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Controllability of Sequential Machines

YAHIKO KAMBAYASHI

Department of Computer Science, University of Illinois, Urbana, Illinois 61801

AND

SHUZO YAJIMA

Department of Information Science, Kyoto University, Kyoto, Japan, 606

A sequential machine is said to be k-controllable if there exists a minimum integer k such that every state transition can be achieved by an input sequence of length k. In this paper properties of controllable sequential machines are discussed. The main results are as follows: (1) An efficient procedure for controllability test is given. (2) It is shown that every uncontrollable strongly connected machine has an autonomous component machine. (3) The upper bound of k in a controllable machine containing a loop of length l is given. (4) The upper and lower bounds of k in k-controllable machines are shown. (5) It is shown that any sequential machine can be realized by a controllable or uncontrollable machine.

1. INTRODUCTION

Recently several fundamental investigations have structured on a unified basis for sequential machines and continuous systems. For example, the studies by Kalman and Arbib (1969), Massey and Sain (1967) and Arimoto (1968) are concerned with such similar properties. We made a study on the controllability of sequential machines which is defined on analogy of the controllability of continuous systems.

A sequential machine is said to be controllable if there exists an integer k such that every state transition can be achieved by an input sequence of length k. Properties of controllable linear sequential machines were discussed by Cohn (1962) and Toda (1966). In this paper, we discuss properties of controllable nonlinear sequential machines. Some of the discussions are contained in our previous paper (1968).

In Section 2, basic definitions including a definition of k-controllability are given. Every controllable sequential machine is strongly connected, but the inverse of this statement is not always true.

In Section 3, basic properties of controllable sequential machines are discussed. As a result an efficient procedure to determine whether or not a sequential machine is controllable is given.

In Section 4, it is shown that every strongly connected uncontrollable sequential machine contains a strongly connected autonomous sequential machine as a component.

In Section 5, we discuss the controllability of a strongly connected graph which contains a loop of length l and the controllability of a graph which contains only two loops. Using these properties, the best upper and lower bounds of k in a k-controllable sequential machine are obtained.

Several relevant topics are discussed, such as (a) realizations of arbitrary sequential machines by controllable or uncontrollable machines and (b) controllable components of sequential machines in Section 6.

2. Definitions

A Mealy-type sequential machine M can be defined as a system consisting of S, X, Y, δ, and λ, i.e., $M = (S, X, Y, \delta, \lambda)$, where S is the set of a finite number of states, i.e., $S = \{s_1, s_2, ..., s_n\}$; X is the set of a finite number of input symbols, i.e., $X = \{x_1, x_2, ..., x_p\}$; Y is the set of a finite number of output symbols, i.e., $Y = \{y_1, y_2, ..., y_q\}$; and δ is the next state function and λ is the output function defined by the following equations:

$$s(t+1) = \delta(s(t), x(t)), \tag{1}$$

$$y(t) = \lambda(s(t), x(t)), \tag{2}$$

where $s(t)$, $x(t)$ and $y(t)$ are the state, the input and the output at time t (which is an integer). If we extend the state function δ and the output function λ over the set X^* of all the input sequences (including the input sequence of length 0) and the set Y^* of all the output sequences (including the output sequence of length 0), respectively, then we have the following:

$$s_i, \quad s_j \in S, \quad \omega \in X^*, \quad s_i = \delta(s_j, \omega), \tag{3}$$

$$s_j \in S, \quad \omega \in X^*, \quad \nu \in Y^*, \quad \nu = \lambda(s_j, \omega), \tag{4}$$

where

$$\| \omega \| = \| \nu \|,$$

and the notation $\| \omega \|$ designates the length of the sequence ω.

In continuous system, a machine is controllable if there exists an input segment which causes a transition from an arbitrary state at time t to an arbitrary state at time $t + t_i$ ($t_i > 0$). In the corresponding definition, a sequential machine is said to be controllable if there exists a nonnegative integer k such that every state transition can be achieved by an input sequence of length k.

DEFINITION 1. A sequential machine M is *weakly k-controllable* if there exists a nonnegative integer k such that

$$\forall s_i, \quad \forall s_j \in S, \quad \exists \omega \in X^*, \quad \| \omega \| = k, \quad s_i = \delta(s_j, \omega) \tag{5}$$

M is *controllable* if it is weakly k-controllable for some finite k.

From Definition 1,

LEMMA 2. *Every controllable machine is strongly connected.*

LEMMA 3. *When a sequential machine M is weakly k-controllable, it is also weakly $(k + i)$-controllable for any nonnegative integer i.*

Proof. For an arbitrary state s_{j0} and an arbitrary input sequence $\omega_1 (\in X^*, \| \omega_1 \| = i)$, there exists a state s_{j1} satisfying

$$s_{j1} = \delta(s_{j0}, \omega_1).$$

As M is weakly k-controllable,

$$\forall s_{j1}, \quad \forall s_{j2} \in S, \quad \exists \omega_2 \in X^*, \quad \| \omega_2 \| = k, \quad s_{j2} = \delta(s_{j0}, \omega_2).$$

Combining these equations we have

$$\forall s_{j0}, \quad \forall s_{j2} \in S, \quad \exists \omega_1\omega_2 \in X^*, \quad \| \omega_1\omega_2 \| = k + i, \quad s_{j2} = \delta(s_{j0}, \omega_1\omega_2).$$

Thus M is weakly $(k + i)$-controllable for any nonnegative integer i.
Q.E.D.

DEFINITION 4. A sequential machine M is k-controllable if and only if it is weakly k-controllable but not weakly $(k - 1)$-controllable, i.e., k is the least integer for which M is controllable.

Figure 1 shows an example of a 3-controllable sequential machine. When two arbitrary states s_i and s_j are given, there exists at least one input sequence ω of length 3 such that

$$s_j = \delta(s_i, \omega).$$

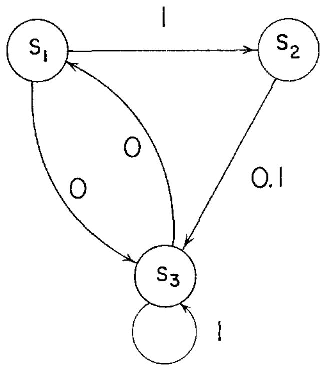


FIG. 1. 3-controllable sequential machine.

For example,

$$\begin{aligned} s_1 &= \delta(s_1, 010), & s_2 &= \delta(s_1, 001), \\ s_3 &= \delta(s_1, 011), & s_1 &= \delta(s_2, 010), \\ s_2 &= \delta(s_2, 001), & s_3 &= \delta(s_2, 011), \\ s_1 &= \delta(s_3, 110), & s_2 &= \delta(s_3, 101), \\ s_3 &= \delta(s_3, 111). \end{aligned}$$

The sequential machine in Fig. 2 is strongly connected but not controllable, for the transition from s_1 to s_2 cannot be achieved by an input sequence of even length and the transition from s_1 to s_3 cannot be achieved by an input sequence of odd length.

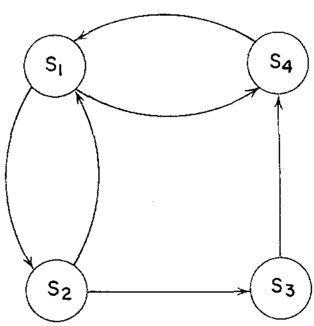


FIG. 2. Uncontrollable strongly connected machine.

A typical example of a sequential machine which is strongly connected but not controllable is a strongly connected autonomous sequential machine (for example see Fig. 3).

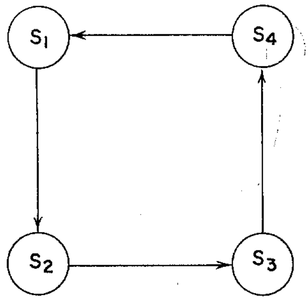


FIG. 3. Strongly connected autonomous sequential machine.

Crudely stated, machines of autonomous type are not controllable because they have no terminals for inputs of control signals. In Section 4, it will be shown that every strongly connected sequential machine which is not controllable contains a strongly connected autonomous sequential machine as a component. Thus the difference between strongly connectedness and controllability is due to whether or not a sequential machine contains an autonomous component.

3. Basic Properties of Controllable Sequential Machines

DEFINITION 5. A set $T_M(s_i, s_j)$ of integers for a sequential machine M is defined by

$$T_M(s_i, s_j) = \{\| \omega \| \mid s_j = \delta(s_i, \omega), \omega \in X^*\}. \tag{6}$$

It should be noted that $T_M(s_i, s_i)$ contains 0. If there exists no sequence which cause a transition from s_i to s_j then $T_M(s_i, s_j)$ is an empty set.

An infinite sequence $\{..., z_i, ...\}$ of elements is said to be ultimately periodic if there exist integers k and h such that $z_j = z_{j+k}$ for every integer $j \geqslant h$. A set $T = \{z_i\}$ ($z_i > z_j$ for $i > j$) of nonnegative integers is said to be ultimately periodic if either T is finite or $z_1, z_2 - z_1, ..., z_{i+1} - z_i, ...$ is an ultimately periodic sequence.

The following lemma is obtained from the property of regular sets.

LEMMA 6. *For any states s_i and s_j in a sequential machine M, $T_M(s_i, s_j)$ is either an ultimately periodic set of nonnegative integers or an empty set.*

THEOREM 7. *For an n-state sequential machine M the following four statements are equivalent.*

(a) *M is controllable.*

(b) *For every $s_i \in S$ there exists k_i such that $\bigcap_{j=1}^{n} T_M(s_i, s_j) \ni k_i$.*

(c) *M is strongly connected and there exists $k_0 \in \bigcap_{j=1}^{n} T_M(s_i, s_j)$ for some $s_i \in S$.*

(d) *M is strongly connected and $k_0, k_0 + 1, \ldots$ are contained in some $T_M(s_i, s_j)$ (s_i and s_j may be identical).*

Proof. We prove these statements in the order (a) → (b) → (c) → (d) → (a).

(a) → (b): If M is controllable then M is k-controllable for some k, so $\bigcap_{j=1}^{n} T_M(s_i, s_j)$ contains k for every state $s_i \in S$ (it is a special case of (b) such that every $k_i = k$).

(b) → (c): If (b) is satisfied, every $T_M(s_i, s_j)$ is a nonempty set, that is, M is strongly connected. Therefore, (c) holds (we can choose any s_i and $k_0 = k_i$).

(c) → (d): For an arbitrary state s_j there exist states $s_j = s_{j0}, s_{j1}, s_{j2}, \ldots$ such that

$$T_M(s_{jh}, s_j) \ni h \qquad (h = 0, 1, 2, \ldots).$$

If (c) is satisfied, for every s_{jh},

$$T_M(s_i, s_{jh}) \ni k_0.$$

Thus

$$T_M(s_i, s_j) \ni k_0 + h \qquad (h = 0, 1, 2, \ldots).$$

Thus (d) holds.

(d) → (a): The minimum element in $T_M(s_h, s_k)$ is denoted by $\min T_M(s_h, s_k)$. For two arbitrary states s_a and s_b, as M is strongly connected,

$$\min T_M(s_a, s_i) \leqslant n - 1,$$

$$\min T_M(s_j, s_b) \leqslant n - 1.$$

Thus

$$k_0 + 2n - 2 - \min T_M(s_a, s_i) - \min T_M(s_j, s_b) \geqslant k_0.$$

If (d) is satisfied,

$$T_M(s_i, s_j) \ni k_0 + 2n - 2 - \min T_M(s_a, s_i) - \min T_M(s_j, s_b).$$

By considering a path $s_a \rightarrow s_i \rightarrow s_j \rightarrow s_b$

$$\begin{aligned} T_M(s_a, s_b) \ni & \min T_M(s_a, s_i) \\ & + \{k_0 + 2n - 2 - \min T_M(s_a, s_i) - \min T_M(s_j, s_b)\} + \min T_M(s_j, s_b) \\ = & k_0 + 2n - 2. \end{aligned}$$

As states s_a and s_b are arbitrary, M is weakly $(k_0 + 2n - 2)$-controllable. Therefore, (a) holds. Q.E.D.

By the last part of the proof, the following corollary is obtained.

COROLLARY 8. *If an n-state sequential machine M is strongly connected and there exists a state s_i such that*

$$\exists x \in X, \qquad s_i = \delta(s_i, x), \qquad (\textit{i.e., } s_i \textit{ is an equilibrium state}),$$

then M is weakly $(2n - 2)$-controllable, and k-controllable for some $k \leqslant (2n - 2)$.

A sequential machine is said to be k-input memory, if k is the least integer satisfying

$$y(t) = f(x(t), x(t-1), \ldots, x(t-k))$$

The following lemma is obvious.

LEMMA 9. *A reduced k-input memory sequential machine is k-controllable.*

DEFINITION 10. A transition matrix $C_M = \{c_{ij}\}$ corresponding to a sequential machine M is defined as follows.

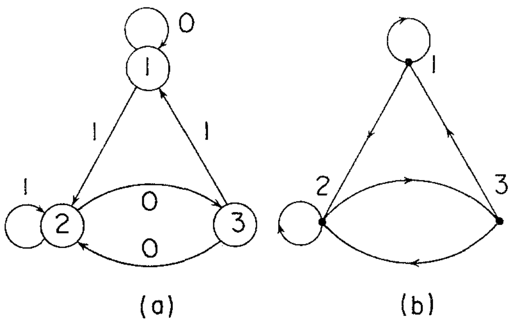
$$\begin{aligned} \exists x \in X, s_j = \delta(s_i, x) \Rightarrow c_{ij} = 1, \\ \forall x \in X, s_j \neq \delta(s_i, x) \Rightarrow c_{ij} = 0. \end{aligned} \tag{7}$$

The oriented graph corresponding to a transition diagram of a sequential machine M is denoted by G_M. Figure 4 shows a transition diagram of M, its corresponding G_M and C_M.

If M is controllable, G_M is said to be a controllable graph. For example, G_M in Fig. 4 (b) is controllable.

Let G be a graph with a node set N and an edge set E. A set E_M of edges for $M \subset N$ is

$$\begin{aligned} E_M = \{e \mid e \in E, e \text{ incidents out from a node } x \in M \\ \text{and incidents into a node } y \in M\}. \end{aligned}$$



$$\begin{array}{c|ccc|} & 1 & 2 & 3 \\ 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 3 & 1 & 1 & 0 \end{array}$$

(c)

FIG. 4. Sequential machine M and its G_M and C_M.

A subgraph of a graph G is defined to be a graph with a node set $M \subset N$ and an edge set $F \subset E_M$. The following result is stronger than Corollary 8.

COROLLARY 11. *If a strongly connected graph G contains a controllable subgraph, G is controllable.*

DEFINITION 12. The matrix product of $C_M^{(1)} = \{c_{ij}^{(1)}\}$ and $C_M^{(2)} = \{c_{ij}^{(2)}\}$ of order n is defined by

$$C_M^{(1)} C_M^{(2)} = \left\{ \bigcup_k c_{ik}^{(1)} \cdot c_{kj}^{(2)} \right\}, \tag{8}$$

where $\cdot$ means a logical product and $\bigcup_k$ means a logical summation over $1 \leqslant k \leqslant n$.

We designate $C_M C_M$ by $C_M{}^2$, etc. C_M is convergent in its powers if there exists k such that

$$C_M{}^k = C_M^{k+1} = \cdots . \tag{9}$$

C_M is oscillatory with period t in its powers if it is not convergent and there exists an integer h and the least integer $t \geqslant 2$ such that

$$C_M{}^k = C_M^{k+t}, \tag{10}$$

for all $k \geqslant h$. As a number of transition matrices of order h is at most 2^{n^2}, C_M is either convergent or oscillatory in its powers.

A matrix whose elements are all 1's is denoted by I.

LEMMA 13. *A necessary and sufficient condition that a sequential machine M is k-controllable is that*

$$\begin{aligned} &C_M{}^k = I, \\ &C_M^{k-j} \neq I, \qquad j \geqslant 1 : \text{integer} \end{aligned} \tag{11}$$

such that $k - j$ is nonnegative.

THEOREM 14. *A necessary and sufficient condition that a sequential machine M is controllable is that M is strongly connected and that C_M is convergent in its powers.*

Proof. If M is not strongly connected, it is not controllable by Lemma 2. If C_M is not convergent in its powers, there exists no k such that $C_M{}^k = C_M^{k+1} = \cdots = I$.

This implies that M is not controllable by Lemma 13.

Conversely, if M is strongly connected and C_M is convergent in its powers,

$$C_M \cup C_M{}^2 \cup \cdots \cup C_M{}^n = I \qquad \text{(strongly connectedness)},$$

$$C_M{}^k = C_M^{k+1} = \cdots = V \qquad \text{(convergent)}.$$

Thus

$$\begin{aligned} V &= C_M{}^k \cup C_M^{k+1} \cup \cdots \cup C_M^{k+n-1} \\ &= C_M^{k-1}(C_M \cup C_M{}^2 \cup \cdots \cup C_M{}^n) = C_M^{k-1} I = I. \end{aligned}$$

As M is strongly connected, every row of C_M^{k-1} contains at least one nonzero element. This implies

$$C_M^{k-1} I = I.$$

Thus

$$V = I = C_M{}^k.$$

From Lemma 13, M is controllable. Q.E.D.

A path in a graph is a sequence $(e_1, e_2, \ldots)$ of edges such that the terminal node of each edge coincides with the initial node of the succeeding edge. A closed path is a finite path in which the initial node of the first edge coincides with the terminal node of the last edge. A loop is a closed path and every vertex which it meets is distinct apart from the coincident initial and terminal vertices. A self-loop is a loop consisting of only one edge.

THEOREM 15. *A sequential machine M is controllable if and only if it is strongly connected and a greatest common divisor d of lengths of all loops in G_M is* 1.

Proof. If C_M is oscillatory with period t in its powers, by a proper permutation matrix P

$$PC_MP' = \begin{pmatrix} 0 & C_{M1} & 0 & \cdots & 0 \\ 0 & 0 & C_{M2} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & C_{Mt-1} \\ C_{Mt} & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where P' is a transposition of P.

This implies that d is a multiple of t. If M is strongly connected and $d = 1$, t must be 1. By Theorem 14, M is controllable. Conversely, if M is controllable, it is strongly connected and from Theorem 7 $T_M(s_i, s_j)$ contains k_i and $k_i + 1$ for an arbitrary s_i. Thus there exist two closed paths Q_1 and Q_2 such that

$$L(Q_1) + 1 = L(Q_2), \tag{i}$$

where $L(Q)$ is a length of a path Q. If d is a greatest common divisor of lengths of all loops in G_M, it can be easily verified that every closed path Q in G_M satisfies

$$L(Q) \equiv O(\text{mod } d).$$

If there exist two paths Q_1 and Q_2 satisfying (i),

$$L(Q_1) \equiv O(\text{mod } d)$$

$$L(Q_2) \equiv L(Q_1) + 1 \equiv O(\text{mod } d).$$

This implies $d = 1$. Q.E.D.

In an n-state machine the upper bound of length of loops is n, so a greatest common divisor of lengths of all loops is 1 if and only if a greatest common divisor of lengths of all closed paths of lengths not greater than n is 1. We have the following procedure of controllability test:

Procedure 16. Controllability test

(i) $r = 1$, $a = 0$, $b = 0$, $B_0 = 0$ (a matrix of order n whose elements are all 0).

(ii) Calculate $C_M{}^r = C_M C_M^{r-1} = \{c_{ij}^{(r)}\}$. If $a = 0$ go to step (iii), otherwise go to step (iv).

(iii) $B_r = C_M{}^r \cup B_{r-1}$. If $B_r = I$ set $a = 1$. If $b = 1$ go to step (vi), otherwise go to step (iv).

(iv) $g_r = \bigcup_{i=1}^{n} c_{ii}^{(r)}$. If $g_r = 1$ go to step (v), otherwise go to step (vii).

(v) If $b = 0$, set $b = r$ otherwise set a new value of b to be a greatest common divisor of b and r. If $b = 1$ go to step (vi) otherwise go to step (vii).

(vi) If $a = 0$ go to step (vii), otherwise go to step (ix).

(vii) If $r < n$ set a new value of r to be $r + 1$ and go to step (ii). Otherwise go to step (viii).

(viii) The given sequential machine M is not controllable.

(ix) The given sequential machine is controllable.

Here,

B_r $B_r = C_M{}^r \cup C_M^{r-1} \cup \cdots \cup C_M$.

a If $B_r = I$ then $a = 1$ otherwise $a = 0$, so if $a = 1$ then M is strongly connected.

g_r $g_r = 1$ if and only if there exists at least one closed path of length r.

b A greatest common divisor of lengths of closed paths whose lengths are not greater than r.

If $a = 1$ and $b = 1$ for some r $(1 \leqslant r \leqslant n)$ then M is controllable.

4. Decomposition of a Strongly Connected Uncontrollable Machine

In this section it will be shown that a strongly connected uncontrollable machine contains an autonomous component.

A partition π on S is a collection of disjoint subsets of S whose set union is S.

DEFINITION 17 [7]. A partition π on the set of states of a sequential machine $M = (S, X, Y, \delta, \lambda)$ is said to have the substitution property (S.P.) if and only if

$$s_1 \equiv s_2(\pi)$$

implies that

$$\forall x \in X, \delta(s_1, x) \equiv \delta(s_2, x)(\pi), \tag{12}$$

where $s_1 \equiv s_2(\pi)$ means that s_1 and s_2 are contained in the same block of π.

The product of two partitions π and τ is defined as follows:

$$s_1 \equiv s_2(\pi \cdot \tau),$$

if and only if

$$s_1 \equiv s_2(\pi), \qquad s_1 \equiv s_2(\tau).$$

If every block of a partition π contains only one element, π is said to be 0 partition denoted by 0. A partition π is nontrivial if it is composed of at least two blocks. A partition π with the substitution property is said to be input independent if

$$\forall s \in S, \forall x_1, \forall x_2 \in X, \delta(s, x_1) \equiv \delta(s, x_2)(\pi) \tag{13}$$

LEMMA 18. *In a strongly connected uncontrollable sequential machine $M = (S, X, Y, \delta, \lambda)$, there exists a nontrivial input independent partition π on S with* S.P.

Proof. As M is strongly connected but not controllable, the greatest common divisor of lengths of all loops in G_M is $d \neq 1$. We divide S into the following blocks by the distances from the state s_1.

$$\begin{aligned} B_1 &= \{s \mid \omega \in X^*, \mid \omega \mid = 0 \pmod d, \delta(s_1, \omega) = s\}, \\ B_2 &= \{s \mid \omega \in X^*, \mid \omega \mid = 1 \pmod d, \delta(s_1, \omega) = s\}, \\ &\vdots \\ B_d &= \{s \mid \omega \in X^*, \mid \omega \mid = d - 1 \pmod d, \delta(s_1, \omega) = s\}. \end{aligned} \tag{14}$$

The partition

$$\pi = \{B_1 ; B_2 ; \cdots ; B_d\}$$

is input independent and nontrivial because $d \neq 1$. Q.E.D.

It two partitions π and τ on S satisfy the following condition, M can be realized by a serial connection of the corresponding machines M_π and M_τ.

$$\text{(i)} \quad \pi \cdot \tau = 0 \qquad \text{(ii)} \quad \pi \text{ has S.P.}$$

In the case when π is nontrivial input independent partition, M_π is an autonomous machine with at least two states.

The partition π given in the proof of Lemma 18 satisfies the above conditions. τ is determined in the following manner:

We denote the j-th element of B_i by b_{ij} (if the number of element in B_i (denoted by $| B_i |$) is less than j, $b_{ij} = \phi$).

$$\begin{aligned} T_j &= \{b_{1j}, b_{2j}, \dots, b_{dj}\} \qquad (j = 1, \dots, h), \\ \tau &= \{T_1; T_2; \cdots; T_h\}. \end{aligned} \tag{15}$$

Using these π and τ, the following theorem is obtained.

THEOREM 19. *A strongly connected uncontrollable sequential machine M can be realized by a serial connection of an autonomous strongly connected sequential machine M_π with at least two states and a controllable sequential machine M_τ (see Fig. 5).*

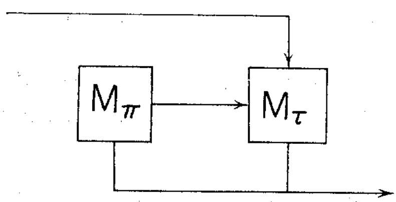


FIG. 5. Decomposition of a strongly connected uncontrollable machine.

M_τ is controllable because the greatest common divisor of lengths of all loops in G_M is 1. This result means that the difference between a strongly connected sequential machine and a controllable sequential machine is due to whether or not a machine contains an autonomous component (i.e., an autonomous component can not be controlled by the input). As in autonomous linear sequential machines the next state of the state (0, 0,..., 0) is also itself, there exist no strongly connected linear autonomous sequential machines with at least two states. If a linear sequential machine is realized by a diagram is Fig. 5 and M_π is an autonomous linear sequential machine with at least two states, the whole machine is not strongly connected. This implies that there exist no linear sequential machines which is strongly connected but not controllable, i.e., in linear sequential machines strongly connectedness and controllability are equivalent (it was shown by Cohn (1962) by a different way).

An example of a sequential machine which is strongly connected but not controllable is shown Table I(a). As the greatest common divisor of lengths of all loops in G_M is 2, the partition π is

$$\pi = \{1, 3; 2, 4\} = \{B_1 ; B_2\},$$

TABLE I

State	Input 0	Input 1
1	2	4
2	3	3
3	4	4
4	1	3

Next state

(a)

	0	1
(B_1, T_1)	(B_2, T_1)	(B_2, T_2)
(B_2, T_1)	(B_1, T_2)	(B_1, T_2)
(B_1, T_2)	(B_2, T_2)	(B_2, T_2)
(B_2, T_2)	(B_1, T_1)	(B_1, T_2)

(b)

	0	1
B_1	B_2	B_2
B_2	B_1	B_1

(c)

	$B_1, 0$	$B_1, 1$	$B_2, 0$	$B_2, 1$
T_1	T_1	T_2	T_2	T_2
T_2	T_2	T_2	T_1	T_2

(d)

and the partition τ is

$$\tau = \{1, 2; 3, 4\} = \{T_1 ; T_2\}.$$

States 1, 2, 3 and 4 are represented by combinations of blocks (B_1, T_1), (B_2, T_1), (B_1, T_2) and (B_2, T_2), respectively. By replacing states in Table I(a) by these combinations of blocks we can obtain Table I(b). By removing T_1 and T_2 from this table and simplify the resulting table, Table I(c) for M_π is obtained. M_τ in Table I(d) can be obtained from Table I(b). For example, as in Table I(b) an input 0 causes a transition from (B_2, T_1) to (B_1, T_2), in M_τ an input $(B_2, 0)$ causes a transition from T_1 to T_2. M_π is a two-state autonomous sequential machine and M_τ is a controllable sequential machine.

5. Controllability of Several Graphs and Bounds of k in k-Controllable Sequential Machines

In this section we discuss controllability of a graph which contains a loop of length l, that of a graph which contains only two loops l_1 and l_2 and the best bounds of k in k-controllable sequential machines.

A function β is defined by

$$S_a = \beta(u), \tag{16}$$

where S_a is a set of nodes to which there exist edges from node u. β is generalized as follows.

$$\beta(S_b) = \{S_a \mid S_a = \beta(u), \forall u \in S_b\}, \tag{17}$$

$$\beta^i(S_b) = \beta(\beta^{i-1}(S_b)). \tag{18}$$

Theorem 20. *If an n-node graph G contains a loop of length l, and it is controllable, then G is weakly $\{(n-2)\,l+n+1\}$-controllable, and k-controllable for some $k \leqslant (n-2)\,l+n+1$.*

Proof. Without loss of generality, we assume that names of nodes in L are $u_1, u_2, \ldots, u_{l-1}$ and u_l such that

$$u_{i+1} \in \beta(u_i) \qquad (1 \leqslant i \leqslant l-1),$$
$$u_1 \in \beta(u_l),$$

and that names of nodes not in L are $u_{l+1}, \ldots, u_{n-1}$ and u_n. Consider the following sequence:

$$u_1 = \beta^0(u_1), \beta(u_1), \ldots, \beta^k(u_1) = \{u_1, \ldots, u_n\},$$
$$\beta^i(u_1) \neq \beta^k(u_1) \qquad \text{for} \quad i < k.$$

Clearly,

$$u_1 \in \beta^l(u_1).$$

Thus for any nonnegative integer $j \leqslant k-l$

$$\beta^{j+l}(u_1) = \beta^j(\beta^l(u_1)) \supset \beta^j(u_1). \tag{i}$$

If $\beta^i(u_1) = \beta^j(u_1)$ for some $i \neq j$, the sequence $\beta^0(u_1), \beta(u_1), \ldots, \beta^k(u_1)$ will be convergent or oscillatory, which contradicts

$$\beta^i(u_1) \neq \beta^k(u_1) \qquad (i < k).$$

Thus

$$\beta^i(u_1) \neq \beta^j(u_1) \qquad \text{for all} \quad i \neq j, \quad i, j \leqslant k. \tag{ii}$$

From (i) and (ii) the following relation holds

$$\beta^j(u_1) \subseteq \beta^{j+l}(u_1), \tag{iii}$$

where j is an integer such that $0 \leqslant j \leqslant k - l$.

This implies

$$|\beta^j(u_1)| < |\beta^{j+l}(u_1)| < n, \qquad 0 \leqslant j < k - l.$$

Thus the number of $\beta^j(u_1)$'s which consist of i nodes $(1 \leqslant i \leqslant n - 1)$ is at most l. As $|\beta^k(u_1)| = n$, k satisfies

$$k \leqslant (n-1)\, l + 1.$$

As u_1 may be replaced by any node in L, there exists a path of length at most $(n-1)\, l + 1$ from any node in L to any node in G. Clearly, there exists a path of length at most $n - l$ from any node in G to some node in L. That is, from Theorem 7, G is weakly $\{(n-2)\, l + n + 1\}$-controllable, and it follows from Definition 4 that it is also k-controllable for some $k \leqslant (n-2)\, l + n + 1$. Q.E.D.

In a controllable graph which contains more than two nodes there exist at least two loops. The graph in Fig. 6 consists of two loops L_1 and L_2 of lengths l_1 and l_2 with l_3 succeeding nodes in common $(l_1 \geqslant l_2)$.

THEOREM 21. *A graph G_M in Fig. 6 is controllable if and only if l_1 and l_2 are relatively prime. If G_M is controllable, it is $\{(l_1 - 1)\, l_2 + l_1 - l_3\}$-controllable $(l_1 \geqslant l_2)$.*

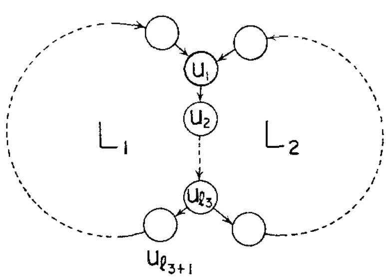


FIG. 6. Graph G_M.

Proof. From Theorem 15, G_M is controllable if and only if l_1 and l_2 are relatively prime. Now we show for any $j(0 \leqslant j \leqslant l_1 - 1)$, there exist nonnegative integers i_1 and i_2 such that

$$(l_1 - 1)\, l_2 = i_1 l_1 + i_2 l_2 + j. \tag{i}$$

When a multiple of l_2 is subtracted from $(l_1 - 1)l_2$, the difference is

$$\begin{array}{lll} 0 & \text{when} & (l_1 - 1)l_2 \quad \text{is subtracted,} \\ l_2 & & (l_1 - 2)l_2\,, \\ \vdots & & \vdots \\ (l_1 - 1)l_2 & & 0. \end{array}$$

As l_1 and l_2 are relatively prime, each difference belongs to a different residue class of modulus l_1, that is, when divided by l_1, the remainders are $0, 1,\ldots, l_1 - 2$ and $l_1 - 1$.

Thus, (i) is proved, and the same statement is not true for any $l < (l_1 - 1)l_2$. This implies that there exists a path of length $(l_1 - 1)l_2$ from the node u_i $(1 \leqslant i \leqslant l_3)$ to an arbitrary node.

The distance from the node u_{l_3+1} to the node u_1 is $l_1 - l_3$ and it is the longest path from a node in G_M to some node u_i $(1 \leqslant i \leqslant l_3)$. Thus G_M is $\{(l_1 - 1)\, l_2 + l_1 - l_3\}$-controllable from the fact that $(l_1 - 1)l_2$ is the minimum number which satisfies (i). Q.E.D.

COROLLARY 22. *There exists an n-state* $(n^2 - 2n + 2)$*-controllable sequential machine for any* $n \geqslant 2$.

Proof. When $l_1 = n$ and $l_2 = l_3 = n - 1$, the value of $\{(l_1 - 1)\, l_2 + l_1 - l_3\}$ is $n^2 - 2n + 2$, which is maximum as easily verified. Q.E.D.

THEOREM 23. *If an n-state sequential machine with p input symbols and q output symbols is k-controllable, k satisfies,*

$$\begin{array}{ll} \ulcorner \log_p n \urcorner \leqslant k \leqslant n^2 - 2n + 2 & (n \geqslant 2) \\ k = 0 & (n = 1) \end{array} \tag{19}$$

Proof. As the number of all possible sequences of length k is p^k,

$$p^k \geqslant n.$$

This implies the lower bound.

If a graph G_M contains a loop of length at most $n - 2 (n \geqslant 3)$, by Theorem 20 it is weakly $(n^2 - 3n + 5)$-controllable. If a controllable graph G_M contains no loop of length less than $n - 1$, it must contain two loops of length n and $n - 1$. By Corollary 22 such a graph is $(n^2 - 2n + 2)$-controllable. For $n \geqslant 3$,

$$n^2 - 2n + 2 \geqslant n^2 - 3n + 5.$$

Thus the upper bound is proved for $n \geqslant 3$. It can be easily verified that this bound is also valid for $n = 2$. Obviously, one-state sequential machine is 0-controllable. Q.E.D.

The existence of the machines which satisfy the upper bound is shown in Corollary 22. The existence of the machines which satisfy the lower bound is shown in the next theorem.

THEOREM 24. *For any n and any p ($\geqslant 2$), there exists a $k(=\lceil \log_p n \rceil)$-controllable n-state sequential machine with p input symbols.*

Proof. Let us consider a shift register machine shown in Fig. 7. As the length of the register is k, this machine is k-controllable. As each combination

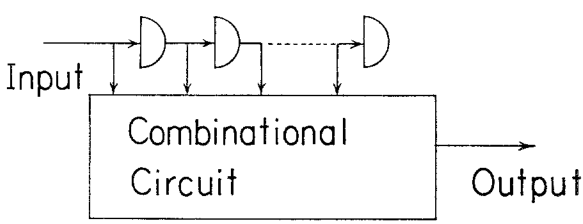


FIG. 7. Minimum controllable machine.

of variables in registers corresponds to each state, the number of states is p^k. As the output of the rightmost register does not connected to the combinational circuit, each state has $p - 1$ equivalent states. There are p^{n-1} equivalence classes such that each equivalence class consists of p states $(a_1, a_2, ..., a_{k-1}, 0)$, $(a_1, a_2, ..., a_{k-1}, 1), ..., (a_1, a_2, ..., a_{k-1}, p - 1)$. If s_i and s_j are in the same equivalence class then for all x in X,

$$\delta(s_i, x) = \delta(s_j, x),$$

$$\lambda(s_i, x) = \lambda(s_j, x).$$

So we can merge these two states. Thus we can replace p states in the same equivalent class by p' states ($1 \leqslant p' \leqslant p$) by combining equivalent states. For any n ($p^{k-1} < n \leqslant p^k$) we can obtain a sequential machine of n states by this method. The n-node graph corresponding to such a machine is $\lceil \log_p n \rceil$-controllable.

An example of transition graphs for $p = 2$ and $k = 3$ is shown in Fig. 8.

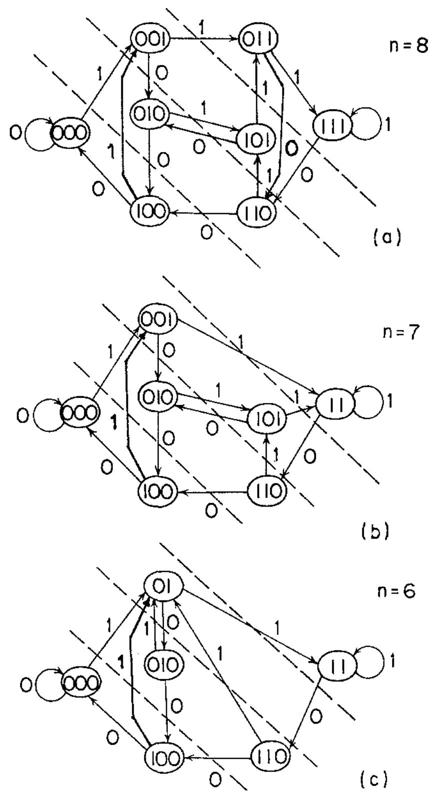


FIG. 8. Examples of minimum controllable machines.

Any pair of states in the same region divided by dotted lines are equivalent in Fig. 8(a), and we can regard any such pair as one state. Thus, for instance, we have 3-controllable machines in Fig. 8(b) and (c) for $n = 7$ and $n = 6$, respectively. Q.E.D.

6. Other Topics

In this section, realizations of arbitrary sequential machines by controllable or uncontrollable sequential machines and controllable components are discussed.

Definition 25. A sequential machine $M = (S, X, Y, \delta, \lambda)$ is said to be realized by $M' = (S', X', Y', \delta', \lambda')$, if there exist onto mappings

$$\phi : S'' \to S, \qquad \psi : X'' \to X \qquad \text{and} \qquad \xi : Y'' \to Y$$

such that

$$\text{(i)} \qquad S'' \subset S', \quad X'' \subset X', \quad Y'' \subset Y',$$

$$\text{(ii)} \qquad s_1, \quad s_2 \in S, \quad x \in X, \quad \delta(s_1, x) = s_2,$$
$$\Downarrow$$
$$s_a \in \phi^{-1}(s_1), \quad \forall s_b \in \phi^{-1}(s_2), \quad \forall x' \in \psi^{-1}(x),$$
$$\delta'(s_a, x') = s_b,$$

$$\text{(iii)} \qquad s_1 \in S, \quad x \in X, \quad y \in Y,$$
$$\lambda(s_1, x) = y$$
$$\Downarrow$$
$$\forall s_a \in \phi^{-1}(s_1), \quad \forall x' \in \psi^{-1}(x)$$
$$\xi(\lambda(s_a, x')) = y.$$

Theorem 26. *For an arbitrary sequential machine M, a controllable sequential machine M' which realizes M can be constructed by adding one input symbol.*

Proof. We can construct $G_M{}'$ which contains a selfloop and a loop which passes through all nodes using edges labeled by a new input symbol and at least one edge to connect the node with selfloop to the new large loop. By Corollary 8, G_M is controllable. Q.E.D.

As a controllable sequential machine is strongly connected, by adding only one state which is isolated the resulting machine realizes the original one and is not controllable.

Theorem 27. *For every controllable sequential machine M, there exists a sequential machine M' which is not controllable and realizes M such that $|S'| = |S| + 1$. Here $|S|$ is the number of elements in the set S.*

THEOREM 28. *For every controllable sequential machine M, there exists a strongly connected sequential machine M' which is not controllable and realizes M such that $|S'| = 2|S|$.*

The sequential machine M' in this theorem is constructed by a direct product of M and a strongly connected autonomous two-state sequential machine (see Section 4). Theorem 28 shows that the controllability is not preserved under state splitting. But this property is preserved under state reducing (procedure for reducing equivalent states).

THEOREM 29. *Controllability of a sequential machine is preserved under state reducing.*

A collection of subsets $\pi = \{B_i\}$ of S is called a cover on S if and only if

$$\bigcup B_i = S \qquad B_i \subset B_j \quad \text{implies } i = j. \tag{20}$$

If π_1 and π_2 are covers on S, we write

$$\pi_1 \geqslant \pi_2 \tag{21}$$

if and only if for each B_j in π_2 there exists a B_h' in π, such that

$$B_h' \supset B_j .$$

DEFINITION 30. A cover π on S is said to be a controllable component of a sequential machine M, if $\pi = \{B_i\}$ satisfies

$$\forall B_i , \quad \forall B_j , \quad \exists s_a \in B_i , \quad \exists s_b \in B_j , \quad \exists \omega \in X^*, \quad \| \omega \| = k, \quad s_b = \delta(s_a , \omega).$$

LEMMA 31. *If π is a controllable component of M, π' $(\geqslant \pi)$ is also a controllable component of M.*

A trivial cover $\pi = \{S\}$ is always a controllable component of M.

CONCLUSION

During the preparation of this paper, Mowle (1970) published a paper on controllable sequential machines. In his paper, procedures of controllability test are given. Procedure I of his paper uses the properties of Corollary 8 and Lemma 13 of this paper. Procedure II of his paper requires searching for two closed paths with relative prime lengths. As the lengths of these closed paths are not bounded by the order of n, our procedure given in the last of Section 3 is easier. For example, the 15-node graph shown in Fig. 9 contains

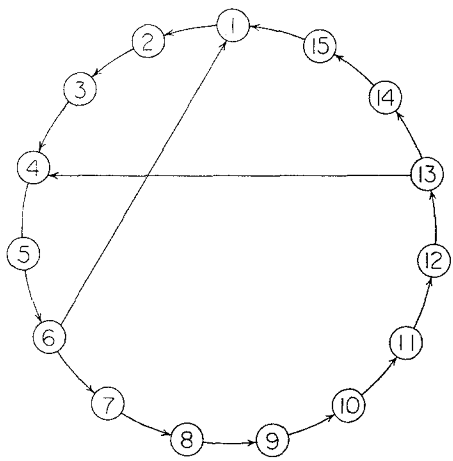


FIG. 9. Example.

loops of lengths 6, 10 and 15. The greatest common divisor of these integers is 1 so we can conclude the graph is controllable by Theorem 15. For an n-node graph we need not check closed paths of length greater than n. By the procedure proposed by Mowle (1970) we have to find two closed paths with relative prime lengths. In this case we can find such paths of lengths 15 and 16. Usually the upper bound of lengths of such closed paths necessary to check is much greater than n.

Properties of observable sequential machines are discussed in Kambayashi and Yajima (1971).

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