Nongeneric SUSY in hot NUT–Kerr–Newman–Kasuya spacetime

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Abstract

We investigate the supersymmetric extension of the hot NUT–Kerr–Newman–Kasuya spacetime. Along with four standard supersymmetries, this type of spacetime admits fermionic symmetries generated by the square root of bosonic constants of motion except the Hamiltonian. Such new supersymmetries correspond to the Killing–Yano tensors, which play an important role in solving the Dirac equation in curved spacetime.

Keywords: Pseudo-classical model; Killing–Yano tensors; Supersymmetries

1. Introduction

In recent years there has been a renewed interest in studying spinning particles, such as Dirac fermions, in curved spacetimes by pseudo-classical mechanics models in which the spin degrees of freedom are characterized in terms of anticommuting Grassmann variables ([1–9]).

The completely integrability of particle motion in the Kerr black-hole spacetime demands the existence of a nontrivial Killing tensor \( K_{\mu\nu} \), which gives rise the associated constant of motion: \( Z = \frac{1}{2} K_{\mu\nu} p^\mu p^\nu \), quadratic in the four-momentum \( p^\mu \) [10]. That is, this constant of motion completes the maximal number of constants of motion in conjunction with the other three well-known constants of motion: the energy coming from the time translation invariance, the angular momentum coming from the axial symmetry, and the Hamiltonian. More surprisingly, the separability of various field equations, e.g., the Dirac equation [11], has a direct consequence of the existence of the Killing–Yano tensor \( f_{\mu\nu\rho} \), which is defined as an antisymmetric second-rank tensor satisfying the following Penrose–Floyd equation [12]:

\[
D_{(\mu} f_{\nu)\rho} = 0
\]  

(1)
and is a square root of the Killing tensor $K^{\mu\nu}$:

$$K^{\mu\nu} = f^\mu_\lambda f^\lambda_\nu.$$  \hspace{1cm} (2)

Here indices are raised and lowered with the spacetime metric $g_{\mu\nu}$ and its inverse.

Recently, Gibbons et al. [9] have been able to show by considering supersymmetric particle mechanics that the Killing–Yano tensor can be understood as an object belonging to a larger class of possible structures which generate generalized supersymmetry (SUSY) algebras. This novel aspect has renewed people’s interest in the Killing–Yano tensor which has long been known to relativists as a mysterious thing.

In this Letter, we apply the formulas of [9] to investigate nongeneric SUSY generated by the Killing–Yano tensor and the corresponding conserved quantity in the NUT–Kerr–Newman–Kasuya–de Sitter spacetime, which is the Kerr–Newman black-hole spacetime generalized with NUT (or magnetic mass) parameter, extra magnetic monopole charge and cosmological constant. The spacetimes endowed with NUT parameter should never be directly physically interpreted [13]. The monopole hypothesis was propounded by Dirac relatively long ago. The ingenious suggestion by Dirac that magnetic monopole does exist was neglected due to the failure to detect such particle. However, in recent years the development of gauge theories has shed new light on it. On the other hand, there is a renewed interest in the cosmological constant as it is found to be present in the inflationary scenario of the early universe. In this scenario the universe undergoes a stage where it is geometrically similar to de Sitter space [14]. Among other things inflation has led to the cold dark matter. If the cold dark matter theory proves correct, it would shed light on the unification of forces [15]. In view of these considerations the work of this Letter is interesting. Since the de Sitter spacetime has been interpreted as being hot [16], we call the spacetime as the hot NUT–Kerr–Newman–Kasuya (HNUTKNK) spacetime.

The organization of this Letter is as follows. In Section 2 we summarize the relevant equations for the motion of spinning particles in curved spacetime. We recall the generalized Killing equations for spinning spaces and derive the constants of motion in terms of the solutions of these equations. We describe the extra SUSY generated by the Killing–Yano tensor $f_{\mu\nu}$. In Section 3 we apply the results of Section 2 to investigate nongeneric SUSY and the corresponding constant of motion in the hot NUT–Kerr–Newman–Kasuya spacetime. Finally, in Section 4 we present our remarks.

2. Spinning space and its symmetries

The configuration space of spinning particles (spinning space) is spanned by the real position variables $x^{\mu}(\tau)$ and the Grassmann-valued spin variables $\psi^a(\tau)$, where $\mu, a = 1, \ldots, d$, with $d$ the dimension of spacetime. Greek and Latin indices refer to world and Lorentz indices, respectively, and are converted into each other by the vielbein $e^\mu_a(x)$ and its inverse $e^\mu_a(x)$. The world-line parameter $\tau$ is the invariant proper time:

$$d\tau^2 = -g_{\mu\nu}(x)dx^\mu dx^\nu.$$  \hspace{1cm} (3)

The Lagrangian of the theory is given by

$$L = \frac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu + \frac{i}{2}\eta_{ab}\psi^a D\psi^b \frac{D\tau}{D\tau},$$  \hspace{1cm} (4)

where $\eta_{ab}$ is the flat-space (Minkowski) metric. The overdot, here and in the following, denotes an ordinary proper-time derivative. The covariant derivative of the spin variable is defined by

$$\frac{D\psi^a}{D\tau} = \dot{\psi}^a - \dot{x}^\mu \omega^a_{\mu b} \psi^b,$$  \hspace{1cm} (5)

where $\omega^a_{\mu b}$ is the spin connection.
The antisymmetric spin variables are related to the standard antisymmetric spin tensor $S^{ab}$ by
\[ S^{ab} = -i \psi^a \psi^b. \] (6)

The space-like components $S^{ij}$ are proportional to the particle’s magnetic dipole moment, while the time-like components $S^{i0}$ represent the electric dipole moment. For free Dirac particles, like free electrons and quarks, spin is space-like. This gives the constraint [7]
\[ Q = e \mu_a \dot{x}^a = 0, \] (7)
which implies that the time-like spin components $S^{i0}$ vanish in the rest frame.

The equations of motion of the theory can be cast in the form
\[ \frac{D^2 x^a}{D\tau^2} = \ddot{x}^a + \Gamma^a_{\lambda\nu} \dot{x}^\lambda \dot{x}^\nu = 0, \] (8)

The world-line Hamiltonian is given by
\[ H = \frac{1}{2} g^{\mu\nu} \Pi_\mu \Pi_\nu, \] (9)
where
\[ \Pi_\mu = g_{\mu\nu} \dot{x}^\nu \] (10)
is the covariant momentum. The constraints (3) and (7) then can be written as follows:
\[ 2H = g^{\mu\nu} \Pi_\mu \Pi_\nu = -1, \quad Q = \Pi \cdot \psi = 0. \] (11)

For any constant of motion $J(x, \Pi, \psi)$, the bracket with $H$ vanishes
\[ \{H, J\} = 0, \] (12)
where the Poisson–Dirac bracket for two functions of the covariant phase-space variables $(x, \Pi, \psi)$ is defined by
\[ \{F, G\} = D_\mu F \frac{\partial G}{\partial \Pi_\mu} - D_\mu G \frac{\partial F}{\partial \Pi_\mu} + \mathcal{R}_{\mu\nu} \frac{\partial F}{\partial \Pi_\mu} \frac{\partial G}{\partial \Pi_\nu} + i(-1)^{a_F} \frac{\partial F}{\partial \psi^a} \frac{\partial G}{\partial \psi^a}, \] (13)
where
\[ D_\mu F = \partial_\mu F + \Gamma_\mu^\lambda \Pi_\lambda \frac{\partial F}{\partial \Pi_\mu} + \omega_{\mu}{}^{ab} \psi^b \frac{\partial F}{\partial \psi^a}, \quad \mathcal{R}_{\mu\nu} = \frac{i}{2} \psi^a \psi^b R_{ab\mu\nu}, \] (14)
and $a_F$ is the Grassmann parity of $F$: $a_F = 0, 1$ for $F$ (even, odd). Eq. (12) is satisfied for $J = Q$ and since the Hamiltonian itself is trivially conserved, the values of $H$ and $Q$ (as given in (11)) are preserved in time.

If we expand $J(x, \Pi, \psi)$ in a power series in the covariant momentum
\[ J = \sum_{n=0}^{\infty} \frac{1}{n!} J^{(n)}(x, \psi) \Pi_{\mu_1} \cdots \Pi_{\mu_n}, \] (15)
then the bracket $\{H, J\}$ vanishes for arbitrary $\Pi_\mu$ if and only if the components of $J$ satisfy the generalized Killing equations [6,7,9]:
\[ D_{(\mu_{n+1}} J^{(n)}_{\mu_1 \cdots \mu_n)} + \omega_{\mu_{n+1} a} \psi^b \frac{\partial J^{(n)}_{\mu_1 \cdots \mu_n}}{\partial \psi^a} = \mathcal{R}_{\nu(\mu_{n+1} J^{(n+1)}_{\mu_1 \cdots \mu_n)}, \] (16)
where $D_\mu$ is an ordinary covariant derivative, and the parentheses denote full symmetrization with norm one over the indices enclosed.
In general, the symmetries of a spinning particle model can be divided into two classes. First, there are four independent generic symmetries, which exist in any theory:

(i) Proper-time translations generated by the Hamiltonian $H$ (11);
(ii) Supersymmetry generated by the supercharge $Q$ (11);
(iii) Chiral symmetry generated by the chiral charge
\[ F_a = -\frac{i|d/2|}{d!} \epsilon_{a_1 \ldots a_d} \psi^{a_1} \ldots \psi^{a_d}; \]  
(17)
(iv) Dual supersymmetry generated by the dual supercharge
\[ Q^* = i \{ Q, F_a \} = -\frac{i|d/2|}{(d-1)!} \epsilon_{a_1 \ldots a_d} e^{\mu a_1} \Pi_\mu \psi^{a_2} \ldots \psi^{a_d}. \]  
(18)

All these quantities have vanishing Poisson–Dirac brackets with the Hamiltonian, and hence are constants of motion.

The second kind of conserved quantities, called nongeneric, depend on the explicit form of the metric $g_{\mu \nu}(x)$. As we mentioned earlier, Killing–Yano tensors of valence 2 play a key role in the Dirac theory on a curved spacetime [17]. The study of the generalized Killing equations strengthens the connection of the Killing–Yano tensors with the supersymmetric classical and quantum mechanics on a curved manifold.

The nongeneric SUSY of the theory is generated by the phase-space function $Q_f$,
\[ Q_f = J^{(1)}_\mu \Pi_\mu + J^{(0)}, \]  
(19)
where $J^{(0,1)}(x, \psi)$ are independent of $\Pi$. This charge generates the SUSY transformation
\[ \delta x^\mu = -i \epsilon f^\mu_a \psi^a = -i \epsilon J^{(1)}_\mu, \]  
(20)
where the infinitesimal parameter $\epsilon$ of the transformation is Grassmann-odd. Inserting the ansatz (19) into the generalized Killing equations (16), one finds
\[ J^{(0)}(x, \psi) = \frac{i}{3!} c_{abc}(x) \psi^a \psi^b \psi^c, \]  
(21)
where $f^\mu_a$ satisfies
\[ D_\mu f_{va} + D_v f_{\mu a} = 0, \]  
(22)
and $c_{abc}$ has the following expression:
\[ c_{abc} = -2 D_{[a} f_{bc]}. \]  
(23)

Here the square brackets denote full antisymmetrization with norm one over the indices enclosed. Let there be $N$ such symmetries specified by $N$ sets of tensors $(f^\mu_{ia}, c_{iabc})$, $i = 1, \ldots, N$. The corresponding generators will be
\[ Q_i = f^\mu_{ia} \Pi_\mu \psi^a + \frac{i}{3!} c_{iabc}(x) \psi^a \psi^b \psi^c. \]  
(24)

Obviously, for $f^\mu_a = e^\mu_a$ and $c_{abc} = 0$, the supercharge in (11) is precisely of this form. It is therefore convenient to assign the index $i = 0$: $Q = Q_0$, $e^\mu_a = f^\mu_{0a}$, etc., when we refer to the quantities defining the standard SUSY.

The covariant form (13) of Poisson–Dirac bracket gives the following algebra for the conserved charges $Q_i$:
\[ \{ Q_i, Q_j \} = -2i Z_{ij}, \]  
(25)
where

\[ Z_{ij} = \frac{1}{2} K_{ij}^{\mu\nu} \Pi_{\mu} \Pi_{\nu} + I_{ij}^{\mu} \Pi_{\mu} + G_{ij}, \]  

(26)

and

\[ K_{ij}^{\mu\nu} = \frac{1}{2} (f_{i a}^{\mu} f_{j a}^{\nu} + f_{i a}^{\nu} f_{j a}^{\mu}). \]  

(27)

\[ I_{ij}^{\mu} = \frac{1}{2} i \psi^{a} \psi^{b} I_{ijab}^{\mu} = \frac{1}{2} i \psi^{a} \psi^{b} \left( f_{i b}^{\nu} D_{\nu} f_{j a}^{\mu} + f_{j b}^{\nu} D_{\nu} f_{i a}^{\mu} + \frac{1}{2} f_{i}^{\mu\nu} c_{jabc} + \frac{1}{2} f_{j}^{\mu\nu} c_{iabc} \right). \]  

(28)

\[ G_{ij} = -\frac{1}{4} \psi^{a} \psi^{b} \psi^{c} \psi^{d} G_{ijabcd} = -\frac{1}{4} \psi^{a} \psi^{b} \psi^{c} \psi^{d} \left( R_{\mu\nu} f_{i}^{\mu} f_{j}^{\nu} + \frac{1}{2} c_{iab} c_{jede} \right). \]  

(29)

The functions \( Z_{ij} \) satisfy the generalized Killing equations. Hence their bracket with the Hamiltonian vanishes and they are constants of motion:

\[ \frac{dZ_{ij}}{d\tau} = 0. \]  

(30)

For \( i = j = 0 \), (25) reduces to the usual SUSY algebra

\[ \{ Q, Q \} = -2i H. \]  

(31)

If \( i \) or \( j \) is not equal to zero, \( Z_{ij} \) correspond to new bosonic symmetries, unless \( K_{ij}^{\mu\nu} = \lambda(ij) h^{\mu\nu} \), with \( \lambda(ij) \), a constant (may be zero). In that case the corresponding Killing vector \( I_{ij}^{\mu} \) and scalar \( G_{ij} \) disappear identically. Further, the supercharges for \( \lambda(ij) \neq 0 \) close on the Hamiltonian. This shows the existence of a second SUSY of the standard type. Thus the theory admits an \( N \)-extended SUSY with \( N \geq 2 \). On the other hand, if there exists a second independent Killing tensor \( K_{\mu\nu} \) not proportional to \( g_{\mu\nu} \), there exists a genuine new type of SUSY.

The quantity \( Q_{i} \) is a superinvariant, that is,

\[ \{ Q_{i}, Q_{j} \} = 0 \]  

(32)

for the bracket defined by (13), if and only if

\[ K_{bi}^{\mu\nu} = f_{i a}^{\mu} e^{a\nu} + f_{i a}^{\nu} e^{a\mu} = 0. \]  

(33)

In this case, the full constant of motion \( Z_{ij} \) can be constructed directly by repeated differentiation of \( f_{\mu a}^{i} \) [9].

Since the \( Z_{ij} \) are symmetric in \( (ij) \) we can diagonalize them. This provides the algebra

\[ \{ Q_{i}, Q_{j} \} = -2i \delta_{ij} Z_{i}, \]  

(34)

with \( N + 1 \) conserved bosonic charges \( Z_{i} \). If all \( Q_{i} \) satisfy condition (33), the first of these diagonal charges (with \( i = 0 \)) is the Hamiltonian: \( Z_{0} = H \).


In this section we apply the results of previous section to investigate a new type of SUSY in the hot NUT–Kerr–Newman–Kasuya spacetime, which has the metric

\[ ds^{2} = \frac{\Sigma}{\Delta_{\theta}} d\theta^{2} + \frac{\Sigma}{\Delta_{r}} dr^{2} + \frac{3^{-2}}{\Sigma} \Delta_{\theta} \sin^{2} \theta (-a dt + \rho d\varphi)^{2} - \frac{3^{-2}}{\Sigma} \Delta_{r} (dt - A d\varphi)^{2}. \]  

(35)
The vierbein $e_{\mu}^{a}(x)$ is determined by the components $f_{\mu a}(x)$
\begin{align*}
\Delta_r &= (r^2 + a^2 + n^2)
\left[ 1 - \frac{1}{3} A(r^2 + 5n^2) \right]
- 2(Mr + n^2) + q^2, \\
\rho &= r^2 + a^2 + n^2, \\
\Sigma &= r^2 + (n + a \cos \theta)^2, \\
q^2 &= q_e^2 + q_m^2, \\
\Delta_0 &= 1 + \frac{1}{3} Aa^2 \cos^2 \theta, \\
\Lambda &= \frac{\Delta_0}{\Delta_r^2}.
\end{align*}
Besides the cosmological constant $\Lambda$, the metric possesses the mass parameter $M$, the NUT (or magnetic mass) parameter $n$, the specific angular momentum parameter $a(= J/M)$, the electric charge parameter $q_e$ and the magnetic monopole charge parameter $q_m$. The electromagnetic field tensor associated with this spacetime is expressed by
\begin{equation}
F = \frac{3^{-1} q}{\Sigma^2} [r^2 - (n + a \cos \theta)^2] d\tau \wedge (dt - A d\phi) = - 2 \frac{3^{-1} q}{\Sigma^4} r(n + a \cos \theta) \sin \theta d\theta \wedge (-a dt + \rho d\phi). \tag{37}
\end{equation}
The nongeneric SUSY in the HNUTKNK spacetime is generated by a supercharge of the form given in (24), with $f_{\mu a} = f_{\mu \nu} e^{\nu a}$ and $c_{abc}$ obtained as in (23).
As was defined in [12] the Killing–Yano tensor $f_{\mu \nu}$ in the HNUTKNK spacetime is given by
\begin{equation}
\frac{1}{2} f_{\mu \nu} dx^\mu \wedge dx^\nu = \frac{(n + a \cos \theta)}{3} dr \wedge (dt - A d\phi) + \frac{r \sin \theta}{3} d\theta \wedge (-a dt + \rho d\phi). \tag{38}
\end{equation}
The vierbein $e_{\mu}^{a}(x)$ corresponding to the metric (35) has the following expressions:
\begin{align*}
e_{0}^{\mu} dx^\mu &= - \frac{\sqrt{\Delta_r}}{3 \sqrt{\Sigma}} (dt - A d\phi), & e_{1}^{\mu} dx^\mu &= \frac{\sqrt{\Sigma}}{\sqrt{\Delta_r}} dr, \\
e_{2}^{\mu} dx^\mu &= \frac{\sqrt{\Sigma}}{\sqrt{\Delta_0}} d\theta, & e_{3}^{\mu} dx^\mu &= \frac{\sqrt{\Delta_0}}{3 \sqrt{\Sigma}} (-a dt + \rho d\phi). \tag{39}
\end{align*}
Using the expressions for the vierbein $e_{\mu}^{a}(x)$ one finds the following components of $f_{\mu a}(x)$:
\begin{align*}
f_{0}^{\mu} dx^\mu &= \frac{\sqrt{\Sigma}}{\sqrt{\Delta_r}} (n + a \cos \theta) dr, & f_{1}^{\mu} dx^\mu &= - \frac{\sqrt{\Delta_r}}{3 \sqrt{\Sigma}} (n + a \cos \theta)(dt - A d\phi), \\
f_{2}^{\mu} dx^\mu &= - \frac{\sqrt{\Delta_0}}{3 \sqrt{\Sigma}} r \sin \theta (-a dt + \rho d\phi), & f_{3}^{\mu} dx^\mu &= \frac{\sqrt{\Sigma}}{\sqrt{\Delta_0}} r d\theta. \tag{40}
\end{align*}
Indeed, this $f_{\mu a}(x)$ satisfies Eq. (22). From (23) the components of $c_{abc}(x)$ are found as follows:
\begin{align*}
c_{012} &= \frac{2a \sqrt{\Delta_0} \sin \theta}{3 \sqrt{\Sigma}}, & c_{013} &= 0 = c_{023}, & c_{123} &= - \frac{2 \sqrt{\Delta_r}}{3 \sqrt{\Sigma}}. \tag{41}
\end{align*}
Inserting the quantities derived in (40), (41) into (24) we obtain the new SUSY generator $Q_{\xi}$ for the HNUTKNK spacetime. From (27)–(29) the Killing tensor, vector, and scalar are constructed as follows:
\begin{align*}
K_{\mu \nu}(x) dx^\mu dx^\nu &= \frac{(n + a \cos \theta)^2 \Sigma}{\Delta_r} dr^2 + \frac{\Delta_r (n + a \cos \theta)^2}{3 \Sigma^2} (dt - A d\phi)^2 \\
&+ \frac{\Delta_0 r^2 \sin^2 \theta}{3 \Sigma^2} (-a dt + \rho d\phi)^2 + \frac{\Sigma}{\Delta_0} r^2 d\theta^2. \tag{42}
\end{align*}
\[
I_\mu(x) \, dx^\mu = -\frac{2}{\Sigma} \left[ a \sqrt{\Delta_\varphi} \sin(\vartheta \sqrt{\Delta_\varphi} (r \sin \vartheta S^\varphi + \sqrt{\Delta_r} \cos \vartheta S^\theta) ) \\
- \sqrt{\Delta_r} (r \sin \vartheta S^\varphi + \sqrt{\Delta_r} \cos \vartheta S^\theta) \right] (-a \, dt + \rho \, d\varphi) \\
+ \sqrt{\Delta_r} \cos \theta \left( a \sqrt{\Delta_\varphi} \sin \theta S^\varphi - \sqrt{\Delta_r} \sin \theta S^\theta \right) d\varphi - \sqrt{\Delta_r} (r \sin \theta S^\varphi + \sqrt{\Delta_r} \cos \theta S^\theta) d\varphi \\
- \frac{a \sqrt{\Delta_\varphi} \sin \theta}{\sqrt{\Delta_r}} [r S^\varphi + (n + a \cos \theta) S^\theta] dr - \frac{\sqrt{\Delta_r}}{\sqrt{\Delta_\varphi}} [(n + a \cos \theta) S^\varphi - r S^\theta] d\theta, 
\]

\[G = \frac{2q}{3\Sigma} (n + a \cos \theta) S^\varphi S^\theta, \]

where the spin tensor \(S^{ab}\) defined in (6) has been used. The expression for \(Q_f\) with (42)–(44) then define the conserved charge

\[Z = i \frac{1}{2} \{Q_f, Q_f\}. \]

The above results reduce to those of the Kerr–Newman black-hole spacetime [9] for \(n = \Lambda = q_m = 0\).

4. Remarks


The Killing tensor \(K_{\mu\nu}\) given in (42) defines a constant of motion (directly) for spinless particles in the hot NUT–Kerr–Newman–Kasuya spacetime, whereas for spinning particles it now requires the nontrivial contributions from spin which involve Killing vector and Killing scalar computed in (43) and (44).

Spacetime SUSY has previously been applied to charged black holes in the context of \(N = 2\) supergravity theory [18,19]. The application of world-line SUSY in this Letter seems at first sight to be unrelated to that work. The results concerning a ‘hidden’ SUSY related to the motion of spinning particles are applicable to all members of KN black-hole solutions as well as to other solutions which are not black-hole solutions but have horizons such as the HNUTKNK solution. On the other hand, the Killing spinors giving rise to symmetries of the solutions of supergravity field equations arise only in the case of extreme solutions (or indeed naked singularities) whose mass and charge in suitable units are equal.

The result of this Letter reduces to those of the KN black-hole spacetime [9] for \(n = \Lambda = q_m = 0\) and to those of the NUT spacetime, which is sometimes considered as unphysical [20], for \(a = a_e = q_m = \Lambda = 0\). So it is interesting to note that our result can be applied not only in the physically interesting black-hole solutions, but also in spacetimes which have no physical interpretation.

Supersymmetry and its local version—supergravity—are relevant in the fundamental theory of particle interactions. In modern particle theory, SUSY is the most general symmetry of the S-matrix consistent with relativistic quantum field theory [21]. So it is not inconceivable that nature might make some use of it. Indeed, superstrings [22,23] are the present best candidates for a consistent quantum theory unifying gravity with all other fundamental interactions, and SUSY appears to play a very important role for the quantum stability of superstring solutions in four-dimensional spacetime. In view of these reasons, the study of the geometry of the graded pseudomanifolds with both real number and anticommuting variables is well justified.

References