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Trigonometric version of quantum–classical duality in integrable systems

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Abstract

We extend the quantum–classical duality to the trigonometric (hyperbolic) case. The duality establishes an explicit relationship between the classical N-body trigonometric Ruijsenaars–Schneider model and the inhomogeneous twisted XXZ spin chain on N sites. Similarly to the rational version, the spin chain data fixes a certain Lagrangian submanifold in the phase space of the classical integrable system. The inhomogeneity parameters are equal to the coordinates of particles while the velocities of classical particles are proportional to the eigenvalues of the spin chain Hamiltonians (residues of the properly normalized transfer matrix). In the rational version of the duality, the action variables of the Ruijsenaars–Schneider model are equal to the twist parameters with some multiplicities defined by quantum (occupation) numbers. In contrast to the rational version, in the trigonometric case there is a splitting of the spectrum of action variables (eigenvalues of the classical Lax matrix). The limit corresponding to the classical Calogero–Sutherland system and quantum trigonometric Gaudin model is also described as well as the XX limit to free fermions.

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1. Introduction

The quantum–classical (QC) duality (correspondence) is an explicit relation between quantum and classical integrable systems of different types. This phenomenon was first observed in [7] for the classical Toda chain. A similar observation was made in [14] for the classical Calogero system and quantum Gaudin model. The classical action variables were assumed to be equal to zero. The case of arbitrary set of action variables was described in [2] using the relation of both models to the KP hierarchy [12]. In a similar way, the QC duality between the classical Ruijsenaars–Schneider (RS) model and the quantum twisted spin chain was proposed in [1,19]. The final version and a direct proof of this relation was presented in [8] via the nested Bethe anzats. Later the duality was extended to the correspondence [18]: it was shown that the RS model is related not to a single quantum model but to a family of supersymmetric spin chains. We do not discuss the supersymmetric case in this paper.

Let us briefly recall the result of [8]. Consider the Lax matrix of the classical N-body RS model[1] [16]

$$L^{\mathrm{RS}}_{ij} = \frac{\nu\,\dot q_j}{q_i - q_j + \eta\nu}\,, \quad i, j = 1\,, \ldots, N \tag{1.1}$$

and quantum transfer matrix of the $GL(n)$ inhomogeneous (generalized) twisted XXX spin chain on N sites[2]

$$\hat T^{\mathrm{XXX}}(z) = \mathrm{tr}\, V + \sum_{j=1}^{N} \frac{\hat H_j^{\mathrm{XXX}}}{z - z_j}\,. \tag{1.2}$$

In the framework of the algebraic nested Bethe ansatz the spectrum H_j^{XXX} of the Hamiltonians $\{\hat H_j^{\mathrm{XXX}}\}$ is constructed in terms of the Bethe roots $\Big\{\{\mu_i^1\}_{N_1}, \ldots, \{\mu_i^{n-1}\}_{N_{n-1}}\Big\}$ which are solution of the system of Bethe equations. Here N_c is the number of Bethe roots at the c-th level of the nested Bethe ansatz.

Substitute

$$\nu\eta = \hbar\,, \tag{1.3}$$

$$q_j = z_j\,, \quad j = 1 \ldots N \tag{1.4}$$

and

$$\dot q_j = \frac{\eta}{\hbar}\, H_j^{\mathrm{XXX}}\Big(\{q_i\}_N; \{\mu_i^1\}_{N_1}, \ldots, \{\mu_i^{n-1}\}_{N_{n-1}}\Big)\,, \quad j = 1\,, \ldots, N\,, \tag{1.5}$$

where $\{\mu_i^a\}$ is any solution of the Bethe equations. Then the spectrum of the classical Lax matrix (1.1) is given by the twist parameters:

$$\Big(\underbrace{V_1\,, \ldots, V_1}_{N-N_1}, \underbrace{V_2\,, \ldots, V_2}_{N_1-N_2}, \ldots, \underbrace{V_{n-1}\,, \ldots, V_{n-1}}_{N_{n-2}-N_{n-1}}, \underbrace{V_n\,, \ldots, V_n}_{N_{n-1}}\Big). \tag{1.6}$$

The multiplicities are defined by the quantum numbers N_c.

[1] In (1.1) the sets of variables $\{\dot q_j\}$ and $\{q_j\}$ are velocities and coordinates of particles respectively, ν is the coupling constant and η is the relativistic deformation parameter (inverse light speed).

[2] In (1.2) $\{\hat H_j^{\mathrm{XXX}}\}$ are the quantum (non-local) Hamiltonians, $\{z_j\}$ are inhomogeneity parameters and $V = \mathrm{diag}(V_1, \ldots, V_n)$ is the twist matrix.

Let us also mention that the QC correspondence appeared also in the framework of gauge theory dualities [15,6,9]. Another relation between classical Lax matrices and quantum R-matrices related to spin chains can be found in [13].

The purpose of this paper is the trigonometric version of the QC duality. We prove an analogue of statement (1.6) for the trigonometric (hyperbolic) RS model and the XXZ twisted inhomogeneous spin chain. We show that in contrast to the rational version, the degeneration of the spectrum of action variables (eigenvalues of the classical Lax matrix) disappears. The identification

$$\eta\nu = \hbar \tag{1.7}$$

and

$$\dot{q}_j = \frac{\eta}{\sinh\hbar}\, H_j^{\mathrm{XXZ}} \tag{1.8}$$

leads to the following eigenvalues of the classical RS Lax matrix (to be compared with (1.6) for the rational case):

$$\begin{aligned}
&\Big\{ \underbrace{e^{-(N-N_1-1)\hbar}\, V_1, \dots, e^{(N-N_1-1)\hbar}\, V_1}_{N-N_1}, \underbrace{e^{-(N_1-N_2-1)\hbar}\, V_2, \dots, e^{(N_1-N_2-1)\hbar}\, V_2}_{N_1-N_2}, \dots, \\
&\dots, \underbrace{e^{-(N_{n-2}-N_{n-1}-1)\hbar}\, V_{n-1}, \dots, e^{(N_{n-2}-N_{n-1}-1)\hbar}\, V_{n-1}}_{N_{n-2}-N_{n-1}}, \\
&\underbrace{e^{-(N_{n-1}-1)\hbar}\, V_n, \dots, e^{(N_{n-1}-1)\hbar}\, V_n}_{N_{n-1}} \Big\}
\end{aligned} \tag{1.9}$$

The eigenvalues of the Lax matrix form “strings” centered at the twist parameters V_a.

2. Trigonometric Ruijsenaars–Schneider model

In this paper we use the following Lax matrix of the trigonometric N-particle RS model:

$$L_{ij}^{\mathrm{RS}} = \frac{\sinh(\eta\nu)}{\sinh(q_i - q_j + \eta\nu)}\, e^{\eta p_j} \prod_{k\neq j}^{N} \frac{\sinh(q_j - q_k - \eta\nu)}{\sinh(q_j - q_k)}\,, \quad i, j = 1, \dots, N. \tag{2.1}$$

The Hamiltonian is defined as

$$H^{\mathrm{RS}} = \mathrm{tr}\, L^{\mathrm{RS}} = \sum_{j=1}^{N} e^{\eta p_j} \prod_{k\neq j}^{N} \frac{\sinh(q_j - q_k - \eta\nu)}{\sinh(q_j - q_k)}\,. \tag{2.2}$$

For the velocities we have

$$\dot{q}_j = \frac{\partial H^{\mathrm{RS}}}{\partial p_i} = \eta\, e^{\eta p_j} \prod_{k\neq j}^{N} \frac{\sinh(q_j - q_k - \eta\nu)}{\sinh(q_j - q_k)}\,. \tag{2.3}$$

Therefore, in terms of velocities the Lax matrix has the form

$$L_{ij}^{\mathrm{RS}} = \frac{\sinh(\eta\nu)}{\sinh(q_i - q_j + \eta\nu)}\, \eta^{-1} \dot{q}_j \tag{2.4}$$

or

$$L^{\mathrm{RS}}_{ij} = \eta^{-1}\, C_{ik}\, \dot{Q}_{kj}\,, \quad Q_{ij} = \delta_{ij}\, q_j\,,$$
$$C_{ij} = \frac{\sinh(\eta\nu)}{\sinh(q_i - q_j + \eta\nu)}\,, \qquad i, j = 1\,, \dots, N. \tag{2.5}$$

Here $\|C_{ij}\|$ is the trigonometric Cauchy matrix.

It is important for our purpose that the classical Lax matrix (2.1) admits the following factorization (see [10,3]):

$$L^{\mathrm{RS}} = D\,\tilde{V}^{-1}(\epsilon)\tilde{V}(\epsilon - \eta\nu)D^{-1}e^{\eta P}\,, \tag{2.6}$$

where $P = \mathrm{diag}(p_1, \dots, p_N)$, $\tilde{V}$ is the (trigonometric) Vandermonde-type matrix

$$\tilde{V}_{ij}(\epsilon) = \exp\big((2i - 1 - N)(q_j + \epsilon)\big) \tag{2.7}$$

and

$$D_{ij} = \delta_{ij} \prod_{k \neq j}^{N} \sinh(q_j - q_k)\,. \tag{2.8}$$

The (spectral) parameter ϵ is fictitious — it does not enter the final answer. Notice that

$$\tilde{V}(\epsilon - \eta\nu) = S(\eta\nu)\tilde{V}(\epsilon)\,, \tag{2.9}$$

where S is the following diagonal matrix:

$$S_{ij}(\zeta) = \delta_{ij} \exp\left(-(2i - 1 - N)\zeta\right)\,. \tag{2.10}$$

It follows from (2.6) and (2.9) that the eigenvalues of the Lax matrix (2.1) become very simple on the Lagrangian submanifold $P = 0$ (i.e. $p_k = 0$ for all $k = 1, \dots, N$). The spectrum of (2.1) is then given by the elements of matrix $S(\eta\nu)$:

$$\mathrm{Spec}(L^{\mathrm{RS}}|_{P=0}) = \left\{e^{-(N-1)\eta\nu}, e^{-(N-3)\eta\nu}, \dots, e^{(N-3)\eta\nu}, e^{(N-1)\eta\nu}\right\}\,. \tag{2.11}$$

The equations of motion $\dot{p}_j = -\dfrac{\partial H^{\mathrm{RS}}}{\partial q_i}$ of the RS model admit the Lax representation $\dot{L}^{\mathrm{RS}} = [B^{\mathrm{RS}}, L^{\mathrm{RS}}]$, where

$$B^{\mathrm{RS}}_{jk} = \Big(\sum_{l \neq j} \dot{q}_l \coth q_{jl} - \sum_{l} \dot{q}_l \coth(q_{jl} + \eta\nu)\Big)\delta_{jk} + \frac{1 - \delta_{jk}}{\sinh q_{jk}}, \qquad q_{jk} \equiv q_j - q_k.$$

Explicitly, the equations of motion read

$$\ddot{q}_j = -\sum_{k \neq j} \frac{2\dot{q}_j \dot{q}_k \sinh^2(\eta\nu)\cosh q_{jk}}{\sinh(q_{jk} - \eta\nu)\sinh q_{jk} \sinh(q_{jk} + \eta\nu)}. \tag{2.12}$$

Two special cases of the trigonometric RS model are $\eta\nu = \pm\infty$ and $\eta\nu = i\pi/2$. In the former case the equations of motion simplify to

$$\eta\nu = \pm\infty : \quad \ddot{q}_j = 2 \sum_{k \neq j} \dot{q}_j \dot{q}_k \coth q_{jk}. \tag{2.13}$$

In the latter case they are:

$$\eta\nu = \frac{i\pi}{2} : \quad \ddot{q}_j = 4 \sum_{k \neq j} \frac{\dot{q}_j \dot{q}_k}{\sinh(2q_{jk})}. \tag{2.14}$$

3. Inhomogeneous $\mathcal{U}_q(\hat{\mathfrak{gl}}_n)$ spin chain

The algebraic structure of the Heisenberg XXZ spin chain is based on the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{gl}}_n)$ [11,5] (see also [4]). The model is defined by the following quantum R-matrix:

$$R_{12}(z) = \frac{\sinh(z+\hbar)}{\sinh z}\sum_{a=1}^{n} \mathrm{e}_{aa}\otimes \mathrm{e}_{aa} + \sum_{1\le a\neq b\le n} \mathrm{e}_{aa}\otimes \mathrm{e}_{bb} + \frac{\sinh\hbar}{\sinh z}\sum_{1\le a\neq b\le n} e^{\mathrm{sign}(b-a)z}\,\mathrm{e}_{ab}\otimes \mathrm{e}_{ba}\,. \tag{3.1}$$

where z is the spectral parameter, $\hbar$ is the anisotropy parameter and e_{ab} denotes the $n\times n$ matrix with 1 in the position (a,b) and 0 otherwise.

The transfer matrix of the twisted inhomogeneous Heisenberg XXZ model on N sites is given by

$$\hat{T}^{\mathrm{XXZ}}(z) = \mathrm{tr}_0\Big[V_0\, R_{01}(z-q_1)\dots R_{0N}(z-q_N)\Big], \tag{3.2}$$

where the diagonal twist matrix

$$V = \mathrm{diag}(V_1, V_2, \dots, V_n) \tag{3.3}$$

acts in the auxiliary n-dimensional vector space labeled by 0. We assume that the parameters q_k are in general position, i.e. $q_j\neq q_k$ and $q_j\neq q_k\pm\hbar$ for $j\neq k$. It follows from the Yang–Baxter equation for the R-matrix that the transfer matrices commute for different values of the spectral parameter: $[\hat{T}^{\mathrm{XXZ}}(z), \hat{T}^{\mathrm{XXZ}}(z')]=0$.

The nested Bethe ansatz gives the following result for eigenvalues of the transfer matrix (3.2):

$$T^{\mathrm{XXZ}}(z) = V_1\prod_{k=1}^{N}\frac{\sinh(z-q_k+\hbar)}{\sinh(z-q_k)}\prod_{\gamma=1}^{N_1}\frac{\sinh(z-\mu_\gamma^1-\hbar)}{\sinh(z-\mu_\gamma^1)} + \sum_{b=2}^{n} V_b\prod_{\gamma=1}^{N_{b-1}}\frac{\sinh(z-\mu_\gamma^{b-1}+\hbar)}{\sinh(z-\mu_\gamma^{b-1})}\prod_{\gamma=1}^{N_b}\frac{\sinh(z-\mu_\gamma^{b}-\hbar)}{\sinh(z-\mu_\gamma^{b})}\,. \tag{3.4}$$

The integer parameters N_b ($N_0=N_n=0$) are the numbers of Bethe roots μ_β^b in the b-th group, $b=1,\dots,n-1$, $\beta=1,\dots,N_b$. They satisfy the system of Bethe equations (BE):

$$V_1\prod_{k=1}^{N}\frac{\sinh(\mu_\beta^1-q_k+\hbar)}{\sinh(\mu_\beta^1-q_k)} = V_2\prod_{\gamma\neq\beta}^{N_1}\frac{\sinh(\mu_\beta^1-\mu_\gamma^1+\hbar)}{\sinh(\mu_\beta^1-\mu_\gamma^1-\hbar)}\prod_{\gamma=1}^{N_2}\frac{\sinh(\mu_\beta^1-\mu_\gamma^2-\hbar)}{\sinh(\mu_\beta^1-\mu_\gamma^2)}$$

$$V_b\prod_{\gamma=1}^{N_{b-1}}\frac{\sinh(\mu_\beta^b-\mu_\gamma^{b-1}+\hbar)}{\sinh(\mu_\beta^b-\mu_\gamma^{b-1})} = V_{b+1}\prod_{\gamma\neq\beta}^{N_b}\frac{\sinh(\mu_\beta^b-\mu_\gamma^b+\hbar)}{\sinh(\mu_\beta^b-\mu_\gamma^b-\hbar)}\prod_{\gamma=1}^{N_{b+1}}\frac{\sinh(\mu_\beta^b-\mu_\gamma^{b+1}-\hbar)}{\sinh(\mu_\beta^b-\mu_\gamma^{b+1})}\,, \tag{3.5}$$

where $b=2,\dots,n-1$. In the last equation it is implied that $N_n=0$. The BE mean that the eigenvalues (3.4) are regular at $z=\mu_\gamma^b$.

It is known that the operators

$$\hat{M}_a = \sum_{j=1}^{N} \mathrm{e}_{aa}^{(j)}, \quad \mathrm{e}_{aa}^{(j)} = \underbrace{1 \otimes \ldots \otimes 1}_{j-1} \otimes \mathrm{e}_{aa} \otimes \underbrace{1 \otimes \ldots \otimes 1}_{N-j}, \tag{3.6}$$

commute with the transfer matrix. The eigenvectors of the latter, built from solutions to the BE, with the number of Bethe roots at level b equal to N_b, are also eigenvectors of the operators $\hat{M}_a$ with the eigenvalues $M_1 = N - N_1$, $M_a = N_{a-1} - N_a$, $a = 2, \ldots, n$.

The transfer matrix (3.2) can be represented as a sum over simple poles at $z = q_k$:

$$\hat{T}^{\mathrm{XXZ}}(z) = \hat{C} + \sum_{k=1}^{N} \hat{H}_i^{\mathrm{XXZ}} \coth(z - q_k) \tag{3.7}$$

(it follows from (3.1) that it is an $i\pi$-periodic function of z). The coefficients

$$\hat{H}_i^{\mathrm{XXZ}} = \operatorname*{Res}_{z=q_i} \hat{T}^{\mathrm{XXZ}}(z) \tag{3.8}$$

are quantum (non-local) Hamiltonians of the inhomogeneous spin chain. They commute with each other, $[\hat{H}_i^{\mathrm{XXZ}}, \hat{H}_j^{\mathrm{XXZ}}] = 0$, and can be simultaneously diagonalized. This ensures integrability of the model. The eigenvalues of the commuting Hamiltonians are given by the formula

$$H_i^{\mathrm{XXZ}} = V_1 \sinh\hbar \prod_{k\neq i}^{N} \frac{\sinh(q_i - q_k + \hbar)}{\sinh(q_i - q_k)} \prod_{\gamma=1}^{N_1} \frac{\sinh(q_i - \mu_\gamma^1 - \hbar)}{\sinh(q_i - \mu_\gamma^1)}, \tag{3.9}$$

where the μ_γ^1's are taken from a solution to the BE. It is easy to see that

$$T^{\mathrm{XXZ}}(\pm\infty) = C \pm \sum_{k=1}^{N} H_k^{\mathrm{XXZ}} = \sum_{a=1}^{n} V_a e^{\pm\hbar M_a}, \tag{3.10}$$

hence we get the "sum rules"

$$C = \sum_{a=1}^{n} V_a \cosh(\hbar M_a), \qquad \sum_{k=1}^{N} H_k^{\mathrm{XXZ}} = \sum_{a=1}^{n} V_a \sinh(\hbar M_a). \tag{3.11}$$

4. Determinant identity

Consider a pair of $N \times N$ and $M \times M$ matrices:

$$\mathcal{L}_{ij}(\{x_i\}_N, \{y_i\}_M, g) = \\ = \frac{g \sinh\hbar}{\sinh(x_i - x_j + \hbar)} \prod_{k\neq j}^{N} \frac{\sinh(x_j - x_k + \hbar)}{\sinh(x_j - x_k)} \prod_{\gamma=1}^{M} \frac{\sinh(x_j - y_\gamma)}{\sinh(x_j - y_\gamma + \hbar)}, \tag{4.1}$$

$i, j = 1, \ldots, N$ and

$$\widetilde{\mathcal{L}}_{\alpha\beta}(\{y_i\}_M, \{x_i\}_N, g) = \\ = \frac{g \sinh\hbar}{\sinh(y_\alpha - y_\beta + \hbar)} \prod_{\gamma\neq\beta}^{M} \frac{\sinh(y_\beta - y_\gamma - \hbar)}{\sinh(y_\beta - y_\gamma)} \prod_{k=1}^{N} \frac{\sinh(y_\beta - x_k)}{\sinh(y_\beta - x_k - \hbar)}, \tag{4.2}$$

$\alpha, \beta = 1, \dots, M$. For definiteness assume that $N \geq M$. Then the following identity holds true:

$$\det_{N\times N}\Big(\mathcal{L}(\{x_i\}_N, \{y_i\}_M, g) - \lambda I\Big) = \det_{(N-M)\times(N-M)}(gS - \lambda I) \det_{M\times M}\Big(\widetilde{\mathcal{L}}(\{y_i\}_M, \{x_i\}_N, g) - \lambda I\Big) \tag{4.3}$$

Here the matrix $S = S(\hbar)$ (2.10) entering the r.h.s. of (4.3) is $(N-M)\times(N-M)$ matrix and I is the unity matrix. The proof of (4.3) is based on (2.6). It is similar to the one given in [8] for the rational case.

5. Quantum–classical duality

Theorem 1. *Under identification of parameters*

$$\eta\nu = \hbar \tag{5.1}$$

and

$$\frac{\dot q_j}{\eta} = \frac{H_j^{XXZ}}{\sinh\hbar} \tag{5.2}$$

where H_j^{XXZ} are eigenvalues of the quantum spin chain Hamiltonians corresponding to any common eigenstate, the spectrum of the classical RS Lax matrix (2.4) is given by

$$\begin{aligned}
&\mathrm{Spec}\, L^{RS}\left(\left\{\dot q_j = \eta\,\frac{H_j^{XXZ}}{\sinh\hbar}\right\}_N \{q_j\}_N, \hbar\right)\Bigg|_{BE} \\
&= \Big\{ \underbrace{e^{-(N-N_1-1)\hbar}\, V_1, \dots, e^{(N-N_1-1)\hbar}\, V_1}_{N-N_1}, \underbrace{e^{-(N_1-N_2-1)\hbar}\, V_2, \dots, e^{(N_1-N_2-1)\hbar}\, V_2}_{N_1-N_2}, \dots, \\
&\dots, \underbrace{e^{-(N_{n-2}-N_{n-1}-1)\hbar}\, V_{n-1}, \dots, e^{(N_{n-2}-N_{n-1}-1)\hbar}\, V_{n-1}}_{N_{n-2}-N_{n-1}}, \\
&\underbrace{e^{-(N_{n-1}-1)\hbar}\, V_n, \dots, e^{(N_{n-1}-1)\hbar}\, V_n}_{N_{n-1}} \Big\}.
\end{aligned} \tag{5.3}$$

Proof. We can reformulate the statement of the theorem as

$$\boxed{\det\left[L\left(\frac{\eta}{\sinh\hbar}\left\{H_j^{\mathrm{XXZ}}\right\}_N, \{q_j\}_N, \hbar\right)\Bigg|_{BE} - \lambda I\right] = \prod_{a=1}^{n} \det\left[V_a\, S_{N_a - N_{a-1}} - \lambda I\right]} \tag{5.4}$$

where $N_0 = N$, $N_n = 0$, $S_M = S_M(\hbar)$ is the matrix (2.10) of size $M \times M$.

The proof of (5.4) is performed by successive usage of the determinant identity (4.3) and BE (3.5). Consider the matrix

$$L^{(0)}_{ij} = L^{\rm RS}_{ij}\left(\frac{\eta}{\sinh\hbar}\left\{H^{\rm XXZ}_k\right\}_N, \{q_k\}_N, \hbar\right)$$
$$= \frac{V_1 \sinh\hbar}{\sinh(q_i - q_j + \hbar)} \prod_{k\neq j}^{N} \frac{\sinh(q_j - q_k + \hbar)}{\sinh(q_j - q_k)} \prod_{\gamma=1}^{N_1} \frac{\sinh(q_j - \mu^1_\gamma - \hbar)}{\sinh(q_j - \mu^1_\gamma)} =$$
$$= \mathcal{L}_{ij}(\{q_k - \hbar\}_N, \{\mu^1_\gamma\}_{N_1}, V_1) \tag{5.5}$$

and

$$L^{(1)}_{\alpha\beta} = \widetilde{\mathcal{L}}_{\alpha\beta}(\{\mu^1_\gamma\}_{N_1}, \{q_i - \hbar\}_N, V_1)$$
$$= \frac{V_1 \sinh\hbar}{\sinh(\mu^1_\alpha - \mu^1_\beta + \hbar)} \prod_{\gamma\neq\beta}^{N_1} \frac{\sinh(\mu^1_\beta - \mu^1_\gamma - \hbar)}{\sinh(\mu^1_\beta - \mu^1_\gamma)} \prod_{k=1}^{N} \frac{\sinh(\mu^1_\beta - q_k + \hbar)}{\sinh(\mu^1_\beta - q_k)}, \tag{5.6}$$

where $\alpha, \beta = 1, ..., N_1$. Identity (4.3) provides the relation

$$\det_{N\times N}\left(L^{(0)} - \lambda I\right) = \det_{(N-N_1)\times(N-N_1)}(V_1 S - \lambda I) \det_{N_1\times N_1}\left(L^{(1)} - \lambda I\right). \tag{5.7}$$

Impose now the BE (3.5). Then we get

$$L^{(1)}\Big|_{BE} = \frac{V_2 \sinh\hbar}{\sinh(\mu^1_\alpha - \mu^1_\beta + \hbar)} \prod_{\gamma\neq\beta}^{N_1} \frac{\sinh(\mu^1_\beta - \mu^1_\gamma + \hbar)}{\sinh(\mu^1_\beta - \mu^1_\gamma)} \prod_{\gamma=1}^{N_2} \frac{\sinh(\mu^1_\beta - \mu^2_\gamma - \hbar)}{\sinh(\mu^1_\beta - \mu^2_\gamma)}, \tag{5.8}$$

i.e.

$$L^{(1)}\Big|_{BE_1} = \mathcal{L}_{ij}(\{\mu^1_\gamma - \hbar\}_{N_1}, \{\mu^2_\gamma\}_{N_2}, V_2)\,. \tag{5.9}$$

At the next step let us define

$$L^{(2)}_{\alpha\beta} = \widetilde{\mathcal{L}}_{\alpha\beta}(\{\mu^2_\gamma\}_{N_2}, \{\mu^1_\gamma - \hbar\}_{N_1}, V_2)\,, \qquad \alpha, \beta = 1\,, \ldots, N_2\,, \tag{5.10}$$

and again we use (4.3) and (3.5) to get:

$$\det_{N_1\times N_1}\left(L^{(1)} - \lambda I\right) = \det_{N_1-N_2\times N_1-N_2}(V_2 S - \lambda I) \det_{N_2\times N_2}\left(L^{(2)} - \lambda I\right), \tag{5.11}$$

$$L^{(2)}\Big|_{BE_2} = \mathcal{L}(\{\mu^2_\gamma - \hbar\}_{N_2}, \{\mu^3_\gamma\}_{N_3}, V_3)\,, \tag{5.12}$$

$$\vdots$$

The process of the subsequent usage of (4.3) and (3.5) is continued until the last step when equation (3.5) is used:

$$L^{(n-1)}\Big|_{BE_{n-1}} = \mathcal{L}_{ij}(\{\mu^{n-1}_\gamma - \hbar\}_{N_{n-1}}, \{\mu^n_\gamma\}_0, V_n)\,. \tag{5.13}$$

Finally, (4.3) with $N = N_{n-1}$, $M = 0$ yields

$$\det_{N_{n-1}\times N_{n-1}}\left(L^{(n-1)} - \lambda I\right) = \det_{N_{n-1}\times N_{n-1}}(V_n S - \lambda I)\,. \qquad \square \tag{5.14}$$

In order to find the characteristic polynomial of the matrix

$$L = L^{\rm RS}\left(\left\{\dot q_j = \eta\,\frac{H_j^{\rm XXZ}}{\sinh\hbar}\right\}_N \{q_j\}_N\,,\,\hbar\right)$$

explicitly, we use the known fact that the coefficient in front of λ^{N-k} in the polynomial $\det\limits_{N\times N}(\lambda I + A)$ equals the sum of all diagonal $k\times k$ minors of the matrix A. All such minors can be found using the explicit expression for the determinant

$$\det_{1\le i,j\le k}\frac{\sinh\hbar}{\sinh(q_i - q_j + \hbar)} = \prod_{1\le i,j\le k} C(q_i - q_j), \qquad C(q) = \frac{\sinh^2 q}{\sinh(q+\hbar)\sinh(q-\hbar)} \tag{5.15}$$

which can be easily proved or taken from [17]. As a result, we get

$$\det_{N\times N}(\lambda I + L) = \sum_{k=0}^{N} J_k \lambda^{N-k},$$

where

$$J_k = (\sinh\hbar)^{-k}\sum_{1\le i_1<\ldots<i_k\le N} H_{i_1}^{\rm XXZ}\ldots H_{i_n}^{\rm XXZ}\prod_{1\le\alpha<\beta\le k} C(q_{i_\alpha} - q_{i_\beta}) \tag{5.16}$$

Therefore, we have the following system of polynomial equations for spectrum of the quantum Hamiltonians:

$$\sum_{1\le i_1<\ldots<i_k\le N} H_{i_1}^{\rm XXZ}\ldots H_{i_k}^{\rm XXZ}\prod_{1\le\alpha<\beta\le k} C(q_{i_\alpha} - q_{i_\beta}) = (\sinh\hbar)^k\sum_{1\le i_1<\ldots<i_k\le N}\lambda_{i_1}\ldots\lambda_{i_k} \tag{5.17}$$

($k = 1,\ldots,N$). Here $\lambda_i\in\operatorname{Spec} L$ are given by (5.3). Setting $q_i = \hbar x_i$, $H_i^{\rm XXZ} = \hbar\tilde H_i$ and tending $\hbar\to 0$, these equations become the equations of the universal spectral variety for models of the XXX type [18].

Equations (5.17) at $k = N$ and $k = 1$ are easy to check without directly appealing to the determinant identity using the side-by-side products of the BE and the “sum rules” (3.11).

At $k = N$ we have the equation

$$\prod_{j=1}^{N} H_j^{\rm XXZ}\cdot\prod_{1\le l<m\le N} C(q_l - q_m) = (\sinh\hbar)^N\prod_{a=1}^{n} V_a^{M_a} \tag{5.18}$$

The Bethe ansatz result gives

$$\prod_{k=1}^{N} H_k^{\rm XXZ} = (V_1\sinh\hbar)^N\prod_{i=1}^{N}\prod_{k\ne i}^{N}\frac{\sinh(q_i - q_k + \hbar)}{\sinh(q_i - q_k)}\prod_{i=1}^{N}\prod_{\gamma=1}^{N_1}\frac{\sinh(q_i - \mu_\gamma^1 - \hbar)}{\sinh(q_i - \mu_\gamma^1)}\,. \tag{5.19}$$

The first double product cancels against the product of the C-factors in (5.18). The side-by-side products of the BE

$$BE_1:\quad V_1^{N_1}\prod_{\beta=1}^{N_1}\prod_{k=1}^{N}\frac{\sinh(q_k - \mu_\beta^1 - \hbar)}{\sinh(q_k - \mu_\beta^1)} = V_2^{N_1}\prod_{\beta=1}^{N_1}\prod_{\gamma=1}^{N_2}\frac{\sinh(\mu_\beta^1 - \mu_\gamma^2 - \hbar)}{\sinh(\mu_\beta^1 - \mu_\beta^2)}\,,$$

$$BE_b: \quad V_b^{N_b} \prod_{\beta=1}^{N_b} \prod_{\gamma=1}^{N_{b-1}} \frac{\sinh(\mu_\gamma^{b-1} - \mu_\beta^b - \hbar)}{\sinh(\mu_\gamma^{b-1} - \mu_\beta^b)} = V_{b+1}^{N_b} \prod_{\beta=1}^{N_b} \prod_{\gamma=1}^{N_{b+1}} \frac{\sinh(\mu_\beta^b - \mu_\gamma^{b+1} - \hbar)}{\sinh(\mu_\beta^b - \mu_\gamma^{b+1})} \,,$$

$$BE_{n-1}: \quad V_{n-1}^{N_{n-1}} \prod_{\beta=1}^{N_{n-1}} \prod_{\gamma=1}^{N_{n-2}} \frac{\sinh(\mu_\gamma^{n-2} - \mu_\beta^{n-1} - \hbar)}{\sinh(\mu_\gamma^{n-2} - \mu_\beta^{n-1})} = V_n^{N_{n-1}} \,, \tag{5.20}$$

form a chain of identities that yields the right-hand side of (5.18).

At $k = 1$ the equation is

$$\sum_{j=1}^{N} H_j^{\mathrm{XXZ}} = \sinh\hbar \sum_{j=1}^{N} \lambda_j \,, \qquad \lambda_j \in \mathrm{Spec}\, L \,. \tag{5.21}$$

According to

$$\sum_{j=0}^{N_{b-1}-N_b-1} e^{-(N_{b-1}-N_b-1)+2j\hbar} = \frac{\sinh\left(\hbar(N_{b-1}-N_b)\right)}{\sinh\hbar}$$

it is exactly the second "sum rule" in (3.11).

6. Limiting cases

6.1. Limit to the Gaudin–Calogero correspondence

Calogero–Sutherland model. The Lax matrix of the Calogero–Sutherland model

$$L_{ij}^{\mathrm{CM}} = \delta_{ij}\, \dot{q}_j + (1-\delta_{ij}) \frac{\nu}{\sinh(q_i - q_j)} \tag{6.1}$$

can be represented as

$$L^{\mathrm{CM}} = P - \nu D V^{-1}(\epsilon) \partial_\epsilon V(\epsilon) D^{-1} \tag{6.2}$$

with matrices P, V and D defined in (2.6)–(2.8) and velocities

$$\dot{q}_j = p_i - \nu \sum_{k \neq i}^{N} \coth(q_i - q_k) \tag{6.3}$$

generated by the Hamiltonian $H^{\mathrm{CM}} = \frac{1}{2}\mathrm{tr}\left(L^{\mathrm{CM}}\right)^2$. The representation (6.2) follows from (2.6) in the non-relativistic limit $\eta \to 0$:

$$L^{\mathrm{RS}} = 1_{N\times N} + \eta\, L^{\mathrm{CM}} + O(\eta^2) \,. \tag{6.4}$$

It follows from (2.7) and (2.10) that

$$\partial_\epsilon V(\epsilon) = -\log S(\nu) V \,, \tag{6.5}$$

where

$$-\log S_{ij}(\zeta) = \delta_{ij} \left((2i - 1 - N)\zeta\right) \,. \tag{6.6}$$

Therefore,

$$\mathrm{Spec}(L^{\mathrm{CM}}\,|_{P=0}) = \{-(N-1)\nu, -(N-3)\nu, \ldots, (N-3)\nu, (N-1)\nu\} \,. \tag{6.7}$$

The trigonometric Gaudin model appears in the limit $\varepsilon \to 0$ from the inhomogeneous XXZ spin chain with the transfer matrix $\hat{T}^{\mathrm{XXZ}}(z;\{q_i\},V^{\varepsilon},\varepsilon\hbar)$, where

$$V^{\epsilon} = 1_{n\times n} + \varepsilon\,\mathrm{diag}(v_1,\dots,v_n) + O(\varepsilon^2)\,.$$

The expansion as $\varepsilon \to 0$,

$$\hat{T}^{\mathrm{XXZ}}(z;\{q_i\},V^{\varepsilon},\varepsilon\hbar) = nI + \varepsilon\hat{T}_1(z;\{q_i\}) + \varepsilon^2\hat{T}_2(z;\{q_i\}) + O(\varepsilon^3), \tag{6.8}$$
$$\hat{T}_1(z;\{q_i\}) = \mathrm{tr}\, v\, I + \hbar\sum_i C_1^{(i)}\coth(z-q_i), \qquad C_1^{(i)} = \sum_a \mathrm{e}_{aa}^{(i)},$$

defines the commuting Gaudin Hamiltonians

$$\hat{H}_i^{\mathrm{G}} = \operatorname*{Res}_{z=q_i} \hat{T}_2(z;\{q_i\})\,, \tag{6.9}$$

$$\hat{H}_i^{\mathrm{G}} = \sum_a v_a \mathrm{e}_{aa}^{(i)} + \sum_{j\neq i}\frac{\hbar}{\sinh(q_i-q_j)}\left(\sum_{a\neq b}\mathrm{e}_{ab}^{(i)}\mathrm{e}_{ba}^{(j)} + \cosh(q_i-q_j)\sum_a \mathrm{e}_{aa}^{(i)}\mathrm{e}_{aa}^{(j)}\right). \tag{6.10}$$

The commutativity of the Gaudin Hamiltonians follows from commutativity of the transfer matrices, taken into account that the term $\hat{T}_1(z;\{q_i\})$ is central. Their eigenvalues can be found using (3.9) and tending $\varepsilon \to 0$. This gives

$$H_i^{\mathrm{G}} = v_1 + \hbar\sum_{k\neq i}^{N}\coth(q_i-q_k) - \hbar\sum_{\gamma=1}^{N_1}\coth(q_i-\mu_\gamma^1) \tag{6.11}$$

with the BE at level b of the form

$$v_b + \delta_{1b}\hbar\sum_{k=1}^{N}\coth(\mu_\beta^b - q_k) + \hbar\sum_{\gamma=1}^{N_{b-1}}\coth(\mu_\beta^b-\mu_\gamma^{b-1}) =$$
$$= v_{b+1} + 2\hbar\sum_{\gamma\neq\beta}^{N_b}\coth(\mu_\beta^b-\mu_\gamma^b) - \hbar\sum_{\gamma=1}^{N_{b+1}}\coth(\mu_\beta^b-\mu_\gamma^{b+1}) \tag{6.12}$$

where $b = 1,\dots,n-1$, $N_0 = N_n = 0$, $\beta = 1,\dots,N_b$. The matrix $v = \mathrm{diag}(v_1,\dots,v_n)$ is the twist matrix of the Gaudin model. Similarly to the (XXZ) spin chain case we use the notation $H_i^{\mathrm{G}}(\{q_i\}_N,\{\mu_\alpha^1\}_{N_1})$ for the function given by the r.h.s. of (6.11). When the set $\{\mu_\alpha^1\}_{N_1}$ is taken from a solution of the system of BE (6.12) this function is equal to some eigenvalue of the Hamiltonian.

Determinant identity. Introduce the following pair of matrices:

$$\mathcal{L}_{ij}(\{x_i\}_N,\{y_i\}_M,\omega)$$
$$= \delta_{ij}\left(\omega + \sum_{k\neq i}^{N}\nu\coth(x_i-x_k) + \sum_{\gamma=1}^{M}\nu\coth(y_\gamma-x_i)\right) + (1-\delta_{ij})\frac{\nu}{\sinh(x_i-x_j)}, \tag{6.13}$$

where $i,j = 1,\dots,N$ and

$$\widetilde{\mathcal{L}}_{\alpha\beta}(\{y_i\}_M, \{x_i\}_N, \omega) = \delta_{\alpha\beta}\left(\omega - \sum_{\gamma\neq\alpha}^{M} \nu\coth(y_\alpha - y_\gamma) - \sum_{k=1}^{N} \nu\coth(x_k - y_\alpha)\right) + \big(1-\delta_{\alpha\beta}\big)\frac{\nu}{\sinh(y_\alpha - y_\beta)}, \tag{6.14}$$

where $\alpha, \beta = 1, \dots, M$. The relation between their determinants is given as follows:

$$\det_{N\times N}\Big(\mathcal{L}(\{x_i\}_N, \{y_i\}_M, \omega) - \lambda I\Big) = \det_{(N-M)\times(N-M)}(\omega I + \log S - \lambda I) \det_{M\times M}\Big(\widetilde{\mathcal{L}}(\{y_i\}_M, \{x_i\}_N, g) - \lambda I\Big), \tag{6.15}$$

where $\log S = \log S(\nu)$ (6.6) entering r.h.s. of (6.15) is the $(N-M)\times(N-M)$ diagonal matrix.

Quantum–classical duality between the classical Calogero–Suthland system and the quantum Gaudin model is given by the following statement:

Theorem 2. *Under identification of the parameters*

$$\nu = \hbar \tag{6.16}$$

and

$$\dot{q}_j = \frac{1}{\hbar} H_j^G(\{q_i\}_N, \{\mu_\alpha^1\}_{N_1}) \tag{6.17}$$

where H_j^G are eigenvalues of the quantum Gaudin Hamiltonians corresponding to any common eigenstate, the spectrum of the Lax matrix (6.1) is equal to

$$\begin{aligned}
&\mathrm{Spec}\, L^{CM}\left(\frac{1}{\hbar}\left\{H_j^G\right\}_N, \left\{q_j\right\}_N, \hbar\right)\Big|_{BE} = \\
&= \Big\{\underbrace{v_1 - (N-N_1-1)\hbar, \, \dots, \, v_1 + (N-N_1-1)\hbar}_{N-N_1}, \\
&\quad \underbrace{v_2 - (N_1-N_2-1)\hbar, \, \dots, \, v_2 + (N_1-N_2-1)\hbar}_{N_1-N_2}, \\
&\quad \dots, \underbrace{v_n - (N_{n-1}-1)\hbar, \, \dots, \, v_n + (N_{n-1}-1)\hbar}_{N_{n-1}}\Big\}.
\end{aligned} \tag{6.18}$$

The proof Theorem 2 is similar to Theorem 1. Similarly to the non-degenerate XXZ case, the eigenvalues of the Lax matrix form "strings" centered at the v_a's. The distance between to subsequent eigenvalues in any string is $2\hbar$.

6.2. Limit to XX model

The XXZ model has a limit $\hbar \to i\pi/2$ called the XX model. The latter is often referred to as the free-fermion model, which is due to the fact that the XX Hamiltonian may be mapped to a creation–annihilation form that corresponds to a system of non-interacting fermions on the 1D lattice. As none of the R-matrix entries vanish at $\hbar = i\pi/2$, the eigenvalues of the transfer matrix simplify insignificantly:

$$T^{\rm XX}(z) = i^{N-N_1}\, V_1 \prod_{k=1}^{N} \coth(z-q_k) \prod_{\gamma=1}^{N_1} \coth(z-\mu_\gamma^1) + \\ + \sum_{b=2}^{n} i^{N_{b-1}-N_b}\, V_b \prod_{\gamma=1}^{N_{b-1}} \coth(z-\mu_\gamma^{b-1}) \prod_{\gamma=1}^{N_b} \coth(z-\mu_\gamma^b). \tag{6.19}$$

So do the eigenvalues of the quantum Hamiltonians:

$$H_j^{\rm XX} = i^{N-N_1}\, V_1 \prod_{k\neq j}^{N} \coth(q_j-q_k) \prod_{\gamma=1}^{N_1} \coth(q_j-\mu_\gamma^1)\,. \tag{6.20}$$

What is special about the free-fermion point is the simplification of the BE (3.5) due to collapse of one of the two products in the right hand sides caused by periodicity of the sinh-function along the imaginary axis:

$$BE_1: \quad i^N\, V_1 \prod_{k=1}^{N} \coth(\mu_\beta^1-q_k) = V_2\,(-1)^{N_1-1}\, i^{-N_2} \prod_{\gamma=1}^{N_2} \coth(\mu_\beta^1-\mu_\gamma^2)\,,$$
$$BE_b: \quad i^{N_{b-1}}\, V_b \prod_{\gamma=1}^{N_{b-1}} \coth(\mu_\beta^b-\mu_\gamma^{b-1}) = V_{b+1}\,(-1)^{N_b-1} i^{-N_{b+1}} \prod_{\gamma=1}^{N_{b+1}} \coth(\mu_\beta^b-\mu_\gamma^{b+1})\,,$$
$$BE_{n-1}: \quad i^{N_{n-2}}\, V_{n-1} \prod_{\gamma=1}^{N_{n-2}} \coth(\mu_\beta^{n-1}-\mu_\gamma^{n-2}) = V_n\,(-1)^{N_{n-1}-1}\,, \tag{6.21}$$

where $b = 2, \dots, n-2$.

The equations for the spectrum (5.17) acquire the form

$$\sum_{1\le i_1<\ldots<i_k\le N} H_{i_1}^{\rm XX} \dots H_{i_k}^{\rm XX} \prod_{1\le\alpha<\beta\le k} \tanh^2(q_{i_\alpha}-q_{i_\beta}) = i^k \sum_{1\le i_1<\ldots<i_k\le N} \lambda_{i_1}\dots\lambda_{i_k} \tag{6.22}$$

($k = 1, \dots, N$). The eigenvalues of the Lax matrix are $i^{-(M_a-1)}(-1)^\alpha V_a$, $a = 1, \dots n$, $\alpha = 0, 1, \dots M_a - 1$.

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