta}_{\phi} f\bigr)(\cdot)\bigr| &= \bigl|\bigl(f \mid \pi\bigl(\exp_M\bigl(\theta(\cdot)\bigr)\bigr)\phi\bigr)\bigr| \leqslant \bigl\|\mathrm{d}\pi(u)\pi\bigl(\exp_M\bigl(\theta(\cdot)\bigr)\phi\bigr)\bigr\| \\ &= \bigl\|\mathrm{d}\pi\bigl(\mathrm{Ad}_{\mathrm{U}(\mathfrak{m}_{\mathbb{C}})}\bigl(\exp_M\bigl(\theta(\cdot)\bigr)\bigr)u\bigr)\phi\bigr\|\end{aligned}$$

and the latter function has polynomial growth by assumption.

By using the method of proof of [2, Prop. 2.27] we can now obtain the following sufficient condition for a symbol to give rise to a trace-class operator.

Proposition 3.21. *Let $\phi_1, \phi_2 \in \mathcal{H}_\infty$ such that $\|\phi_j\| = 1$ and for every $u \in \mathrm{U}(\mathfrak{m}_{\mathbb{C}})$ the function $\|\mathrm{d}\pi(u)\pi(\exp_M(\theta(\cdot)))\phi_j\|$ has polynomial growth, for $j = 1, 2$, and assume the following hypotheses*:

(1) *The representation π satisfies both the density condition and the orthogonality relations along the linear mapping $\theta : \Xi \to \mathfrak{m}$.*
(2) *The localized Weyl calculus for the representation π along θ is regular.*

Then for every $a \in M^{1,1}_{\mathcal{W}(\phi_1,\phi_2)}(\pi^\#, \theta \times \theta)$ we have $\mathrm{Op}^\theta(a) \in \mathfrak{S}_1(\mathcal{H})$, and the linear mapping $\mathrm{Op}^\theta : M^{1,1}_{\mathcal{W}(\phi_1,\phi_2)}(\pi^\#, \theta \times \theta) \to \mathfrak{S}_1(\mathcal{H})$ is continuous.

Proof. It follows by Lemmas 3.8(2), 3.9(2) and Remark 3.4 that the representation $\pi^\# : M \ltimes M \to \mathcal{B}(L^2(\Xi^*))$ satisfies both the orthogonality relations and the growth condition along the linear mapping $\theta \times \theta : \Xi \times \Xi \to \mathfrak{m} \ltimes \mathfrak{m}$. Moreover, it is easily seen that the function $\Phi_0 := \mathcal{W}(\phi_1, \phi_2) \in \mathcal{S}(\Xi^*)$ has the property that for every $u \in \mathrm{U}((\mathfrak{m} \ltimes \mathfrak{m})_{\mathbb{C}})$ the norm of $\mathrm{d}\pi^\#(u)\pi^\#(\exp_{M \ltimes M}((\theta \times \theta)(\cdot)))\Phi_0$ has polynomial growth on $\Xi \times \Xi$, since a similar property has the rank-one operator $\mathrm{Op}^\theta(\Phi_0) = (\cdot \mid \phi_2)\phi_1 \in \mathfrak{S}_2(\mathcal{H})$ with respect to the representation $\pi^\ltimes$, as a direct consequence of the calculation (3.12) below. Therefore we can use Lemma 3.19(4) for the representation $\pi^\#$ to see that for arbitrary $a \in \mathcal{S}'(\Xi^*)$ we have

$$a = \iint\limits_{\Xi\times\Xi} \big(\mathcal{A}_{\Phi_0}^{\pi^\#,\theta\times\theta} a\big)(X,Y)\cdot \pi^\#\big(\exp_{M\ltimes M}\big(\theta(X),\theta(Y)\big)\big)\Phi_0 \,\mathrm{d}X\,\mathrm{d}Y,$$

whence by (3.6) we get

$$\mathrm{Op}^\pi(a) = \iint\limits_{\Xi\times\Xi} \big(\mathcal{A}_{\Phi_0}^{\pi^\#,\theta\times\theta} a\big)(X,Y)\cdot \mathrm{Op}^\theta\big(\pi^\#\big(\exp_{M\ltimes M}\big(\theta(X),\theta(Y)\big)\big)\Phi_0\big) \,\mathrm{d}X\,\mathrm{d}Y \tag{3.11}$$

where the latter integral is weakly convergent in $\mathcal{L}(\mathcal{H}_\infty,\mathcal{H}_{-\infty})$ ($\simeq \mathcal{L}(\mathcal{H}_{-\infty},\mathcal{H}_\infty)'$ by Proposition 2.5(2)). On the other hand, for arbitrary $X, Y \in \Xi$ we get by Remarks 3.14 and 3.11

$$\begin{aligned}
&\mathrm{Op}^\theta\big(\pi^\#\big(\exp_{M\ltimes M}\big(\theta(X),\theta(Y)\big)\big)\Phi_0\big)\\
&\quad= \pi\big(\exp_M\big(\theta(X)+\theta(Y)\big)\big)\circ \mathrm{Op}^\theta(\Phi_0)\circ \pi\big(\exp_M\big(\theta(X)\big)\big)^{-1}\\
&\quad= \big(\cdot \mid \pi\big(\exp_M\big(\theta(X)\big)\big)\phi_2\big)\pi\big(\exp_M\big(\theta(X+Y)\big)\big)\phi_1.
\end{aligned} \tag{3.12}$$

In particular, $\mathrm{Op}^\theta(\pi^\#(\exp_{M\ltimes M}(\theta(X),\theta(Y)))\Phi_0) \in \mathfrak{S}_1(\mathcal{H})$ and

$$\begin{aligned}
\big\|\mathrm{Op}^\theta\big(\pi^\#\big(\exp_{M\ltimes M}\big(\theta(X),\theta(Y)\big)\big)\Phi_0\big)\big\|_1 &= \big\|\pi\big(\exp_M\big(\theta(X+Y)\big)\big)\phi_1\big\|\cdot\big\|\pi\big(\exp_M\big(\theta(X)\big)\big)\phi_2\big\|\\
&= 1.
\end{aligned}$$

It then follows that the integral in (3.11) is absolutely convergent in $\mathfrak{S}_1(\mathcal{H})$ for every symbol $a \in M_{\Phi_0}^{1,1}(\pi^\#,\theta\times\theta)$ and moreover we have

$$\big\|\mathrm{Op}^\theta(a)\big\|_1 \leqslant \iint\limits_{\Xi\times\Xi} \big|\big(\mathcal{A}_{\Phi_0}^{\pi^\#,\theta\times\theta} a\big)(X,Y)\big| \,\mathrm{d}X\,\mathrm{d}Y = \|a\|_{M_\Phi^{1,1}(\pi^\#,\theta\times\theta)}$$

which concludes the proof. □

4. Applications to the magnetic Weyl calculus

We proved in [1] that the magnetic Weyl calculus on $\mathbb{R}^n$ constructed in [24] can be alternatively described as the localized Weyl calculus for a suitable representation. This point of view actually allowed us to construct magnetic Weyl calculi on any simply connected nilpotent Lie group G, by using an appropriate representation $\pi : M = \mathcal{F} \rtimes G \to \mathcal{B}(L^2(G))$ and linear mappings $\theta^A : \mathfrak{g}\times\mathfrak{g}^* \to \mathfrak{m}$.

We shall see in the present section that all of the conditions studied in Sections 2 and 3 are met by π and θ^A (see Corollary 4.7 below), provided the coefficients of the magnetic potential $A \in \Omega^1(G)$ have polynomial growth. Therefore, the abstract results of the previous sections can be used for obtaining continuity and nuclearity properties for the magnetic Weyl calculus (see Corollaries 4.8–4.10 below).

Notation 4.1. For any Lie group G we denote by $\lambda : G \to \mathrm{End}(\mathcal{C}^\infty(G))$, $g \mapsto \lambda_g$, the left regular representation defined by $(\lambda_g\phi)(x) = \phi(g^{-1}x)$ for every $x, g \in G$ and $\phi \in \mathcal{C}^\infty(G)$. Moreover, we denote by $\mathbf{1}$ the constant function which is identically equal to 1 on G. (This should not be confused with the unit element of G, which is denoted in the same way.)

We now recall the following notion from [1].

Definition 4.2. Let G be a finite-dimensional Lie group. A linear space $\mathcal{F}$ of real functions on G is said to be *admissible* if it is endowed with a sequentially complete, locally convex topology and satisfies the following conditions:

(1) The linear space $\mathcal{F}$ is invariant under the representation of G by left translations, that is, if $\phi \in \mathcal{F}$ and $g \in G$ then $\lambda_g \phi \in \mathcal{F}$.
(2) We have a continuous inclusion mapping $\mathcal{F} \hookrightarrow \mathcal{C}^\infty(G)$.
(3) The mapping $G \times \mathcal{F} \to \mathcal{F}$, $(g, \phi) \mapsto \lambda_g \phi$ is smooth. For every $\phi \in \mathcal{F}$ we denote by $\dot{\lambda}(\cdot)\phi : \mathfrak{g} \to \mathcal{F}$ the differential of the mapping $g \mapsto \lambda_g \phi$ at the point $\mathbf{1} \in G$.
(4) For every $g_1, g_2 \in G$ with $g_1 \neq g_2$ there exists $\phi \in \mathcal{F}$ with $\phi(g_1) \neq \phi(g_2)$.
(5) We have $\{\phi'_g \mid \phi \in \mathcal{F}\} = T^*_g G$ for every $g \in G$.

For instance, the function space $\mathcal{C}^\infty_{\mathbb{R}}(G)$ is admissible.

Proposition 4.3. *Let G be a finite-dimensional simply connected nilpotent Lie group with the inverse of the exponential map denoted by $\log_G : G \to \mathfrak{g}$. If we define*

$$\mathcal{F}_G := \mathrm{span}_{\mathbb{R}}\big(\big\{\lambda_g(\xi \circ \log_G) \bigm| \xi \in \mathfrak{g}^*,\ g \in G\big\}\big), \tag{4.1}$$

then the following assertions hold:

(1) *$\mathcal{F}_G$ is a finite-dimensional linear subspace of $\mathcal{C}^\infty(G)$ which is invariant under the left regular representation and contains the constant functions.*
(2) *The semi-direct product $M_0 := \mathcal{F}_G \rtimes_\lambda G$ is a finite-dimensional simply connected nilpotent Lie group.*

Proof. Since G is a simply connected nilpotent Lie group, we may assume that $G = (\mathfrak{g}, *)$.

(1) It is clear that the linear space $\mathcal{F}_G$ is invariant under the left regular representation. On the other hand, for every $V, X \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$ we have

$$(\lambda_V \xi)(X) = \langle \xi, (-V) * X\rangle = \left\langle \xi, -V + X + \frac{1}{2}[-V, X] + \cdots \right\rangle.$$

Thus, if we denote by N the nilpotency index of $\mathfrak{g}$, then we see that $\mathcal{F}_G$ consists of polynomial functions on $\mathfrak{g}$ of degree $\leqslant N$, hence $\dim \mathcal{F}_G < \infty$. Moreover, if $\mathfrak{z}$ denotes the center of $\mathfrak{g}$ and we pick $V \in \mathfrak{z}$ and $\xi \in \mathfrak{g}^*$, then $\lambda_V \xi = -\langle \xi, V\rangle \mathbf{1} + \xi$. We thus see that the constant functions belong to $\mathcal{F}_G$.

(2) On the Lie algebra level we have $\mathfrak{m}_0 := \mathcal{F}_G \rtimes_{\dot{\lambda}} \mathfrak{g}$, and both $\mathcal{F}_G$ and $\mathfrak{g}$ are nilpotent Lie algebras. Therefore Engel's theorem shows that, for proving that $\mathfrak{m}_0$ is nilpotent, it is enough to check that the adjoint action $\mathrm{ad}_{\mathfrak{m}_0}$ gives a representation of $\mathfrak{g}$ on $\mathcal{F}_G$ by *nilpotent* endomorphisms. This representation is just $\dot{\lambda} : \mathfrak{g} \to \mathrm{End}(\mathcal{F}_G)$ hence, by the theorem on weight space decompositions for representations of nilpotent Lie algebras (see for instance [6, Th. 2.9]), it suffices to prove the following fact: *If $\alpha \in \mathfrak{g}^*$, $\phi \in \mathcal{F}_G \setminus \{0\}$, and for every $X \in \mathfrak{g}$ we have $\dot{\lambda}(X)\phi = \alpha(X)\phi$, then $\alpha = 0$.*

To this end, let $X_0 \in \mathfrak{g}$ arbitrary. Since $\dot{\lambda}(X_0)\phi = \alpha(X_0)\phi$, it follows that for every $Y \in \mathfrak{g}$ and $t \in \mathbb{R}$ we have $\phi((-tX_0)*Y) = \mathrm{e}^{t\alpha(X_0)}\phi(Y)$. We have seen above that $\mathcal{F}_G$ consists of polynomial functions on $\mathfrak{g}$ of degree $\leqslant N$, therefore for every $Y \in \mathfrak{g}$ there exists a constant $C_{\phi,Y} > 0$ such that

$$(\forall t \in \mathbb{R}) \quad \mathrm{e}^{t\alpha(X_0)}\left|\phi(Y)\right| = \left|\phi\big((-tX_0)*Y\big)\right| \leqslant C_{\phi,Y}\big(1+|t|\big)^{N^2}.$$

On the other hand, since $\phi \in \mathcal{F}_G \setminus \{0\}$, there exists $Y \in \mathfrak{g}$ such that $\phi(Y) \neq 0$, and then the above inequality shows that $\alpha(X_0) = 0$. This holds for arbitrary $X_0 \in \mathfrak{g}$, hence $\alpha = 0$, as we wished for. □

Theorem 4.4. *Let G be a finite-dimensional simply connected nilpotent Lie group with an admissible function space $\mathcal{F}$ such that there exist the continuous inclusion maps $\mathfrak{g}^* \hookrightarrow \mathcal{F} \hookrightarrow \mathcal{C}^\infty_{\mathrm{pol}}(G)$, where the embedding $\mathfrak{g}^* \hookrightarrow \mathcal{F}$ is given by $\xi \mapsto \xi \circ \log_G$. Denote $M = \mathcal{F} \rtimes_\lambda G$, fix $\epsilon \in \mathbb{R} \setminus \{0\}$, and consider the unitary representation $\pi : M \to \mathcal{B}(L^2(G))$, $\pi(\phi, g)f = \mathrm{e}^{\mathrm{i}\epsilon\phi}\lambda_g f$ for all $\phi \in \mathcal{F}$, $g \in G$, and $f \in L^2(G)$. Then π is a nuclearly smooth representation and its space of smooth vectors is the Schwartz space $\mathcal{S}(G)$.*

Proof. Let us denote $\mathcal{H} = L^2(G)$ and let $\mathcal{H}_\infty$ be the space of smooth vectors for the representation π. We first check that $\mathcal{S}(G) = \mathcal{H}_\infty$.

For proving that $\mathcal{S}(G) \subseteq \mathcal{H}_\infty$, let $f \in \mathcal{S}(G)$ arbitrary. Since $\mathcal{F} \hookrightarrow \mathcal{C}^\infty_{\mathrm{pol}}(G)$, it follows at once that for every $\phi \in \mathcal{F}$ and $g \in G$ we have $\pi(\phi,\cdot)f \in \mathcal{C}^\infty(G,\mathcal{H})$ and $\pi(\cdot,g)f \in \mathcal{C}^\infty(\mathcal{F},\mathcal{H})$. It then follows by [27, Sect. I] (see also [20, Th. 3.4.3]) that $\pi(\cdot)f \in \mathcal{C}^\infty(M,\mathcal{H})$, hence $f \in \mathcal{H}_\infty$.

To prove the converse inclusion $\mathcal{S}(G) \subseteq \mathcal{H}_\infty$ we need the function space $\mathcal{F}_G$ defined in (4.1). Since $\mathcal{F}$ contains $\{\xi \circ \log_G \mid \xi \in \mathfrak{g}^*\}$ and is invariant under the left regular representation of G, we get $\mathcal{F}_G \hookrightarrow \mathcal{F}$. Now Proposition 4.3 shows that $M_0 := \mathcal{F}_G \rtimes G$ is a finite-dimensional nilpotent Lie group. Since $\mathfrak{g}^* \hookrightarrow \mathcal{F}_G$, it is easily seen that the unitary representation $\pi_0 := \pi|_{M_0} : M_0 \to \mathcal{B}(\mathcal{H})$ is irreducible. Let $\mathcal{H}_{\infty,\pi_0}$ be its space of smooth vectors. If $\delta_{\mathbf{1}} : \mathcal{C}^\infty(G) \to \mathbb{C}$ is the Dirac distribution at $\mathbf{1} \in G$, then the discussion in [1, Subsect. 2.4] shows that $\mathcal{F}_G \times \{0\}$ is a polarization for the functional $(\delta_{\mathbf{1}}|_{\mathcal{F}_G}, 0) \in \mathfrak{m}_0^*$, and the corresponding induced representation is just π_0. Now $\mathcal{H}_{\infty,\pi_0} = \mathcal{S}(G)$ by [9, Cor. to Th. 3.1]. Therefore we get the continuous inclusion $\mathcal{H}_\infty \hookrightarrow \mathcal{S}(G)$, which completes the proof for the equality $\mathcal{S}(G) = \mathcal{H}_\infty$.

Furthermore, it easily follows by [8, Cor. A.2.4] that $\mathcal{H}_\infty = \mathcal{S}(G) = \mathcal{S}(\mathfrak{g})$ as locally convex spaces. On the other hand, it is well known that $\mathcal{S}(\mathfrak{g})$ is a nuclear Fréchet space; see for instance [35]. Finally, both mappings $M \times \mathcal{S}(G) \to \mathcal{S}(G)$, $(m,\phi) \mapsto \pi(m)\phi$, and $\mathfrak{m} \times \mathcal{S}(G) \to \mathcal{S}(G)$, $(X,\phi) \mapsto \mathrm{d}\pi(X)\phi$ are continuous as a direct consequence of [8, Th. A.2.6], and this concludes the proof of the fact that π is a nuclearly smooth representation. □

We now prove that the conclusion of Theorem 4.4 actually holds under a much stronger form.

Corollary 4.5. *In the setting of Theorem* 4.4*, the unitary representation π is twice nuclearly smooth.*

Proof. The proof has two stages. For the sake of simplicity we assume $\epsilon = 1$, however it is clear that the following reasonings carry over to the general case.

1° We first make the following remark: For $j = 1, 2$, let G_j be a finite-dimensional simply connected nilpotent Lie group with an admissible function space $\mathcal{F}_j$ such that $\mathfrak{g}_j^* \hookrightarrow \mathcal{F}_j \hookrightarrow \mathcal{C}^\infty_{\rm pol}(G_j)$ as in Theorem 4.4. Also define the group $M_j = \mathcal{F}_j \rtimes_\lambda G_j$ and the unitary representation $\pi_j : M_j \to \mathcal{B}(L^2(G_j))$, $\pi_j(\phi, g)f = {\rm e}^{{\rm i}(-1)^{j-1}\phi}\lambda_g f$ for all $\phi \in \mathcal{F}_j$, $g \in G_j$, and $f \in L^2(G_j)$. Now consider the direct product group $G_0 := G_1 \times G_2$, the function space

$$\mathcal{F}_0 := (\mathcal{F}_1 \otimes \mathbf{1}) + (\mathbf{1} \otimes \mathcal{F}_2) \hookrightarrow \mathcal{C}^\infty_{\rm pol}(G_0),$$

and the representation $\pi_0 : M_0 \to \mathcal{B}(L^2(G_0))$, $\pi_0(\phi, g)f = {\rm e}^{{\rm i}\phi}\lambda_g f$ for all $\phi \in \mathcal{F}_0$, $g \in G_0$, and $f \in L^2(G_0)$, where $M_0 := \mathcal{F}_0 \rtimes_\lambda G_0$. Then $\mathcal{F}_0$ is an admissible function space on G_0 and there exists a 1-dimensional central subgroup $N \subseteq M_1 \times M_2$ such that $N \subseteq {\rm Ker}(\pi_1 \otimes \pi_2)$, and we have $M_0 = (M_1 \times M_2)/N$. Moreover, the representation π_0 is equal to $\pi_1 \otimes \pi_2$ factorized modulo N.

In fact, let us define the linear map

$$\Delta : \mathcal{F}_1 \times \mathcal{F}_2 \to \mathcal{F}_0, \quad (\phi_1, \phi_2) \mapsto \phi_1 \otimes \mathbf{1} - \mathbf{1} \otimes \phi_2.$$

Then ${\rm Ran}\,\Delta = \mathcal{F}_0$ and ${\rm Ker}\,\Delta = \{(t\mathbf{1}, t\mathbf{1}) \mid t \in \mathbb{R}\} \simeq \mathbb{R}$, hence we get a linear isomorphism $\mathcal{F}_0 \simeq (\mathcal{F}_1 \times \mathcal{F}_2)/{\rm Ker}\,\Delta$, and this can be used to define the topology of $\mathcal{F}_0$. Moreover, it is clear that ${\rm Ker}\,\Delta$ is contained in the center of $\mathfrak{m}_1 \times \mathfrak{m}_2 \simeq \mathfrak{m}_0$ and ${\rm Ker}\,\Delta \subseteq {\rm Ker}({\rm d}(\pi_1 \otimes \pi_2))$, hence the above remark holds for $N = \exp_{M_0}({\rm Ker}\,\Delta)$.

2° We now come back to the proof of the corollary. We already know from Theorem 4.4 that the representation π is nuclearly smooth. Moreover, by using the remark of stage 1° for $G_1 = G_2 = G$ along with Theorem 4.4 for the group $G \times G$, we easily see that the space of smooth vectors for the representation $\pi \otimes \bar{\pi}$ is linear and topologically isomorphic to $\mathcal{S}(G \times G)$, which in turn is isomorphic to $\mathcal{S}(G) \mathbin{\widehat{\otimes}} \mathcal{S}(G)$ (see for instance [35]). On the other hand, $\mathcal{S}(G)$ is the space of smooth vectors for π, by Theorem 4.4. Thus the representation π also satisfies the second condition in the definition of a twice nuclearly smooth representation (see Definition 2.2), and we are done. □

Notation 4.6. Let G be any Lie group with the Lie algebra $\mathfrak{g}$ and with the space of globally defined smooth vector fields (that is, global sections in its tangent bundle) denoted by $\mathfrak{X}(G)$ and the space of globally defined smooth 1-forms (that is, global sections in its cotangent bundle) denoted by $\Omega^1(G)$. Then there exists a natural bilinear map

$$\langle\cdot,\cdot\rangle : \Omega^1(G) \times \mathfrak{X}(G) \to \mathcal{C}^\infty(G)$$

defined as usually by evaluations at every point of G.

Moreover, for arbitrary $g \in G$, we denote the corresponding right-translation mapping by $R_g : G \to G$, $h \mapsto hg$. Then we define the injective linear mapping

$$\iota^{\rm R} : \mathfrak{g} \to \mathfrak{X}(G)$$

by $(\iota^{\rm R}X)(g) = (T_{\mathbf{1}}(R_g))X \in T_gG$ for all $g \in G$ and $X \in \mathfrak{g}$.

Corollary 4.7. *Assume the setting of Theorem 4.4. If we have $A \in \Omega^1(G)$ such that $\langle A, \iota^{\mathrm{R}} X\rangle \in \mathcal{F}$ whenever $X \in \mathfrak{g}$, then we define the linear mapping*

$$\theta^A : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{m} = \mathcal{F} \ltimes_{\dot\lambda} \mathfrak{g}, \quad (X, \xi) \mapsto \bigl(\xi \circ \log_G + \langle A, \iota^{\mathrm{R}} X\rangle, X\bigr).$$

Then for every $\epsilon \in \mathbb{R} \setminus \{0\}$ the representation $\pi_\epsilon : M \to \mathcal{B}(L^2(G))$ has the following properties:

(1) *The representation π_ϵ satisfies the orthogonality relations along the mapping θ^A.*
(2) *The representation π_ϵ satisfies the growth condition along θ^A.*
(3) *The localized Weyl calculus for π_ϵ along θ^A is regular and defines a unitary operator $\mathrm{Op}^{\theta^A} : L^2(\mathfrak{g} \times \mathfrak{g}^*) \to \mathfrak{S}_2(L^2(G))$.*
(4) *If $u \in \mathrm{U}(\mathfrak{m}_{\mathbb{C}})$ and $\phi \in \mathcal{S}(G)$, the function $\|\mathrm{d}\pi(\mathrm{Ad}_{\mathrm{U}(\mathfrak{m}_{\mathbb{C}})}(\exp_M(\theta^A(\cdot)))u)\phi\|$ has polynomial growth on $\mathfrak{g} \times \mathfrak{g}^*$.*

Proof. Throughout the proof we assume $\epsilon = 1$ and we denote $\pi_1 = \pi$ for the sake of simplicity. The case of an arbitrary $\epsilon \in \mathbb{R} \setminus \{0\}$ can be handled by a similar method. Since G is simply connected, we may assume $G = (\mathfrak{g}, *)$. Then the space of smooth vectors for π_ϵ is equal to $\mathcal{S}(\mathfrak{g})$ by Theorem 4.4.

(1) The assertion follows by [3, Th. 2.8(1)].

(2) To check the growth condition (3.2) we shall denote for every $X \in \mathfrak{g}$,

$$\Psi_X : \mathfrak{g} \to \mathfrak{g}, \quad \Psi_X(Y) = \int_0^1 Y * (sX)\,\mathrm{d}s$$

and also

$$\tau_A(X, Y) = \exp\left(\mathrm{i}\int_0^1 \langle A, \iota^{\mathrm{R}} X\rangle\bigl((-sX) * Y\bigr)\,\mathrm{d}s\right)$$

for $X, Y \in \mathfrak{g}$. It then follows by [3, Prop. 2.9(1)] that for every $f, \phi \in \mathcal{S}(\mathfrak{g})$ we have

$$\bigl(\mathcal{A}_\phi^{\pi,\theta^A} f\bigr)(X, \xi) = \int_{\mathfrak{g}} \mathrm{e}^{\mathrm{i}\langle\xi, Y\rangle}\,\overline{\tau_A\bigl(X, -\Psi_X^{-1}(Y)\bigr)}\, f\bigl(-\Psi_X^{-1}(Y)\bigr)\,\overline{\phi\bigl((-X) * \bigl(-\Psi_X^{-1}(Y)\bigr)\bigr)}\,\mathrm{d}Y.$$

Therefore the function $\mathcal{A}_\phi^{\pi,\theta^A} f : \mathfrak{g} \times \mathfrak{g}^* \to \mathbb{C}$ is a partial inverse Fourier transform of the function defined on $\mathfrak{g} \times \mathfrak{g}$ by

$$(X, Y) \mapsto \overline{\tau_A\bigl(X, -\Psi_X^{-1}(Y)\bigr)}\, f\bigl(-\Psi_X^{-1}(Y)\bigr)\,\overline{\phi\bigl((-X) * \bigl(-\Psi_X^{-1}(Y)\bigr)\bigr)} : \mathfrak{g} \to \mathbb{C}.$$

On the other hand, it was noted in the proof of [1, Th. 4.4(4)] that each of the mappings Σ_1, $\Sigma_2 : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ defined by

$$\Sigma_1(Y, Z) = \bigl(-Y, Y * (-Z)\bigr) \quad \text{and} \quad \Sigma_2(V, W) = \bigl(-\Psi_W(V), W\bigr)$$

is a polynomial diffeomorphism whose inverse is a polynomial. Since

$$\Sigma_2^{-1}(Y,X)=\big(\Psi_X^{-1}(-Y),X\big)$$

and $\tau_A\in\mathcal{C}^\infty_{\mathrm{pol}}(\mathfrak{g}\times\mathfrak{g})$, it then easily follows by [8, Lemma A.2.1(a)] that we have a well-defined continuous sesquilinear mapping

$$\mathcal{S}(\mathfrak{g})\times\mathcal{S}(\mathfrak{g})\to\mathcal{S}\big(\mathfrak{g}\times\mathfrak{g}^*\big),\quad (f,\phi)\mapsto\mathcal{A}_\phi^{\pi,\theta^A}f.$$

Thus the representation π satisfies the growth condition along the mapping θ^A.

(3) Use the above assertion (3) along with [1, Th. 4.4(4)].

(4) The assertion follows as a direct consequence of [3, Lemma 2.5]. □

In the next corollaries we denote by π the representation π_ϵ in Theorem 4.4 for $\epsilon=1$. Recall that we work with a finite-dimensional simply connected nilpotent Lie group G with an admissible function space $\mathcal{F}$ such that there exist the continuous inclusion maps $\mathfrak{g}^*\hookrightarrow\mathcal{F}\hookrightarrow\mathcal{C}^\infty_{\mathrm{pol}}(G)$, where the embedding $\mathfrak{g}^*\hookrightarrow\mathcal{F}$ is given by $\xi\mapsto\xi\circ\log_G$. Moreover $M=\mathcal{F}\rtimes_\lambda G$, and the aforementioned unitary representation $\pi:M\to\mathcal{B}(L^2(G))$ is defined by $\pi(\phi,g)f=\mathrm{e}^{\mathrm{i}\phi}\lambda_g f$ for all $\phi\in\mathcal{F}$, $g\in G$, and $f\in L^2(G)$.

If we have $A\in\Omega^1(G)$ such that $\langle A,\iota^{\mathrm{R}}X\rangle\in\mathcal{F}$ whenever $X\in\mathfrak{g}$, and we define the linear mapping

$$\theta^A:\mathfrak{g}\times\mathfrak{g}^*\to\mathfrak{m}=\mathcal{F}\ltimes_{\dot\lambda}\mathfrak{g},\quad (X,\xi)\mapsto\big(\xi\circ\log_G+\langle A,\iota^{\mathrm{R}}X\rangle,X\big)$$

as in Corollary 4.7, then one can consider the modulation spaces of symbols for the localized Weyl calculus for the representation π along the linear mapping θ^A. These are just the modulation spaces for the representation $\pi^\#:M\ltimes M\to\mathcal{B}(L^2(\mathfrak{g}\times\mathfrak{g}^*))$ with respect to the linear mapping $(\theta^A,\theta^A):(\mathfrak{g}\times\mathfrak{g}^*)\times(\mathfrak{g}\times\mathfrak{g}^*)\to\mathfrak{m}\ltimes\mathfrak{m}$. It follows by Remark 3.14 that for arbitrary $\Phi\in\mathcal{S}(\mathfrak{g}\times\mathfrak{g}^*)$ and $F\in\mathcal{S}'(\mathfrak{g}\times\mathfrak{g}^*)$ the corresponding ambiguity function $\mathcal{A}_\Phi^{\pi^\#,\theta^A\times\theta^A}F:(\mathfrak{g}\times\mathfrak{g}^*)\times(\mathfrak{g}\times\mathfrak{g}^*)\to\mathbb{C}$ is given by the formula

$$\begin{aligned}\big(\mathcal{A}_\Phi^{\pi^\#,\theta^A\times\theta^A}F\big)\big((X_1,\xi_1),(X_2,\xi_2)\big)&=\big(\pi^\#\big(\exp_{M\ltimes M}\big(\theta^A(X_1,\xi_1),\theta^A(X_2,\xi_2)\big)\big)F\mid\Phi\big)_{L^2(\mathfrak{g}\times\mathfrak{g}^*)}\\&=\iint\limits_{\mathfrak{g}\times\mathfrak{g}^*}\big(\mathrm{e}^{\mathrm{i}\langle\cdot,(X_1+X_2,\xi_1+\xi_2)\rangle}\#^{\theta^A}F\#^{\theta^A}\mathrm{e}^{-\mathrm{i}\langle\cdot,(X_1,\xi_1)\rangle}\big)\overline{\Phi(\cdot)}\end{aligned}$$

where $\#^{\theta^A}$ stands for the Moyal product on $\mathfrak{g}\times\mathfrak{g}^*$ defined by means of the magnetic potential A. For $r,s\in[1,\infty]$ and the window function $\Phi\in\mathcal{S}(\mathfrak{g}\times\mathfrak{g}^*)$ we have the modulation space of symbols

$$M_\Phi^{r,s}\big(\pi^\#,\theta^A\times\theta^A\big)=\big\{F\in\mathcal{S}'\big(\mathfrak{g}\times\mathfrak{g}^*\big)\mid\mathcal{A}_\Phi^{\pi^\#,\theta^A\times\theta^A}F\in L^{r,s}\big(\big(\mathfrak{g}\times\mathfrak{g}^*\big)\times\big(\mathfrak{g}\times\mathfrak{g}^*\big)\big)\big\}.$$

Corollary 4.8. *In the above setting, pick $\phi_1,\phi_2\in\mathcal{S}(G)\setminus\{0\}$. If $r,s,r_1,s_1,r_2,s_2\in[1,\infty]$ satisfy the conditions*

$$r \leqslant s, \qquad r_2, s_2 \in [r, s], \quad \textit{and} \quad \frac{1}{r_1} - \frac{1}{r_2} = \frac{1}{s_1} - \frac{1}{s_2} = 1 - \frac{1}{r} - \frac{1}{s},$$

then for every symbol $a \in M^{r,s}_{\mathcal{W}(\phi_1,\phi_2)}(\pi^\#, \theta^A \times \theta^A)$ *we have a bounded linear operator*

$$\mathrm{Op}^{\theta^A}(a) : M^{r_1,s_1}_{\phi_1}(\pi, \theta^A) \to M^{r_2,s_2}_{\phi_2}(\pi, \theta^A).$$

Moreover, the linear mapping

$$\mathrm{Op}^{\theta^A} : M^{r,s}_{\mathcal{W}(\phi_1,\phi_2)}(\pi^\#, \theta^A \times \theta^A) \to \mathcal{B}\big(M^{r_1,s_1}_{\phi_1}(\pi, \theta^A), M^{r_2,s_2}_{\phi_2}(\pi, \theta^A)\big)$$

is continuous.

Proof. It follows by Theorem 4.4 that the space of smooth vectors for the representation π is the Schwartz space $\mathcal{S}(G)$. Moreover, Corollary 4.7 shows that we can apply Corollary 3.17 for the representation π. Now the conclusion follows by using the latter corollary. □

Corollary 4.9. *Assume the setting of Corollary* 4.7, *let* $\phi_1, \phi_2 \in \mathcal{S}(G) \setminus \{0\}$, *and* $r, s \in [1, \infty]$ *such that* $\frac{1}{r} + \frac{1}{s} = 1$. *Then for every* $a \in M^{r,s}_{\mathcal{W}(\phi_1,\phi_2)}(\pi^\#, \theta^A \times \theta^A)$ *we have* $\mathrm{Op}^{\theta^A}(a) \in \mathcal{B}(L^2(G))$. *Moreover,* $\mathrm{Op}^{\theta^A} : M^{r,s}_{\mathcal{W}(\phi_1,\phi_2)}(\pi^\#, \theta^A \times \theta^A) \to \mathcal{B}(L^2(G))$ *is a continuous linear mapping.*

Proof. This is the special case of Corollary 4.8 with $r_1 = s_1 = r_2 = s_2 = 2$, since Remark 3.3 shows that $M^{2,2}_{\phi_j}(\pi, \theta^A) = L^2(G)$ for $j = 1, 2$. □

Corollary 4.10. *Assume the setting of Corollary* 4.7 *and let* $\phi_1, \phi_2 \in \mathcal{S}(G) \setminus \{0\}$. *Then for every* $a \in M^{1,1}_{\mathcal{W}(\phi_1,\phi_2)}(\pi^\#, \theta^A \times \theta^A)$ *we have* $\mathrm{Op}^{\theta}(a) \in \mathfrak{S}_1(L^2(G))$, *and the linear mapping* $\mathrm{Op}^{\theta^A} : M^{1,1}_{\mathcal{W}(\phi_1,\phi_2)}(\pi^\#, \theta^A \times \theta^A) \to \mathfrak{S}_1(L^2(G))$ *is continuous.*

Proof. Recall from Theorem 4.4 that the space of smooth vectors for the representation π is the Schwartz space $\mathcal{S}(G)$. Moreover, Corollary 4.7 shows that we can use Proposition 3.21, and the conclusion follows. □

Remark 4.11. In the special case when G is the abelian group $(\mathbb{R}^n, +)$ and we have the magnetic potential $A \in \Omega^1(\mathbb{R}^n)$, the magnetic Weyl calculus

$$\mathrm{Op}^{\theta^A} : \mathcal{S}'\big(\mathbb{R}^n \times (\mathbb{R}^n)^*\big) \to \mathcal{L}\big(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n)\big)$$

is just the one constructed in [24]. In this setting, we note the following:

(1) In the case when the coefficients of the magnetic field $B := \mathrm{d}A \in \Omega^2(\mathbb{R}^n)$ belong to the Fréchet space $\mathrm{BC}^\infty(\mathbb{R}^n)$ of smooth functions on $\mathbb{R}^n$ which are bounded along with all of their partial derivatives, one established in [22] some sufficient conditions on a symbol $a \in \mathcal{S}'(\mathbb{R}^n \times (\mathbb{R}^n)^*)$ that ensure that the magnetic pseudo-differential operator $\mathrm{Op}^{\theta^A}(a)$ is bounded on $L^2(\mathbb{R}^n)$. In this connection, we note that the previous Corollary 4.9 provides another type of sufficient conditions for L^2-boundedness when the coefficients of the magnetic field B belong to the larger LF-space $\mathcal{C}^\infty_{\mathrm{pol}}(\mathbb{R}^n)$ of smooth functions on $\mathbb{R}^n$ that grow

polynomially together with their partial derivatives of arbitrary order. This follows since for every closed 2-form $B \in \Omega^2(\mathbb{R}^n)$ whose coefficients belong to $\mathcal{C}^\infty_{\text{pol}}(\mathbb{R}^n)$, one can construct in the usual way a 1-form $A \in \Omega^1(\mathbb{R}^n)$ whose coefficients belong to $\mathcal{C}^\infty_{\text{pol}}(\mathbb{R}^n)$ again such that $\mathrm{d}A = B$.

(2) It follows by the comments preceding Corollary 4.8 that the modulation spaces of symbols $M^{r,s}_\Phi(\pi^\#, \theta^A \times \theta^A)$ can be alternatively described in terms of the modulation mapping which was introduced in [25] in the case of the abelian group $G = (\mathbb{R}^n, +)$ by using the magnetic Moyal product $\#^A$. It had been already noted in [24] that the magnetic Moyal product on $(\mathbb{R}^n, +)$ actually depends only on the magnetic field $B = \mathrm{d}A$. This assertion holds true for the two-step nilpotent Lie groups, as an easy consequence of the formula established in Th. 4.7 in [1].

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