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Bipartite bihypergraphs: A survey and new results

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Abstract

Let H^0 and H^1 be hypergraphs with the same vertex-set V. The ordered pair $H=(H^0, H^1)$ is called a *bihypergraph*. A set $S \subseteq V$ is *stable* in H^i if S contains no hyperedges of H^i, $i=0,1$. A bihypergraph $H=(H^0, H^1)$ is called *bipartite* if there exists an ordered partition $S^0 \cup S^1 = V(H)$ such that the set S^i is stable in H^i for $i=0,1$.

In Section 1, we survey numerous applications of bipartite bihypergraphs. In Section 3, we show that recognizing bipartite bihypergraphs within classes of k-complete bihypergraphs can be done in polynomial time. A bihypergraph $H=(H^0, H^1)$ is called *k-complete*, $k \geqslant 0$, if each k-subset of $V(H)$ contains a hyperedge of H, i.e., a hyperedge of H^0 or H^1. Moreover, we can construct all bipartitions of a k-complete bihypergraph, if any, in polynomial time.

A bihypergraph $H=(H^0, H^1)$ is called *strongly bipartite* if each maximal stable set of H^0 is a transversal of H^1. We show that recognizing strongly bipartite bihypergraphs (H^0, H^1) is a co-NP-complete problem even in the case where H^0 is a graph and H^1 has exactly one hyperedge. Some examples of strongly bipartite bihypergraphs are given.

Keywords: Bipartite bihypergraphs; Bipartite hypergraphs; k-complete bihypergraphs; Satisfiability problem

1. Bipartite bihypergraphs

We use standard graph-theoretic terminology, see Melnikov et al. [20]. A *hypergraph* is an ordered pair $H=(V, E)$, where $V=V(H)$ is a finite set, called *vertex-set* of H, and $E=E(H)$ is multi-set of some subsets of V, called *hyperedge-set* of H. Thus, a hypergraph may contain *multiple hyperedges*, which coincide as subsets of V but have different names. Also, the empty set may be a hyperedge. A set $S \subseteq V(H)$ in a hypergraph H is *stable* in H if S contains no hyperedges of H.

Definition 1. Let H^0 and H^1 be hypergraphs with the same vertex-set V. The ordered pair $H=(H^0, H^1)$ is called a *bihypergraph*.

Every hyperedge of either H^0 or H^1 is considered as a *hyperedge* of H.

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Definition 2. A bihypergraph $H=(H^0, H^1)$ is called *bipartite* if there exists an ordered partition $S^0 \cup S^1 = V(H)$, called a *bipartition*, such that the set S^i is stable in H^i for $i=0, 1$. We denote by $\mathscr{BIPBIH}$ the class of all bipartite bihypergraphs.

Equivalently, a bihypergraph is bipartite if it has the following Property S.

Property S. *There exists an ordered partition $T^0 \cup T^1 = V(H)$ such that the set T^i is a transversal in H^i for $i=0, 1$.*

Recall that a *transversal* in a hypergraph is a vertex subset that intersects all hyperedges. It is obvious that a set $S \subseteq V(H)$ is stable in a hypergraph H if and only if $V(H)\backslash S$ is a transversal in H. Property S is named for Schrijver who considered disjoint transversals of a bihypergraph in Schrijver [26].

Decision Problem 1 (*Bipartite bihypergraph*).
Instance: A bihypergraph $H=(H^0, H^1)$.
Question: Is H bipartite?

The problem can be formulated as a system of Boolean equations. Let $H=(H^0, H^1)$ be a bihypergraph on vertex-set $V=\{1, 2, \ldots, n\}$. If there exists a bipartition $S^0 \cup S^1$ of H, then we define Boolean variables $x_1, x_2, \ldots, x_n$:

$$x_i^* = \begin{cases} 0 & \text{if } i \in S^0, \\ 1 & \text{if } i \in S^1. \end{cases}$$

The point $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ is a solution to the system

$$\begin{cases} \bigvee_{i\in e} x_i = 1 & \text{for each } e \in E(H^0), \\ \prod_{i\in e} x_i = 0 & \text{for each } e \in E(H^1). \end{cases} \tag{1}$$

Conversely, each solution to the system (1) determines a bipartition of H.

Proposition 1. *A bihypergraphs is bipartite if and only if the system* (1) *is consistent.*

Since $\prod_{i\in e} x_i = 0$ if and only if $\bigvee_{i\in e} \overline{x}_i = 1$, the system (1) can be written as

$$\begin{cases} \bigvee_{i\in e} x_i = 1 & \text{for each } e \in E(H^0), \\ \bigvee_{i\in e} \overline{x}_i = 1 & \text{for each } e \in E(H^1). \end{cases} \tag{2}$$

In general, recognizing bipartite bihypergraphs is NP-complete, see Theorem 1. However, if both H^0 and H^1 are graphs, it is easy to recognize whether (H^0, H^1) is bipartite. Gavril [14] called this problem 2-colors graph partition. His Theorem 1 and the algorithm in Even et al. [12] give a linear-time sequential algorithm for the problem. Also, Gavril [14] constructed a parallel algorithm for it requiring $O(\log n)$ time and $O(n^3/(\log_4 n)^{1.5})$ processors on a CRCW PRAM.

2. Applications of bipartite bihypergraphs

In this section, we give a survey of known applications of bipartite bihypergraphs to bipartite hypergraphs, Satisfiability Problem, graph colorings, distinct representatives, and graph vertex bipartitions with prescribed properties. Also, we propose a new connection with Boolean dualization.

2.1. Bipartite hypergraphs

A hypergraph $H=(V, E)$ is *bipartite* if there exists a partition $S^0 \cup S^1 = V$, called a *bipartition*, such that the sets S^0 and S^1 are stable in H.

Decision Problem 2 (*Bipartite hypergraph*).
Instance: A hypergraph H.
Question: Is H bipartite?

Bipartite Hypergraph is a particular case of Decision Problem 1. Indeed, a hypergraph H is bipartite if and only if the corresponding bihypergraph (H, H) is bipartite. The Bipartite Hypergraph Problem is also NP-complete, see Garey and Johnson [13]. In fact, Lovász [18] proved that deciding whether a hypergraph H is bipartite is as hard as to determine the chromatic number.

A hypergraph H is bipartite if and only if

Property B. *There exists a transversal in H which is a stable set.*

Here, "B" stands for Felix Bernstein who noted in 1908 that a countable system of infinite sets has Property B. Sufficient conditions for a hypergraph to be bipartite and many other related properties were obtained by Erdős [8–10], Erdős and Hajnal [11], Miller [21], and Woodall [31]. Woodall [31] and Stein [28] have found interesting connections between bipartite hypergraphs and planar graphs. For example, Four Color Conjecture is equivalent to the following: the family of all odd circuits of a planar graph (considered as sets of edges) has property B. Some sufficient conditions for bipartite hypergraphs were extended to bihypergraphs by Cowen [5].

2.2. Connections with SAT

Let $X=\{x_1, x_2, \dots, x_n\}$ be a set of 0–1 variables. We define the set of literals over X, $L_X=\{x_i, \overline{x}_i : i=1, 2, \dots, n\}$, where $\overline{x}_i = 1 - x_i$ is the *negation* of x_i. A *truth assignment to* X is a mapping $t : X \to \{0, 1\}$ that assigns a value $t(x_i) \in \{0, 1\}$ to each variable $x_i \in X$. We extend t to L_X putting $t(\overline{x}) = \overline{t(x)}$. A literal $l \in L_X$ is *true under* t if $t(l)=1$. A *clause* over X is a conjunction of some literals of L_X. Let $C = \{c_1, c_2, \dots, c_m\}$ be a set of clauses over X. A truth assignment t to X *satisfies* a clause $c_j \in C$ if c_j involves at least one true literals under t. The following Satisfiability Problem, or SAT, is well-known.

Decision Problem 3 (*SAT*).
Instance: A set of clauses C over X.
Question: Is there a truth assignment to X that satisfies all clauses in C?

In other words, we are asked whether the conjunctive normal form (CNF) defined by C can take value 1 or it equals zero identically.

Theorem 1 (*see Cowen [5]*). SAT *is polynomial-time reducible to the* Bipartite Bihypergraph Problem.

Proof. Given an instance (C, X) to SAT with $C=\{c_1, c_2, \dots, c_m\}$ and $X=\{x_1, x_2, \dots, x_n\}$, we define the following *sat-bihypergraph* $H_C = (G, H)$ on vertex-set L_X:

- $E(G)=\{x_i\overline{x}_i : i=1, 2, \dots, n\}$, and
- $E(H)=C$, where each clause in C is considered as a set of literals.

It is easy to see that there exists a truth assignment satisfying C if and only if the corresponding sat-bihypergraph H_C is bipartite. □

Since 3-SAT is NP-complete, Bipartite Bihypergraph is NP-complete for bihypergraphs $H = (H^0, H^1)$ such that H^0 is a 1-*regular* graph (that is, H_0 consists of pairwise disjoint edges covering all vertices), and each hyperedge of H^1 has at most three vertices. Cowen [2–5] and Kolany [17] extended known methods for SAT (Analytic Tableaux of Smullyan [27], Resolution Proof Procedure, Davis–Putnam [6]) to so-called satisfiability on hypergraphs; see also Pretolani [23].

Corollary 1 (*Cowen [5]*). *Recognizing bipartite bihypergraphs and* SAT *are reducible to each other in linear time*.

Proof. The system (2) defines an instance to SAT problem. The result follows from Proposition 1, since (1) and (2) are equivalent. □

2.3. Applications to coloring problems

A vertex coloring $c : V(G) \to \{1, 2, \ldots, k\}$ of a graph G is called *proper* if adjacent vertices always have different colors. Suppose that a list $L(u) \subseteq \{1, 2, \ldots, k\}$ of colors is assigned to every vertex u of a graph G. A *list coloring from* L of G is a proper vertex coloring $c : V(G) \to \{1, 2, \ldots, k\}$ such that $c(u) \in L(u)$ for each vertex u of G, see West [30]. A graph G is *L-list colorable* if G admits a list coloring from L.

Decision Problem 4 (*List coloring*).
Instance: A graph G with lists $L(u)$ for all $u \in V(G)$.
Question: Is G an L-list colorable graph?

Given an instance (G, L) to List Coloring Problem, we construct a bihypergraph $H_{G,L} = (H^0, H^1)$ on the set $V = \{(u, i) : u \in V(G), i \in L(u)\}$ as follows:

- E^0 contains a hyperedge $\{(u, i) : i \in L(u)\}$ for each $u \in V(G)\}$, and
- E^1 consists of all the pairs $\{(u, i), (v, i)\}$, where $uv \in E(G)$ and $i \in L(u) \cap L(v)$.

Proposition 2 (*Cowen [5]*). *A graph G is L-list colorable if and only if the bihypergraph $H_{G,L}$ is bipartite*.

Taking $L(u) = \{1, 2, \ldots, k\}$ for all u, we may apply this result to ordinary vertex k-colorings.

2.4. Connections with distinct representatives

Let S be a finite set, and let $F = \{S_1, S_2, \ldots, S_k\}$ be a family of subsets $S_i \subseteq S$. We say that F has a *system of distinct representatives*, or *SDR*, if there is an injective mapping $\phi : \{1, 2, \ldots, k\} \to S$ such that $\phi(i) \in S_i$ for all $i = 1, 2, \ldots, k$.

Decision Problem 5 (*Distinct representatives*).
Instance: A family $F = \{S_1, S_2, \ldots, S_k\}$ of subsets $S_i \subseteq S$ of a finite set S.
Question: Does F have an SDR?

Given an instance $F = \{S_1, S_2, \ldots, S_k\}$ to Distinct Representatives Problem, we construct a bihypergraph $H_F = (H^0, H^1)$ on the set $V = \{(s, i) : s \in S_i, i = 1, 2, \ldots, k\}$ as follows:

- E^0 contains a hyperedge $\{(s, i) : s \in S_i\}$ for all $i \in \{1, 2, \ldots, k\}$, and
- E^1 consists of edges $\{(s, j), (s, k)\}$ for all $s \in S$ and distinct $j, k \in \{1, 2, \ldots, k\}$.

Proposition 3 (*Cowen [5]*). *A family F has an SDR if and only if the bihypergraph H_F is bipartite*.

2.5. Applications to graph bipartitions

Zverovich [32] found that bipartite bihypergraphs constitute a natural model for studying hereditary classes of graphs defined in terms of vertex bipartitions. For a set of graphs Z, a graph G is called *Z-free* if no graph of Z is an induced subgraph of G. A class of graphs is *hereditary* if and only if it consists of all Z-free graphs for some set Z. For a set $X \subseteq V(G)$, the subgraph of G *induced* by X, denoted by $G(X)$, has X as its vertex-set, and vertices $x, x' \in X$ are adjacent in $G(X)$ if and only if they are adjacent in G.

Let $\mathscr{P}_i$ be a hereditary class of all Z_i-free graphs, $i = 0, 1$. We assume that each $\mathscr{P}_i$ is given by the set Z_i.

Decision Problem 6 (*Graph* $(\mathscr{P}_0, \mathscr{P}_1)$*-bipartition*).
Instance: A graph G.
Question: Is there a bipartition $V_0 \cup V_1 = V(G)$ such that the induced subgraph $G(V_i)$ belongs to $\mathscr{P}_i$ for $i = 0, 1$?

The sets Z_0 and Z_1 define two hypergraphs H^0 and H^1 on $V(G)$, namely $E(H^i) = \{X \subseteq V(G) : G(X) \in Z_i\}$, $i = 0, 1$.

Proposition 4. *A graph G has a* $(\mathscr{P}_0, \mathscr{P}_1)$*-bipartition if and only if the bihypergraph* (H^0, H^1) *is bipartite.*

Zverovich [32] applied Proposition 4 to (p, q)-split graphs of Gyárfás [15] and (α, β)-polar graphs of Tyshkevich and Chernyak [29]. Actually, Zverovich [32] considered a family of hereditary classes of bipartite bihypergraphs. It was shown that each class in the family has a finite forbidden induced subhypergraph characterization. Namely this result and Proposition 4 were applied to hereditary classes of graphs. For further development see Zverovich and Zverovich [33] and Zverovich and Zverovich [34]. This approach can easily be extended to other hereditary systems, such as subgraphs, homeomorphic subgraphs, minors, etc.

2.6. Applications to Boolean dualization

Let $\mathbb{B}^n$ denote the n-dimensional Boolean cube. For $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{B}^n$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{B}^n$, we write $\mathbf{x} \leqslant \mathbf{y}$ (respectively, $\mathbf{x} \geqslant \mathbf{y}$) if $x_i \leqslant y_i$ (respectively, $x_i \geqslant y_i$) for $i = 1, 2, \ldots, n$. Also, $\mathbf{x} < \mathbf{y}$ (respectively, $\mathbf{x} > \mathbf{y}$) means that $\mathbf{x} \leqslant \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$ (respectively, $\mathbf{x} \geqslant \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$).

Definition 3. A Boolean function $f : \mathbb{B}^n \to \mathbb{B}$ is called *monotone* if $\mathbf{x} \leqslant \mathbf{y}$ and $f(\mathbf{x}) = 1$ imply $f(\mathbf{y}) = 1$.

For a Boolean function $f : \mathbb{B}^n \to \mathbb{B}$, $T(f) = \{\mathbf{x} : f(\mathbf{x}) = 1\}$ is the set of *true points*. Similarly, $F(f) = \{\mathbf{x} : f(\mathbf{x}) = 0\}$ is the set of *false points*. If *f* is monotone, then $\mathrm{MT}(f) = \{\mathbf{x} \in T(f) : f(\mathbf{y}) = 0 \text{ for each } \mathbf{y} < \mathbf{x}\}$ is the set of *minimal true points*. Similarly, $\mathrm{MF}(f) = \{\mathbf{x} \in F(f) : f(\mathbf{y}) = 1 \text{ for each } \mathbf{y} > \mathbf{x}\}$ is the set of *maximal false points*. Note that a monotone Boolean function *f* is uniquely determined by the set $\mathrm{MT}(f)$.

As usual, $\overline{x} = 1 - x$ is the *negation* of $x \in \{0, 1\}$. Accordingly, $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ for $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{B}^n$. The operation $\mathbf{x} \to \overline{\mathbf{x}}$ is also known as *complementation*. We define the *complement* $\overline{f}$ of a Boolean function $f : \mathbb{B}^n \to \mathbb{B}$ by $\overline{f}(\mathbf{x}) = \overline{f(\mathbf{x})}$ for all $\mathbf{x} \in \mathbb{B}^n$.

Definition 4. The *dual* of a Boolean function $f : \mathbb{B}^n \to \mathbb{B}$ is the Boolean function $f^{\mathrm{d}} : \mathbb{B}^n \to \mathbb{B}$ defined by $f^{\mathrm{d}}(\mathbf{x}) = \overline{f}(\overline{\mathbf{x}})$.

It is well-known and easy to see that the dual of a monotone Boolean function is also a monotone Boolean function. Suppose that a monotone Boolean function *f* is given by the set $\mathrm{MT}(f)$ of its minimal true points. We want to construct the set $\mathrm{MT}(f^{\mathrm{d}})$ to determine the dual of *f*. A related decision problem can be formulated as follows.

Decision Problem 7 (*Boolean dualization*).
Instance: A monotone Boolean function *f* given by the set $\mathrm{MT}(f)$, and a subset *S* of $\mathrm{MT}(f^d)$.
Question: Is *S* equal to $\mathrm{MT}(f^d)$?

Complexity of this problem is unknown. Boolean Dualization Problem has a simple hypergraph interpretation. A *clutter* is a hypergraph $H = (V, E)$ such that $e \subseteq e'$ for $e, e' \in E$ implies $e = e'$. There is a natural bijection between monotone Boolean functions and clutters. Let $V = \{1, 2, \ldots, n\}$. The *characteristic vector* of a subset $X \subseteq V$ is $\mathrm{char}(X) = (x_1, x_2, \ldots, x_n) \in \mathbb{B}^n$, where $x_i = 1$ if and only if $i \in X$. Conversely, the *set corresponding to* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{B}^n$ is $\mathrm{set}(\mathbf{x}) = \{i \in V : x_i = 1\}$. A clutter $C = (V, E)$ defines a monotone Boolean function f_C on $\mathbb{B}^n$: $\mathrm{MT}(f) = \{\mathrm{char}(e) : e \in E\}$. Conversely, minimal true points of a monotone Boolean function f defines hyperedge-set of a clutter C_f on *V*. The following results are folklore and straightforward.

Proposition 5. $\mathrm{MT}(f^{\mathrm{d}}) = \{\overline{\mathbf{x}} : \mathbf{x} \in \mathrm{MF}(f)\}$ *for each monotone Boolean function f.*

Let $\mathrm{Trans}(C)$ be the set of all minimal transversals of a clutter C.

Proposition 6. *For each monotone Boolean function f,* $\{\mathrm{set}(\overline{\mathbf{x}}) : \mathbf{x} \in \mathrm{MF}(f)\}$ *is* $\mathrm{Trans}(C_f)$.

Thus, Boolean Dualization is the same problem as to find all minimal transversals of a given clutter.

Decision Problem 8 (*Clutter transversal*).
Instance: A clutter C and a set $T \subseteq \mathrm{Trans}(C)$.
Question: Is T equal to $\mathrm{Trans}(C)$?

Proposition 7. Boolean Dualization *and* Clutter Transversal *are polynomial-time equivalent decision problems*.

Now, we establish a connection with bipartite bihypergraphs.

Proposition 8. Boolean Dualization *is a particular case of* Bipartite Bihypergraph Problem.

Proof. According to Proposition 6 we may deal with Clutter Transversal Problem. Let a clutter C and a set $T \subseteq \mathrm{Trans}(C)$ be an instance to Decision Problem 8. We define a bihypergraph $H = (C, C')$ on $V(C)$, where $E(C') = T$.

Suppose (S, S') is a bipartition of H. Since S is a stable set in C, $S' = V(C)\backslash S$ is a transversal of C. The set S' is stable in C', therefore S' does not contain any member of T. Thus, S' contains minimal transversals which are not in T, i.e., the answer to Clutter Transversal is "no" [$T \neq \mathrm{Trans}(C)$].

Now, suppose that $T \neq \mathrm{Trans}(C)$. We consider a minimal transversal $S' \in \mathrm{Trans}(C)\backslash T$. Clearly, S' is a stable set in C'. Since S' is a transversal of C, the set $S = V(C)\backslash S'$ is a stable set in C. Thus, (S, S') is a bipartition of H. □

3. Recognizing k-complete bipartite bihypergraphs

Since Bipartite Bihypergraph is a hard problem, it is natural to impose additional conditions to obtain polynomial-time recognizible classes of bipartite bihypergraphs. We define a parametric family of such classes depending on a single parameter k. If $X \subseteq Y$ and $|X| = k$, then we say that X is a *k-subset* of Y.

Definition 5 (*Zverovich [32]*). A bihypergraph $H = (H^0, H^1)$ is called *k-complete*, $k \geqslant 0$, if each k-subset of $V(H)$ contains a hyperedge of H. We denote by $\mathscr{COBI}(k)$ the class of all k-complete bihypergraphs.

The condition of being k-complete is very natural in view of Ramsey's theorem (Ramsey [24]). Indeed if $k = r(m, n)$ is the Ramsey number, then each vertex k-subset in a graph contains either an m-clique or a stable n-set. Thus, bounded cliques and stable sets may be considered as hyperedges of a k-complete bihypergraph. Clearly,

$$\mathscr{COBI}(0) \subseteq \mathscr{COBI}(1) \subseteq \cdots \subseteq \mathscr{COBI}(k) \subseteq \cdots$$

and $\bigcup_{i=0}^{\infty} \mathscr{COBI}(k)$ contains all bihypergraphs having at least one hyperedge. We show that recognizing bipartite bihypergraphs within each class $\mathscr{COBI}(k)$ can be done in polynomial time. Moreover, we can construct all bipartitions, if any, also in polynomial time. We start with two auxiliary results.

Claim 1. *Let* $H = (H^0, H^1) \in \mathscr{COBI}(k)$. *If* S^i *is a stable set in* H^i, $i = 0, 1$, *then* $|S^0 \cap S^1| < k$.

Proof. The set $S^0 \cap S^1$ does not contain a hyperedges of H^0 as being a subset of a stable set S^0 of H^0. Similarly, it cannot contain any hyperedge of H^1. Since H is a k-complete bihypergraph, the result follows. □

Claim 2. *Let* $H = (H^0, H^1)$ *be a bipartite k-complete bihypergraph. Suppose that* S^0 *and* S^1 *are disjoint stable sets of* H^0 *and* H^1, *respectively. If* $S^0 \cup S^1 \neq V(H)$, *then there exist sets* $X^i \subseteq S^i$, $i = 0, 1$, *and disjoint sets*

$Y^0, Y^1 \subseteq (V(H)\backslash(S^0 \cup S^1)) \cup (X^0 \cup X^1)$ *such that*

(B1) $|X^i| < k$, $i = 0, 1$,
(B2) $(S^i\backslash X^i) \cup Y^i$ *is a stable set in* H^i, $i = 0, 1$, *and*
(B3) $|Y^0 \cup Y^1| = |X^0 \cup X^1| + 1$.

Proof. Since H is a bipartite bihypergraph, there exists a bipartition $A^0 \cup A^1$ of $V(H)$. We denote $X^0 = S^0 \cap A^1$ and $X^1 = S^1 \cap A^0$. By Claim 1, $|X^i| < k$ for $i = 0, 1$, i.e., (B1) holds.

Since $S^0 \cup S^1 \neq V(H)$, the set $R = (V(H)\backslash(S^0 \cup S^1)) \cup (X^0 \cup X^1)$ contains at least $N = |X^0 \cup X^1| + 1$ vertices. We choose an N-subset Y in R. It remains to define $Y^i = Y \cap A^i$, $i = 0, 1$, and note that $(S^i\backslash X^i) \cup Y^i$ is a stable set in H^i as being a subset of A^i. Both (B2) and (B3) follow. □

Here is our main result.

Theorem 2. *Recognizing bipartite bihypergraphs within each class* $\mathscr{COBI}(k)$ *can be done in polynomial time. Moreover, it is possible to construct at least one bipartition of a k-complete bipartite bihypergraph in polynomial time.*

Proof. Let $H = (H^0, H^1)$ be a k-complete bihypergraph. If $\emptyset$ is a hyperedge of H, then H is not bipartite, since either H^0 or H^1 has no stable sets. Suppose it is not the case. To recognize whether H is bipartite, we choose initial stable sets $S_0^0 = \emptyset$ in H^0 and $S_0^1 = \emptyset$ in H^1. Suppose that we have already constructed disjoint stable sets S_i^0 in H^0 and S_i^1 in H^1 for some $i \geqslant 0$. If $S_i^0 \cup S_i^1 = V(H)$, then we have a bipartition of H.

If $S_i^0 \cup S_i^1 \neq V(H)$, then we apply Claim 2, that is we consider the set $\mathscr{T}$ of all 4-tuples $(X_i^0, X_i^1, Y_i^0, Y_i^1)$ such that

- $X_i^0 \subseteq S_i^0$ and $X_i^1 \subseteq S_i^1$,
- $|X_i^0| < k$ and $|X_i^1| < k$,
- $Y_i^0, Y_i^1 \subseteq (V(H)\backslash(S_i^0 \cup S_i^1)) \cup (X_i^0 \cup X_i^1)$ are disjoint sets, and
- $|Y_i^0 \cup Y_i^1| = |X_i^0 \cup X_i^1| + 1$.

Since $|X_i^0| < k$, $|X_i^1| < k$, and $|Y_i^0| + |Y_i^1| = |X_i^0| + |X_i^1| + 1 < 2k$, we can generate the set $\mathscr{T}$ in polynomial time. For each $(X_i^0, X_i^1, Y_i^0, Y_i^1)$ in $\mathscr{T}$, we check in polynomial time whether $(S_i^0\backslash X_i^0) \cup Y_i^0$ is a stable set in H^0 and $(S_i^1\backslash X_i^1) \cup Y_i^1$ is a stable set in H^1. If yes, we have found new disjoint stable sets in H^0 and H^1, namely $S_{i+1}^0 = (S_i^0\backslash X_i^0) \cup Y_i^0$ and $S_{i+1}^1 = (S_i^1\backslash X_i^1) \cup Y_i^1$. In this case, we continue the construction with S_{i+1}^0 and S_{i+1}^1. If not, H is not a bipartite bihypergraph by Claim 2.

Let $n = |V(H)|$. Since $|S_0^0 \cup S_0^1| = 0$ and $|S_{i+1}^0 \cup S_{i+1}^1| = |S_i^0 \cup S_i^1| + 1$ for all $i \geqslant 0$, either we can construct $S_n^0 \cup S_n^1$, thus obtaining a bipartition of H, or we reject H as being non-bipartite on some step $i < n$. Thus, our algorithm runs in polynomial time. □

Let us estimate complexity of the algorithm in the proof of Theorem 2. For each $i = 0, 1, \ldots, n-1$, we can generate all variants for the sets X_i^0 and X_i^1 in time $O(n^{2k-2})$. There are at most $O(n^{2k-1})$ possibilities to choose $Y_i^0 \cup Y_i^1$. To subdivide $Y_i^0 \cup Y_i^1$ into Y_i^0 and Y_i^1, we consider all possible 2^{2k-1} (which is a constant) variants. In $O(m)$ time, where m is the total number of hyperedges, we check whether $(S_i^0\backslash X_i^0) \cup Y_i^0$ is a stable set in H^0 and $(S_i^1\backslash X_i^1) \cup Y_i^1$ is a stable set in H^1. Thus, total complexity can be estimated as $O(mn^{4k-2})$.

Corollary 2. *It is possible to construct all bipartitions of a k-complete bihypergraph in polynomial time.*

Proof. Let $H = (H^0, H^1)$ be a k-complete bihypergraph. According to Theorem 2, we can recognize bipartiteness of H in polynomial time. Moreover, if H is bipartite, we can construct at least one bipartition $A^0 \cup A^1$ of H also in polynomial time.

Let $B^0 \cup B^1$ be an arbitrary bipartition of H. By Claim 1, $|A^0 \cap B^1| < k$ and $|A^1 \cap B^0| < k$. Hence to obtain an arbitrary bipartition from the initial bipartition $A^0 \cup A^1$, it is sufficient to consider all pairs (X^0, X^1) such that $X^i \subseteq A^i$, and $|X^i| < k$, $i = 0, 1$. For each pair (X^0, X^1), we check, whether the sets $B^0 = (A^0\backslash X^0) \cup X^1$ and $B^1 = (A^1\backslash X^1) \cup X^0$

are stable in H^0 and H^1, respectively. The result follows from the obvious fact that there are polynomially many pairs (X^0, X^1). □

We have $O(n^{2k-2})$ variants to choose $X^i \subseteq A^i$ with $|X^i| < k$, $i = 0, 1$. Then we check stability of the sets $B^0 = (A^0 \backslash X^0) \cup X^1$ and $B^1 = (A^1 \backslash X^1) \cup X^0$ in time $O(m)$. Thus, total complexity of the algorithm of Corollary 2 is $O(mn^{4k-2} + mn^{2k-2}) = O(mn^{4k-2})$.

4. Strongly bipartite bihypergraphs

Here, we consider a subclass of bipartite bihypergraphs.

Definition 6. A bihypergraph $H = (H^0, H^1)$ is called *strongly bipartite* if each maximal stable set of H^0 is a transversal of H^1.

Given a bihypergraph $H = (H^0, H^1)$, it is easy to find a maximal stable set S^0 in H^0 in a greedy way. If S^0 is a transversal of H^1, then $S^1 = V(H) \backslash S^0$ is a stable set of H^1. Thus, it is easy to recognize bipartiteness of strongly bipartite bihypergraphs.

Decision Problem 9 (*Strongly bipartite bihypergraph*).
Instance: A bihypergraph $H = (H^0, H^1)$.
Question: Is H strongly bipartite?

Theorem 3. *Recognizing strongly bipartite bihypergraphs* (H^0, H^1) *is a co-NP-complete problem even in the case where* H^0 *is a graph and* H^1 *has exactly one hyperedge.*

Proof. We use a polynomial-time reduction from SAT. Let (C, X) be an instance to SAT with $C = \{c_1, c_2, \ldots, c_m\}$ and $X = \{x_1, x_2, \ldots, x_n\}$. Recall that $L_X = \{x_i, \overline{x}_i : i = 1, 2, \ldots, n\}$. We define a bihypergraph (G, H) on $C \cup L_X$:

- in G, L_X induces a matching $x_i\overline{x}_i$, $i = 1, 2, \ldots, n$; vertices $l \in L_X$ and $c_j \in C$ are adjacent if and only if the clause C_j involves the literal l, and
- C is the only hyperedge of H.

Suppose that there exists a truth assignment ϕ that satisfies C. The set S of all true literals, considered as a vertex subset of L_X, is a maximal stable set in G. However, S is not a transversal of H. By definition, the bihypergraph (G, H) is not strongly bipartite.

Conversely, suppose that C is not satisfiable. It means that G does not have a stable set $S \subseteq L_X$ that dominates C. We say that S *dominates* C if each vertex of C is adjacent to a vertex of S. It follows that each maximal stable set in G intersects C, implying that (G, H) is a strongly bipartite bihypergraph. □

Below, we propose two examples of strongly bipartite bihypergraphs.

4.1. Triangle graphs

An interesting class of intersection graphs was introduced by McAvaney et al. [19], see also Anbeek et al. [1] and DeTemple et al. [7]. A *general partition graph* is the intersection graph G of a family of subsets of a set S with the property that every maximal independent set in G corresponds to a partition of S. All general partition graphs satisfy the triangle condition below. If an edge e connects vertices u and v, then we simply write $e = uv$.

Definition 7. A graph G is called a *triangle graph* if it satisfies the following condition:
Triangle condition: For every maximal stable set I and every edge $e = uv$ in $G - I$, there exists a vertex $w \in I$ such that $\{u, v, w\}$ induces a triangle in G.

Many interesting properties of triangle graphs were found. Recently Orlovich and Zverovich [22] proved that the Independent Domination Problem is NP-complete within $K_{1,4}$-free triangle graphs. Kloks et al. [16] showed that the triangle condition can be checked in polynomial time for AT-free graphs, planar graphs and for circle graphs.

Conjecture 1 (*Kloks et al. [16]*). Recognizing triangle graphs is a co-NP-complete problem.

Note that recognizing triangle graphs is a particular case of the Strongly Bipartite Bihypergraph Problem. We define the *proper neighborhood* of an edge $e=uv$ in a graph G as the set $\mathrm{PN}_G[e]$, consisting of u, v, and all vertices in G that are adjacent to both u and v.

Definition 8. Given a graph $G=(V, E)$, we define the *proper hypergraph* of G, denoted by $\mathrm{PH}(G)=(V, E')$, where $E'=\{\mathrm{PN}_G[e] : e \in E\}$. The *proper bihypergraph* of G is $(G, \mathrm{PH}(G))$.

Proposition 9. *A graph G satisfies the* Triangle Condition *if and only if the proper bihypergraph* $(G, \mathrm{PH}(G))$ *is strongly bipartite*.

Proof (*Necessity*). For every maximal stable set I and every edge uv in $G-I$, there exists a vertex $w \in I$ such that $\{u, v, w\}$ induces a triangle in G. In other words, I intersects all sets $\mathrm{PN}_G[e]$, i.e., I is a transversal of $\mathrm{PH}(G)$.

Sufficiency is similar. □

Definition 9. An edge e of a graph G is *tristable* if each maximal stable set in G intersects $\mathrm{PN}_G[e]$, otherwise it is *non-tristable*.

Thus, in a triangle graph each edge is tristable. Conjecture 1 states that it is NP-complete to decide whether a graph has a non-tristable edge.

Decision Problem 10 (*Tristable edge*).
Instance: A graph G and an edge e of G.
Question: Is e a tristable edge?

Now, we extend the construction of Theorem 3 to the Tristable Edge Problem.

Corollary 3. Tristable Edge *is a co*-NP-*complete problem*.

Proof. We consider the bihypergraph (G, H) constructed in the proof of Theorem 3 for an instance (C, X) to SAT. Without loss of generality, we may assume that $n=|X|\geqslant 2$. Let G' be a graph obtained from G by adding adjacent vertices u and v, and edges

- ux_i, $u\overline{x}_i$ for all even $i\leqslant n$,
- vx_i, $v\overline{x}_i$ for all odd $i\leqslant n$, and
- uc_j, vc_j for all $c_j \in C$.

Let $e=uv$. Clearly, $\mathrm{PN}_{G'}[e]=C\cup\{u, u\}$. If the bihypergraph (G, H) is strongly bipartite, then e is a tristable edge in G'. Indeed, suppose there exists a maximal stable set S' in G' which is disjoint from $\mathrm{PN}_{G'}[e]$. We have $S' \subseteq L_X$, and S' is a maximal stable set in G. Since S' is not a transversal of H, we obtain a contradiction to the assumption that (G, H) is a strongly bipartite bihypergraph.

Conversely, let e be a tristable edge in G'. Suppose that the bihypergraph (G, H) is not strongly bipartite, i.e., there exists a maximal stable set S in G which is not a transversal of H. Clearly, $S \subseteq L_X$. Maximality of S implies that $|S|=n$, therefore each of the vertices u, v is adjacent to some vertex of S. In other words, S is a maximal stable set in G'. We have a contradiction: $S \cap \mathrm{PN}_{G'}[e]=\emptyset$, i.e., e is not a tristable edge in G'. □

4.2. Stable graphs

Ravindra [25] observed that in a P_4-free graph every maximal stable set meets every maximal clique. A *clique* in a graph is a vertex set that induces a complete subgraph.

Definition 10. A graph G is called a *stable graph* if each maximal stable set in G intersects all maximal cliques of G.

For a graph G, we define the *clique hypergraph* $\mathrm{Cl}(G)$ on $V(G)$ by $E(\mathrm{Cl}(G)) = \{X \subseteq V(G) : X$ induces a maximal clique in $G\}$. Clearly, a graph G is stable if and only if the corresponding bihypergraph $(G, \mathrm{Cl}(G))$ is strongly bipartite. Thus, recognizing stable graphs is a particular case of the Strongly Bipartite Bihypergraph Problem.

Conjecture 2. Recognizing stable graphs is a co-NP-complete problem.

5. Conclusion

Since bipartite bihypergraphs have many applications, it is important to find new classes of bihypergraphs where bipartiteness can be tested in polynomial time. Here, we defined a family of such classes consisting of k-complete bihypergraphs for a fixed k. Moreover, it is easy to recognize k-complete bihypergraphs. Another interesting class, the strongly bipartite bihypergraphs, arises in many situations. In a greedy way, we can construct a bipartition of such a bihypergraph. However, our negative result is that recognizing strongly bipartite bihypergraphs (H^0, H^1) is co-NP-complete even in the case where H^0 is a graph and H^1 has exactly one hyperedge.

Besides developing the general direction, it is interesting to resolve particular conjectures proposed in the paper.

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