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A new characterization of nonisotropic chaotic vibrations of the one-dimensional linear wave equation with a van der Pol boundary condition

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Abstract

The one-dimensional linear wave equation with a van der Pol nonlinear boundary condition is one of the simplest models that may cause isotropic or nonisotropic chaotic vibrations (Trans. Amer. Math. Soc. 350 (1998) 4265–4311, Internat. J. Bifur. Chaos 8 (1998) 423–445, Internat. J. Bifur. Chaos 8 (1998) 447–470, J. Math. Phys. 39 (1998) 6459–6489, Internat. J. Bifur. Chaos 12 (2002) 535–559). In this paper, we characterize nonisotropic chaotic vibration by means of the total variation theory. We obtain the classification results on the growth of the total variation of the snapshots on the spatial interval in the long-time horizon with respect to two parameters entering different regimes in R^2.

1. Introduction

There has been increasing interest in the phenomena of chaos in mechanics and physics in the last two decades. Chaos has been observed in many mechanics and electronic circuits systems, but it is challenging to give rigorously mathematical proofs, especially for the systems governed by partial differential equations. In their series of papers [4–7], Chen et al. first studied chaotic vibrations of one-dimensional (1D) wave equation on a bounded interval with a van der Pol boundary condition. They have rigorously proven the existence

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of isotropic chaotic vibrations when a parameter enters a certain regime. Recently, Chen et al. [8] discussed nonisotropic spatiotemporal chaotic vibrations of the 1D wave equation with mixing energy transport and a van der Pol boundary condition. The aforementioned works are all for the simplest infinite-dimensional systems that may admit chaos. However, those works can provide some motivation and directions for the future study of chaos for more general partial differential equation systems and for the recently emerged study of anticontrol (cf. [3,11–13] and references therein).

We point out that there are several alternative definitions of chaos, each of which reflects its own background in its appropriate setting. Recently, Chen et al. [9] first characterized the chaotic oscillation of the same 1D wave equation as in [5] by means of the unbounded growth of total variations. In an earlier work [14], the author classified the growth rates of total variations of the snapshots of the Riemann invariants for the same system.

In this paper, we consider the initial-boundary value problem

$$\begin{cases} w_{xx}(x,t) - \nu w_{xt}(x,t) - w_{tt}(x,t) = 0, & 0 < x < 1,\ t > 0, \\ w_x(0,t) = 0, & t > 0, \\ w_x(1,t) = \alpha w_t(1,t) - \beta w_t^3(1,t), & \alpha, \beta > 0,\ t > 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), & 0 < x < 1. \end{cases} \tag{1.1}$$

We will study the dynamical behavior of this system by means of total variation theory as the two parameters (ν, α) vary in $[0, +\infty) \times [0, +\infty)$.

This paper can be viewed as a continuation of the earlier work [14]. But the difference here is twofold: (i) we will classify the growth of the total variations of the snapshots of the Riemann invariants of (1.1) in the long time horizon as the two parameters (ν, α) vary in the plane instead of just varying one parameter; (ii) the mixed partial derivative linear energy transport term in Eq. (1.1) can lead to strong mixing of waves and nonisotropic spatiotemporal chaos, contrary to the isotropic case in [14].

As in [8], we let

$$\rho_1(\nu) \equiv \frac{-\nu + \sqrt{4+\nu^2}}{2}, \tag{1.2}$$

$$\rho_2(\nu) \equiv \frac{\nu + \sqrt{4+\nu^2}}{2}. \tag{1.3}$$

We then have

$$\rho_1(\nu)\rho_2(\nu) = 1, \quad \rho_2(\nu) - \rho_1(\nu) = \nu > 0, \quad \rho_1(\nu) + \rho_2(\nu) = \sqrt{4+\nu^2}. \tag{1.4}$$

Letting

$$\begin{cases} u = \frac{1}{\rho_1(\nu)+\rho_2(\nu)}[\rho_2(\nu) w_x + w_t], \\ v = \frac{1}{\rho_1(\nu)+\rho_2(\nu)}[\rho_1(\nu) w_x - w_t], \end{cases} \tag{1.5}$$

we can convert $(1.1)_1$ into the equivalent uncoupled first order hyperbolic system

$$\frac{\partial}{\partial t}\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} = \begin{bmatrix} \rho_1(\nu) & 0 \\ 0 & -\rho_2(\nu) \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix}, \quad 0 < x < 1,\ t > 0. \tag{1.6}$$

The initial conditions for u and v are

$$\begin{cases} u(x,0) = u_0(x) = \frac{1}{\rho_1(\nu)+\rho_2(\nu)}[\rho_2(\nu) w_0'(x) + w_1(x)], \\ v(x,0) = v_0(x) = \frac{1}{\rho_1(\nu)+\rho_2(\nu)}[\rho_1(\nu) w_0'(x) - w_1(x)]. \end{cases} \tag{1.7}$$

The boundary condition at $x = 0$ is

$$v(0,t) = -u(0,t) \equiv G\big(u(0,t)\big), \quad t > 0, \tag{1.8}$$

and the boundary condition at $x = 1$ is

$$u(1,t) = F_{\nu,\alpha}\big(v(1,t)\big). \tag{1.9}$$

For given $x \in R$,

$$F_{\nu,\alpha}(x) = \rho_2\big[\rho_2 x + g_{\nu,\alpha}(x)\big], \tag{1.10}$$

where $y = g_{\nu,\alpha}(x)$ is the unique real solution of the cubic equation

$$\beta y^3 + (\rho_2 - \alpha)y + \big(\rho_2^2 + 1\big)x = 0, \tag{1.11}$$

provided that

$$0 < \alpha \leqslant \rho_2 \ \big(= \rho_2(\nu)\big). \tag{1.12}$$

Since the parameter β in the equation only plays the role of "scaling" (see [8]), it does not affect the properties of the functions $g_{\nu,\alpha}$ and $F_{\nu,\alpha}$. Thus we can view $g_{\nu,\alpha}$ and $F_{\nu,\alpha}$ as functions only dependent on the two parameters ν and α.

If (1.12) is violated, then $g_{\nu,\alpha}(x)$ is multi-valued. From now on, we always assume that $\alpha, \beta, \nu > 0$ satisfy condition (1.12). By the method of characteristics, the solution u and v of (1.6)–(1.9) can be expressed explicitly as follows: for $t = k(\rho_1 + \rho_2) + \tau$, $k = 0, 1, 2, \ldots,$ $0 \leqslant \tau \leqslant \rho_1 + \rho_2$, and $0 \leqslant x \leqslant 1$,

$$u(x,t) = \begin{cases} (F_{\nu,\alpha} \circ G)^k(u_0(x + \rho_1\tau)), \quad \tau \leqslant \rho_2(1-x), \\ F_{\nu,\alpha} \circ (G \circ F_{\nu,\alpha})^k(v_0(1 + \rho_2^2 - \rho_2^2(x + \rho_1\tau))), \\ \quad \rho_2(1-x) < \tau \leqslant \rho_2(1 + \rho_1^2 - x), \\ (F_{\nu,\alpha} \circ G)^{k+1}(u_0(x + \rho_1\tau - 1 - \rho_1^2)), \\ \quad \rho_2(1 + \rho_1^2 - x) < \tau \leqslant \rho_1 + \rho_2, \end{cases} \tag{1.13}$$

and

$$v(x,t) = \begin{cases} (G \circ F_{\nu,\alpha})^k(v_0(x - \rho_2\tau)), \quad \tau \leqslant \rho_1 x, \\ G \circ (F_{\nu,\alpha} \circ G)^k(u_0(-\rho_1^2(x - \rho_2\tau))), \\ \quad \rho_1 x < \tau \leqslant \rho_1(x + \rho_2^2), \\ (F_{\nu,\alpha} \circ G)^{k+1}(v_0(x - \rho_2\tau + 1 + \rho_2^2)), \\ \quad \rho_1(x + \rho_2^2) < \tau \leqslant \rho_1 + \rho_2. \end{cases} \tag{1.14}$$

Here, for example, $(G \circ F_{\nu,\alpha})^k$ denotes the k-fold iterative composition of $G \circ F_{\nu,\alpha}$.

From these explicit representations, we can estimate the growth rates of the total variations of $u(\cdot,t)$ and $v(\cdot,t)$ on $[0,1]$ as t goes to infinity by means of the estimation of those of $(G \circ F_{\nu,\alpha})^n(\cdot)$ and $(F_{\nu,\alpha})^n(\cdot)$ on some spatial intervals as n goes to infinity.

The organization of the paper is as follows. In Section 2, we list some basic properties of the map $G \circ F_{\nu,\alpha}$ and find invariant intervals of $G \circ F_{\nu,\alpha}(\cdot)$ when the parameters ν and α vary in R^+. In Section 3, we first estimate the growth rate of the total variations of the iterates $(G \circ F_{\nu,\alpha})^n(\cdot)$ as $n \to \infty$ and further obtain an estimate of the total variation of $u(\cdot,t)$ and of $v(\cdot,t)$ on $[0,1]$ as t goes to infinity. In the final Section 4, some numerical simulations are given.

2. Properties of the map $G \circ F_{\nu,\alpha}$

Most of the basic properties of the map $G \circ F_{\nu,\alpha}$ have been established by Chen et al. in [8]. But they only studied the properties of $G \circ F_{\nu,\alpha}$ when the parameter ν varies but α and β were held fixed. Contrary to this, we will regard both ν and α as varying parameters and discuss the basic properties of $G \circ F_{\nu,\alpha}$.

Let

$$S = \{(\nu,\alpha) \in R^2 \mid 0 < \nu < +\infty,\ 0 < \alpha \leqslant \rho_2(\nu)\}, \tag{2.1}$$

where $\rho_2(\nu)$ is given by (1.3). Since $F_{\nu,\alpha}$ is well defined if and only if $(\nu,\alpha) \in S$, we always assume that $(\nu,\alpha) \in S$ throughout this paper.

Lemma 2.1. *Let $\beta > 0$. Then the map $G \circ F_{\nu,\alpha}$ has the following properties*:

(i) *$G \circ F_{\nu,\alpha}(\cdot)$ is odd*;

(ii) *$G \circ F_{\nu,\alpha}$ has exactly three fixed points 0, v_0 and $-v_0$, where*

$$v_0 = v_0(\nu,\alpha) = \frac{1}{\rho_1(\nu)+\rho_2(\nu)}\sqrt{\frac{\alpha}{\beta}}, \tag{2.2}$$

and the origin is a repelling fixed point;

(iii) *$-G \circ F_{\nu,\alpha}$ $(= F_{\nu,\alpha})$ has exactly three fixed points $-v_1$, 0 and v_1, where*

$$v_1 = v_1(\nu,\alpha) = \frac{1}{\rho_2-\rho_1}\sqrt{\frac{1+\alpha(\rho_2-\rho_1)}{\beta(\rho_2-\rho_1)}} = \frac{1}{\nu}\sqrt{\frac{1+\alpha\nu}{\beta\nu}}, \tag{2.3}$$

where the last equality in (2.3) *follows from* (1.4);

(iv) *The equation $G \circ F_{\nu,\alpha}(v) = 0$ has exactly three roots 0, v_I and $-v_I$, where*

$$v_I = v_I(\nu,\alpha) = \frac{1}{\rho_2}\sqrt{\frac{1+\alpha\rho_2}{\beta\rho_2}}; \tag{2.4}$$

(v) *$G \circ F_{\nu,\alpha}$ has local extremal values*

$$m = G \circ F_{\nu,\alpha}(-v_c) = -\frac{2}{3}\frac{1+\alpha\rho_2}{\rho_1+\rho_2}\sqrt{\frac{1+\alpha\rho_2}{3\beta\rho_2}}, \tag{2.5}$$

$$M = G \circ F_{\nu,\alpha}(v_c) = \frac{2}{3}\frac{1+\alpha\rho_2}{\rho_1+\rho_2}\sqrt{\frac{1+\alpha\rho_2}{3\beta\rho_2}} = -m, \tag{2.6}$$

where

$$v_c = v_c(\nu,\alpha) = \frac{3\rho_2^2 - 2\alpha\rho_2 + 1}{3\rho_2(\rho_2^2+1)}\sqrt{\frac{1+\alpha\rho_2}{3\beta\rho_2}}, \tag{2.7}$$

v_c and $-v_c$ are critical points of $G \circ F_{\nu,\alpha}$. Here m and M are, respectively, the local minimum and maximum of $G \circ F_{\nu,\alpha}$. The function $G \circ F_{\nu,\alpha}$ is strictly decreasing on $(-\infty,-v_c)$ and $(v_c,+\infty)$, but strictly increasing on $(-v_c,v_c)$;

(vi) $$(G \circ F_{\nu,\alpha})'(v) = -\rho_2^2 + \rho_2 \frac{\rho_2^2+1}{D}, \tag{2.8}$$

$$(G \circ F_{\nu,\alpha})''(v) = \frac{6\beta\rho_2(\rho_2^2+1)^2 g_{\nu,\alpha}(v)}{D^3}, \tag{2.9}$$

where

$$D = 3\beta g_{\nu,\alpha}^2(v) + \rho_2 - \alpha,$$

and $g_{\nu,\alpha}(\cdot)$ *is defined through* (1.11).

Proof. See Section 2 in [8]. □

Now we need first to study the bounded invariant intervals of the map $G \circ F_{\nu,\alpha}$. We shall show that they depends heavily on the choice of the parameters ν and α in S. First, we divide the regime S into three sub-domains in each of which $G \circ F_{\nu,\alpha}$ has different invariant intervals.

Solving the simultaneous equations

$$\begin{cases} \alpha = \rho_2(\nu), \\ \alpha = \frac{1}{\rho_2(\nu)}\left[\frac{3\sqrt{3}}{2\rho_2^2(\nu)} + \frac{3\sqrt{3}}{2} - 1\right] \equiv h_1(\nu), \end{cases} \tag{2.10}$$

we obtain a pair of unique positive solutions

$$\begin{cases} \alpha_1 = \sqrt{\frac{3\sqrt{3}}{2}}, \\ \nu_1 = \alpha_1 - \frac{1}{\alpha_1}. \end{cases} \tag{2.11}$$

By direct computation, we obtain that the equations

$$\begin{cases} \alpha = \rho_2(\nu), \\ \frac{2}{3}\frac{1+\alpha\rho_2}{\rho_1+\rho_2}\sqrt{\frac{1+\alpha\rho_2}{3\beta\rho_2}} = \frac{1}{\rho_2-\rho_1}\sqrt{\frac{1+\alpha(\rho_2-\rho_1)}{\beta(\rho_2-\rho_1)}}, \end{cases} \tag{2.12}$$

have a pair of unique positive solutions

$$\begin{cases} \alpha_2 = \sqrt{\frac{27/4+\sqrt{(27/4)^2+4}}{2}}, \\ \nu_2 = \alpha_2 - \frac{1}{\alpha_2}. \end{cases} \tag{2.13}$$

For any $\nu > 0$, by Cardan's formula, we can deduce that the second equation in (2.12) has a unique positive solution

$$\alpha = 3\frac{1+\rho_1^2}{\nu}\cos\theta - \rho_1 \equiv h_2(\nu), \tag{2.14}$$

where

$$\theta = \frac{1}{3}\arccos\frac{1}{1+\rho_2^2}.$$

From (1.2)–(1.4), we obtain

$$\frac{\pi}{9} \leqslant \theta \leqslant \frac{\pi}{6}.$$

So

$$h_1(\nu) < h_2(\nu), \quad \forall \nu > 0.$$

It is easy to prove that

$$M(\nu,\alpha) > v_1(\nu,\alpha) \quad \text{if } \alpha > h_2(\nu) \tag{2.15}$$

and

$$\frac{1}{\nu} < h_1(\nu) \quad \text{if } \nu > \frac{1}{\sqrt{2}}. \tag{2.16}$$

Here $M(\nu,\alpha)$ and $v_1(\nu,\alpha)$ are, respectively, the local maximum value and the intersection point with the diagonal line $u+v=0$ of the map $G \circ F_{\nu,\alpha}$ defined by (2.6) and (2.3) in Lemma 2.1, respectively.

Define

$$\begin{aligned} S_1 = {} & \bigl\{(\nu,\alpha) \mid 0 < \nu < \nu_1,\ 0 < \alpha \leqslant \rho_2(\nu)\bigr\} \\ & \cup \bigl\{(\nu,\alpha) \mid \nu_1 \leqslant \nu < +\infty,\ 0 < \alpha \leqslant h_1(\nu)\bigr\}, \end{aligned} \tag{2.17}$$

$$\begin{aligned} S_2 = {} & \bigl\{(\nu,\alpha) \mid \nu_1 < \nu < \nu_2,\ h_1(\nu) \leqslant \alpha \leqslant \rho_2(\nu)\bigr\} \\ & \cup \bigl\{(\nu,\alpha) \mid \nu_2 \leqslant \nu < +\infty,\ h_1(\nu) \leqslant \alpha \leqslant h_2(\nu)\bigr\}, \end{aligned} \tag{2.18}$$

$$S_3 = \bigl\{(\nu,\alpha) \mid \nu_2 \leqslant \nu < +\infty,\ h_2(\nu) < \alpha \leqslant \rho_2(\nu)\bigr\}, \tag{2.19}$$

where ν_1 and ν_2 are given by (2.11) and (2.13), respectively.

A routine check shows that

$$\begin{aligned} & M(\nu,\alpha) < v_I(\nu,\alpha) \quad \text{if } (\nu,\alpha) \in S_1, \\ & v_I(\nu,\alpha) \leqslant M(\nu,\alpha) \leqslant v_1(\nu,\alpha) \quad \text{if } (\nu,\alpha) \in S_2, \\ & v_1(\nu,\alpha) < M(\nu,\alpha) \quad \text{if } (\nu,\alpha) \in S_3, \end{aligned}$$

where $M(\nu,\alpha)$, $v_I(\nu,\alpha)$ and $v_1(\nu,\alpha)$ are given by Lemma 2.1. Thus, we have

Theorem 2.1 (Bounded invariant intervals of the map $G \circ F_{\nu,\alpha}$).

(i) *If $(\nu,\alpha) \in S_1$, then $I_1 \equiv [0, v_I]$ and $-I_1 \equiv [-v_I, 0]$ are bounded invariant intervals of $G \circ F_{\nu,\alpha}$. Furthermore, $G \circ F_{\nu,\alpha}$ is unimodal on I_1;*

(ii) *If $(\nu,\alpha) \in S_2$, then $I_2 \equiv [-M, M]$ is a bounded invariant interval of $G \circ F_{\nu,\alpha}$;*

(iii) *If $(\nu,\alpha) \in S_3$, then the map $G \circ F_{\nu,\alpha}$ has no any bounded invariant interval.*

It is easy to check that the pair of equations

$$\begin{cases} \alpha = \rho_2(\nu), \\ \alpha = \frac{1}{\nu}, \end{cases}$$

has the pair

$$\begin{cases} \nu_0 = \frac{1}{\sqrt{2}}, \\ \alpha_0 = \sqrt{2}, \end{cases} \tag{2.20}$$

as its unique solution with positive components.

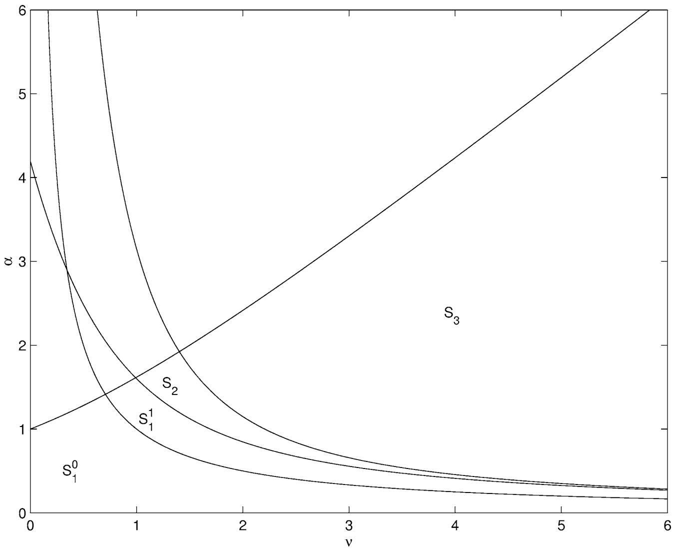

Fig. 1. The regime S is divided into four sub-regimes S_1^0, S_1^1, S_2, and S_3.

We divide the regime S_1 defined by (2.17) into the two parts

$$\begin{aligned} S_1^0 &= \{(\nu,\alpha) \mid 0<\nu<\nu_0,\ 0<\alpha\leqslant\rho_2(\nu)\} \\ &\quad\cup \left\{(\nu,\alpha) \mid \nu_0\leqslant\nu<+\infty,\ 0<\alpha\leqslant\frac{1}{\nu}\right\}, \end{aligned} \tag{2.21}$$

$$\begin{aligned} S_1^1 &= \left\{(\nu,\alpha) \mid \nu_0<\nu<\nu_1,\ \frac{1}{\nu}<\alpha\leqslant\rho_2(\nu)\right\} \\ &\quad\cup \left\{(\nu,\alpha) \mid \nu_1\leqslant\nu<+\infty,\ \frac{1}{\nu}<\alpha<h_1(\nu)\right\}. \end{aligned} \tag{2.22}$$

So we have divided the regime S into four sub-regimes S_1^0, S_1^1, S_2 and S_3, see Fig. 1. Figures 2–5 give examples of graphs of $G\circ F_{\nu,\alpha}$ when the parameters (ν,α) belong to each of the sub-regimes, respectively. We shall see in the next section that the growth of total variations of $(G\circ F_{\nu,\alpha})^n$ as n goes to infinity depends on the parameters (ν,α) in the different sub-regimes.

We need the following lemma from Block and Coppel [2, Proposition 1, Chapter VI].

Lemma 2.2. *Let I be a compact interval and $f: I\to I$ be a continuous map. If f has no periodic point of period 2, then, for every $x\in I$, $f^k(x)$ converges to a fixed point of f as $k\to\infty$.*

Theorem 2.2. *Let $\beta>0$. Then*

(i) *If $(\nu,\alpha)\in S_1^0$, then $G\circ F_{\nu,\alpha}$ has no periodic point of period 2 on the invariant interval I_1. Moreover,*

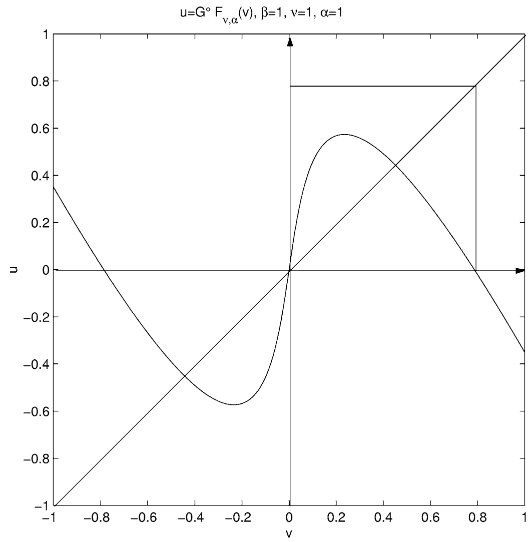

Fig. 2. Graph of $G \circ F_{\nu,\alpha}(v)$ when $(\nu, \alpha) = (1, 1) \in S_1^0$.

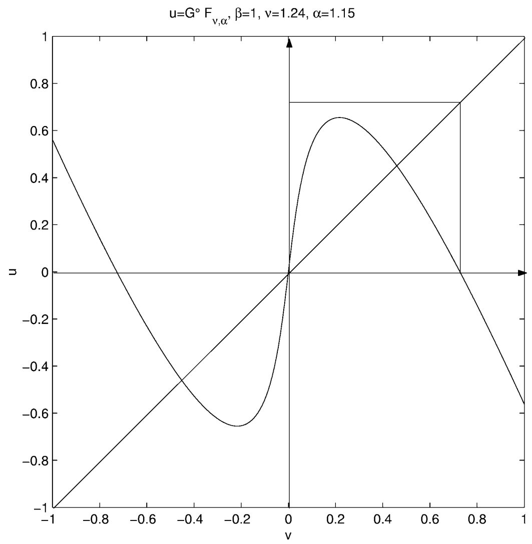

Fig. 3. Graph of $G \circ F_{\nu,\alpha}(v)$ when $(\nu, \alpha) = (1.24, 1.15) \in S_1^1$.

$$\lim_{n\to\infty} (G \circ F_{\nu,\alpha})^n(v) = v_0, \quad \forall v \in (0, v_I) = I_1, \tag{2.23}$$

$$\lim_{n\to\infty} (G \circ F_{\nu,\alpha})^n(v) = -v_0, \quad \forall v \in (-v_I, 0) = -I_1; \tag{2.24}$$

(ii) *If* $(\nu, \alpha) \in S_1^1$, *then* $G \circ F_{\nu,\alpha}$ *has at least a periodic point on the invariant interval* I_2 *with period greater than or equals to* 2;

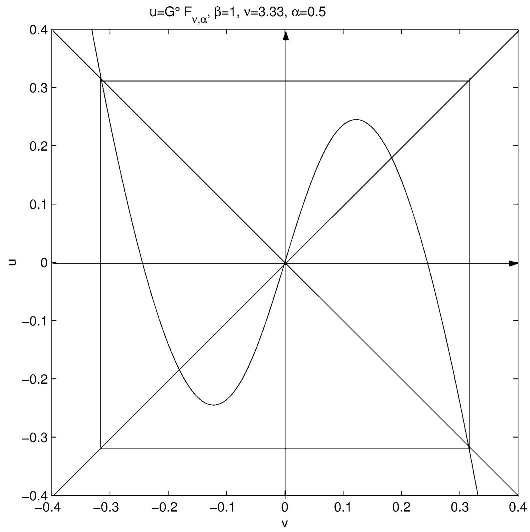

Fig. 4. Graph of $G \circ F_{\nu,\alpha}(v)$ when $(\nu, \alpha) = (3.33, 0.5) \in S_2$.

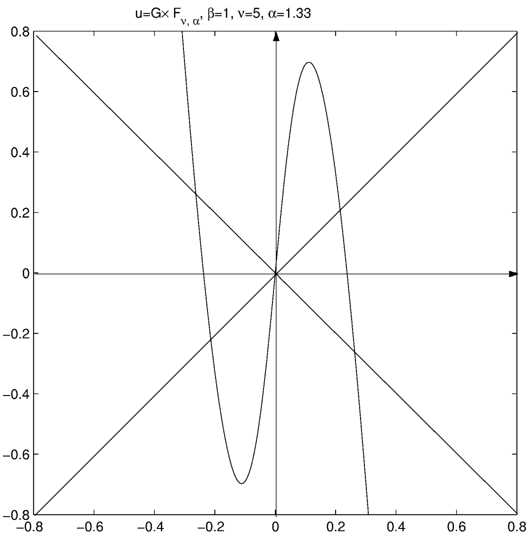

Fig. 5. Graph of $G \circ F_{\nu,\alpha}(v)$ when $(\nu, \alpha) = (5, 1.33) \in S_3$.

(iii) *If* $(\nu, \alpha) \in S_2$, *then* $G \circ F_{\nu,\alpha}$ *has periodic points on the invariant interval* I_3 *with period not a power of* 2 (*where we include* $1 = 2^0$ *as a power of* 2);

(iv) *If* $(\nu, \alpha) \in S_3$, *then* $G \circ F_{\nu,\alpha}$ *has an invariant Cantor set* $\wedge$ *with measure zero and* $G \circ F_{\nu,\alpha}$ *is chaotic in the sense of Li–Yorke on* $\wedge$. *Furthermore, for any small interval* J, *there is* $v \in J$, *such that*

$$\lim_{n\to\infty} (G \circ F_{\nu,\alpha})^n(v) = \infty.$$

Proof. For (i), suppose that $G \circ F_{\nu,\alpha}$ has a periodic point $v \in I_1$ with period 2. That is

$$(G \circ F_{\nu,\alpha})^2(v) = v, \qquad G \circ F_{\nu,\alpha}(v) \neq v.$$

Let

$$z = G \circ F_{\nu,\alpha}(v) = -\rho_2\big[\rho_2 v + g_{\nu,\alpha}(v)\big]. \tag{2.25}$$

Then

$$v = G \circ F_{\nu,\alpha}(z) = -\rho_2\big[\rho_2 z + g_{\nu,\alpha}(z)\big]. \tag{2.26}$$

Thus

$$g_{\nu,\alpha}(v) = -\rho_1 z - \rho_2 v \equiv -w, \tag{2.27}$$

$$g_{\nu,\alpha}(z) = -\rho_1 v - \rho_2 z \equiv -y. \tag{2.28}$$

Since $v \in I_1$ and $v \neq z$, w and y are all positive and

$$w \neq y. \tag{2.29}$$

By the definition of $g_{\nu,\alpha}$, w and y satisfy the equations

$$-\beta w^3 - (\rho_2 - \alpha)w + (1 + \rho_2)v = 0 \tag{2.30}$$

and

$$-\beta y^3 - (\rho_2 - \alpha)y + (1 + \rho_2)z = 0. \tag{2.31}$$

It follows from (2.27) and (2.28) that

$$v = \frac{1}{\rho_2^2 - \rho_1^2}(\rho_2 w - \rho_1 y) \tag{2.32}$$

and

$$z = -\frac{1}{\rho_2^2 - \rho_1^2}(\rho_1 w - \rho_2 y). \tag{2.33}$$

Substituting (2.32) and (2.33) into (2.30) and (2.31), respectively, after simplifying by (1.4), we obtain

$$\nu\beta w^3 - (1 + \nu\alpha)w + y = 0 \tag{2.34}$$

and

$$\nu\beta y^3 - (1 + \nu\alpha)y + w = 0. \tag{2.35}$$

From the last two equations and (2.29), we have

$$\nu\beta(w^2 + wy + y^2) - (2 + \nu\alpha) = 0 \tag{2.36}$$

and

$$\nu\beta(w^2 - wy + y^2) - \nu\alpha = 0. \tag{2.37}$$

Thus

$$2\nu\beta wy - 2 = 0,$$

so

$$w = \frac{1}{\nu\beta y}. \tag{2.38}$$

From (2.38) and (2.35), we have

$$\nu\beta y^4 - (1+\nu\alpha)y^2 + \frac{1}{\nu\beta} = 0. \tag{2.39}$$

Since $(\nu,\alpha) \in S_1^0$, we have

$$0 < \nu\alpha < 1.$$

So

$$(1+\nu\alpha)^2 - 4\nu\beta\frac{1}{\nu\beta} < 0.$$

Thus Eq. (2.39) has no real solution, a contradiction. That means that $G \circ F_{\nu,\alpha}$ has no periodic point of period 2 in I_1.

On the other hand, since $(\nu,\alpha) \in S_1^0$, we have

$$\nu\alpha < 1. \tag{2.40}$$

By (iv) in Lemma 2.1,

$$(G \circ F_{\nu,\alpha})'(v_0) = -\rho_2^2 + \rho_2\frac{\rho_2^2+1}{D} = -\rho_2^2 + \rho_2\frac{\rho_2^2+1}{3\beta g_{\nu,\alpha}^2(v_0)+\rho_2-\alpha}.$$

Since v_0 is a fixed point of $G \circ F_{\nu,\alpha}$, from (1.10), we have

$$\begin{aligned}
-v_0 &= F_{\nu,\alpha}(v_0) = \rho_2\big[\rho_2 v_0 + g_{\nu,\alpha}(v_0)\big],\\
g_{\nu,\alpha}(v_0) &= -\frac{v_0}{\rho_2} - \rho_2 v_0\\
&= -(\rho_1+\rho_2)\frac{1}{\rho_1+\rho_2}\sqrt{\frac{\alpha}{\beta}} \quad \text{by (2.3)}\\
&= \frac{\alpha}{\beta}.
\end{aligned}$$

So

$$(G \circ F_{\nu,\alpha})'(v_0) = -\rho_2^2 + \rho_2\frac{\rho_2^2+1}{3\alpha+\rho_2-\alpha} = \frac{1-2\alpha\rho_2}{1+2\alpha\rho_1}. \tag{2.41}$$

Here we have used the relation that $\rho_1(\nu)\rho_2(\nu) = 1$ in (1.4). From (2.40), we have

$$-1 < (G \circ F_{\nu,\alpha})'(v_0) < 0 \quad \text{if } (\nu,\alpha) \in S_1^0. \tag{2.42}$$

Thus the v_0 is the unique attracting fixed point of $G \circ F_{\nu,\alpha}$ on $I_1 = [0, v_I]$. (0 is a repelling fixed point of $G \circ F_{\nu,\alpha}$ by (ii) in Lemma 2.1.) From Lemma 2.2, we get (2.23). Since $G \circ F_{\nu,\alpha}(-v) = -G \circ F_{\nu,\alpha}(v)$, we have (2.24).

For (ii), if $(\nu,\alpha) \in S_1^1$, then, from (2.41), we get

$$(G \circ F_{\nu,\alpha})'(v_0) < -1. \tag{2.43}$$

Setting

$$f_{\nu,\alpha}(v) = (G \circ F_{\nu,\alpha})^2(v) - v,$$

we obtain

$$f'_{\nu,\alpha}(v_0) > 0 \quad \text{and} \quad f_{\nu,\alpha}(v_0) = 0. \tag{2.44}$$

Thus, by the smoothness of $f_{\nu,\alpha}$, there is a point v' in the small left neighborhood of v_0 such that

$$f_{\nu,\alpha}(v') < 0. \tag{2.45}$$

On the other hand, since $G \circ F_{\nu,\alpha}(\cdot)$ is strictly increasing in $(0, v_c)$, where v_c is the local maximum point given by (2.7) in Lemma 2.1. Also

$$\frac{1}{\rho_2(\nu)} < \frac{1}{\nu} < \alpha$$

implies $v_c < M$, so there exists a point v'' with $0 < v'' < v_c$ such that

$$v'' < G \circ F_{\nu,\alpha}(v'') = v_c.$$

Thus

$$(G \circ F_{\nu,\alpha})^2(v'') = M > v''$$

or

$$f_{\nu,\alpha}(v'') = (G \circ F_{\nu,\alpha})^2(v'') - v'' > 0. \tag{2.46}$$

From (2.45) and (2.46), there is v''' with $v'' < v''' < v'$ such that

$$f_{\nu,\alpha}(v''') = 0.$$

This implies that v''' is a periodic point of $G \circ F_{\nu,\alpha}$ with period 2.

For (iii), if $(\nu, \alpha) \in S_2$, then $M \geqslant v_I$. This shows that $G \circ F_{\nu,\alpha}$ has a homoclinic point in the invariant set $I_2 = [-v_1, v_1]$. Thus $G \circ F_{\nu,\alpha}$ has a periodic point with period not a power of 2 (see [1]).

Finally, we prove (iv). The first part of (iv) follows from Theorem 3.3 in [8]. Since $(\nu, \alpha) \in S_3$, $G \circ F_{\nu,\alpha}$ does not have any invariant bounded intervals. Thus, for any interval J, if $J \cap (-v_1, v_1) = \phi$, where v_1 is defined by (iii) in Lemma 2.1, then

$$\lim_{n\to\infty} (G \circ F_{\nu,\alpha})^n(v) = \infty$$

for any $v \in J$ by (v) in Lemma 2.1. If $J \cap (-v_1, v_1) \neq \phi$, then there exist a positive integer k and a point $v_2 \in J \cap (-v_1, v_1)$ such that

$$(G \circ F_{\nu,\alpha})^k(v_2) = M.$$

Thus, again by (v) in Lemma 2.1, we have

$$\lim_{n\to\infty} (G \circ F_{\nu,\alpha})^n(v_2) = \infty. \qquad \square$$

Remark 2.1. The map $G \circ F_{\nu,\alpha}$ has a period-doubling cascade when only one parameter ν varies but α is fixed, see Theorem 3.1 in [8].

Remark 2.2. From Theorem 2.2, for the discrete dynamical system $\{(G \circ F_{\nu,\alpha})^n\}_{n \in N^+}$ bifurcation occurs on the curve $\Gamma := \{(\nu, \alpha) \in S \mid \nu\alpha = 1\}$.

3. The growth rates of the total variation of $u(\cdot, t)$ and of $v(\cdot, t)$ as $t \to \infty$

First, we study the growth rate of total variations of $(G \circ F_{\nu,\alpha})^n$ as n goes to infinity when the parameters (ν, α) vary in the regime S. We shall obtain the following results: the growth of the total variation of the map $(G \circ F_{\nu,\alpha})^n$

(1) remains bounded on I_1 if $(\nu, \alpha) \in S_1^0$, (Lemma 3.1);
(2) is unbounded on I_1 if $(\nu, \alpha) \in S_1^1$, (Lemma 3.3);
(3) is exponential on I_2 if $(\nu, \alpha) \in S_2$, (Lemma 3.5)

as $n \to \infty$. It is meaningless to consider the case that $(\nu, \alpha) \in S_2$, since the map has no bounded invariant intervals of $G \circ F_{\nu,\alpha}$ by Theorem 2.1.

Let f be a continuous map from an interval J into itself. Throughout this section, we denote by $V_J(f)$ the total variation of f on J.

For simplicity and clarity, write

$$f_{\nu,\alpha} \equiv G \circ F_{\nu,\alpha}.$$

Lemma 3.1. *Let $\beta > 0$. If $(\nu, \alpha) \in S_1^0$, then*

$$V_{[-v_I, v_I]}\big((f_{\nu,\alpha})^n\big) \leqslant C, \quad \forall n = 1, 2, \ldots, \tag{3.1}$$

for some constant C.

Proof. Since $f_{\nu,\alpha}$ is odd and unimodal, it suffices to prove that

$$V_{I_1}\big((f_{\nu,\alpha})^n\big) \leqslant C, \quad \forall n = 1, 2, \ldots, \tag{3.2}$$

for some constant C. If

$$(\nu, \alpha) \in \left\{(\nu, \alpha) \in S \;\middle|\; 0 < \alpha \leqslant \frac{1}{2\rho_2(\nu)}\right\} \subset S_1^0,$$

then we can deduce that the local maximum value M of $f_{\nu,\alpha}$ on I_1 satisfies

$$M = f(v_c) \leqslant v_c,$$

and the fixed point v_0 of $f_{\nu,\alpha}$ satisfies

$$0 < v_0 \leqslant v_c.$$

Let

$$J_1 = [0, v_0], \qquad J_2 = [v_0, v_c], \qquad J_3 = [v_c, \tilde{v}_0], \quad \text{and} \quad J_4 = [\tilde{v}_0, v_I],$$

where $\tilde{v}_0 \in (v_c, v_I)$ is the unique point with

$$f_{\nu,\alpha}(\tilde{v}_0) = v_0.$$

Since $f_{\nu,\alpha}$ is monotone on J_i, $i = 1, 2, 3, 4$, and

$$f_{\nu,\alpha}(J_1) = J_1, \qquad f_{\nu,\alpha}(J_2) \subset J_2, \tag{3.3}$$

$$f_{\nu,\alpha}(J_3) = J_2, \qquad f_{\nu,\alpha}(J_4) = J_1, \tag{3.4}$$

we have

$$V_{J_1}\big((f_{\nu,\alpha})^n\big) = v_0$$

for any positive integer n, and

$$\lim_{n\to\infty} V_{J_2}\big((f_{\nu,\alpha})^n\big) = 0.$$

Thus

$$V_{I_1}\big((f_{\nu,\alpha})^n\big) = \sum_{i=1}^{4} V_{J_i}\big((f_{\nu,\alpha})^n\big) \to 2v_0 \quad \text{as } n \to \infty.$$

When

$$(\nu,\alpha) \in S_1^0 - \left\{(\nu,\alpha) \in S \;\middle|\; 0 < \alpha \leqslant \frac{1}{2\rho_2(\nu)}\right\},$$

a direct computation shows that

$$v_c < v_0 < M.$$

Different from the case above in which $f^n_{\nu,\alpha}(v_c)$ decreases to v_0, $f^n_{\nu,\alpha}(v_c)$ tends in an oscillatory manner to the fixed point v_0 as $n \to \infty$. Thus, $f^n_{\nu,\alpha}$ is piecewise monotone with $2n - 1$ extremal points in I_1, with the corresponding extremal values being

$$\begin{aligned} M = {} & f_{\nu,\alpha}(v_c), f^2_{\nu,\alpha}(v_c), \ldots, f^{n-1}_{\nu,\alpha}(v_c), f^n_{\nu,\alpha}(v_c), f^{n-1}_{\nu,\alpha}(v_c), \ldots, \\ & f^2_{\nu,\alpha}(v_c), f_{\nu,\alpha}(v_c) = M, \end{aligned}$$

from left to right in I, respectively. Therefore

$$V_I\big(f^{2n}_{\nu,\alpha}\big) = 2\left[M + \sum_{k=1}^{n}\big(f^{2k-1}_{\nu,\alpha}(v_c) - f^{2k}_{\nu,\alpha}(v_c)\big) + \sum_{k=2}^{n}\big(f^{2k-1}_{\nu,\alpha}(v_c) - f^{2k-2}_{\nu,\alpha}(v_c)\big)\right] \tag{3.5}$$

and

$$V_I\big(f^{2n+1}_{\nu,\alpha}\big) = 2\left[M + \sum_{k=1}^{n}\big(f^{2k-1}_{\nu,\alpha}(v_c) - f^{2k}_{\nu,\alpha}(v_c)\big) + \sum_{k=2}^{n+1}\big(f^{2k-1}_{\nu,\alpha}(v_c) - f^{2k-2}_{\nu,\alpha}(v_c)\big)\right]. \tag{3.6}$$

On the other hand, again since $-1 < f'_{\nu,\alpha}(v_0(\eta)) < 0$ for $(\nu,\alpha) \in S_1$ and $f''_{\nu,\alpha}(v) < 0$ in I_1, there exists $u \in I_1$ such that $u > v_0$ and

$$\big|f'_{\nu,\alpha}(v)\big| < \big|f'_{\nu,\alpha}(u)\big| \equiv \delta_0 < 1, \quad \forall v \in [v_c, u]. \tag{3.7}$$

Moreover, since

$$\lim_{k\to\infty} f^k_{\nu,\alpha}(v_c) = v_0,$$

by (i) in Theorem 2.2, there exists a positive integer K, such that for any $k \geqslant K$,

$$f^k_{\nu,\alpha}(v_c) \in [v_c, u].$$

It follows from the mean value theorem that for any positive integer k,

$$\left| f^{k+K+1}_{\nu,\alpha}(v_c) - f^{k+K}_{\nu,\alpha}(v_c)\right| \leqslant \delta_0^k \left| f^{K+1}_{\nu,\alpha}(v_c) - f^{K}_{\nu,\alpha}(v_c)\right|. \tag{3.8}$$

Thus, the right-hand sides in (3.5) and (3.6) are dominated by the positive series

$$C\sum_{k=0}^{\infty}\delta_0^k,$$

which is convergent for some positive constant C. Therefore, there exists a constant $C' > 0$ such that

$$V_{I_1}\big(f^n_{\nu,\alpha}\big) \leqslant C', \quad \forall n = 1, 2, \ldots.$$

Thus the proof of Lemma 3.1 is complete. □

For the case that $(\nu, \alpha) \in S_1^1$, we need the following results.

Lemma 3.2 [14, Lemma 3.1 and Remark 3.1]. *Let $a > 0$. Suppose that g is unimodal map from $[0, a]$ to itself with a periodic point p of period 2. We have*

$$\lim_{n\to\infty} V_{[0,p]}(g^n) = \infty.$$

For more general results about the total variations and periodic orbits for a map on interval, see [10].

Lemma 3.3. *Let $\beta > 0$. If $(\nu, \alpha) \in S_1^1$, then for any $\varepsilon_0 > 0$, we have*

$$\lim_{n\to\infty} V_{[0,\varepsilon_0]}\big(f^n_{\nu,\alpha}\big) = \infty \quad \textit{and} \quad \lim_{n\to\infty} V_{[-\varepsilon_0,0]}\big(f^n_{\nu,\alpha}\big) = \infty. \tag{3.9}$$

Proof. If $(\nu, \alpha) \in S_1^1$, then the unimodal map $f_{\nu,\alpha}$ on I_1 has a periodic point (denoted by p) with period 2 by (ii) in Theorem 2.2. Thus, from Lemma 3.2, we get

$$\lim_{n\to\infty} V_{[0,p]}\big(f^n_{\nu,\alpha}\big) = \infty. \tag{3.10}$$

On the other hand, we know from the proof of (ii) of Theorem 2.2 that the periodic point p above belongs to the interval $(0, v_0)$. Since $f_{\nu,\alpha}$ increases strictly in $[0, v_c]$, for any given $\varepsilon_0 > 0$, there exists a positive integer K such that

$$[0, p] \subset f^K_{\nu,\alpha}\big([0, \varepsilon_0]\big).$$

Thus, by (3.10), we get the first equation of (3.9). The second equation of (3.9) follows from the fact that $f_{\nu,\alpha}$ is odd. □

Finally, we discuss the case that $(\nu, \alpha) \in S_2$.

Lemma 3.4 [10]. *Let J be a bounded closed interval and let $f: J \to J$ be a continuous map. If f has a periodic point whose period is not a power of 2, then there exists a positive constant c such that*

$$V_J(f^n) \geqslant O\big(\exp(cn)\big) \quad \text{as } n \to \infty. \tag{3.11}$$

Lemma 3.5. *Let $\beta > 0$. If $(\nu, \alpha) \in S_2$, then for any sufficiently small $\varepsilon_0 > 0$,*

$$V_{[0,\varepsilon_0]}\big(f^n_{\nu,\alpha}\big) \geqslant c_1 \exp(c_2 n) \quad \text{and} \quad V_{[-\varepsilon_0,0]}\big(f^n_{\nu,\alpha}\big) \geqslant c_1 \exp(c_2 n), \quad \forall n = 1, 2, \ldots, \tag{3.12}$$

for some positive constants c_1 and c_2.

Proof. If $(\nu, \alpha) \in S_2$, from Lemma 2.1 and Theorem 2.2, we have that $f_{\nu,\alpha}$ has periodic points in $(-M, 0)$ and $(0, M)$, respectively, with the periods not a power of 2. Lemma 3.4 shows that

$$V_{[-m,0]}\big(f^n_{\nu,\alpha}\big) \geqslant O\big(\exp(cn)\big) \quad \text{as } n \to \infty, \tag{3.13}$$

$$V_{[0,M]}\big(f^n_{\nu,\alpha}\big) \geqslant O\big(\exp(cn)\big) \quad \text{as } n \to \infty, \tag{3.14}$$

for some positive constant c.

On the other hand, since $f_{\nu,\alpha}$ is strictly increasing in $(-v_c, v_c)$ by Lemma 2.1, we have, for any small $\varepsilon > 0$, there exists a positive integer K such that

$$f^K_{\nu,\alpha}\big([-\varepsilon, 0]\big) \supset [-M, 0], \qquad f^K_{\nu,\alpha}\big([0, \varepsilon]\big) \supset [0, M]. \tag{3.15}$$

Therefore, (3.12) follows from (3.13)–(3.15). □

Lemma 3.6 [14, Lemma 2.4]. *Assume that*

(1) *g is a continuous function with bounded variation on a bounded interval L_1;*
(2) *ϕ is a piecewise monotone continuous function with finitely many extremal points on a bounded interval L_2;*
(3) *The range of ϕ on L_2 is a subset of L_1.*

Then the composition $g \circ \phi$ is a continuous function of bounded variation on L_2.

Theorem 3.1. *Consider the initial-boundary value problem* (1.1)*. Assume that w_0 and w_1 in $(1.1)_4$ are sufficiently smooth such that u_0 and v_0 in* (1.7) *are continuous and piecewise monotone and satisfy the compatibility conditions*

$$v_0(0) = -u_0(0), \qquad u_0(1) = f_{\nu,\alpha}\big(v_0(1)\big). \tag{3.16}$$

Then, we have

(i) *If $(\nu, \alpha) \in S^0_1$ and*

$$\big|u_0(x)\big| \leqslant v_I, \quad \big|v_0(x)\big| \leqslant v_I, \quad \forall x \in [0, 1],$$

then, for u and v in (1.6), *we have*

$$V_{[0,1]}\big(u(\cdot,t)\big) \leqslant C \quad \textit{and} \quad V_{[0,1]}\big(v(\cdot,t)\big) \leqslant C, \quad \forall t \geqslant 0, \tag{3.17}$$

for some positive constant C independent of t;

(ii) *If $(\nu,\alpha) \in S_1^1$ and there is a small positive ε_0 such that either*

$$[0,\varepsilon_0] \subset \operatorname{Range} u_0 \cap \operatorname{Range} v_0 \quad \textit{or} \quad [-\varepsilon_0,0] \subset \operatorname{Range} u_0 \cap \operatorname{Range} v_0, \tag{3.18}$$

then, for u and v in (1.6), *we have*

$$\lim_{t\to\infty} V_{[0,1]}\big(u(\cdot,t)\big) = \infty \quad \textit{and} \quad \lim_{t\to\infty} V_{[0,1]}\big(v(\cdot,t)\big) = \infty; \tag{3.19}$$

(iii) *If $(\nu,\alpha) \in S_2$ and there is a small positive constant ε_0 such that* (3.18) *holds, then, for u and v in* (1.6), *the growth rates of $V_{[0,1]}(u(\cdot,t))$ and $V_{[0,1]}(v(\cdot,t))$ are at least exponential as $t \to \infty$.*

Proof. Part (i) follows from Lemmas 3.1 and 3.6 and the continuity of the total variations $V_{[0,1]}(u(\cdot,t))$ and $V_{[0,1]}(v(\cdot,t))$ with respect to the variable t. For the same reason, parts (ii) and (iii) follow from Lemmas 3.3 and 3.5, respectively. □

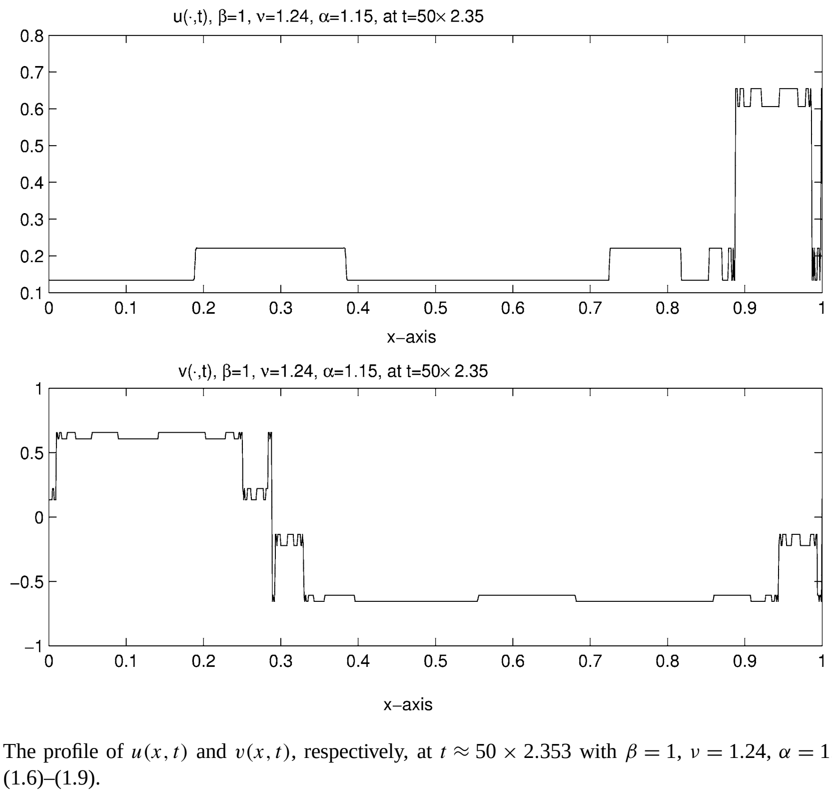

Fig. 6. The profile of $u(x,t)$ and $v(x,t)$, respectively, at $t \approx 50 \times 2.353$ with $\beta = 1$, $\nu = 1.24$, $\alpha = 1.15$, for system (1.6)–(1.9).

4. Numerical simulation results

We offer a few computer simulation results in this section as snapshots of the vibrations of a solution pair $u(\cdot,t)$ and $v(\cdot,t)$ of (1.6)–(1.9).

Throughout this section, we choose $\beta = 1$ and the initial conditions as

$$w_0(x) = 0.2\sin\left(\frac{\pi}{2}x\right), \quad w_1(x) = 0.2\sin(\pi x), \quad x \in [0,1].$$

Then, by (1.7), we have

$$u_0(x) = \frac{0.2}{\rho_1(\nu)+\rho_2(\nu)}\left[\rho_2(\nu)\frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right) + \sin(\pi x)\right], \quad x \in [0,1], \tag{4.1}$$

$$v_0(x) = \frac{0.2}{\rho_1(\nu)+\rho_2(\nu)}\left[\rho_1(\nu)\frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right) - \sin(\pi x)\right], \quad x \in [0,1]. \tag{4.2}$$

When the parameter pair $(\nu,\alpha) = (1.24, 1.1) \in S_1^1$, $\rho_1(\nu) \approx 0.556$ and $\rho_2(\nu) \approx 1.797$. The snapshots of $u(\cdot,t)$ and $v(\cdot,t)$ at $t = 50(\rho_1+\rho_2) \approx 50 \times 2.353$ are displayed in Fig. 6. In this case, the map $G \circ F_{\nu,\alpha}$ seems to have a periodic point of period 4. Regarding the profiles of u and v, we see that they are slowly oscillatory. There is no chaotic occurrence, but the total variation of each of them still grows infinitely as $t \to \infty$.

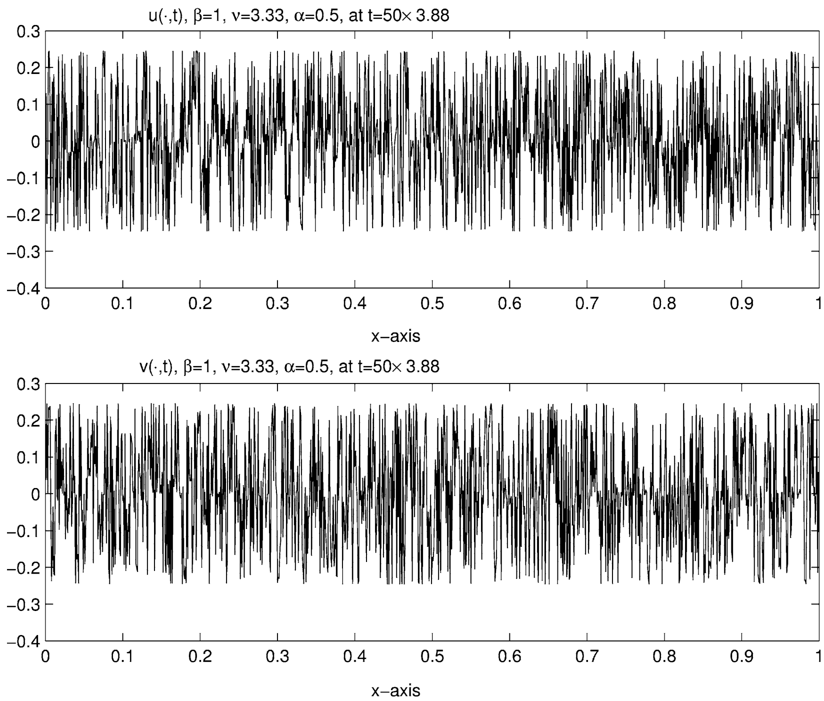

Fig. 7. The profile of $u(x,t)$ and $v(x,t)$, respectively, at $t \approx 50 \times 3.88$ with $\beta = 1$, $\nu = 3.33$, $\alpha = 0.5$, for system (1.6)–(1.9).

When the parameter pair $(\nu, \alpha) = (3.33, 0.5) \in S_2$, $\rho_1(\nu) \approx 0.277$ and $\rho_2(\nu) \approx 3.607$. The snapshots of $u(\cdot, t)$ and $v(\cdot, t)$ at $t = 50(\rho_1 + \rho_2) \approx 50 \times 3.884$ are displayed in Fig. 7. In this case, the map $G \circ F_{\nu,\alpha}$ has a periodic point with period not a power of 2 and the total variation of $u(\cdot, t)$ and of $v(\cdot, t)$ grow exponentially. From the profiles of u and v, we can see that they are extremely oscillatory at any interval in $[0, 1]$. Moreover, in contrast to Fig. 6, macroscopically coherent structures no longer exist.

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