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Quasiminimal crystals with a volume constraint and uniform rectifiability

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Abstract

We establish here, in a quite general context, uniform rectifiability properties for quasiminimal crystals with a volume constraint. Namely we prove that to any quasiminimal crystal with a volume constraint corresponds a unique equivalent open set whose boundary is Ahlfors-regular and which satisfies the so-called condition B. Moreover implicit bounds in these properties, which imply the uniform rectifiability of the boundary, can be chosen universal. As a consequence we give a universal upper bound for the number of connected components of reduced quasiminimizers and we also prove that quasiminimal crystals with a volume constraint actually satisfy, in some universal way, an apparently stronger quasiminimality condition where admissible perturbations are not required to be volume-preserving anymore.

Résumé

On démontre ici, dans un contexte assez général, des propriétés de rectifiabilité uniforme pour des cristaux quasiminimaux à volume fixé. Plus précisément, on montre qu'à tout tel quasiminimiseur correspond un unique ouvert équivalent dont la frontière est Ahlfors-régulière et qui satisfait à la condition B. De plus, les constantes implicites intervenant dans ces deux propriétés entraînant l'uniforme rectifiabilité de la frontière, peuvent être choisies universelles. Comme conséquence de ces résultats on peut par exemple obtenir une borne universelle sur le nombre de composantes connexes des cristaux quasiminimaux réduits. On obtient aussi qu'ils satisfont à une condition de quasiminimalité apparemment plus forte où l'on s'est totalement affranchi de la contrainte de volume.

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1. Introduction

In this paper we are concerned with regularity results for quasiminimal crystals with a volume constraint. These quasiminimizers are sets for which one controls the variation of a surface-like energy under volume-preserving perturbations. Roughly speaking one knows that this variation is, at least for small perturbations, negligible compared to the initial surface energy. Our main goal is to prove in a quite general context that to any quasiminimizer corresponds a unique equivalent open set whose boundary enjoys strong quantitative rectifiability properties.

The study of quasiminimal crystals is motivated by the study of variational problems where surface and volume energies are competing and some classes of local minimizers and quasiminimizers for surface-like energies have already been studied in the literature. However one does not impose in general a volume constraint and one often considers instead localized versions of the quasiminimality condition we shall work with in this paper together with a larger class of admissible perturbations that are not required to be volume-preserving. In such a case and with suitable assumptions on the surface energy, it is now well known that the boundary of a quasiminimizer is a regular (say C^1 or $C^{1,\alpha}$, depending also on the degree of quasiminimality) hypersurface out of a small singular set, see, e.g., [5,17,18] for quasiminimizers for the standard perimeter, [1,2,4,6] for quasiminimizers for more general anisotropic surface energies, and the references given in these papers, this list not being exhaustive. However it may be appropriate for several applications, for instance when one works with incompressible fluids, to impose a volume constraint. In this setting, regularity results for local minimizers and quasiminimizers for the standard perimeter are also known (see, e.g., [15,16]). One of the aims of the present paper is to extend the study to more general surface energies on which we shall impose only very few conditions.

The above mentioned papers give in general regularity results that are of local and asymptotic nature. We want to stress that the kind of regularity properties we will consider here are of a quite different flavor. We shall indeed prove quantitative rectifiability results, namely uniform rectifiability with the terminology of G. David and S. Semmes. This approach and some of our general arguments have been inspired by [11] where the same kind of properties are shown in a different context. Uniform rectifiability is a variant of the notion of rectifiability which comes with uniform and scale-invariant estimates. This condition implies ordinary rectifiability and is actually much stronger because of the uniform bounds (see, e.g., [9] and the references therein for more details). Moreover we shall prove that in the present situation these bounds can be chosen universal, that is, depending only on the general data of the problem. Besides the regularity properties, we stress that this universal control may be considered as the central and main new information here (especially compared with the kind of results in the above mentioned papers). This was actually one of the main motivation for the present work and turns out to be the most delicate point to obtain. As a consequence we will furthermore get nontrivial universal control on other geometrical quantities such as the number of connected components of a quasiminimizer (see also the comment after Theorem 1.4). Note that with the quite general setting adopted here (when (1) is essentially the only assumption on the defining integrand for the surface energy) one cannot hope to have much more in the way of regularity than uniform rectifiability just for reasons of bilipschitz invariance.

On the other hand, in more specific cases, one may consider this kind of rectifiability properties as a first step in the study of the regularity of quasiminimal crystals with a volume constraint. It turns out that the kind of properties we will obtain here are exactly what one needs to handle properly the volume constraint. As a consequence we will be able to prove that quasiminimal crystals with a volume constraint also satisfy another apparently stronger quasiminimality condition where admissible perturbations are not required to be volume-preserving anymore. Then one can apply in some cases former results about unconstrained local quasiminimizers to get further regularity results when suitable assumptions are made on the defining integrand of the surface energy.

Let us now define more precisely quasiminimal crystals and state the main results of this paper. We denote by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$ and fix once and for all a continuous function $\Gamma : \mathbb{S}^{n-1} \to \mathbb{R}^+$ such that

$$\alpha \leqslant \Gamma(\nu) \leqslant \beta \quad \text{for all } \nu \in \mathbb{S}^{n-1}, \tag{1}$$

for some $\alpha > 0$ and $\beta > 0$. Then the surface energy is defined by:

$$P_\Gamma(F, \mathbb{R}^n) := \int_{\partial^* F} \Gamma(\nu_F)\, \mathrm{d}\mathcal{H}^{n-1},$$

where F is a subset of $\mathbb{R}^n$ with finite perimeter, $\partial^* F$ denotes its reduced boundary, ν_F is its generalized unit inner normal (see Section 2 for precise definitions) and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. We will call this measure the Γ-perimeter of F. We also fix some $a > 0$, the prescribed measure of the quasiminimal crystals, and a map $g : [0, +\infty] \to [0, +\infty]$ such that

$$\lim_{v \to 0^+} v^{-(n-1)/n} g(v) = 0.$$

Definition 1.1 (*Quasiminimal crystals with a volume constraint*). We say that a subset E of $\mathbb{R}^n$ with finite perimeter is a quasiminimal crystal with a volume constraint (and with prescribed measure a) if $|E| = a$ and

$$P_\Gamma(E, \mathbb{R}^n) \leqslant P_\Gamma(F, \mathbb{R}^n) + g\big(|F \triangle E|\big) \tag{2}$$

for any set F with finite perimeter such that $|F| = |E|$. We will denote by $\mathcal{QM}$ the class of all such quasiminimal crystals.

In this definition and in the rest of this paper, $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^n$ and $F \triangle E := (F \setminus E) \cup (E \setminus F)$ the symmetric difference between F and E.

Note that the quasiminimality condition (2) gives significant information only when $|F \triangle E|$ is small. Then the way the energy can be decreased through the admissible modification F of the quasiminimal crystal E is controlled by $g(|F \triangle E|)$ hence, by assumption on g, is negligible compared to $|F \triangle E|^{(n-1)/n}$. It turns out, as we shall see later, that this last quantity can be generally related to, and shown to be negligible compared to the initial surface energy.

When $g \equiv 0$, quasiminimal crystals with a volume constraint are simply sets that minimize the Γ-perimeter among all sets with prescribed Lebesgue measure. It is well known that, up to a null set, solutions of this variational problem are the so-called Wulff sets. This case will play a central role in the constructions of this paper and we shall spend some time to give a detailed analysis of Wulff sets (see Section 2).

When Γ is constant, the surface energy reduces, up to a multiplicative constant, to the standard perimeter. This case has already been studied in [16]. However we shall give here, even in the case of the standard perimeter, new and simpler constructions.

To state the main results of this paper we need some more definitions. For any $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B_r(x)$ the open ball with center x and radius r.

Definition 1.2 (*Ahlfors-regularity*). Let $S \subset \mathbb{R}^n$ be closed. We say that S is Ahlfors-regular (of codimension 1) if there exists a Borel measure μ supported in S and a constant $C > 1$ such that

$$C^{-1} r^{n-1} \leqslant \mu\big(B_r(x)\big) \leqslant C r^{n-1} \tag{3}$$

for all $x \in S$ and $r \leqslant 1$. We shall often refer to such a constant C as an Ahlfors-regularity constant for S.

This is a uniform and scale-invariant version of the property of having upper and lower densities with respect to $\mathcal{H}^{n-1}$ that are positive and finite (one can indeed prove that if μ is a measure that satisfies (3) then μ is equivalent to the measure $\mathcal{H}^{n-1}$ restricted to S).

Definition 1.3 (*Condition B*). Let $F \subset \mathbb{R}^n$ be open. We say that F satisfies the condition B if there exists a constant $C > 0$ such that for any ball B centered on ∂F with radius $r \leqslant 1$ there exists two balls B_1 and B_2 with radius Cr such that $B_1 \subset F \cap B$ and $B_2 \subset B \setminus \overline{F}$. We shall often refer to such a constant C as a condition B constant.

This condition is a quantitative, uniform and scale-invariant way of saying that the topological boundary ∂F of F separates well F from its complement. It turns out that sets satisfying the condition B and whose boundary is Ahlfors-regular have strong rectifiability properties. Namely their boundary contains “Big Pieces of Lipschitz Graphs” and thus is uniformly rectifiable (see [7] for the original proof or [8,10] for simpler proofs). The aim of this paper not being to speak about the theory of uniform rectifiability, we will not enter the details and refer to [9] and the references therein for more information.

As a convention, we say that a constant is universal if its value can be chosen depending only on (some of) the given data of the problem, namely the dimension n, the bounds α and β of the function Γ, the prescribed measure a and the function g but on nothing else.

We can now state the main result of the present paper.

Theorem 1.4. *Let $E \in \mathcal{QM}$. There exists a unique set E_1 equivalent to E such that*

$$E_1 \text{ is open},$$
$$\partial E_1 \text{ is Ahlfors-regular},$$
$$E_1 \text{ satisfies the condition B}.$$

Moreover the Ahlfors-regularity and the condition B constants can be chosen universal.

We also refer to Theorem 3.3 for a refined version of this result. Throughout this paper, we say that two measurable sets are equivalent if the Lebesgue measure of their symmetric difference is zero. Note that by definition of quasiminimality we are only concerned here by equivalent classes of sets. If $E \in \mathcal{QM}$ and E' is equivalent to E then E' is still in $\mathcal{QM}$. Thus it is natural to have first to clean up quasiminimal crystals before stating properties that hold everywhere on the topological boundary. And the set E_1 given by Theorem 1.4 is still a quasiminimal crystal with a volume constraint exactly in the same way as E. We fix the terminology with the following definition.

Definition 1.5. We say that a set $E \in \mathcal{QM}$ is a reduced quasiminimal crystal (with a volume constraint) if E is open, ∂E is Ahlfors-regular and E satisfies the condition B.

Note that by uniqueness in Theorem 1.4 the Ahlfors-regularity and condition B constants of a reduced quasiminimal crystal can always be chosen universal.

As already mentioned, we would like to stress once again that, besides the uniform rectifiability property, this universal control on the Ahlfors-regularity and condition B constants is one of the key main new information in Theorem 1.4. This gives some kind of geometric a priori estimates that hold true uniformly for all quasiminimal crystals in the class $\mathcal{QM}$. As an application this might be for instance of particular interest when proving the existence of minimizers for variational problems where surface and volume energies are competing under a volume constraint. One usually considers an approximating minimizing sequence of sets and one would like to get from this sequence a limiting set that still satisfy the volume constraint. This can be a quite difficult issue without any suitable a priori estimates on the elements in the minimizing sequence. We refer to [16, Chapter 5] for such kind of existence problems where this applies. Let us also note that the surface energy involved in [16] is the standard perimeter. Another point of interest of Theorem 1.4 is that it holds for quasiminimal crystals for general Γ-perimeter, thus allowing to extend some applications to more general settings. Recall that we only require for the defining integrand Γ the nondegeneracy condition (1) but no further or more involved regularity assumptions.

The general strategy to prove Theorem 1.4 is to construct suitable deformations of the quasiminimal crystal E and then deduce from the quasiminimality condition the required conclusions. The main issue here is to handle properly the volume constraint especially because we want to get universal constants in the Ahlfors-regularity and the condition B. Wulff sets will play a central role for this purpose. Because they minimize the Γ-perimeter among all sets with prescribed Lebesgue measure, one can easily compare the Γ-perimeter of a set with that of its intersection or union with any Wulff set (see Lemmas 2.5 and 2.6).We shall use this to adjust the Lebesgue measure of deformations of E in order to get at the end admissible candidates with exactly the same measure than E. The delicate point is then to find suitable Wulff sets to add or to remove. This strategy is close to that adopted in [15] where local minimizers for the perimeter with a volume constraint are studied. However, because of the universal regularity constants we are looking for, one needs here to find suitable Wulff sets with size independent of the local geometry of E and the existence and position of interior and exterior points which depends strongly of the geometry of E as shown in [15] will not fit our needs. We refer to Sections 3 and 4 for more details and complete proofs.

To conclude this introduction we state two consequences of Theorem 1.4 (and of the arguments of its proof) that will be also proved in this paper. First we will be able to refine the study of the regularity of reduced quasiminimal crystals proving the Ahlfors-regularity and the condition B with universal constants for each one of their connected components in their own. As an immediate consequence one gets the already mentioned universal upper bound on their number.

Theorem 1.6. *Let $E \in \mathcal{QM}$ be a reduced quasiminimal crystal and A be a connected component of E. Then ∂A is Ahlfors-regular and A satisfies the condition B with universal constants. In particular E has at most C connected components for some universal constant $C > 0$.*

Finally, as already mentioned, once one has Theorem 1.4 in hand, it is much easier to find suitable volume-preserving deformations and one can definitely get rid of the volume constraint.

Theorem 1.7. *Assume that g is nondecreasing and let $E \in \mathcal{QM}$. There exist a universal function $\omega : [0, +\infty] \to [0, +\infty]$ with $\lim_{r\to 0} \omega(r) = 0$ and a universal radius $R \leqslant 1$ such that*

$$P_\Gamma\big(E, B_r(x)\big) \leqslant P_\Gamma\big(F, B_r(x)\big) + r^{n-1}\omega(r)$$

for any $x \in \mathbb{R}^n$, $r \leqslant R$ and any set F with finite perimeter such that $F \triangle E \Subset B_r(x)$.

The assumption on g to be nondecreasing is here mostly for technical convenience and is not really restrictive. As already mentioned in the beginning of this introduction one can then apply already known regularity results for sets that satisfy the quasiminimality condition given in Theorem 1.7 and one gets further regularity for quasiminimal crystals with a volume constraint. Precise statements depending strongly on further assumptions on the functions Γ and ω we will not enter this in detail here and refer to the already mentioned references.

The rest of this paper is organized as follows. In Section 2 we recall some background material, mainly about the theory of sets with finite perimeter and about the so-called Wulff sets, and show for further reference a list of preliminary results. In Section 3 we prove the upper estimate in the Ahlfors-regularity (see Lemma 3.1) and reduce the proof of the other properties to a lemma about the behavior of the proportion of a quasiminimal crystal and of its complement inside Wulff sets (see Lemma 3.2). We prove this lemma in Section 4. Finally we will prove Theorems 1.6 and 1.7 in Section 5.

2. Preliminaries

As a general convention the letter C will always denote in what follows a positive constant whose value, unless otherwise stated, can change at each occurrence.

2.1. *Sets with finite perimeter and Γ-perimeter*

We recall here well-known results about the theory of sets with finite perimeter and refer to, e.g., [3,14] or [19] for more details. We shall use this to give useful properties of the Γ-perimeter to be used later.

For any set $F \subset \mathbb{R}^n$ we denote by $\mathbf{1}_F$ its characteristic function. If F is a measurable set and Ω is open, the perimeter of F in Ω, denoted by $P(F,\Omega)$, is defined by:

$$P(F,\Omega) := \sup\left\{\int_\Omega \mathbf{1}_F \operatorname{div}\phi \,\mathrm{d}x\colon\ \phi \in C_0^1(\Omega,\mathbb{R}^n),\ \|\phi\|_\infty \leqslant 1\right\},$$

and we say that F is a set with finite perimeter if $P(F,\mathbb{R}^n) < +\infty$.

If F is a set with finite perimeter then it turns out that the set function $\Omega \mapsto P(F,\Omega)$ defined above for Ω open is actually the restriction of a finite Borel measure, which will be called the (standard) perimeter of F and denoted by $P(F,\cdot)$. Equivalently a measurable set F has finite perimeter if and only if the distributional gradient $\nabla\mathbf{1}_F$ of its characteristic function can be represented by a vector-valued measure. Moreover the total variation $|\nabla\mathbf{1}_F|$ of this measure coincides with $P(F,\cdot)$.

If F is a set with finite perimeter it is well known that its perimeter coincides with the restriction of the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ to its so-called reduced boundary $\partial^* F$,

$$P(F,B) = \mathcal{H}^{n-1}(\partial^* F \cap B)$$

for any Borel set B. The reduced boundary of a set F with finite perimeter is defined as the set of points $x \in \mathbb{R}^n$ such that

$$\int_{B_r(x)} |\nabla\mathbf{1}_F| > 0 \quad \text{for all } r > 0,$$

the limit

$$\nu_F(x) := \lim_{r\to 0} \frac{\int_{B_r(x)} \nabla\mathbf{1}_F}{\int_{B_r(x)} |\nabla\mathbf{1}_F|}$$

exists, and $|\nu_F(x)| = 1$. Note that $\partial^* F \subset \partial F$. Moreover it follows from the theorem of Besicovitch on differentiation of measures that $\nu_F(x)$ exists and $|\nu_F(x)| = 1$ for $|\nabla \mathbf{1}_F|$-a.e. $x \in \mathbb{R}^n$ and furthermore that $\nabla \mathbf{1}_F = \nu_F |\nabla \mathbf{1}_F|$ as an equality between measures. In particular it follows that

$$\nabla \mathbf{1}_F = \nu_F \mathbf{1}_{\partial^* F}\, \mathrm{d}\mathcal{H}^{n-1}. \tag{4}$$

For any measurable set F and $t > 0$, we set:

$$F(t) := \left\{ x \in \mathbb{R}^n \colon \lim_{r \to 0} \frac{|F \cap B_r(x)|}{|B_r(x)|} = t \right\}$$

and define the essential boundary $\partial_* F$ of F as the set of points where the volume density of F is neither 0 nor 1, $\partial_* F = \mathbb{R}^n \setminus (F(0) \cup F(1))$. Note that $\partial_* F \subset \partial F$. It is well known that if F is a set with finite perimeter, then

$$\partial^* F \subset F(1/2) \subset \partial_* F \quad \text{and} \quad \mathcal{H}^{n-1}\big(\mathbb{R}^n \setminus \big(F(0) \cup F(1) \cup \partial^* F\big)\big) = 0. \tag{5}$$

The next lemma is a simple consequence of the above mentioned results. It will be useful to get further properties of the Γ-perimeter.

Lemma 2.1. *Let A and F be two sets with finite perimeter such that $A \subset F$. Then*

$$\begin{gathered}
\partial^* F \cap \partial^* A \cap \partial^*(F \setminus A) = \emptyset, \\
\mathcal{H}^{n-1}\big(\partial^* F \setminus \big(\partial^* A \cup \partial^*(F \setminus A)\big)\big) = 0, \\
\nu_F(x) = \nu_A(x) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial^* F \cap \partial^* A, \\
\nu_F(x) = \nu_{F \setminus A}(x) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial^* F \cap \partial^*(F \setminus A).
\end{gathered}$$

Proof. According to (5) we have $\partial^* A \cap \partial^*(F \setminus A) \subset A(1/2) \cap (F \setminus A)(1/2) \subset F(1)$ and $F(1) \cap \partial^* F = \emptyset$, hence $\partial^* F \cap \partial^* A \cap \partial^*(F \setminus A) = \emptyset$. Next, we have $\mathbf{1}_F = \mathbf{1}_A + \mathbf{1}_{F \setminus A}$, hence

$$\nabla \mathbf{1}_F = \nabla \mathbf{1}_A + \nabla \mathbf{1}_{F \setminus A}$$

as an equality between measures (note that $F \setminus A$ has finite perimeter). Then it follows from (4) that

$$\nu_F \mathbf{1}_{\partial^* F} = \nu_A \mathbf{1}_{\partial^* A} + \nu_{F \setminus A} \mathbf{1}_{\partial^*(F \setminus A)} \tag{6}$$

$\mathcal{H}^{n-1}$-a.e. and, since $|\nu_F(x)| = 1$ for all $x \in \partial^* F$, we get:

$$\mathcal{H}^{n-1}\big(\partial^* F \setminus \big(\partial^* A \cup \partial^*(F \setminus A)\big)\big) = 0.$$

Finally the last two claims follow easily from the fact that $\partial^* F \cap \partial^* A \cap \partial^*(F \setminus A) = \emptyset$ together with (6). □

We now turn our attention to the Γ-perimeter as defined in the introduction. Recall that $\Gamma:\mathbb{S}^{n-1}\to\mathbb{R}^+$ is a fixed continuous function and that the associated Γ-perimeter of a set F with finite perimeter is defined as

$$P_\Gamma(F,B):=\int_{\partial^* F\cap B}\Gamma(\nu_F)\,\mathrm{d}\mathcal{H}^{n-1}$$

for any Borel set B. When dealing with quasiminimal crystals we shall also assume in this paper that Γ satisfies (1). However, for the time being, we do not need this additional assumption to state and prove some general properties of the Γ-perimeter. Note that when $\Gamma\equiv 1$, the Γ-perimeter of F coincides simply with the (standard) perimeter of F. Note also that, according to (5), ν_F is well defined $\mathcal{H}^{n-1}$-a.e. on $F(1/2)$ and $\partial_* F$, hence one can replace $\partial^* F$ by $F(1/2)$ or $\partial_* F$ in the definition of the Γ-perimeter. We will freely use this remark in the rest of this paper, choosing in each specific situation the most convenient definition to work with.

First it follows easily from the definition that, if F and G are any two sets with finite perimeter and Ω is open, then

$$P_\Gamma(F,\Omega)=P_\Gamma(G,\Omega)\quad\text{whenever }\big|(F\,\triangle\,G)\cap\Omega\big|=0.$$

The following lemma generalizes to the Γ-perimeter a well-known property of the perimeter. It will be of frequent use throughout this paper.

Lemma 2.2. *Let F and G be two sets with finite perimeter. Then we have*:

$$P_\Gamma(F\cup G,\mathbb{R}^n)+P_\Gamma(F\cap G,\mathbb{R}^n)\leqslant P_\Gamma(F,\mathbb{R}^n)+P_\Gamma(G,\mathbb{R}^n).$$

Proof. Let F and G be two sets with finite perimeter. Then $F\cup G$ and $F\cap G$ have finite perimeter. We first estimate $P_\Gamma(F\cup G,\mathbb{R}^n)$. It follows from (5) that

$$P_\Gamma(F\cup G,\mathbb{R}^n)=P_\Gamma\big(F\cup G,G(1)\big)+P_\Gamma\big(F\cup G,G(0)\big)+P_\Gamma\big(F\cup G,G(1/2)\big).$$

We have $(F\cup G)(1/2)\cap G(1)=\emptyset$, hence,

$$P_\Gamma\big(F\cup G,G(1)\big)=0.$$

Next, $(F\cup G)(1/2)\cap G(0)=F(1/2)\cap G(0)$ and $\nu_{F\cup G}(x)=\nu_F(x)$ for $\mathcal{H}^{n-1}$-a.e. $x\in(F\cup G)(1/2)\cap F(1/2)$ according to Lemma 2.1 together with (5), hence,

$$P_\Gamma\big(F\cup G,G(0)\big)=P_\Gamma\big(F,G(0)\big).$$

We have, once again by (5),

$$P_\Gamma\big(F\cup G,G(1/2)\big)=P_\Gamma\big(F\cup G,G(1/2)\cap F(1)\big)+P_\Gamma\big(F\cup G,G(1/2)\cap F(0)\big)\\+P_\Gamma\big(F\cup G,G(1/2)\cap F(1/2)\big).$$

Similarly as before we have $(F\cup G)(1/2)\cap F(1)=\emptyset$ and $(F\cup G)(1/2)\cap G(1/2)\cap F(0)=G(1/2)\cap F(0)$ with $\nu_{F\cup G}(x)=\nu_G(x)$ for $\mathcal{H}^{n-1}$-a.e. $x\in(F\cup G)(1/2)\cap G(1/2)$, hence,

$$P_\Gamma\big(F\cup G, G(1/2)\cap F(1)\big)+P_\Gamma\big(F\cup G, G(1/2)\cap F(0)\big)=P_\Gamma\big(G, F(0)\big).$$

Finally, since $\nu_{F\cup G}(x)=\nu_F(x)$ for $\mathcal{H}^{n-1}$-a.e. $x\in(F\cup G)(1/2)\cap F(1/2)$, we have:

$$P_\Gamma\big(F\cup G, G(1/2)\cap F(1/2)\big)\leqslant P_\Gamma\big(F, G(1/2)\big).$$

It follows that

$$P_\Gamma(F\cup G,\mathbb{R}^n)\leqslant P_\Gamma\big(F, G(0)\big)+P_\Gamma\big(G, F(0)\big)+P_\Gamma\big(F, G(1/2)\big). \tag{7}$$

Arguing in a similar way, one also gets that

$$\begin{gathered}P_\Gamma\big(F\cap G, G(0)\big)=0,\\ P_\Gamma\big(F\cap G, G(1)\big)=P_\Gamma\big(F, G(1)\big),\\ P_\Gamma\big(F\cap G, G(1/2)\cap\big(F(0)\cup F(1)\big)\big)=P_\Gamma\big(G, F(1)\big),\\ P_\Gamma\big(F\cap G, G(1/2)\cap F(1/2)\big)\leqslant P_\Gamma\big(G, F(1/2)\big),\end{gathered}$$

hence

$$P_\Gamma(F\cap G,\mathbb{R}^n)\leqslant P_\Gamma\big(F, G(1)\big)+P_\Gamma\big(G, F(1)\big)+P_\Gamma\big(G, F(1/2)\big). \tag{8}$$

To conclude, we add up (7) and (8) and use once again (5) to recover $P_\Gamma(F,\mathbb{R}^n)$ and $P_\Gamma(G,\mathbb{R}^n)$ in the right-hand side. □

The next lemma will also essentially follow from Lemma 2.1. It will be used at the end of this paper in Section 5.

Lemma 2.3. *Let F be an open set with finite perimeter. Assume that $\mathcal{H}^{n-1}(\partial F\setminus\partial^*F)=0$ and let $A\subset F$ be a set with finite perimeter. Then*

$$P_\Gamma(A,\mathbb{R}^n)+P_\Gamma(F\setminus A,\mathbb{R}^n)=P_\Gamma(F,\mathbb{R}^n)+P_\Gamma(A,F)+P_\Gamma(F\setminus A,F).$$

Proof. Let F and A be as in the statement. We compute separately $P_\Gamma(F,\mathbb{R}^n)$, $P_\Gamma(A,\mathbb{R}^n)$ and $P_\Gamma(F\setminus A,\mathbb{R}^n)$. Thanks to Lemma 2.1, we have:

$$\begin{aligned}P_\Gamma(F,\mathbb{R}^n)=\int_{\partial^*F}\Gamma(\nu_F)\,\mathrm{d}\mathcal{H}^{n-1}&=\int_{\partial^*F\cap\partial^*A}\Gamma(\nu_F)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\partial^*F\cap\partial^*(F\setminus A)}\Gamma(\nu_F)\,\mathrm{d}\mathcal{H}^{n-1}\\ &=\int_{\partial^*F\cap\partial^*A}\Gamma(\nu_A)\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\partial^*F\cap\partial^*(F\setminus A)}\Gamma(\nu_{F\setminus A})\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned}$$

On the other hand, since $A \subset F$, hence $\partial^* A \subset \overline{F}$, since F is open, hence $F \cap \partial F = \emptyset$, and since $\mathcal{H}^{n-1}(\partial F \setminus \partial^* F) = 0$, we have:

$$P_\Gamma(A, \mathbb{R}^n) = P_\Gamma(A, F) + \int_{\partial^* A \cap \partial F} \Gamma(\nu_A)\,\mathrm{d}\mathcal{H}^{n-1} = P_\Gamma(A, F) + \int_{\partial^* A \cap \partial^* F} \Gamma(\nu_A)\,\mathrm{d}\mathcal{H}^{n-1},$$

and similarly,

$$P_\Gamma(F \setminus A, \mathbb{R}^n) = P_\Gamma(F \setminus A, F) + \int_{\partial^*(F \setminus A) \cap \partial^* F} \Gamma(\nu_{F \setminus A})\,\mathrm{d}\mathcal{H}^{n-1}.$$

It follows that

$$\begin{aligned} P_\Gamma(A, \mathbb{R}^n) + P_\Gamma(F \setminus A, \mathbb{R}^n) &= P_\Gamma(A, F) + P_\Gamma(F \setminus A, F) + \int_{\partial^* A \cap \partial^* F} \Gamma(\nu_A)\,\mathrm{d}\mathcal{H}^{n-1} \\ &\quad + \int_{\partial^*(F \setminus A) \cap \partial^* F} \Gamma(\nu_{F \setminus A})\,\mathrm{d}\mathcal{H}^{n-1} \\ &= P_\Gamma(A, F) + P_\Gamma(F \setminus A, F) + P_\Gamma(F, \mathbb{R}^n) \end{aligned}$$

as wanted. □

Finally, let us point out that if Γ satisfies (1), we have for any set F with finite perimeter and any Borel set B,

$$\alpha P(F, B) \leqslant P_\Gamma(F, B) \leqslant \beta P(F, B). \tag{9}$$

In particular the Γ-perimeter of F is equivalent to the measure $\mathcal{H}^{n-1}$ restricted to one of the sets $\partial^* F$, $F(1/2)$ or $\partial_* F$.

2.2. Wulff sets

From now on we assume that $\Gamma : \mathbb{S}^{n-1} \to \mathbb{R}^+$ is continuous and satisfies (1). We recall in this section the definition of the so-called Wulff sets which are the sets that minimize the Γ-perimeter among all sets with given Lebesgue measure (see Theorem 2.4 stated below). They will play a central role in this paper and we also give here useful properties of these sets to be used later. We shall in particular study regularity properties directly related to those of quasiminimal crystals as stated in Theorem 1.4.

First we extend the function Γ to $\mathbb{R}^n$ as an homogeneous function of degree one and we still denote this extension by Γ,

$$\Gamma(x) := \begin{cases} \|x\|\,\Gamma\big(x/\|x\|\big) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We also set:

$$\Gamma^*(x) := \sup_{y\in\mathbb{R}^n\setminus\{0\}} \frac{\langle x, y\rangle}{\Gamma(y)} = \max_{\nu\in\mathbb{S}^{n-1}} \frac{\langle x, \nu\rangle}{\Gamma(\nu)}.$$

For any $x \in \mathbb{R}^n$ and $r > 0$, the Wulff set with center x and radius r is defined as

$$W_r(x) := \{y \in \mathbb{R}^n\colon\ \Gamma^*(y - x) < r\}.$$

More generally we say that a set is a Wulff set if it is of the previous form for some $x \in \mathbb{R}^n$ and $r > 0$. We set $W_\Gamma := W_1(0)$. It is actually the closure of W_Γ that is sometimes called the Wulff set (or crystal) of Γ in the literature rather than W_Γ itself. This does not make any difference for the main minimality property of Wulff sets (see Theorem 2.4) because W_Γ and $\overline{W}_\Gamma$ are equivalent. On the other hand, it will be more convenient for some of our purposes to work with open sets.

We first collect for further reference simple but useful properties of Γ^* and of Wulff sets. The main point is that, for most applications, Wulff sets behave like Euclidean balls. Note however that Γ, and thus also Γ^*, are not assumed to be even and, strictly speaking, one cannot identify Wulff sets with balls associated to some norm equivalent to the Euclidean one. The function Γ^* is homogeneous of degree one and convex hence continuous and subadditive. It follows in particular that Wulff sets are open, convex and bounded. Moreover, for any $x \in \mathbb{R}^n$ and $r > 0$, $W_r(x)$ is then simply the translation of vector x of the dilation by a factor r of W_Γ. It follows that

$$|W_r(x)| = |W_\Gamma| r^n, \tag{10}$$

$$P(W_r(x), \mathbb{R}^n) = \mathcal{H}^{n-1}(\partial W_r(x)) = \mathcal{H}^{n-1}(\partial W_\Gamma) r^{n-1}. \tag{11}$$

Note also that, the Γ-perimeter being invariant under translations and homogeneous of degree $(n-1)$ with respect to the dilations, we have $P_\Gamma(W_r(x), \mathbb{R}^n) = P_\Gamma(W_\Gamma, \mathbb{R}^n) r^{n-1}$. Next, thanks to (1) and by definition of Γ^*, we have for all $x \in \mathbb{R}^n$,

$$\beta^{-1}\|x\| \leqslant \Gamma^*(x) \leqslant \alpha^{-1}\|x\|. \tag{12}$$

It follows that

$$B_{\alpha r}(x) \subset W_r(x) \subset B_{\beta r}(x). \tag{13}$$

On the other hand, using the subadditivity of Γ^*, we also get that

$$|\Gamma^*(y) - \Gamma^*(x)| \leqslant \max(\Gamma^*(y - x), \Gamma^*(x - y)) \leqslant \alpha^{-1}\|y - x\|$$

for any $x, y \in \mathbb{R}^n$, hence Γ^* is an α^{-1}-Lipschitz function. Then, noting that by definition of Wulff sets and continuity of Γ^* we have:

$$\partial W_t(x) = \{y \in \mathbb{R}^n\colon\ \Gamma^*(y - x) = t\}$$

for any $x \in \mathbb{R}^n$ and $t > 0$, one can apply the coarea formula with the Lipschitz function $y \mapsto \Gamma^*(y - x)$ and one gets that

$$\int_r^s \mathcal{H}^{n-1}\big(F \cap \partial W_t(x)\big)\,\mathrm{d}t \leqslant C\big|F \cap \big(W_s(x) \setminus W_r(x)\big)\big| \tag{14}$$

for any $0 < r < s$ and with a constant $C > 0$ which depends only on the dimension n and α. Finally, denoting by F^c the complement of a set F, we have:

$$\alpha(s - r) \leqslant \operatorname{dist}\big(W_r(x), W_s(x)^c\big) \leqslant \beta(s - r). \tag{15}$$

Indeed, if $y \in \partial W_r(x)$ and $z \in \partial W_s(x)$ with $r < s$, we have,

$$(s - r) = \Gamma^*(z - x) - \Gamma^*(y - x) \leqslant \Gamma^*(z - y) \leqslant \alpha^{-1}\|z - y\|$$

from which the left inequality follows. On the other hand, if $y \in \partial W_r(x)$, then $(s/r)(y - x) + x \in \partial W_s(x)$ and thus

$$\operatorname{dist}\big(W_r(x), W_s(x)^c\big) \leqslant \big\|(s/r)(y - x) + x - y\big\| \leqslant \big(\|y - x\|/r\big)(s - r) \leqslant \beta(s - r).$$

We turn now to the main characterization of Wulff sets. We refer to [12,13] and the references therein for proofs and more details. We set:

$$C_\Gamma := P_\Gamma(W_\Gamma, \mathbb{R}^n)|W_\Gamma|^{(1-n)/n}.$$

Theorem 2.4 (Wulff Theorem). *Let F be a set with finite perimeter and finite Lebesgue measure. Then*

$$C_\Gamma|F|^{(n-1)/n} \leqslant P_\Gamma(F, \mathbb{R}^n), \tag{16}$$

and equality holds if and only if F is equivalent to some Wulff set.

We deduce from the minimality of Wulff sets a comparison between the Γ-perimeter of a set and that of its intersection or union with any Wulff set. This will be needed later.

Lemma 2.5. *For any set F with finite perimeter and finite Lebesgue measure and any Wulff set W, we have*:

$$P_\Gamma(F \cap W, \mathbb{R}^n) \leqslant P_\Gamma(F, \mathbb{R}^n).$$

Proof. Let F and W be as in the statement. It follows from Theorem 2.4 that

$$P_\Gamma(W, \mathbb{R}^n) = C_\Gamma|W|^{(n-1)/n} \leqslant C_\Gamma|F \cup W|^{(n-1)/n} \leqslant P_\Gamma(F \cup W, \mathbb{R}^n).$$

Then, using Lemma 2.2, we get:

$$P_\Gamma(W,\mathbb{R}^n)+P_\Gamma(F\cap W,\mathbb{R}^n)\leqslant P_\Gamma(F\cup W,\mathbb{R}^n)+P_\Gamma(F\cap W,\mathbb{R}^n)$$
$$\leqslant P_\Gamma(F,\mathbb{R}^n)+P_\Gamma(W,\mathbb{R}^n),$$

and the lemma follows. □

Lemma 2.6. *For any set F with finite perimeter and any Wulff set W, we have*:

$$P_\Gamma(F\cup W,\mathbb{R}^n)\leqslant P_\Gamma(F,\mathbb{R}^n)+\frac{C_\Gamma}{|W|^{1/n}}|W\setminus F|.$$

Proof. Let W be some fixed Wulff set and let us define:

$$\mathcal{F}(G):=P_\Gamma(G,\mathbb{R}^n)-\frac{C_\Gamma}{|W|^{1/n}}|G|$$

for any set G with finite perimeter. If $G\subset W$, thanks to (16) we have:

$$\frac{C_\Gamma}{|W|^{1/n}}|G|\leqslant C_\Gamma|G|^{(n-1)/n}\leqslant P_\Gamma(G,\mathbb{R}^n),$$

hence $\mathcal{F}(G)\geqslant 0$. On the other hand, $\mathcal{F}(W)=0$. It follows that, for any set $G\subset W$, we have $\mathcal{F}(W)\leqslant\mathcal{F}(G)$ and thus,

$$P_\Gamma(W,\mathbb{R}^n)\leqslant P_\Gamma(G,\mathbb{R}^n)+\frac{C_\Gamma}{|W|^{1/n}}|W\setminus G|.$$

Now we let F be any set with finite perimeter and we apply this inequality with $G=F\cap W$ to get:

$$P_\Gamma(W,\mathbb{R}^n)\leqslant P_\Gamma(F\cap W,\mathbb{R}^n)+\frac{C_\Gamma}{|W|^{1/n}}|W\setminus F|.$$

Combining this with Lemma 2.2, it follows that

$$P_\Gamma(F\cup W,\mathbb{R}^n)\leqslant P_\Gamma(F\cup W,\mathbb{R}^n)+P_\Gamma(F\cap W,\mathbb{R}^n)-P_\Gamma(W,\mathbb{R}^n)+\frac{C_\Gamma}{|W|^{1/n}}|W\setminus F|$$
$$\leqslant P_\Gamma(F,\mathbb{R}^n)+\frac{C_\Gamma}{|W|^{1/n}}|W\setminus F|$$

as wanted. □

We now discuss regularity properties of Wulff sets in terms of Ahlfors-regularity and condition B as stated in Theorem 1.4. Recall that Wulff sets are quasiminimal crystals with a volume constraint corresponding to a map $g\equiv 0$. The set W_Γ is open and convex and because of (13), it is then bilipschitz equivalent to the unit Euclidean ball $B_1(0)$, that is, there exists a bilipschitz function $f:\mathbb{R}^n\to\mathbb{R}^n$ such that $f(B_1(0))=W_\Gamma$. Moreover the bilipschitz constant for f depends only on n, α and β. On the other hand, $\partial B_1(0)$ is Ahlfors-regular and $B_1(0)$ satisfies the condition B with Ahlfors-regularity and condition B constants depending only on the dimension. Ahlfors-regularity and condition B being invariant under bilipschitz equivalence, we get that ∂W_Γ is Ahlfors-regular and W_Γ satisfies the condition B as well and with constants depending only on n, α and β. Next, using translations and dilations, we get that any Wulff set satisfies these properties together with some uniform control on the Ahlfors-regularity and condition B constants.

Proposition 2.7. *There exist two constants $C > 1$ and $C' > 0$ depending only on n, α and β, such that, if W is a Wulff set with radius $r > 0$, then, for any $x \in \partial W$ and $t \leqslant r$, we have*

$$C^{-1}t^{n-1} \leqslant \mathcal{H}^{n-1}\big(\partial W \cap B_t(x)\big) \leqslant Ct^{n-1},$$

and there exist two balls B_1 and B_2 with radius $C't$ such that $B_1 \subset B_t(x) \cap W$ and $B_2 \subset B_t(x) \setminus \overline{W}$.

Another consequence of the bilipschitz equivalence between W_Γ and $B_1(0)$ and the equivalence (9) between the standard perimeter and the Γ-perimeter is that Wulff sets are domains of isoperimetry as well as balls are (in that case, this is just the relative isoperimetric inequality for balls).

Proposition 2.8. *There exists a constant $C > 0$ depending only on n, α and β, such that, for any $x \in \mathbb{R}^n$, $r > 0$ and any set F with finite perimeter, we have*:

$$\min\big\{\big|F \cap W_r(x)\big|, \big|W_r(x) \setminus F\big|\big\}^{(n-1)/n} \leqslant C P_\Gamma\big(F, W_r(x)\big).$$

We end this section with two simple consequences of the condition B for Wulff sets to be used in the main constructions in Section 4. For any Wulff set W and $\lambda > 0$, we denote by λW the Wulff set with the same center as W and with radius λ times the radius of W. Roughly speaking the first lemma tells us that if a large proportion (in measure) of a Wulff set W' is contained in another Wulff set W and if the ratio between the radii of W' and W is controlled, then a slightly smaller Wulff set is entirely contained in W.

Lemma 2.9. *There exists a constant $\theta > 0$, depending only on n, α and β, such that, if W and W' are two Wulff sets with radius respectively r and r' such that*

$$r' \leqslant 2\alpha^{-1}r \quad \text{and} \quad |W' \setminus W| \leqslant \theta |W'|,$$

then

$$(\alpha/2\beta)W' \subset W.$$

Proof. Let $\theta > 0$ be a constant to be fixed later and W and W' be as in the statement. Let x be the center of W'. We have

$$\operatorname{dist}(x, \partial W) \geqslant (\alpha/2)r'.$$

Otherwise there would exist $y \in \partial W \cap B_{(\alpha/2)r'}(x)$ and, since $(\alpha/2)r' \leqslant r$, Proposition 2.7 would give a ball B with radius $C'(\alpha/2)r'$ where C' depends only on n, α and β, such that

$$B \subset B_{(\alpha/2)r'}(y) \setminus \overline{W}.$$

Since, by (13), $B_{(\alpha/2)r'}(y) \subset B_{\alpha r'}(x) \subset W'$, we would have $|B| \leqslant |W' \setminus W| \leqslant \theta|W'|$ which is impossible if θ is small enough, depending only on n, α and β (remember (10)). Next it follows that $x \in W$ because otherwise we would have $B_{(\alpha/2)r'}(x) \subset W' \setminus W$ which is also impossible if θ is small enough. Hence, using once again (13), we get

$$W_{(\alpha/2\beta)r'}(x) \subset B_{(\alpha/2)r'}(x) \subset W,$$

as wanted. □

Arguing in a similar way, we also have the next lemma.

Lemma 2.10. *There exists a constant $\theta > 0$, depending only on n, α and β, such that, if W and W' are two Wulff sets with radius respectively r and r' such that*

$$r' \leqslant 2\alpha^{-1} r \quad \textit{and} \quad |W' \cap W| \leqslant \theta |W'|,$$

then

$$(\alpha/2\beta)W' \subset \overline{W}^{\,c}.$$

2.3. Approximation of Γ-quasi-isoperimetric sets

We prove in this section an approximation lemma for sets $F \subset \mathbb{R}^n$ that are Γ-quasi-isoperimetric in the sense that their isoperimetric ratio $|F|^{(1-n)/n} P_\Gamma(F, \mathbb{R}^n)$ is close to the Γ-isoperimetric constant C_Γ. This approximation will be done by means of Wulff sets in the L^1 sense. The main point is that it comes with universal control. This will be one of the key ingredients in the main constructions in Section 4.

Lemma 2.11. *For any $0 < \delta < 1$, there exists $\eta > 0$ depending only on n, α, β and δ such that if F is a set with finite perimeter and finite Lebesgue measure such that*

$$P_\Gamma(F, \mathbb{R}^n) \leqslant C_\Gamma (1+\eta) |F|^{(n-1)/n},$$

then there exists a Wulff set W such that $|W| = |F|$ and

$$|F \triangle W| \leqslant \delta |F|.$$

Proof. To prove the lemma we will argue by contradiction and use a concentration—compactness type argument. First we note that it is sufficient to prove the lemma with Γ convex. In that case the Γ-perimeter is lower semi-continuous with respect to the L^1

topology and this will be needed later. Indeed one can always consider the lower convex envelope $\widehat{\Gamma}$ of Γ,

$$\widehat{\Gamma} := \sup\{f:\ f \text{ is convex and } f \leqslant \Gamma\}.$$

It turns out that

$$\widehat{\Gamma}(x) = \sup\{\langle y, x\rangle:\ y \in W_\Gamma\}.$$

In particular $\widehat{\Gamma}$ is homogeneous of degree one, convex and still satisfies (1) (remember (13)). Moreover $P_{\widehat{\Gamma}}(F,\cdot) \leqslant P_\Gamma(F,\cdot)$ for any set F with finite perimeter and $W_{\widehat{\Gamma}} = W_\Gamma$ (see [12, Proposition 3.5]). Hence it is sufficient to prove the lemma for $\widehat{\Gamma}$ in place of Γ and, for simplicity of notations, we assume in the rest of this proof that Γ is convex.

Let $\delta \in (0,1)$ be fixed. Arguing by contradiction we assume that there exists a sequence of sets with finite perimeter $(F_k)_{k\geqslant 1}$ such that, for all $k \geqslant 1$,

$$P_\Gamma(F_k, \mathbb{R}^n) \leqslant C_\Gamma(1+1/k)|F_k|^{(n-1)/n},$$

but $|F_k \vartriangle W| > \delta|F_k|$ for any Wulff set W with $|W| = |F_k|$. We set:

$$G_k := \{y \in \mathbb{R}^n:\ |F_k|^{1/n}\, y \in F_k\}.$$

For all $k \geqslant 1$, we have

$$|G_k| = 1 \tag{17}$$

and, remembering that the Γ-perimeter is homogeneous of degree $(n-1)$ with respect to the dilations,

$$P_\Gamma(G_k, \mathbb{R}^n) \leqslant C_\Gamma(1+1/k), \tag{18}$$

and also,

$$|G_k \vartriangle W| > \delta \tag{19}$$

for any Wulff set such that $|W| = 1$.

We would like to go to the limit as $k \uparrow +\infty$ and get, at least up to a subsequence, some limit, say G, for the sequence G_k. Classical embeddings theorems only ensure convergence in L^1_{loc} (in the sense of convergence of the corresponding characteristic functions) and one could have a limit set $G = \emptyset$. To avoid this situation we first need to modify the sequence G_k before passing to a subsequence. The point is that one can find a constant $\gamma > 0$, depending only on n, α and β, and for all $k \geqslant 1$, some $x_k \in \mathbb{R}^n$ such that $|G_k \cap B_1(x_k)| \geqslant \gamma$. This follows from Lemma 2.13 to be proved a few lines below together with (17) and (18) which imply that $|G_k| P(G_k, \mathbb{R}^n)^{-1} \geqslant \alpha/(2C_\Gamma)$ for all $k \geqslant 1$. Then, considering the sequence $(G_k - x_k)_{k\geqslant 1}$ that we still denote by G_k for simplicity, we have

$$\big|G_k \cap B_1(0)\big| \geqslant \gamma. \tag{20}$$

Note that (17), (18) and (19) still hold (remember in particular that the Γ-perimeter is invariant under translations). Next we have $\sup_{k\geqslant 1}\{|G_k| + P(G_k, \mathbb{R}^n)\} < +\infty$ and we can extract a subsequence, still denoted by $(G_k)_{k\geqslant 1}$, which converges to some set G in L^1_{loc} (see [3, Theorem 3.38]). Let us prove that the convergence actually holds in L^1.

For simplicity of notations we set $B_t := B_t(0)$ for any $t > 0$. Let $\varepsilon < 1$ be fixed. We have,

$$|G| \leqslant \liminf_{k\to+\infty} |G_k| < +\infty,$$

hence one can find $t > 1$ such that $|G \setminus B_t| \leqslant \varepsilon$. Then, by convergence in L^1_{loc}, we have $|G_k \cap (B_{t+1} \setminus B_t)| \leqslant 2\varepsilon$ if k is large enough. Next, using Tchebytchev's inequality and the coarea formula, one can find $t_k \in (t, t+1)$ such that

$$\mathcal{H}^{n-1}\big(\partial B_{t_k} \setminus G_k(0)\big) \leqslant C\big|G_k \cap (B_{t+1} \setminus B_t)\big| \leqslant C\varepsilon$$

for some suitable-dimensional constant $C > 0$. Remember that $G_k(0)$ is the set of Lebesgue points of G_k^c and hence is equivalent to G_k^c. Then, since $G_k \cap B_{t_k}$ and G_k coincide on the open set B_{t_k} and since $\partial_*(G_k \cap B_{t_k}) \cap G_k(0) = \emptyset$, we have (remember also (9))

$$P_\Gamma(G_k \cap B_{t_k}, \mathbb{R}^n) \leqslant P_\Gamma(G_k, B_{t_k}) + \beta\mathcal{H}^{n-1}\big(\partial B_{t_k} \setminus G_k(0)\big) \leqslant P_\Gamma(G_k, B_{t_k}) + C\varepsilon.$$

Similarly,

$$P_\Gamma(G_k \setminus B_{t_k}, \mathbb{R}^n) \leqslant P_\Gamma\big(G_k, \overline{B}^c_{t_k}\big) + \beta\mathcal{H}^{n-1}\big(\partial B_{t_k} \setminus G_k(0)\big) \leqslant P_\Gamma\big(G_k, \overline{B}^c_{t_k}\big) + C\varepsilon.$$

We set $\gamma_k := |G_k \cap B_{t_k}|$. Applying the isoperimetric inequality (16) to both sets $G_k \cap B_{t_k}$ and $G_k \setminus B_{t_k}$, we get:

$$\begin{aligned} C_\Gamma\big(\gamma_k^{(n-1)/n} + (1-\gamma_k)^{(n-1)/n}\big) &\leqslant P_\Gamma(G_k \cap B_{t_k}, \mathbb{R}^n) + P_\Gamma(G_k \setminus B_{t_k}, \mathbb{R}^n) \\ &\leqslant P_\Gamma(G_k, \mathbb{R}^n) + C\varepsilon \leqslant C_\Gamma(1 + 1/k) + C\varepsilon, \end{aligned}$$

where the last inequality follows from (18). Thus, if k is large enough, we have

$$f(\gamma_k) \leqslant 1 + C\varepsilon,$$

where $f : [0,1] \to [0,1]$ is defined by $f(u) = u^{(n-1)/n} + (1-u)^{(n-1)/n}$ and C depends only on n, α and β. We have $f(u) = f(1-u)$, $f(0) = f(1) = 1$, and f is increasing on $[0, 1/2]$. On the other hand, according to (20) we have $\gamma_k \geqslant \gamma$ and it then follows that one must have

$$\gamma_k \geqslant h(\varepsilon)$$

for some function h which goes to 1 when ε goes to zero. It follows that

$$|G_k \setminus B_{t+1}| \leqslant 1 - \gamma_k \leqslant 1 - h(\varepsilon) \quad \text{and} \quad |G \setminus B_{t+1}| \leqslant \varepsilon,$$

hence,

$$\limsup_{k\to+\infty}\int_{\mathbb{R}^n}|\mathbf{1}_{G_k}-\mathbf{1}_G|\leqslant\limsup_{k\to+\infty}\int_{B_{t+1}}|\mathbf{1}_{G_k}-\mathbf{1}_G|+1-h(\varepsilon)+\varepsilon=1-h(\varepsilon)+\varepsilon$$

for all $\varepsilon<1$. Then we take the limit when ε goes to zero to get that $(G_k)_{k\geqslant1}$ converges to G in L^1 as claimed.

Now, passing to the limit when $k\uparrow+\infty$ in (17) and (19), it follows from the convergence in L^1 that $|G|=1$ and that

$$|G\bigtriangleup W|\geqslant\delta \tag{21}$$

for any Wulff set with $|W|=|G|=1$. On the other hand, by lower semi-continuity of P_Γ when Γ is convex (see [12, Theorem 4.5] and Remark 2.12), it follows from (18) that

$$P_\Gamma(G,\mathbb{R}^n)\leqslant\liminf_{k\to+\infty}P_\Gamma(G_k,\mathbb{R}^n)\leqslant C_\Gamma.$$

Combining this with the isoperimetric inequality (16) we get that $P_\Gamma(G,\mathbb{R}^n)=C_\Gamma$. Hence G is, according to Theorem 2.4, equivalent to some Wulff set and this contradicts (21). □

Remark 2.12. Theorem 4.5 in [12] about the lower semi-continuity of P_Γ is given only for sequences of bounded sets with finite perimeter. However it turns out that standard truncation arguments imply that the result still holds even when the elements of the sequence are not necessarily bounded. More precisely, we consider a sequence $(G_k)_{k\geqslant1}$ of sets with finite perimeter such that $\sup_{k\geqslant1}\{|G_k|+P(G_k,\mathbb{R}^n)\}<+\infty$ and we assume that $(G_k)_{k\geqslant1}$ converges to some set G in L^1. Then one can construct an increasing sequence $(r_k)_{k\geqslant1}$ with $r_k\uparrow+\infty$ and such that

$$P_\Gamma(G_k\cap B_{r_k},\mathbb{R}^n)\leqslant P_\Gamma(G_k,\mathbb{R}^n)+1/k.$$

Indeed one first fix for instance $r(k)$ large enough so that $|G_k\setminus B_{r(k)}|\leqslant C/k$ for some suitable small universal constant $C>0$ and then one chooses $r_k\in(r(k),r(k)+1)$ in the same way as t_k in Lemma 2.11 so that

$$\mathcal{H}^{n-1}\big(\partial B_{t_k}\setminus G_k(0)\big)\leqslant\beta^{-1}/k.$$

Then, since $r_k\uparrow+\infty$ and because $|G|\leqslant\liminf_{k\to+\infty}|G_k|<+\infty$, the sequence $(G_k\cap B_{r_k})_{k\geqslant1}$ still converges in L^1 to G and it follows from [12, Theorem 4.5] applied to that latter sequence of bounded sets that

$$P_\Gamma(G,\mathbb{R}^n)\leqslant\liminf_{k\to+\infty}P_\Gamma(G_k\cap B_{r_k},\mathbb{R}^n)\leqslant\liminf_{k\to+\infty}P_\Gamma(G_k,\mathbb{R}^n).$$

Of course the same arguments apply to more general anisotropic perimeters as considered in [12].

Lemma 2.13. *Let F be a set with finite perimeter and finite Lebesgue measure. Assume that $\gamma \in (0, |B_1(0)|/2)$ is such that $|F \cap B_1(x)| \leqslant \gamma$ for all $x \in \mathbb{R}^n$. Then*

$$C|F|^n P(F, \mathbb{R}^n)^{-n} \leqslant \gamma$$

for some constant $C > 0$ that depends only on n.

Proof. Let F and γ be as in the statement. Let $\mathcal{A}$ be a maximal family of points in $\mathbb{R}^n$ at mutual distance $\geqslant 1/2$ and such that $|F \cap B_{1/2}(x)| > 0$ for all $x \in \mathcal{A}$. Then $\bigcup_{x\in\mathcal{A}} B_1(x)$ covers almost all of F. Otherwise there would exist a point $y \in \mathbb{R}^n$ such that

$$\left|\left(F \Big\backslash \bigcup_{x\in\mathcal{A}} B_1(x)\right) \cap B_{1/2}(y)\right| > 0.$$

By maximality of $\mathcal{A}$ we would have $y \in B_{1/2}(x)$ for some $x \in \mathcal{A}$ and then $B_{1/2}(y) \subset B_1(x)$ which gives a contradiction. Hence we have

$$|F| \leqslant \sum_{x\in\mathcal{A}} \bigl|F \cap B_1(x)\bigr| \leqslant \gamma^{1/n} \sum_{x\in\mathcal{A}} \bigl|F \cap B_1(x)\bigr|^{(n-1)/n} \leqslant C\gamma^{1/n} \sum_{x\in\mathcal{A}} P\bigl(F, B_1(x)\bigr),$$

where the last inequality follows from the relative isoperimetric inequality for balls (note that $|F \cap B_1(x)| \leqslant \gamma \leqslant |B_1(0)|/2$ hence $\min\{|F \cap B_1(x)|, |B_1(x) \setminus F|\} = |F \cap B_1(x)|$). Now the balls $B_1(x)$, $x \in \mathcal{A}$, have bounded overlap because the balls $B_{1/4}(x)$, $x \in \mathcal{A}$, are disjoint. Thus we have

$$\sum_{x\in\mathcal{A}} P\bigl(F, B_1(x)\bigr) \leqslant C P(F, \mathbb{R}^n)$$

and it follows

$$|F| \leqslant C\gamma^{1/n} P(F, \mathbb{R}^n)$$

as claimed. □

3. Ahlfors-regularity and condition B

This section and the following one are entirely devoted to the proof of Theorem 1.4. In this section we prove in Lemma 3.1 the upper estimate in the Ahlfors-regularity and reduce the proof of the other properties to a lemma, Lemma 3.2, which analyzes the behavior of the proportion of a quasiminimal crystal and of its complement inside Wulff sets. We fix for the rest of this section and the following one a quasiminimal crystal with a volume constraint $E \in \mathcal{QM}$ with prescribed measure a as in Definition 1.1. Recall that in this definition $\varGamma : \mathbb{S}^{n-1} \to \mathbb{R}^+$ is a fixed continuous function which satisfies (1), $g : [0, +\infty] \to [0, +\infty]$ is fixed such that $\lim_{v\to 0^+} v^{-(n-1)/n} g(v) = 0$ and $a > 0$ is fixed.

Note that because of (13) one can replace in the definitions of the Ahlfors-regularity (Definition 1.2) and of the condition B (Definition 1.3) balls by Wulff sets and get equivalent definitions. We shall freely use this in what follows, choosing in each situation the most convenient definition to work with.

Lemma 3.1. *There exists a universal constant $C > 0$ such that*

$$P_\Gamma\big(E, W_r(x)\big) \leqslant C r^{n-1}$$

for all $x \in \mathbb{R}^n$ and $r \leqslant 1$.

Proof. Let $x \in \mathbb{R}^n$ and $r \leqslant 1$ be fixed. We set $F = (E \setminus W_r(x)) \cup W$ where W is a Wulff set contained in $W_r(x)$ and such that $|W| = |E \cap W_r(x)|$. Then $|F| = |E|$ and F is an admissible candidate for E. We have $\partial_* F \cap \overline{W}_r(x) \subset \partial W \cup \partial W_r(x)$ and F coincides with E on the open set $\overline{W}_r(x)^c$. Combining this with (9) and (11), we get:

$$\begin{aligned} P_\Gamma(F, \mathbb{R}^n) &\leqslant P_\Gamma\big(E, \overline{W}_r(x)^c\big) + \beta\mathcal{H}^{n-1}(\partial W) + \beta\mathcal{H}^{n-1}\big(\partial W_r(x)\big) \\ &\leqslant P_\Gamma(E, \mathbb{R}^n) - P_\Gamma\big(E, W_r(x)\big) + C r^{n-1}, \end{aligned}$$

for some universal constant $C > 0$. Moreover we have $|F \triangle E| \leqslant 2|E \cap W_r(x)| \leqslant C r^n$. Thus if r is small enough, $r \leqslant r_1$ say for some universal constant $r_1 \leqslant 1$, we have

$$g\big(|F \triangle E|\big) \leqslant r^{n-1},$$

by assumption on g. Then it follows from the quasiminimality of E (see (2)) that

$$P_\Gamma(E, \mathbb{R}^n) \leqslant P_\Gamma(F, \mathbb{R}^n) + g\big(|F \triangle E|\big) \leqslant P_\Gamma(E, \mathbb{R}^n) - P_\Gamma\big(E, W_r(x)\big) + C r^{n-1},$$

hence,

$$P_\Gamma\big(E, W_r(x)\big) \leqslant C r^{n-1},$$

which gives the required conclusion provided $r \leqslant r_1$. The conclusion for radii $r \in (r_1, 1]$ and with a slightly different constant which depends on r_1 follows easily by a covering argument (one can for instance cover $W_r(x)$ with at most $C(r/r_1)^n$ Wulff sets with radius r_1). □

The lower estimate in the Ahlfors-regularity and the condition B are more delicate to prove, especially because we want to get universal constants in these properties. The main step in the proof is given by Lemma 3.2 stated below. It says that if the proportion of E (in measure) inside some Wulff set, say W, is small enough then $(1/2)W$ is essentially contained in the complement of E, and similarly for E^c. For $x \in \mathbb{R}^n$ and $r > 0$, we set

$$h(x, r) := r^{-n} \min\big\{\big|E \cap W_r(x)\big|, \big|W_r(x) \setminus E\big|\big\}.$$

Lemma 3.2. *There exist two universal constants $\varepsilon_0 > 0$ and $R \leqslant 1$ such that, for any $x \in \mathbb{R}^n$ and $r \leqslant R$, if $h(x,r) \leqslant \varepsilon_0$, then*

$$\left|E \cap W_{r/2}(x)\right| = 0 \quad \text{or} \quad \left|W_{r/2}(x) \setminus E\right| = 0.$$

We will prove this lemma in Section 4. The proof relies strongly on the quasiminimality of E. Then it turns out that once one has this lemma in hand, the required properties follow essentially by quite standard covering arguments that do not use the quasiminimality of E anymore. The same kind of arguments have been already used in the literature, see, e.g., [11] for a different situation and [16] for the case of the standard perimeter. For sake of completeness we give some more details about this in the rest of this section (this will also be useful later).

We set:

$$E_1 := \left\{x \in \mathbb{R}^n\colon \text{ there exists } r > 0 \text{ such that } \left|W_r(x) \setminus E\right| = 0\right\}, \tag{22}$$

$$E_0 := \left\{x \in \mathbb{R}^n\colon \text{ there exists } r > 0 \text{ such that } \left|E \cap W_r(x)\right| = 0\right\}, \tag{23}$$

$$S := \left\{x \in \mathbb{R}^n\colon\ h(x,r) > \varepsilon_0 \text{ for all } r \leqslant R\right\}, \tag{24}$$

where ε_0 and R are given by Lemma 3.2. One can actually deduce from Lemmas 3.1 and 3.2 the following refined version of Theorem 1.4.

Theorem 3.3. *With E_1, E_0 and S as above, the following conclusions hold*:

E_1, E_0 and S form a partition of $\mathbb{R}^n$,

E_1 and E_0 are open and equivalent to E and E^c respectively,

$S = \partial E_1 = \partial E_0$,

$\partial E_1 = \partial_* E_1 = \partial_* E$ and $\partial E_0 = \partial_* E_0 = \partial_*(E^c)$,

S is Ahlfors-regular,

E_1 is bounded,

E_1 and E_0 satisfy the condition B.

Moreover the Ahlfors-regularity and condition B constants can be chosen universal.

Proof. The first claim is an immediate consequence of Lemma 3.2. The sets E_1 and E_0 are clearly open. Moreover E_1 coincides with the set $E(1)$ of Lebesgue points of E. Indeed if $x \in E(1)$ then $h(x,r) = r^{-n}|W_r(x) \setminus E|$ if r is small enough and $h(x,r)$ tends to zero when r goes to zero. Then it follows from Lemma 3.2 that $x \in E_1$. The other inclusion follows from the definition of E_1. Hence E and E_1 are equivalent. Similar arguments with E replaced by its complement show that E_0 is the set of Lebesgue points of E^c and then E^c and E_0 are equivalent.

By definition of S and since E and E_1, respectively E^c and E_0, are equivalent, we clearly have $S \subset \partial E_1 \cap \partial E_0$. On the other hand, E_1 and E_0 are open and disjoint and, since E_1, E_0 and S form a partition of $\mathbb{R}^n$, it follows that $\partial E_1 \cup \partial E_0 \subset S$. Thus $S = \partial E_1 = \partial E_0$. We clearly have $E_1 = E_1(1) = E(1) = E_0(0) = E^c(0)$ and $E_0 = E_1(0) = E(0) = E_0(1) = E^c(1)$. Then the equalities between S and the various essential boundaries is an immediate consequence of the definitions together with the fact that E_1, E_0 and S form a partition of $\mathbb{R}^n$.

Next we prove that S is Ahlfors-regular with $\mu = P_\Gamma(E, \cdot) = P_\Gamma(E_1, \cdot)$ in (3). First we have $\operatorname{spt}(\mu) \subset \partial E_1 = S$. The upper estimate follows from Lemma 3.1. Next if $x \in S$ and $r \leqslant R$, we have by definition of S and by the relative isoperimetric inequality for Wulff sets (Proposition 2.8),

$$C\varepsilon_0^{(n-1)/n} r^{n-1} \leqslant C\big(r^n h(x,r)\big)^{(n-1)/n} \leqslant P_\Gamma\big(E, W_r(x)\big).$$

This gives the required conclusion provided $r \leqslant R$. The conclusion for radii $r \in (R, 1]$ follows easily with a slightly different constant depending now also on R.

To prove that E_1 is bounded, we consider a maximal family $\mathcal{A}_1$ of points in ∂E_1 at mutual distance $\geqslant 1$. The balls $B_{1/2}(x)$, $x \in \mathcal{A}_1$, are pairwise disjoint and since ∂E_1 is Ahlfors-regular we have:

$$\operatorname{card}(\mathcal{A}_1) \leqslant C \sum_{x\in\mathcal{A}_1} P_\Gamma\big(E_1, B_{1/2}(x)\big) \leqslant C P_\Gamma\left(E_1, \bigcup_{x\in\mathcal{A}_1} B_{1/2}(x)\right) \leqslant C P_\Gamma(E_1, \mathbb{R}^n) < +\infty.$$

Since $\partial E_1 \subset \bigcup_{x\in\mathcal{A}_1} B_1(x)$ we get that ∂E_1 is bounded. Hence, since $|E_1| < +\infty$, we have $\operatorname{diam}(E_1) = \operatorname{diam}(\partial E_1) < +\infty$.

We now prove that E_1 and E_0 satisfy the condition B. Let $x \in S$ and $r \leqslant \min(1/2, \beta R)$ be fixed. Set:

$$Z := \big\{z \in B_{r/2}(x)\colon \operatorname{dist}(z, S) \leqslant sr/2\big\},$$

where $0 < s < 1$ will be fixed small later. We have $|Z| \leqslant Csr^n$. To see this, we take a maximal family $\mathcal{A}$ of points in $S \cap B_r(x)$ at mutual distance $\geqslant sr/2$. The balls $B_{sr/4}(y)$, $y \in \mathcal{A}$, are pairwise disjoint, S is Ahlfors-regular, $\bigcup_{y\in\mathcal{A}} B_{sr/4}(y) \subset B_{2r}(x)$, then, arguing as above, we get:

$$\operatorname{card}(\mathcal{A}) \leqslant C(sr)^{-n+1} \sum_{y\in\mathcal{A}} P_\Gamma\big(E, B_{sr/4}(y)\big) \leqslant C(sr)^{-n+1} P_\Gamma\big(E, B_{2r}(x)\big) \leqslant Cs^{-n+1}.$$

Going back to Z, we have

$$Z \subset \bigcup_{y\in\mathcal{A}} B(y, sr)$$

and then $|Z| \leqslant C \operatorname{card}(\mathcal{A})(sr)^n \leqslant Csr^n$ as claimed. Since $x \in S$, we also have:

$$\begin{aligned}\left|E_1 \cap B_r(x)\right| &\geqslant \left|E_1 \cap W_{\beta^{-1}r}(x)\right| \geqslant \varepsilon_0 \beta^{-n} r^n,\\ \left|E_0 \cap B_r(x)\right| &\geqslant \left|E_0 \cap W_{\beta^{-1}r}(x)\right| \geqslant \varepsilon_0 \beta^{-n} r^n.\end{aligned}$$

Then, if s is small enough, depending on n, ε_0 and β, one can find $z_1 \in (E_1 \cap B_r(x)) \setminus Z$ and $z_0 \in (E_0 \cap B_r(x)) \setminus Z$ and then

$$\begin{aligned}B_{sr/2}(z_1) &\subset E_1 \cap B_r(x) = B_r(x) \setminus \overline{E}_0,\\ B_{sr/2}(z_0) &\subset E_0 \cap B_r(x) = B_r(x) \setminus \overline{E}_1.\end{aligned}$$

This gives the required conclusion for any $x \in S = \partial E_0 = \partial E_1$ and $r \leqslant \min(1/2, \beta R)$. And the conclusion follows easily for any $r \leqslant 1$ (with a slightly different constant in the condition B). This concludes the proof of the theorem. □

To prove Theorem 1.4 it only remains to prove uniqueness. This follows from the fact that two open sets that are equivalent and both satisfy the condition B coincide.

4. Behavior of the proportion of E and E^c inside Wulff sets

This section is devoted to the proof of Lemma 3.2. The basic idea is the following. When the proportion of the complement of E inside some Wulff set is very small, it is natural to try to add this Wulff set (or at least a slightly smaller one) to E. Similarly if the proportion of E is very small, one can to try to remove the Wulff set from E. Then, to make use of the quasiminimality of E through a suitable comparison argument, one needs, because of the volume constraint, to adjust the measure of this first modification to get an admissible candidate with exactly the same Lebesgue measure than E. Moreover, remembering also that we want to get at the end universal regularity constants, one must find a way to do these adjustments while keeping some universal control in all the constructions.

In the first case, adjusting the measure (that is, removing some mass) will not be too complicated because Lemma 2.5 gives a way to do this while keeping a suitable control on the variation of the Γ-perimeter. This will be done in Section 4.1.

The second case, when one needs to add some mass after having removed a Wulff set, is more complicated and will occupy Sections 4.2, 4.3, 4.4 and 4.5. We shall not get directly the conclusion but rather argue by contradiction. Roughly speaking we will consider a point, say x, around which the proportion of E inside Wulff sets is small but for which the conclusion of Lemma 3.2 fails. We shall prove that near such a point one can always find some Wulff set essentially contained in E that one can moreover move around to add some mass (see Lemma 4.3). The point is that its size and the way it can be moved around will be controlled in a uniform and universal way. The mass that can then be added and the associated variation of the Γ-perimeter (remember Lemma 2.6) will consequently be also suitably controlled. This will be used to prove through a direct comparison argument that the conclusion of Lemma 3.2 does hold far away from x. In particular it will follow that the condition B holds on a substantial part of ∂E. Then, using this condition B property, one can perform suitable constructions that allow in turn to remove definitely the set E around x and give the final contradiction.

4.1. *Behavior of E^c*

We prove in this section Lemma 3.2 when the proportion of the complement of E around some point is very small. The proof is divided into two parts. First we show that in that case the proportion of E^c decreases geometrically.

Lemma 4.1. *There exists a universal constant $\varepsilon_1 > 0$ such that, for any $x \in \mathbb{R}^n$ and $r \leqslant 1$,*

$$\left|W_r(x) \setminus E\right| \leqslant \varepsilon_1 r^n \quad \Rightarrow \quad h(x, r/2) \leqslant h(x, r)/2.$$

Proof. Let $\varepsilon_1 > 0$ be a small constant that will be fixed later. Let $x \in \mathbb{R}^n$ and $r \leqslant 1$ be such that $|W_r(x) \setminus E| \leqslant \varepsilon_1 r^n$. First, if ε_1 is small enough, $\varepsilon_1 \leqslant |W_\Gamma|/2$, we have:

$$h(x, r) = r^{-n}\left|W_r(x) \setminus E\right|.$$

Next, using Tchebytchev's inequality and (14), one can always find $t \in (r/2, r)$ such that

$$\mathcal{H}^{n-1}\big(\partial W_t(x) \setminus E(1)\big) \leqslant Cr^{-1}\left|W_r(x) \setminus E\right| = Cr^{n-1}h(x, r)$$

for some universal constant $C > 0$. Recall that $E(1)$ is the set of Lebesgue points of E and thus is equivalent to E. We set $F = (E \cup W_t(x)) \cap W$ where W is a Wulff set chosen in such a way that $|F| = |E|$ (obviously we take $W = \mathbb{R}^n$ if $|W_t(x) \setminus E| = 0$). Using Lemma 2.5 and the fact that E and $E \cup W_t(x)$ coincide on the open set $\overline{W}_t(x)^c$, we get:

$$P_\Gamma(F, \mathbb{R}^n) \leqslant P_\Gamma\big(E \cup W_t(x), \mathbb{R}^n\big) = P_\Gamma\big(E, \overline{W}_t(x)^c\big) + P_\Gamma\big(E \cup W_t(x), \partial W_t(x)\big).$$

Furthermore $\partial_*(E \cup W_t(x)) \cap E(1) = \emptyset$, hence by choice of t we have:

$$P_\Gamma\big(E \cup W_t(x), \partial W_t(x)\big) \leqslant \beta\mathcal{H}^{n-1}\big(\partial W_t(x) \setminus E(1)\big) \leqslant Cr^{n-1}h(x, r),$$

and finally,

$$P_\Gamma(F, \mathbb{R}^n) \leqslant P_\Gamma(E, \mathbb{R}^n) - P_\Gamma\big(E, W_t(x)\big) + Cr^{n-1}h(x, r).$$

On the other hand,

$$|F \triangle E| \leqslant 2\left|W_t(x) \setminus E\right| \leqslant 2r^n h(x, r) \leqslant 2\varepsilon_1$$

(recall that $r \leqslant 1$). Hence, if ε_1 is small enough, we get thanks to the relative isoperimetric inequality for Wulff sets (Proposition 2.8) that, for some suitable universal constant $C > 0$,

$$g\big(|F \triangle E|\big) \leqslant C\left|W_t(x) \setminus E\right|^{(n-1)/n} \leqslant P_\Gamma\big(E, W_t(x)\big)/2.$$

Since $|F| = |E|$, one gets by quasiminimality of E,

$$\begin{aligned} P_\Gamma(E,\mathbb{R}^n) &\leqslant P_\Gamma(F,\mathbb{R}^n) + g\big(|F \vartriangle E|\big) \\ &\leqslant P_\Gamma(E,\mathbb{R}^n) - P_\Gamma\big(E, W_t(x)\big)/2 + Cr^{n-1}h(x,r), \end{aligned}$$

that is,

$$P_\Gamma\big(E, W_t(x)\big) \leqslant Cr^{n-1}h(x,r)$$

for some suitable universal constant $C > 0$. On the other hand, once again by the relative isoperimetric inequality for Wulff sets, we have

$$C\big(r^n h(x,r/2)\big)^{(n-1)/n} \leqslant P_\Gamma\big(E, W_{r/2}(x)\big) \leqslant P_\Gamma\big(E, W_t(x)\big)$$

hence,

$$h(x,r/2) \leqslant Ch(x,r)^{n/(n-1)} \leqslant C\varepsilon_1^{1/(n-1)}h(x,r) \leqslant h(x,r)/2,$$

provided ε_1 is chosen small enough. □

Now the conclusion is an automatic consequence of the previous lemma.

Lemma 4.2. *There exists a universal constant $\varepsilon_2 > 0$ such that, for any $x \in \mathbb{R}^n$ and $r \leqslant 1$,*

$$\big|W_r(x) \setminus E\big| \leqslant \varepsilon_2 r^n \quad \Rightarrow \quad \big|W_{r/2}(x) \setminus E\big| = 0.$$

Proof. Let $\varepsilon_2 > 0$ be a small constant to be fixed later. Let $x \in \mathbb{R}^n$ and $r \leqslant 1$ be such that $|W_r(x) \setminus E| \leqslant \varepsilon_2 r^n$ and let y be any point in $W_{r/2}(x)$. We have $W_{r/2}(y) \subset W_r(x)$ and, if ε_2 is small enough,

$$h(y,r/2) = (r/2)^{-n}\big|W_{r/2}(y) \setminus E\big| \leqslant (r/2)^{-n}\big|W_r(x) \setminus E\big| \leqslant 2^n\varepsilon_2.$$

Then, using Lemma 4.1 and an induction procedure, one can easily show that

$$h\big(y, 2^{-k}r\big) = \big(2^{-k}r\big)^{-n}\big|W_{2^{-k}r}(y) \setminus E\big|$$

and

$$h\big(y, 2^{-(k+1)}r\big) \leqslant h\big(y, 2^{-k}r\big)/2$$

for all $k \geqslant 1$, provided ε_2 is small enough. It follows that

$$\lim_{k\to+\infty}\big(2^{-k}r\big)^{-n}\big|W_{2^{-k}r}(y) \setminus E\big| = 0,$$

and thus y is not a Lebesgue point of the complement of E. Since this holds for any $y \in W_{r/2}(x)$, we get that $|W_{r/2}(x) \setminus E| = 0$ as wanted. □

4.2. The main constructions

Following the strategy sketched at the beginning of this section, we turn now our attention to points around which the proportion of E is very small. The proof in that case is divided in several steps and will be achieved in Section 4.5. Arguing by contradiction we shall first analyze the behavior of E around such points for which the conclusion of Lemma 3.2 fails.

Lemma 4.3. *There exists a universal constant $\varepsilon_3 > 0$ such that, for any $x \in \mathbb{R}^n$ and $r \leqslant 1$, if*

$$\left|E \cap W_r(x)\right| \leqslant \varepsilon_3 r^n \quad \text{and} \quad \left|E \cap W_{r/2}(x)\right| > 0,$$

then there exist $r_1 \in (r/2, 5r/8)$ and $r_2 \in (7r/8, r)$ such that

$$\max_{i=1,2}\left\{P\left(E \cap W_{r_i}(x), \partial W_{r_i}(x)\right), P\left(E \setminus W_{r_i}(x), \partial W_{r_i}(x)\right)\right\} = 0,$$

$$\left|E \cap \left(W_{r_2}(x) \setminus \overline{W}_{r_1}(x)\right)\right| = 0,$$

and one can find a Wulff set $W \subset W_{r_1}(x)$ such that $|W \setminus E| = 0$ and $|W| = C|E \cap W_{r_1}(x)|$ for some universal constant $C > 0$.

Note that all the conclusions in this lemma come with universal and scale-invariant bounds. The proof will be achieved in Section 4.4. It will be a consequence of several suitable uses of the main lemma to be proved now, see Lemma 4.4. Let us stress that the various constructions of the present section give the main comparison arguments of this paper. They will be quite constantly re-used later (in slight different context and with slight technical differences though).

For the rest of this section, let $x \in \mathbb{R}^n$ be fixed. For simplicity of notations, set $W_s := W_s(x)$ for $s > 0$. For any fixed $0 \leqslant s_0 < s_1 < s_2 \leqslant 1$ and $i = 1, 2$, set

$$E^i := E \cap (W_{s_i} \setminus \overline{W}_{s_{i-1}}) \quad \text{and} \quad m_i := \left|E^i\right|,$$
$$\rho := \max_{i=0,1,2}\left\{P(E \cap W_{s_i}, \partial W_{s_i}), P(E \setminus W_{s_i}, \partial W_{s_i})\right\}.$$

For any $\varepsilon > 0$, we say that (H_ε) holds if

$$\max\{m_1, m_2\} \leqslant \begin{cases} \varepsilon \min\left\{s_1^n, (s_2 - s_1)^n\right\} & \text{if } s_0 = 0, \\ \varepsilon \min\left\{s_0^n, (s_1 - s_0)^n, (s_2 - s_1)^n\right\} & \text{if } s_0 > 0. \end{cases} \qquad (H_\varepsilon)$$

Note that when $s_0 = 0$, then $W_{s_0} = \emptyset$ and the convention is that $E^1 = E \cap W_{s_1}$ and $P(E \cap W_{s_0}, \partial W_{s_0}) = P(E \setminus W_{s_0}, \partial W_{s_0}) = 0$.

Lemma 4.4. *There exist two universal constants $\varepsilon_4 > 0$ and $C > 0$ such that, if (H_{ε_4}) holds, then*

$$\min\{m_1, m_2\} \leqslant C\rho^{n/(n-1)}. \tag{25}$$

In what follows, when saying that a constant depends only on some given data, we mean that its value can be chosen depending only on these data and also possibly on n, α, β, g and a but on nothing else.

We begin with the proof of Lemma 4.4 in two special cases. The first one deals with the situation where E^1 and E^2 are of comparable size.

Lemma 4.5. *For any $\tau \in (0,1)$, there exists $\varepsilon > 0$ depending only on τ such that, if (H_ε) holds and $\tau m_1 \leqslant m_2 \leqslant \tau^{-1} m_1$, then* (25) *holds with a constant C which depends only on τ.*

Proof. Let $\tau \in (0,1)$ be fixed, $\varepsilon > 0$ be a small constant to be fixed later, assume that (H_ε) holds and that $\tau m_1 \leqslant m_2 \leqslant \tau^{-1} m_1$. We want to replace $E^1 \cup E^2$ by a single Wulff set. According to (H_ε) we have $m_1 + m_2 \leqslant 2\varepsilon(s_2 - s_0)^n$. Hence, thanks to Lemma 4.6 (to be proved below), one can find a Wulff set $W \Subset W_{s_2} \setminus \overline{W}_{s_0}$ with $|W| = m_1 + m_2$, at least if ε is small enough, how small depending only on n, α and β. We set $F = (E \setminus (E^1 \cup E^2)) \cup W$. Then $|F| = |E|$ and F is an admissible candidate. We first estimate its Γ-perimeter. We have (see Lemma 2.2)

$$P_\Gamma(F, \mathbb{R}^n) \leqslant P_\Gamma\big(E \setminus \big(E^1 \cup E^2\big), \mathbb{R}^n\big) + P_\Gamma(W, \mathbb{R}^n).$$

The set $E \setminus (E^1 \cup E^2)$ coincides with E on the open set $(\overline{W}_{s_2} \setminus W_{s_0})^c$, is equivalent to the empty set inside the open set $W_{s_2} \setminus \overline{W}_{s_0}$, coincides with $E \setminus W_{s_2}$ on a neighborhood of ∂W_{s_2} and with $E \cap W_{s_0}$ on a neighborhood of ∂W_{s_0}. It follows that

$$\begin{aligned}
&P_\Gamma\big(E \setminus \big(E^1 \cup E^2\big), \mathbb{R}^n\big)\\
&\quad = P_\Gamma(E, \mathbb{R}^n) - P_\Gamma(E, \overline{W}_{s_2} \setminus W_{s_0}) + P_\Gamma(E \setminus W_{s_2}, \partial W_{s_2}) + P_\Gamma(E \cap W_{s_0}, \partial W_{s_0})\\
&\quad \leqslant P_\Gamma(E, \mathbb{R}^n) - P_\Gamma(E, W_{s_1} \setminus \overline{W}_{s_0}) - P_\Gamma(E, W_{s_2} \setminus \overline{W}_{s_1}) + 2\beta\rho.
\end{aligned}$$

On the other hand, arguing in a similar way, we have

$$\begin{aligned}
P_\Gamma\big(E^i, \mathbb{R}^n\big) &= P_\Gamma(E, W_{s_i} \setminus \overline{W}_{s_{i-1}}) + P_\Gamma(E \setminus W_{s_{i-1}}, \partial W_{s_{i-1}}) + P_\Gamma(E \cap W_{s_i}, \partial W_{s_i})\\
&\leqslant P_\Gamma(E, W_{s_i} \setminus \overline{W}_{s_{i-1}}) + 2\beta\rho
\end{aligned}$$

for $i = 1, 2$. Hence we get that

$$P_\Gamma\big(E \setminus \big(E^1 \cup E^2\big), \mathbb{R}^n\big) \leqslant P_\Gamma(E, \mathbb{R}^n) - P_\Gamma\big(E^1, \mathbb{R}^n\big) - P_\Gamma\big(E^2, \mathbb{R}^n\big) + 6\beta\rho$$

and going back to F, it follows:

$$
\begin{aligned}
P_\Gamma(F,\mathbb{R}^n) &\leqslant P_\Gamma(E,\mathbb{R}^n) - P_\Gamma\big(E^1,\mathbb{R}^n\big) - P_\Gamma\big(E^2,\mathbb{R}^n\big) + P_\Gamma(W,\mathbb{R}^n) + 6\beta\rho \\
&\leqslant P_\Gamma(E,\mathbb{R}^n) - C_\Gamma\big(m_1^{(n-1)/n} + m_2^{(n-1)/n} - (m_1+m_2)^{(n-1)/n}\big) + 6\beta\rho \\
&\leqslant P_\Gamma(E,\mathbb{R}^n) - C_\Gamma m_1^{(n-1)/n}\big(1+u^{(n-1)/n} - (1+u)^{(n-1)/n}\big) + 6\beta\rho,
\end{aligned}
$$

by the isoperimetric inequality for the Γ-perimeter (see Theorem 2.4) and by choice of W, and where $u = m_2/m_1$. We have $u \in [\tau, \tau^{-1}]$ by assumption and

$$
\min_{u\in[\tau,\tau^{-1}]}\big\{1+u^{(n-1)/n} - (1+u)^{(n-1)/n}\big\} > 0,
$$

hence,

$$
P_\Gamma(F,\mathbb{R}^n) \leqslant P_\Gamma(E,\mathbb{R}^n) - C_\tau m_1^{(n-1)/n} + 6\beta\rho
$$

for some constant $C_\tau > 0$ which depends only on τ. On the other hand, $|F \vartriangle E| \leqslant 2(m_1+m_2) \leqslant 2(1+\tau^{-1})m_1 \leqslant 2(1+\tau^{-1})\varepsilon$ by (H_ε) (recall that $s_i \leqslant 1$ for $i = 0, 1, 2$), and we choose ε small enough, depending only on τ, so that

$$
g\big(|F \vartriangle E|\big) \leqslant C_\tau m_1^{(n-1)/n}/2.
$$

Then we get by quasiminimality of E,

$$
P_\Gamma(E,\mathbb{R}^n) \leqslant P_\Gamma(F,\mathbb{R}^n) + g\big(|F \vartriangle E|\big) \leqslant P_\Gamma(E,\mathbb{R}^n) - C_\tau m_1^{(n-1)/n}/2 + 6\beta\rho,
$$

and finally,

$$
\min\{m_1, m_2\} \leqslant m_1 \leqslant C\rho^{n/(n-1)},
$$

where C depends only on τ, as required. □

Lemma 4.6. *There exists a constant $c > 0$, depending only on n, α and β, such that, for all $0 \leqslant s < s' \leqslant 1$ and $m > 0$ such that $m \leqslant c(s'-s)^n$, one can find a Wulff set $W \Subset W_{s'} \setminus \overline{W}_s$ with $|W| = m$.*

Proof. With the notations of the statement, we apply the condition B to some point in $\partial W_{s'}$ to find a Wulff set W' with radius comparable to $\mathrm{dist}(W_s, W_{s'}^c)$ and strictly contained in $W_{s'} \setminus \overline{W}_s$ (see Proposition 2.7 and recall also that because of (13), one can replace balls by Wulff sets in the condition B). Then we have

$$
|W'| \geqslant C\,\mathrm{dist}\big(W_s, W_{s'}^c\big)^n \geqslant c(s'-s)^n
$$

according to (10) and (15) and for some suitable constants C and c depending only on n, α and β. Then, if $m \leqslant c(s'-s)^n$, one can obviously find a Wulff set $W \subset W'$ with $|W| = m$ which gives the conclusion. □

We now prove Lemma 4.4 when either E^1 or E^2 are not Γ-quasi-isoperimetric in the sense of Lemma 2.11.

Lemma 4.7. *For any $\eta > 0$, there exists $\varepsilon > 0$ depending only on η such that, if (H_ε) holds and*

$$P_\Gamma\big(E^i, \mathbb{R}^n\big) \geqslant C_\Gamma(1+\eta)\big|E^i\big|^{(n-1)/n}$$

for $i = 1$ or $i = 2$, then (25) *holds with a constant C which depends only on η.*

Proof. Let $\eta > 0$ be fixed and $\varepsilon > 0$ be a small constant to be chosen later. Assume that (H_ε) holds and that

$$P_\Gamma\big(E^i, \mathbb{R}^n\big) \geqslant C_\Gamma(1+\eta)\big|E^i\big|^{(n-1)/n}$$

for $i = 1$ or $i = 2$. We replace E^i by a Wulff set $W_i \Subset W_{s_i} \setminus \overline{W}_{s_{i-1}}$ with $|W_i| = m_i$ setting $F = (E \setminus E^i) \cup W_i$. This is always possible according to Lemma 4.6 together with (H_ε) provided ε is small enough. We have $|F| = |E|$. Arguing as in Lemma 4.5, we have:

$$\begin{aligned} P_\Gamma(F, \mathbb{R}^n) &\leqslant P_\Gamma(E, \mathbb{R}^n) - P_\Gamma(E^i, \mathbb{R}^n) + P_\Gamma(W_i, \mathbb{R}^n) + 4\beta\rho \\ &\leqslant P_\Gamma(E, \mathbb{R}^n) - C_\Gamma \eta m_i^{(n-1)/n} + 4\beta\rho \end{aligned}$$

by assumption on E^i and choice of W_i. On the other hand, we have $|F \triangle E| \leqslant 2m_i \leqslant 2\varepsilon$ because of (H_ε). Then, if ε is small enough, depending only on η,

$$g\big(|F \triangle E|\big) \leqslant C_\Gamma \eta m_i^{(n-1)/n}/2,$$

and we conclude using the quasiminimality of E similarly as before,

$$P_\Gamma(E, \mathbb{R}^n) \leqslant P_\Gamma(F, \mathbb{R}^n) + g\big(|F \triangle E|\big) \leqslant P_\Gamma(E, \mathbb{R}^n) - C_\Gamma \eta m_i^{(n-1)/n}/2 + 4\beta\rho,$$

hence,

$$\min\{m_1, m_2\} \leqslant m_i \leqslant C\rho^{n/(n-1)},$$

where C depends only on η. □

We now turn to Lemma 4.4 in its full generality.

Proof of Lemma 4.4. Let $\varepsilon_4 > 0$ be a small universal constant to be fixed later and assume that (H_{ε_4}) holds. Let $\tau < 1$ be a small constant to be fixed universal later. If $\tau m_1 \leqslant m_2 \leqslant \tau^{-1} m_1$, Lemma 4.5 gives the required conclusion provided ε_4 is small enough. Thus we only need to consider the cases where $m_2 < \tau m_1$ or $m_1 < \tau m_2$. To fix the ideas we assume that we are in the first case. The other one can be proved exactly in the same way exchanging the role of E^1 and E^2. The idea is to remove E^2 from E and add the corresponding mass to E^1 using a suitable Wulff set.

Step 1. We first want to find a Wulff set W_1 with $|W_1|$ comparable to m_1 and essentially contained in E^1. Let $\delta > 0$ be a small constant to be fixed universal in a moment and $\eta > 0$ be associated to δ by Lemma 2.11. If

$$P_\Gamma\big(E^1, \mathbb{R}^n\big) \geqslant C_\Gamma (1+\eta) m_1^{(n-1)/n},$$

then Lemma 4.7 gives the conclusion provided ε_4 is small enough. Thus one can assume that

$$P_\Gamma\big(E^1, \mathbb{R}^n\big) \leqslant C_\Gamma (1+\eta) m_1^{(n-1)/n},$$

and, according to Lemma 2.11, one can then find a Wulff set W such that $|W| = m_1$ and

$$\big|E^1 \triangle W\big| \leqslant \delta m_1.$$

We have $|W \setminus E| \leqslant |W \setminus E^1| \leqslant \delta |W|$ and the radius of W is less than 1 because $|W| = |E^1|$ and $E^1 \subset W_{s_1}$. Hence, if δ is small enough, we get from Lemma 4.2 that

$$|W' \setminus E| = 0,$$

where $W' = (1/2)W$. On the other hand, we have

$$|W' \setminus W_{s_1}| \leqslant \big|W \setminus E^1\big| \leqslant C\delta |W'|,$$
$$|W' \cap W_{s_0}| \leqslant \big|W \setminus E^1\big| \leqslant C\delta |W'|,$$

and, thanks to (H_{ε_4}), $r' \leqslant C m_1^{1/n} \leqslant C \varepsilon_4^{1/n} \min\{s_0, s_1\}$ where r' denotes the radius of W'. Then it follows from Lemmas 2.9 and 2.10 that

$$W_1 := (\alpha/2\beta) W' \subset W_{s_1} \setminus \overline{W}_{s_0},$$

provided δ and ε_4 are chosen small enough. Since $E^1 = E \cap (W_{s_1} \setminus \overline{W}_{s_0})$, we finally get that

$$\big|W_1 \setminus E^1\big| = |W_1 \setminus E| = 0.$$

Step 2. Next we show that it is always possible to move W_1 strictly inside $W_{s_2} \setminus \overline{W}_{s_0}$ until it reaches a new position W_2 such that $|W_2 \setminus E^1| = m_2$. First we note that because $|W_1| = C m_1$ for some universal constant $C > 0$ and because of (H_{ε_4}), we have

$$|W_1| \leqslant C \varepsilon_4 (s_2 - s_1)^n,$$

hence one can always find a Wulff set $W_1' \Subset W_{s_2} \setminus \overline{W}_{s_1}$ such that $|W_1'| = |W_1|$ (see Lemma 4.6) provided ε_4 is small enough. Moreover $m_2 < \tau m_1 \leqslant |W_1|$ if τ is small enough. Then, since $|W_1 \setminus E^1| = 0$ and $E^1 \cap W_1' = \emptyset$, one can move W_1 continuously inside $W_{s_2} \setminus \overline{W}_{s_0}$ (at least if ε_4 is small enough, depending only on the dimension, α and β, to make sure to stay inside $W_{s_2} \setminus \overline{W}_{s_0}$) until it reaches an intermediate position $W_2 \Subset W_{s_2} \setminus \overline{W}_{s_0}$ between W_1 and W_1' such that

$$\left|W_2 \setminus E^1\right| = m_2.$$

Step 3. We set $F = (E \setminus E^2) \cup W_2$. By construction we have $|F| = |E|$ and F is an admissible candidate. Arguing as in Lemma 4.5 and using Lemma 2.6, we have:

$$\begin{aligned} P_\Gamma(F,\mathbb{R}^n) &\leqslant P_\Gamma\big(E \setminus \big(E^1 \cup E^2\big), \mathbb{R}^n\big) + P_\Gamma\big(E^1 \cup W_2, \mathbb{R}^n\big) \\ &\leqslant P_\Gamma(E,\mathbb{R}^n) - P_\Gamma\big(E^2,\mathbb{R}^n\big) + \frac{C_\Gamma}{|W_2|^{1/n}}\left|W_2 \setminus E^1\right| + 6\beta\rho \\ &\leqslant P_\Gamma(E,\mathbb{R}^n) - C_\Gamma m_2^{(n-1)/n} + C m_1^{-1/n} m_2 + 6\beta\rho \\ &\leqslant P_\Gamma(E,\mathbb{R}^n) - C_\Gamma m_2^{(n-1)/n} + C\tau^{1/n} m_2^{(n-1)/n} + 6\beta\rho \end{aligned}$$

for some universal constant $C > 0$. Then we choose τ small enough so that

$$P_\Gamma(F,\mathbb{R}^n) \leqslant P_\Gamma(E,\mathbb{R}^n) - C_\Gamma m_2^{(n-1)/n}/2 + 6\beta\rho.$$

To conclude we have $|F \vartriangle E| \leqslant 2m_2 \leqslant 2\varepsilon_4$ by (H_{ε_4}) and, if ε_4 is small enough,

$$g\big(|F \vartriangle E|\big) \leqslant C_\Gamma m_2^{(n-1)/n}/4.$$

Then we use similarly as before the quasiminimality of E,

$$P_\Gamma(E,\mathbb{R}^n) \leqslant P_\Gamma(F,\mathbb{R}^n) + g\big(|F \vartriangle E|\big) \leqslant P_\Gamma(E,\mathbb{R}^n) - C_\Gamma m_2^{(n-1)/n}/4 + 6\beta\rho,$$

to get that

$$\min\{m_1, m_2\} = m_2 \leqslant C\rho^{n/(n-1)}$$

for some universal constant $C > 0$. □

4.3. Vanishing traces

The first conclusion of Lemma 4.3 will be given by Lemma 4.8 proved in this section. It essentially follows from an iterative use of Lemma 4.4.

Lemma 4.8. *There exists a universal constant $\varepsilon_5 > 0$ such that, for any $x \in \mathbb{R}^n$ and $r \leqslant 1$, if $|E \cap W_r(x)| \leqslant \varepsilon_5 r^n$, then there exist $r_1 \in (r/2, 5r/8)$ and $r_2 \in (7r/8, r)$ such that*

$$\max_{i=1,2}\big\{P\big(E \cap W_{r_i}(x), \partial W_{r_i}(x)\big), P\big(E \setminus W_{r_i}(x), \partial W_{r_i}(x)\big)\big\} = 0.$$

Proof. Let $\varepsilon_5 > 0$ be a small constant to be fixed universal later and assume that $x \in \mathbb{R}^n$ and $r \leqslant 1$ are as in the statement. We prove that one can find $r_1 \in (r/2, 5r/8)$ such that

$$P\big(E \cap W_{r_1}(x), \partial W_{r_1}(x)\big) = P\big(E \setminus W_{r_1}(x), \partial W_{r_1}(x)\big) = 0.$$

One can argue exactly in the same way to get the existence of $r_2 \in (7r/8, r)$. To simplify the notations, we set as before $W_s := W_s(x)$ for $s > 0$.

First we note that it is sufficient to build two nonconstant sequences $(a_j)_{j \geqslant 0}$ increasing and $(b_j)_{j \geqslant 0}$ decreasing such that, for all $j \geqslant 0$,

$$r/2 \leqslant a_j \leqslant b_j \leqslant 5r/8, \tag{26}$$

$$\lim_{j \to +\infty} b_j - a_j = 0, \tag{27}$$

$$\lim_{j \to +\infty} P(E \cap W_{a_j}, \partial W_{a_j}) = \lim_{j \to +\infty} P(E \setminus W_{b_j}, \partial W_{b_j}) = 0. \tag{28}$$

Indeed, if r_1 denotes the common limit of these two sequences, we have $r_1 \in (r/2, 5r/8)$. Moreover,

$$P(E, W_{r_1}) \leqslant P(E \cap W_{r_1}, \mathbb{R}^n) \leqslant \liminf_{j \to +\infty} P(E \cap W_{a_j}, \mathbb{R}^n)$$

by lower semicontinuity of the perimeter and because $E \cap W_{a_j}$ converges in L^1 to $E \cap W_{r_1}$. On the other hand, we have

$$P(E \cap W_{a_j}, \mathbb{R}^n) = P(E, W_{a_j}) + P(E \cap W_{a_j}, \partial W_{a_j}).$$

Then, since $(W_{a_j})_{j \geqslant 0}$ is an increasing sequence of sets such that $\bigcup_j W_{a_j} = W_{r_1}$ and because of (28), we get

$$\lim_{j \to +\infty} P(E \cap W_{a_j}, \mathbb{R}^n) = P(E, W_{r_1}).$$

Thus, going back to $E \cap W_{r_1}$, we finally get

$$P(E \cap W_{r_1}, \mathbb{R}^n) = P(E, W_{r_1}),$$

which implies that

$$P(E \cap W_{r_1}, \partial W_{r_1}) = 0.$$

To prove that $P(E \setminus W_{r_1}, \partial W_{r_1}) = 0$, one argue in a similar way, using the sequence $(E \setminus W_{b_j})_{j \geqslant 0}$ to compare $P(E \setminus W_{r_1}, \mathbb{R}^n)$ and $P(E, \mathbb{R}^n \setminus \overline{W}_{r_1})$.

We construct now these two sequences $(a_j)_{j \geqslant 0}$ and $(b_j)_{j \geqslant 0}$. This will done by an induction procedure and an iterative use of Lemma 4.4. We set $a_0 = r/2$ and $b_0 = 5r/8$. Assume that we have constructed a_j and b_j such that (26) holds and set:

$$l_j = b_j - a_j.$$

Using Tchebytchev's inequality and (14) (we argue here in a similar way than in Lemma 2.11, see also Lemma 4.1), one can always find:

$$s_0^j \in (a_j, a_j + l_j/4),$$
$$s_1^j \in (a_j + 3l_j/8, a_j + 5l_j/8),$$
$$s_2^j \in (b_j - l_j/4, b_j),$$

such that

$$\max_{i=0,1,2}\left\{\mathcal{H}^{n-1}\left(\partial W_{s_i^j} \setminus E(0)\right)\right\} \leqslant C_1 \frac{m_j}{l_j}$$

for some universal constant $C_1 > 0$ and where $m_j = |E \cap (W_{b_j} \setminus \overline{W}_{a_j})|$. Then, remembering that $\partial_*(E \cap W_{s_i^j}) \cap E(0) = \emptyset$ and $\partial_*(E \setminus W_{s_i^j}) \cap E(0) = \emptyset$, we get:

$$\rho_j := \max_{i=0,1,2}\left\{P(E \cap W_{s_i^j}, \partial W_{s_i^j}), P(E \setminus W_{s_i^j}, \partial W_{s_i^j})\right\} \leqslant \max_{i=0,1,2}\left\{\mathcal{H}^{n-1}\left(\partial W_{s_i^j} \setminus E(0)\right)\right\}$$
$$\leqslant C_1 \frac{m_j}{l_j}. \tag{29}$$

We set:

$$a_{j+1} := s_0^j \quad \text{and} \quad b_{j+1} := s_1^j \quad \text{if } \left|E \cap (W_{s_1^j} \setminus \overline{W}_{s_0^j})\right| \leqslant \left|E \cap (W_{s_2^j} \setminus \overline{W}_{s_1^j})\right|,$$
$$a_{j+1} := s_1^j \quad \text{and} \quad b_{j+1} := s_2^j \quad \text{otherwise.}$$

The sequence $(a_j)_{j\geqslant 0}$ is clearly increasing, $(b_j)_{j\geqslant 0}$ is clearly decreasing, (26) clearly holds by construction, and we have:

$$b_{j+1} - a_{j+1} \leqslant \frac{5}{8}(b_j - a_j),$$

thus (27) follows. To prove (28), we first show that, if ε_5 is small enough, then, for all $j \geqslant 0$,

$$m_j \leqslant \varepsilon \left(\frac{l_j}{8}\right)^n \tag{30}$$

for some small universal constant $\varepsilon > 0$ which depends essentially only on the constant ε_4 given by Lemma 4.4, and

$$\rho_j \leqslant \left(\left(\prod_{k=0}^{j} l_k^{N^{-k}}\right)^{-1} C_1^{\sum_{k=0}^{j} N^{-k}} C^{\sum_{k=1}^{j} N^{-k}} m_0\right)^{N^j}, \tag{31}$$

where $N = n/(n-1)$ and C is given by Lemma 4.4. To see this let $\varepsilon > 0$ be a small constant to be fixed soon. When $j = 0$, we have

$$m_0 \leqslant |E \cap W_r| \leqslant \varepsilon_5 r^n,$$

and $l_0 = r/8$, hence (30) holds provided ε_5 is small enough. And (31) is exactly (29) with $j = 0$. Assume that (30) and (31) hold for some $j \geqslant 0$. We have

$$\max_{i=1,2}\left\{\left|E \cap (W_{s_i^j} \setminus \overline{W}_{s_{i-1}^j})\right|\right\} \leqslant m_j \leqslant \varepsilon\left(\frac{l_j}{8}\right)^n$$

and, on the other hand,

$$\frac{l_j}{8} \leqslant s_i^j - s_{i-1}^j \leqslant \frac{r}{8} \leqslant s_0^j.$$

Hence (H_{ε_4}) is satisfied with $s_0 = s_0^j$, $s_1 = s_1^j$ and $s_2 = s_2^j$ provided $\varepsilon \leqslant \varepsilon_4$ and Lemma 4.4 implies that

$$m_{j+1} \leqslant C\rho_j^N. \tag{32}$$

This combined with (29) and (30) gives

$$m_{j+1} \leqslant CC_1^N\left(\frac{m_j}{l_j}\right)^N \leqslant C'\varepsilon^N (l_j)^n,$$

for some universal constant $C' > 0$. On the other hand, by construction, we have $l_{j+1} \geqslant l_j/8$ and, if ε is small enough (recall that $N > 1$), we get:

$$m_{j+1} \leqslant \varepsilon\left(\frac{l_{j+1}}{8}\right)^n.$$

Next, thanks to (29), (32) and (31), we have:

$$\rho_{j+1} \leqslant C_1\frac{m_{j+1}}{l_{j+1}} \leqslant \frac{C_1 C}{l_{j+1}}\rho_j^N \leqslant \left(\left(\prod_{k=0}^{j+1} l_k^{N^{-k}}\right)^{-1} C_1^{\sum_{k=0}^{j+1} N^{-k}} C^{\sum_{k=1}^{j+1} N^{-k}} m_0\right)^{N^{j+1}}.$$

By induction it follows that (30) and (31) hold for all $j \geqslant 0$ as claimed. Finally we note that since $N > 1$ we have:

$$\sup_{j\geqslant 0}\left(C_1^{\sum_{k=0}^{j} N^{-k}} C^{\sum_{k=1}^{j} N^{-k}}\right) < +\infty.$$

Furthermore, $1 \geqslant l_k \geqslant l_0/8^k = r/8^{k+1}$ for all $k \geqslant 0$, hence

$$\prod_{k=0}^{j} l_k^{N^{-k}} \geqslant \prod_{k=0}^{+\infty} l_k^{N^{-k}} \geqslant \left(\prod_{k=0}^{+\infty} 8^{-(k+1)N^{-k}}\right) r^{\sum_{k=0}^{+\infty} N^{-k}} \geqslant \left(\prod_{k=0}^{+\infty} 8^{-(k+1)N^{-k}}\right) r^n$$

for all $j \geqslant 0$. Then, (31) implies that

$$\rho_j \leqslant \left(C \frac{m_0}{r^n}\right)^{N^j}$$

for some universal constant $C > 0$ (which does not denote anymore the constant given by Lemma 4.4). On the other hand, by assumption, we have $m_0 \leqslant \varepsilon_5 r^n$, and if ε_5 if small enough, we get

$$\lim_{j \to +\infty} \rho_j = 0,$$

from which (28) follows and this concludes the proof of the lemma. □

4.4. *Proof of Lemma 4.3*

We now complete the proof of Lemma 4.3. Let $\varepsilon_3 > 0$ be a small constant to be fixed universal later, assume that $x \in \mathbb{R}^n$ and $r \leqslant 1$ are as in the statement and set as before $W_s := W_s(x)$ for $s > 0$. If ε_3 is small enough, Lemma 4.8 gives $r_1 \in (r/2, 5r/8)$ and $r_2 \in (7r/8, r)$ such that

$$\rho := \max_{i=1,2} \left\{ P(E \cap W_{r_i}, \partial W_{r_i}),\, P(E \setminus W_{r_i}, \partial W_{r_i}) \right\} = 0.$$

On the other hand, we have:

$$|E \cap W_{r_1}| \leqslant |E \cap W_r| \leqslant \varepsilon_3 r^n \leqslant C\varepsilon_3 \min\left\{r_1^n, (r_2 - r_1)^n\right\},$$

$$\left|E \cap (W_{r_2} \setminus \overline{W}_{r_1})\right| \leqslant |E \cap W_r| \leqslant \varepsilon_3 r^n \leqslant C\varepsilon_3 \min\left\{r_1^n, (r_2 - r_1)^n\right\},$$

because $r_1 \geqslant r/2$ and $(r_2 - r_1) \geqslant r/4$. Hence, if ε_3 is small enough, one can apply Lemma 4.4 with $s_0 = 0$, $s_1 = r_1$ and $s_2 = r_2$, and one gets that

$$\min\left\{|E \cap W_{r_1}|, \left|E \cap (W_{r_2} \setminus \overline{W}_{r_1})\right|\right\} = 0.$$

Since by assumption $|E \cap W_{r_1}| \geqslant |E \cap W_{r/2}| > 0$, it follows that

$$\left|E \cap (W_{r_2} \setminus \overline{W}_{r_1})\right| = 0$$

which proves the first part of the lemma.

Next we prove the existence of the Wulff set W. We set $\widehat{E} = E \cap W_{r_1}$ and let $\delta > 0$ be a small constant that will be fixed small and universal soon and η be associated to δ by Lemma 2.11. We have

$$P_\Gamma(\widehat{E}, \mathbb{R}^n) \leqslant C_\Gamma(1+\eta)|\widehat{E}|^{(n-1)/n}$$

provided ε_3 is small enough. Otherwise, we argue as in Lemma 4.7 and set $F = (E \setminus \widehat{E}) \cup W'$ where W' is a Wulff set contained in W_{r_1} with $|W'| = |\widehat{E}|$. Taking into account the fact that ρ as defined above vanishes and arguing as in Lemma 4.7, we have:

$$P_\Gamma(F, \mathbb{R}^n) \leqslant P_\Gamma(E, \mathbb{R}^n) - P_\Gamma(\widehat{E}, \mathbb{R}^n) + P_\Gamma(W', \mathbb{R}^n) \leqslant P_\Gamma(E, \mathbb{R}^n) - C_\Gamma \eta |\widehat{E}|^{(n-1)/n}$$

(one has actually equality on the first line because one even knows that E is equivalent to the empty set inside $W_{r_2} \setminus \overline{W}_{r_1}$). Then we use as usual the quasiminimality of E to conclude, choosing ε_3 small enough so that

$$g\big(|F \triangle E|\big) \leqslant C_\Gamma \eta |\widehat{E}|^{(n-1)/n}/2$$

(note that $|F \triangle E| \leqslant 2|\widehat{E}| \leqslant C\varepsilon_3$), and we get

$$P_\Gamma(E, \mathbb{R}^n) \leqslant P_\Gamma(F, \mathbb{R}^n) + g\big(|F \triangle E|\big) \leqslant P_\Gamma(E, \mathbb{R}^n) - C|\widehat{E}|^{(n-1)/n}.$$

This implies that $|\widehat{E}| = 0$ and gives the contradiction because $|\widehat{E}| \geqslant |E \cap W_{r/2}| > 0$. Now we argue in a similar way than in Step 1 of the proof of Lemma 4.4. By choice of η and thanks to Lemma 2.11, one can find a Wulff set $\widehat{W}$ such that $|\widehat{W}| = |\widehat{E}|$ and $|\widehat{W} \triangle \widehat{E}| \leqslant \delta |\widehat{E}|$. Then Lemma 4.2 implies that $|(1/2)\widehat{W} \setminus E| = 0$ whenever δ is small enough. On the other hand, we have:

$$\big|(1/2)\widehat{W} \setminus W_{r_1}\big| \leqslant \big|(1/2)\widehat{W} \setminus \widehat{E}\big| \leqslant C\delta\big|(1/2)\widehat{W}\big| \quad \text{and} \quad s' \leqslant C\varepsilon_3^{1/n} r_1,$$

where s' denotes the radius of $(1/2)\widehat{W}$. Thus, if δ and ε_3 are small enough, we get from Lemma 2.9 that $W := (\alpha/4\beta)\widehat{W} \subset W_{r_1}$ and this concludes the proof.

4.5. *Behavior of E*

We conclude this section with the end of the proof of Lemma 3.2. We fix $R \leqslant 1$ universal and small enough so that $|(3R)W_\Gamma| < a$. Recall that a denotes the prescribed Lebesgue measure of the quasiminimal crystal E. Remembering Lemma 4.2, it remains to prove that for any $x \in \mathbb{R}^n$ and $r \leqslant R$, if $h(x,r) = r^{-n}|E \cap W_r(x)| \leqslant \varepsilon_6$ then $|E \cap W_{r/2}(x)| = 0$, where $\varepsilon_6 > 0$ is a small suitable universal constant. Thus let $\varepsilon_6 > 0$ to be fixed later. Arguing by contradiction, we assume that one can find $x \in \mathbb{R}^n$ and $r \leqslant R$ such that

$$\big|E \cap W_r(x)\big| \leqslant \varepsilon_6 r^n \quad \text{and} \quad \big|E \cap W_{r/2}(x)\big| > 0.$$

The contradiction will follow from the same kind of comparison arguments as before together with a suitable use of Lemma 4.3. The point is that, for such points x, the traces of E on the boundary of the associated Wulff sets given there vanish. Furthermore there will be some space available around them (namely the annulus-like set $W_{r_2}(x) \setminus \overline{W}_{r_1}(x)$ with the notation of Lemma 4.3) and this will be quite useful to add some mass when needed, moving around the Wulff set also given by Lemma 4.3 that is essentially contained in $E \cap W_{r_1}(x)$.

Thus, assuming that ε_6 is small enough, let $r_1 \in (r/2, 5r/8)$ and $r_2 \in (7r/8, r)$ be associated to x by Lemma 4.3 and set

$$E^1 := E \cap W_{r_1}(x).$$

Step 1. We first prove that the conclusion of Lemma 3.2 holds outside of $W_{2R}(x)$, that is,

$$\begin{aligned} &y \in W_{2R}(x)^c,\ t \leqslant \alpha\beta^{-1}R \text{ and } h(y,t) \leqslant \varepsilon_6 \\ &\quad \Rightarrow \quad \left|E \cap W_{t/2}(y)\right| = 0 \text{ or } \left|W_{t/2}(y) \setminus E\right| = 0, \end{aligned} \tag{33}$$

provided ε_6 is small enough. Taking into account Lemma 4.2 and arguing as before, it is sufficient to assume that one can find $y \in W_{2R}(x)^c$ and $t \leqslant \alpha\beta^{-1}R$ such that

$$\left|E \cap W_t(y)\right| \leqslant \varepsilon_6 t^n \quad \text{and} \quad \left|E \cap W_{t/2}(y)\right| > 0,$$

and to find a contradiction. We let $t_1 \in (t/2, 5t/8)$ and $t_2 \in (7t/8, t)$ be associated to such a y by Lemma 4.3 and set:

$$E^2 := E \cap W_{t_1}(y).$$

Note that E^1 and E^2 do not denote here the same sets as in Section 4.2 but they will play similar roles in the comparison arguments. First since $y \notin W_{2R}(x)$ and $\max\{r_2, \beta\alpha^{-1}t_2\} \leqslant R$, we have $W_{r_2}(x) \cap W_{t_2}(y) = \emptyset$. Indeed, otherwise one could find $z \in W_{r_2}(x) \cap W_{t_2}(y)$ and then one would have (remember in particular (12)):

$$\Gamma^*(y-x) \leqslant \Gamma^*(y-z) + \Gamma^*(z-x) \leqslant \beta\alpha^{-1}\Gamma^*(z-y) + \Gamma^*(z-x) < 2R,$$

which gives a contradiction. Then, arguing as in the proof of Lemma 4.5, it easily follows from the construction of r_i and t_i, $i = 1, 2$, especially from the fact that

$$\begin{aligned} &\max_{i=1,2}\left\{P\left(E \cap W_{r_i}(x), \partial W_{r_i}(x)\right), P\left(E \setminus W_{r_i}(x), \partial W_{r_i}(x)\right)\right\} \\ &\quad = \max_{i=1,2}\left\{P\left(E \cap W_{t_i}(y), \partial W_{t_i}(y)\right), P\left(E \setminus W_{t_i}(y), \partial W_{t_i}(y)\right)\right\} = 0, \end{aligned}$$

that

$$P_\Gamma\left(E \setminus \left(E^1 \cup E^2\right), \mathbb{R}^n\right) \leqslant P_\Gamma(E, \mathbb{R}^n) - P_\Gamma\left(E^1, \mathbb{R}^n\right) - P_\Gamma\left(E^2, \mathbb{R}^n\right). \tag{34}$$

(One can even show the equality because E is equivalent to the empty set inside $(W_{r_2}(x) \setminus \overline{W}_{r_1}(x)) \cup (W_{t_2}(y) \setminus \overline{W}_{t_1}(y))$.) Now let $\tau > 0$ be a small constant to be fixed universal in a moment. We have $|E^2| < \tau |E^1|$ or $|E^1| < \tau |E^2|$ provided ε_6 is small enough. Otherwise, one can essentially follow the proof of Lemma 4.5. We replace $E^1 \cup E^2$ by a single Wulff set W with Lebesgue measure $|E^1| + |E^2|$ strictly contained in $W_{r_2}(x)$ if $r_2 \geqslant t_2$ or in $W_{t_2}(y)$ otherwise. This is always possible provided ε_6 is small enough because $|E^1| + |E^2| \leqslant C\varepsilon_6 \max\{r_2, t_2\}^n$ (remember that $r_2 \geqslant 7r/8$ and $t_2 \geqslant 7t/8$), see Lemma 4.6. Then, with $F = (E \setminus (E^1 \cup E^2)) \cup W$ and remembering (34), we argue as in the proof of Lemma 4.5 and we would get by quasiminimality of E that $|E^1| = 0$ provided ε_6 is small enough. This gives a contradiction because $|E^1| \geqslant |E \cap W_{r/2}(x)| > 0$.

Thus let us assume that $|E^2| < \tau |E^1|$. The other case is similar, exchanging the role of E^1 and E^2. The argument is now close to that of the proof of Lemma 4.4. By Lemma 4.3 one can find a Wulff set $W \subset W_{r_1}(x)$ so that $|W \setminus E^1| = |W \setminus E| = 0$ and $|W| = C|E^1|$ for some universal constant $C > 0$. Next we have $|W| \leqslant C\varepsilon_6 (r_2 - r_1)^n$ (remember that $r_2 - r_1 \geqslant r/4$) hence, if ε_6 is small enough, one can find thanks to Lemma 4.6 a Wulff set $W' \Subset W_{r_2}(x) \setminus \overline{W}_{r_1}(x)$ such that $|W'| = |W|$. In particular $W' \cap E^1 = \emptyset$. On the other hand, we have $|E^2| < \tau |E^1| \leqslant |W|$ if τ is small enough. Thus one can move W strictly inside $W_{r_2}(x)$ until it reaches a new position W_2 between W and W' so that $|W_2 \setminus E^1| = |W_2 \setminus E| = |E^2|$ (remember that $E \cap W_{r_2}(x)$ and $E^1 \cap W_{r_2}(x)$ are equivalent). Then we set $F = (E \setminus E^2) \cup W_2$. By construction we have $|F| = |E|$. Arguing as in the Step 3 of the proof of Lemma 4.4 and remembering (34), one gets by quasiminimality of E that $|E^2| = 0$ provided τ and then ε_6 are chosen small enough. But $|E^2| \geqslant |E \cap W_{t/2}(y)| > 0$ and this gives a contradiction and concludes the proof of (33).

Step 2. We set $\Omega = \overline{W}_{2R}(x)^c$. It is not hard to see that because of (33), essentially the same conclusions as in Theorem 3.3 hold inside Ω. One must however check that we are not in a degenerate situation where one would have $E_1(\Omega) = \emptyset$ and/or $E_0(\Omega) = \emptyset$ and/or where the corresponding set $S(\Omega)$ would be empty. Here $E_1(\Omega) = E_1 \cap \Omega$, $E_0(\Omega) = E_0 \cap \Omega$ where E_1 and E_0 are defined in (22) and (23). Similarly $S(\Omega) = S \cap \Omega$ where S is defined as in (24) with ε_6 in place of ε_0 and $\alpha\beta^{-1}R$ instead of R. This follows from the choice of R. In fact we even have:

$$\left|E \setminus \overline{W}_{3R}(x)\right| \geqslant |E| - \left|(3R)W_\Gamma\right| = a - \left|(3R)W_\Gamma\right| > 0,$$

hence $|E_1(\Omega)| \geqslant |E \setminus \overline{W}_{3R}(x)| > 0$ because $E_1(\Omega)$ turns out to be equivalent to $E \cap \Omega$. Similarly $E_0(\Omega)$ is equivalent to $E^c \cap \Omega$ hence $|E_0(\Omega)| = +\infty$. On one hand, it follows that $E_1(\Omega)$ and $E_0(\Omega)$ are nonempty open and disjoint sets that both meet $\overline{W}_{3R}(x)^c$ hence $\partial E_1(\Omega) \setminus \overline{W}_{3R}(x) \neq \emptyset$ and $\partial E_0(\Omega) \setminus \overline{W}_{3R}(x) \neq \emptyset$. On the other hand, arguing as in Theorem 3.3, one can prove that $S(\Omega) = \partial E_1(\Omega) \cap \Omega = \partial E_0(\Omega) \cap \Omega$ hence $S(\Omega) \neq \emptyset$ and even $S(\Omega) \setminus \overline{W}_{3R}(x) \neq \emptyset$. Then one can argue as in the rest of the proof of Theorem 3.3 to get the Ahlfors-regularity and the condition B. Note however that, strictly speaking, one must handle carefully the localization inside Ω and one gets the condition B property only inside a slightly smaller set, that is, around points in $S(\Omega) \setminus \overline{W}_{3R}(x)$ say. That is the main reason for the choice of R which ensures by the previous argument that $S(\Omega) \setminus \overline{W}_{3R}(x) \neq \emptyset$.

Thus let $y \in S(\Omega) \setminus \overline{W}_{3R}(x)$. Note that $W_{\alpha\beta^{-1}R}(y) \subset \Omega$. Then let $W_0 \Subset E_0 \cap W_{\alpha\beta^{-1}R}(y)$ and $W_1 \Subset E_1 \cap W_{\alpha\beta^{-1}R}(y)$ be two Wulff sets given by the condition B with radius CR where C is a suitable universal constant (recall, as already used, that one can replace balls by Wulff sets in the definition of the condition B). Then we have $|W_1 \setminus E| = 0$, $|W_0 \cap E| = 0$ and $|E^1| \leqslant |W_1|$ provided ε_6 is small enough (by choice of W_1, we have that $|W_1|$ is some universal number). Hence, arguing as before, one can move W_1 strictly inside $W_{\alpha\beta^{-1}R}(y)$ so that it reaches an intermediate position W between W_1 and W_0 with $|W \setminus E| = |E^1|$. Then we set $F = (E \setminus E^1) \cup W$. We have $|F| = |E|$ and

$$\begin{aligned}
P_\Gamma(F, \mathbb{R}^n) &\leqslant P_\Gamma(E, \mathbb{R}^n) - P_\Gamma\big(E^1, \mathbb{R}^n\big) + \frac{C_\Gamma}{|W|^{1/n}}|W \setminus E| \\
&\leqslant P_\Gamma(E, \mathbb{R}^n) - C_\Gamma\big|E^1\big|^{(n-1)/n} + C\big|E^1\big| \\
&\leqslant P_\Gamma(E, \mathbb{R}^n) - \big(C_\Gamma - C\varepsilon_6^{1/n}\big)\big|E^1\big|^{(n-1)/n} \\
&\leqslant P_\Gamma(E, \mathbb{R}^n) - C_\Gamma\big|E^1\big|^{(n-1)/n}/2
\end{aligned}$$

provided ε_6 is small enough. Note that here we do not have any information about the trace of E on the boundary of Wulff sets around y and one must argue in a slightly more careful way than before to get the first inequality. This follows for instance from the fact that $F = E \setminus E^1$ on a neighborhood of $W_{\alpha\beta^{-1}R}(y)^c$, $F = E \cup W$ inside $W_{\alpha\beta^{-1}R}(y)$ and that, on the other hand, $E \setminus E^1 = E$ inside $W_{\alpha\beta^{-1}R}(y)$ and $E \cup W = E$ on a neighborhood of $W_{\alpha\beta^{-1}R}(y)^c$, hence

$$\begin{aligned}
P_\Gamma(F, \mathbb{R}^n) &= P_\Gamma\big(E \setminus E^1, W_{\alpha\beta^{-1}R}(y)^c\big) + P_\Gamma\big(E \cup W, W_{\alpha\beta^{-1}R}(y)\big) \\
&= P_\Gamma\big(E \setminus E^1, \mathbb{R}^n\big) + P_\Gamma(E \cup W, \mathbb{R}^n) - P_\Gamma(E, \mathbb{R}^n).
\end{aligned}$$

Then one uses the usual arguments to estimate $P_\Gamma(E \setminus E^1, \mathbb{R}^n)$ and $P_\Gamma(E \cup W, \mathbb{R}^n)$. Then we conclude as usual, using the quasiminimality of E to get that $|E^1| = 0$ if ε_6 is chosen small enough. This is not possible because $|E \cap W_{r/2}(x)| > 0$ and give the final contradiction.

5. Consequences

We end this paper with the proof of Theorems 1.6 and 1.7.

5.1. Connected components

We prove in this section Theorem 1.6 which mainly says that each connected component of a reduced quasiminimal crystal has an Ahlfors-regular boundary and satisfies the condition B on its own and with universal constants. Thus let us fix a reduced quasiminimal crystal $E \in \mathcal{QM}$ as in Definition 1.5 and let A be a connected component of E. The general scheme of the proof is the same as for the proof of Theorem 3.3.

First we have $\partial A \subset \partial E$ and $\partial E = \partial_* E$ because E is reduced (see Theorem 3.3 and remember that by uniqueness in Theorem 1.4, E coincides with the set E_1 defined in (22)). Hence it follows from the Ahlfors-regularity of ∂E (we actually need here only Lemma 3.1) that

$$P_\Gamma\bigl(A, W_r(x)\bigr) \leqslant \beta \mathcal{H}^{n-1}\bigl(\partial A \cap W_r(x)\bigr) \leqslant \beta \mathcal{H}^{n-1}\bigl(\partial E \cap W_r(x)\bigr) \\ \leqslant \beta\alpha^{-1} P_\Gamma\bigl(E, W_r(x)\bigr) \leqslant C r^{n-1}$$

for any $x \in \mathbb{R}^n$ and $r \leqslant 1$ and for some suitable universal constant $C > 0$.

For $x \in \mathbb{R}^n$ and $r > 0$, we set:

$$h_A(x,r) := r^{-n} \min\bigl\{\bigl|A \cap W_r(x)\bigr|, \bigl|W_r(x) \setminus A\bigr|\bigr\}.$$

Using similar arguments than in the proof of Theorem 3.3, it is not hard to see that the Ahlfors-regularity of ∂A and the condition B for A (note that A is open because E is), will follow as soon as we show that $\partial A = S_A$, where

$$S_A := \bigl\{x \in \mathbb{R}^n\colon\ h_A(x,r) > \varepsilon_7 \text{ for all } r \leqslant R\bigr\}$$

for some suitable universal constants $\varepsilon_7 > 0$ and $R \leqslant 1$. We obviously have $S_A \subset \partial A$.

We note that the value of R here may be slightly different from that given by Lemma 3.2 (even though a suitable choice could be used for both cases) and we only assume to begin with that R is smaller than the value given there. Then, since $\partial A \subset \partial E$, it follows from Theorem 3.3 that

$$\bigl|W_r(x) \setminus A\bigr| \geqslant \bigl|W_r(x) \setminus E\bigr| > \varepsilon_0 r^n$$

whenever $x \in \partial A \subset \partial E$ and $r \leqslant R$.

To bound from below $|A \cap W_r(x)|$ we argue in a similar way than in Section 4.1 and first prove the following lemma:

Lemma 5.1. *There exists a universal constant $\varepsilon_8 > 0$ such that, for any $x \in \mathbb{R}^n$ and $r \leqslant R$,*

$$\bigl|A \cap W_r(x)\bigr| \leqslant \varepsilon_8 r^n \quad \Rightarrow \quad h_A(x, r/2) \leqslant h_A(x,r)/2.$$

Proof. The proof will be achieved as usual thanks to a suitable comparison argument. One will try to remove the component A from$E$ inside $W_r(x)$ (or in a slightly smaller Wulff set) and then, arguing as in Section 4.5 Step 2, we shall use the condition B outside of $W_R(x)$ to add the corresponding mass. The main difference with the previous constructions is that we only remove here a part of E.

Thus let $\varepsilon_8 \leqslant |W_\Gamma|/2$ be a small constant to be fixed later and assume that $x \in \mathbb{R}^n$ and $r \leqslant R$ are as in the statement. Using Tchebytchev's inequality and (14), we choose $t \in (r/2, r)$ such that

$$\mathcal{H}^{n-1}\bigl(A \cap \partial W_t(x)\bigr) \leqslant C r^{n-1} h_A(x,r)$$

for some universal constant $C > 0$. We first estimate the Γ-perimeter of $E \setminus (A \cap W_t(x))$. Since E is reduced, we know that E is open and $\partial E = \partial_* E$, hence one can apply Lemma 2.3 and it follows:

$$\begin{aligned} P_\Gamma\big(E \setminus \big(A \cap W_t(x)\big), \mathbb{R}^n\big) &= P_\Gamma(E, \mathbb{R}^n) - P_\Gamma\big(A \cap W_t(x), \mathbb{R}^n\big) \\ &\quad + P_\Gamma\big(A \cap W_t(x), E\big) + P_\Gamma\big(E \setminus \big(A \cap W_t(x)\big), E\big) \\ &\leqslant P_\Gamma(E, \mathbb{R}^n) - P_\Gamma\big(A, W_t(x)\big) \\ &\quad + P_\Gamma\big(A \cap W_t(x), E\big) + P_\Gamma\big(E \setminus \big(A \cap W_t(x)\big), E\big). \end{aligned}$$

Since E and A are open and $\partial A \cap E = \emptyset$, we have $\partial(A \cap W_t(x)) \cap E \subset A \cap \partial W_t(x)$ and $\partial(E \setminus (A \cap W_t(x))) \cap E \subset A \cap \partial W_t(x)$. By choice of t, it follows:

$$\begin{aligned} P_\Gamma\big(A \cap W_t(x), E\big) + P_\Gamma\big(E \setminus \big(A \cap W_t(x)\big), E\big) &\leqslant 2\beta \mathcal{H}^{n-1}\big(A \cap \partial W_t(x)\big) \\ &\leqslant C r^{n-1} h_A(x, r), \end{aligned}$$

and finally

$$P_\Gamma\big(E \setminus \big(A \cap W_t(x)\big), \mathbb{R}^n\big) \leqslant P_\Gamma(E, \mathbb{R}^n) - P_\Gamma\big(A, W_t(x)\big) + C r^{n-1} h_A(x, r).$$

Next, arguing as in Section 4.5 Step 2, one can always choose R universal and small enough so that one can find $y \in \partial E$ with $W_R(y) \cap W_R(x) = \emptyset$. Then, rephrasing the argument in Section 4.5 Step 2, one can move strictly inside $W_R(y)$ some Wulff set given by the condition B to find a Wulff set $W \Subset W_R(y)$ whose Lebesgue measure is some universal number and such that $|W \setminus E| = |A \cap W_t(x)|$, at least if ε_8 is small enough. Then we set $F = E \setminus (A \cap W_t(x)) \cup W$. By construction we have $|F| = |E|$ and, arguing as before,

$$\begin{aligned} P_\Gamma(F, \mathbb{R}^n) &\leqslant P_\Gamma\big(E \setminus \big(A \cap W_t(x)\big), \mathbb{R}^n\big) + \frac{C_\Gamma}{|W|^{1/n}} |W \setminus E| \\ &\leqslant P_\Gamma(E, \mathbb{R}^n) - P_\Gamma\big(A, W_t(x)\big) + C r^{n-1} h_A(x, r) + C\big|A \cap W_t(x)\big| \\ &\leqslant P_\Gamma(E, \mathbb{R}^n) - P_\Gamma\big(A, W_t(x)\big) + C r^{n-1} h_A(x, r) \end{aligned}$$

for some universal constant $C > 0$ (recall that $r \leqslant 1$). On the other hand, we have $|F \triangle E| \leqslant 2|A \cap W_t(x)| \leqslant C r^n h_A(x, r) \leqslant C\varepsilon_8$. Then, if ε_8 is small enough, we get by the relative isoperimetric inequality for Wulff sets (Proposition 2.8)

$$g\big(|F \triangle E|\big) \leqslant C\big|A \cap W_t(x)\big|^{(n-1)/n} \leqslant P_\Gamma\big(A, W_t(x)\big)/2,$$

and we argue as in Lemma 4.1 to get the conclusion. □

Then it follows automatically that, for all $x \in \mathbb{R}^n$ and $r \leqslant R$,

$$\big|A \cap W_r(x)\big| \leqslant \varepsilon_9 r^n \quad \Rightarrow \quad \big|A \cap W_{r/2}(x)\big| = 0$$

for some suitable universal constant $\varepsilon_9 > 0$ (see the argument in Lemma 4.2). On the other hand, since A is open, one also automatically has

$$(\overline{A})^c = \{x \in \mathbb{R}^n\colon \text{ there exists } r > 0 \text{ such that } |A \cap W_r(x)| = 0\}.$$

Hence we get that for all $x \in \mathbb{R}^n$ and $r \leqslant R$,

$$|A \cap W_r(x)| \leqslant \varepsilon_9 r^n \quad \Rightarrow \quad x \in (\overline{A})^c.$$

Then, taking $\varepsilon_7 = \min\{\varepsilon_0, \varepsilon_9\}$, it finally easily follows that $\partial A = S_A$ as wanted.

Remark 5.2. Note that one can also easily see that

$$A = \{x \in \mathbb{R}^n\colon \text{ there exists } r > 0 \text{ such that } |W_r(x) \setminus A| = 0\}.$$

To conclude the proof of Theorem 1.6 it remains to give an upper bound for the number of connected components of E. To get this one can for instance apply the condition B to a point $x \in \partial A$ to obtain the existence of a Wulff set W contained in $A \cap W_1(x)$ whose radius is a universal constant. This implies that $|A| \geqslant |W| \geqslant C$ for some universal constant $C > 0$. Since $|E|$ is fixed, $|E| = a$, we get the required conclusion.

5.2. *Unconstrained local quasiminimality condition*

This section is devoted to the proof of Theorem 1.7 which asserts that any quasiminimal crystal with a volume constraint actually satisfies a stronger quasiminimality condition where admissible perturbations are not required to be volume-preserving. We assume here that g is nondecreasing and let E be a fixed reduced quasiminimal crystal as in Definition 1.5 (see Remark 5.3 for the general case). Similarly as before we also let $R' \leqslant 1$ be fixed universal and small enough so that, for any $x \in \mathbb{R}^n$, one has $\partial E \setminus B_{2R'}(x) \neq \emptyset$. Then let $R \leqslant R'$ to be fixed universal later and let $x \in \mathbb{R}^n$ and $r \leqslant R$ be fixed. We consider a compact perturbation F of E inside $B_r(x)$ so that $F \bigtriangleup E \Subset B_r(x)$.

If $|F| \geqslant |E|$, we choose a Wulff set W such that $|F \cap W| = |E|$ (with $W = \mathbb{R}^n$ if $|F| = |E|$) and it follows from the quasiminimality of E and from Lemma 2.5 that

$$P_\Gamma(E, \mathbb{R}^n) \leqslant P_\Gamma(F \cap W, \mathbb{R}^n) + g\big(|(F \cap W) \bigtriangleup E|\big) \leqslant P_\Gamma(F, \mathbb{R}^n) + g\big(2|B_r(x)|\big),$$

because

$$|(F \cap W) \bigtriangleup E| \leqslant |F \bigtriangleup E| + |E \setminus W| \leqslant 2|B_r(x)|.$$

If $|F| < |E|$, we pick some ball B centered on $\partial E \setminus B_{2R'}(x)$ with radius R' and let $W_0 \Subset B \setminus \overline{E}$ and $W_1 \Subset E \cap B$ be two Wulff sets given by the condition B and with radius CR' for some suitable universal constant $C > 0$. We have:

$$0 < |E| - |F| \leqslant |F \bigtriangleup E| \leqslant |B_r(x)| \leqslant |B_R(x)| \leqslant |W_1|$$

provided $R \leqslant R'$ is chosen small enough. Then we argue as in the previous sections (see for instance Section 4.5 Step 2) to find a Wulff set $W' \Subset B$ in between W_1 and W_0 with $|W'| = |W_1| = |W_0|$ and $|W' \setminus E| = |E| - |F|$. Since $B \cap B_r(x) = \emptyset$ and thus in particular E and F coincide on B, we have $|F \cup W'| = |E|$. According to Lemma 2.6 and by construction, we have:

$$P_\Gamma(F \cup W', \mathbb{R}^n) \leqslant P_\Gamma(F, \mathbb{R}^n) + \frac{C_\Gamma}{|W'|^{1/n}} |W' \setminus E| \leqslant P_\Gamma(F, \mathbb{R}^n) + C\bigl|B_r(x)\bigr|$$

for some universal constant $C > 0$. On the other hand, we have:

$$\bigl|(F \cup W') \triangle E\bigr| \leqslant |F \triangle E| + |W' \setminus E| \leqslant 2\bigl|B_r(x)\bigr|,$$

and it follows from the quasiminimality of E that

$$\begin{aligned} P_\Gamma(E, \mathbb{R}^n) &\leqslant P_\Gamma(F \cup W', \mathbb{R}^n) + g\bigl(\bigl|(F \cup W') \triangle E\bigr|\bigr) \\ &\leqslant P_\Gamma(F, \mathbb{R}^n) + C\bigl|B_r(x)\bigr| + g\bigl(2\bigl|B_r(x)\bigr|\bigr). \end{aligned}$$

Then, taking into account the fact that E and F coincide on a neighborhood of $B_r(x)^c$, we get in both cases:

$$P_\Gamma\bigl(E, B_r(x)\bigr) \leqslant P_\Gamma\bigl(F, B_r(x)\bigr) + r^{n-1}\omega(r),$$

where

$$\omega(r) := r^{-(n-1)}\bigl(C\bigl|B_r(0)\bigr| + g\bigl(2\bigl|B_r(0)\bigr|\bigr)\bigr).$$

We have $\lim_{r\to 0}\omega(r) = 0$. This is exactly what we want and concludes the proof of Theorem 1.7.

Remark 5.3. The assumption on E to be reduced is not a serious issue here. Otherwise we consider the equivalent reduced quasiminimal crystal E_1 given by Theorem 1.4. Then F and E_1 essentially coincide on a neighborhood of $B_r(x)^c$, that is, $|(F \triangle E_1) \setminus B_{r'}(x)| = 0$ for some $r' < r$, and the same construction as before applied with E_1 in place of E gives the required conclusion.

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