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A connection between orthogonal polynomials on the unit circle and matrix orthogonal polynomials on the real line

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Abstract

Szegő's procedure to connect orthogonal polynomials on the unit circle and orthogonal polynomials on $[-1,1]$ is generalized to nonsymmetric measures. It generates the so-called semi-orthogonal functions on the linear space of Laurent polynomials Λ, and leads to a new orthogonality structure in the module $\Lambda \times \Lambda$. This structure can be interpreted in terms of a 2×2 matrix measure on $[-1,1]$, and semi-orthogonal functions provide the corresponding sequence of orthogonal matrix polynomials. This gives a connection between orthogonal polynomials on the unit circle and certain classes of matrix orthogonal polynomials on $[-1,1]$. As an application, the strong asymptotics of these matrix orthogonal polynomials is derived, obtaining an explicit expression for the corresponding Szegő's matrix function.

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1. Introduction. Semi-orthogonal functions

From long time ago, it is well known that there exists a simple relation between orthogonal polynomials (OP) on the unit circle ($\mathbb{T}$) and OP on $[-1,1]$ (see [9,24]). This close relationship provides a method to translate results from OP on $\mathbb{T}$ to OP on $[-1,1]$. For instance, this idea was largely exploited to get asymptotic properties of OP on $[-1,1]$ starting from the asymptotics of OP on $\mathbb{T}$ [18–20,24]. However, this relation is valid only for symmetric measures on $\mathbb{T}$. Recently it has been shown that above procedure can be generalized to arbitrary measures on $\mathbb{T}$, giving a connection between any sequence of OP on $\mathbb{T}$ and the so-called semi-orthogonal functions [1,4].

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As we will see, semi-orthogonal functions are given in terms of a sequence of two-dimensional matrix polynomials. The orthogonality properties of semi-orthogonal functions implies that these matrix polynomials are quasi-orthogonal with respect to some two-dimensional matrix measure related to the measure on $\mathbb{T}$. A sequence of matrix OP with respect to this matrix measure can be explicitly constructed from above quasi-orthogonal matrix polynomials. This gives a connection between OP on $\mathbb{T}$ and a class of two-dimensional matrix OP on the real line.

Matrix OP on the real line appear in the Lanczos method for block matrices [12,13], in scattering theory [11], in the spectral theory of doubly infinite Jacobi matrices [23] and discrete Sturm-Liouville operators [2,3], in the analysis of sequences of polynomials satisfying higher order recurrence relations [8], rational approximation and system theory [10]. Unfortunately their study is much more complicated and few things are known if compared with the scalar case (some nice surveys are [17,21,23]).

Previous connections between scalar and matrix OP appear in [16,14] where a relation is derived between scalar OP on an algebraic harmonic curve (lemniscata) and matrix OP on the real line (unit circle). Using a similar technique, a connection between scalar OP with respect to a discrete Sobolev inner product and matrix OP is presented in [8] (which is a consequence of the fact that OP with respect to a discrete Sobolev inner product satisfy a higher order recurrence relation, see [15]). These connections are more general than the one given in this paper because they deal with matrix OP of arbitrary dimension. However, they link matrix OP with "unknown worlds", in the sense that not too much is known about the different kind of OP that they connect with matrix OP. So, they are not too useful to get new results for matrix OP.

On the contrary, the connection presented in this paper, although much more restricted, allows us to translate results from the more "known word" of scalar OP on the unit circle to a great variety of two-dimensional matrix OP. So, it provides many models of matrix OP where many things can be known and that, therefore, can be used to get or check some ideas about new results for matrix OP. Here we have to point out that for certain applications, like the study of the doubly infinite matrices that appear in discrete Sturm–Liouville problems on the real line, only two-dimensional matrix OP are needed [3,23].

As an example of the utility of the present connection we derive the strong asymptotics of these matrix OP when the corresponding matrix measure belongs to the Szegő's class. General results about this situation can be found in [2], where a generalization of the connection between the real line and $\mathbb{T}$ for matrix OP is used again to obtain the asymptotics in the real line from the asymptotics in $\mathbb{T}$. However, the problem is far from being closed since there is no explicit expression for the Szegő's matrix function that gives the asymptotic behaviour and only some general properties are known. The connection given here let us obtain explicitly this Szegő's matrix function for a class of two-dimensional matrix measures. Other results about asymptotics of matrix OP, such as ratio and relative asymptotics, appear in [7,25] respectively.

Now, we proceed to introduce the starting point of our discussion, the semi-orthogonal functions, summarizing some results in [1,4] with a sketch of some proofs there for the convenience of the reader.

First of all we fix some notations. The real vector space of polynomials with real coefficients is denoted by $\mathscr{P}$, the subspace of $\mathscr{P}$ of polynomials with degree less than or equal to n is $\mathscr{P}_n$ and $\mathscr{P}_n^{\#}$ is the subset of $\mathscr{P}_n$ constituted by those polynomials whose degree is exactly n. Also, Λ is the complex vector space of Laurent polynomials, that is, $\Lambda = \bigcup_{n=0}^{\infty} \Lambda_{-n,n}$ where $\Lambda_{m,n} = \{\sum_{k=m}^{n} \alpha_k z^k \,|\, \alpha_k \in \mathbb{C}\}$ for

$m \leqslant n$. The elements of $\Lambda_{m,n}$ such that $\alpha_m, \alpha_n \neq 0$ form the subset $\Lambda^{\#}_{m,n}$. For an arbitrary complex number α, their real and imaginary parts are denoted $\Re\alpha$ and $\Im\alpha$ respectively.

Taking into account the usual identification between the unit circle $\mathbb{T} = \{e^{i\theta} | \theta \in [0, 2\pi)\}$ and the interval $[0, 2\pi)$, we talk about a measure on $\mathbb{T}$ when we deal with a measure with support on $[0, 2\pi)$. With this convention, in what follows $d\mu$ is a measure on $\mathbb{T}$ with finite moments. Unless we say explicitly that it is an arbitrary measure on $\mathbb{T}$, we suppose that $d\mu$ is a positive measure with infinite support. Then, the sesquilinear functional $\langle \cdot, \cdot \rangle_{d\mu}$ on Λ defined by

$$\langle f, g \rangle_{d\mu} = \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \, d\mu(\theta), \quad f, g \in \Lambda,$$

is an inner product and, hence, there exists a unique sequence $(\phi_n)_{n \geqslant 0}$ of monic OP with respect to $\langle \cdot, \cdot \rangle_{d\mu}$. If, as it is usual, ϕ_n^* denotes the reversed polynomial of ϕ_n ($\phi_n^*(z) = z^n \bar{\phi}_n(z^{-1})$), then, it is well known that OP are determined by the so-called Schur parameters $a_n = \phi_n(0)$ through the recurrence

$$\begin{aligned} &\phi_0(z) = 1, \\ &\phi_n(z) = z\phi_{n-1}(z) + a_n \phi_{n-1}^*(z), \quad n \geqslant 1. \end{aligned} \tag{1}$$

If we denote by b_n the coefficient of z^{n-1} in $\phi_n(z)$, from (1) we have that

$$b_n = b_{n-1} + a_n \bar{a}_{n-1}, \quad n \geqslant 1. \tag{2}$$

Notice that $b_0 = 0$ and

$$b_n = \sum_{k=1}^{n} a_k \bar{a}_{k-1}, \quad n \geqslant 1. \tag{3}$$

We can use (1) to show that the positive constants $\varepsilon_n = \langle \phi_n, \phi_n \rangle_{d\mu}$ are related to the Schur parameter by

$$\frac{\varepsilon_n}{\varepsilon_{n-1}} = 1 - |a_n|^2, \quad n \geqslant 1. \tag{4}$$

This relation implies that $|a_n| < 1$ for $n \geqslant 1$ and that the sequence $(\varepsilon_n)_{n \geqslant 0}$ must be strictly decreasing. Besides, (4) gives the following expression for ε_n:

$$\varepsilon_n = \prod_{k=1}^{n} (1 - |a_k|^2) \varepsilon_0, \quad n \geqslant 1. \tag{5}$$

Orthonormal polynomials are defined up to a factor with unit module, but they can be fixed if we ask for their leading coefficients to be real and positive. In this case we denote the n-th orthonormal polynomial by φ_n, and the corresponding leading coefficient by κ_n. It is clear that $\kappa_n = \varepsilon_n^{-1/2}$ and, thus, $(\kappa_n)_{n \geqslant 0}$ is strictly increasing.

The symmetric measure of $d\mu$ is

$$d\tilde{\mu}(\theta) = -d\mu(2\pi - \theta), \quad \theta \in [0, 2\pi),$$

and the measure $d\mu$ is said to be symmetric iff $d\tilde{\mu}=d\mu$. This is equivalent to affirm that the monic OP have real coefficients, which, in sight of (1), is in fact equivalent to state that the Schur parameters are real.

With the intention of connecting $\mathbb{T}$ with the interval $[-1,1]$, for $z\in\mathbb{C}\setminus\{0\}$ we write $x=(z+z^{-1})/2$ and $y=(z-z^{-1})/2i$ (therefore $z=x+iy, z^{-1}=x-iy$ and $x^2+y^2=1$). Both expressions give a transformation in the complex plane that maps $\mathbb{T}$ on the interval $[-1,1]$. Moreover, they map bijectively onto $\mathbb{C}\setminus[-1,1]$ the exterior of $\mathbb{T}$ as well as its interior excepting the origin. So, when restricted to these domains we can invert the transformations giving, for example, $z=x+\sqrt{x^2-1}$ (the choice of the square root must be done according to the location of z: exterior or interior to $\mathbb{T}$). Also, the transformation $x=(z+z^{-1})/2$ maps bijectively the upper as well as the lower closed half $\mathbb{T}$ onto $[-1,1]$ (in this case, writing $z=e^{i\theta}$, it is $x=\cos\theta$). So, by composition with the corresponding inverse transformations, the measure $d\mu$ provides two projected measures $d\nu_1$, $d\nu_2$ on $[-1,1]$, being

$$\begin{aligned} d\nu_1(x) &= -d\mu(\arccos x),\\ d\nu_2(x) &= -d\tilde{\mu}(\arccos x). \end{aligned} \tag{6}$$

The condition of symmetry for $d\mu$ is equivalent to the equality $d\nu_1=d\nu_2$.

Now, we wish to arrive at a family of polynomials with real coefficients, orthogonal with respect to an inner product defined through the measures $d\nu_1$, $d\nu_2$. To this end, and following [1,4], we start by introducing previously the so-called semi-orthogonal functions.

Definition 1. The semi-orthogonal functions (SOF) associated to the measure $d\mu$ are the functions $f_n^{(k)}:\mathbb{C}\setminus\{0\}\to\mathbb{C}$, $n\geqslant 1$, $k=1,2$, defined by

$$f_n^{(1)}(z)=\frac{z\phi_{2n-1}(z)+\phi^*_{2n-1}(z)}{2^n z^n},$$

$$f_n^{(2)}(z)=\frac{z\phi_{2n-1}(z)-\phi^*_{2n-1}(z)}{i2^n z^n},$$

where ϕ_n, $n\geqslant 1$, are the monic OP with respect to $\langle\cdot,\cdot\rangle_{d\mu}$.

The expressions in above definition are the same used in Szegő's method, with the difference that we consider here monic OP with complex instead of real coefficients. Let us go to summarize some interesting properties of SOF [1,4]:

Proposition 1. *The SOF associated to* $d\mu$ *satisfy*:

(i) $\bar{f}_n^{(k)}(z^{-1})=f_n^{(k)}(z)$ *and there is a unique decomposition*

$$f_n^{(k)}(z)=f_n^{(k1)}(x)+yf_n^{(k2)}(x),\qquad f_n^{(kj)}\in\mathscr{P}.$$

More precisely, $f_n^{(11)}\in\mathscr{P}_n^{\#}$, $f_n^{(22)}\in\mathscr{P}_{n-1}^{\#}$, *both monic polynomials, and* $f_n^{(21)}\in\mathscr{P}_{n-1}$, $f_n^{(12)}\in\mathscr{P}_{n-2}$.

(ii) *The family of functions* $\mathscr{B}_n = \{1\} \cup (\bigcup_{m=1}^{n} \{f_m^{(1)}, f_m^{(2)}\})$ *is a basis of* $\Lambda_{-n,n}$ *for all* $n \geqslant 1$. *The matrix of* $\langle \cdot, \cdot \rangle_{\mathrm{d}\mu}$ *with respect to the basis* $\mathscr{B} = \cup_{n \geqslant 1} \mathscr{B}_n$ *of* Λ *is a diagonal-block one*

$$\begin{pmatrix} \varepsilon_0 & 0 & 0 & \ldots \\ 0 & C_1 & 0 & \ldots \\ 0 & 0 & C_2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{7}$$

where

$$C_n = \frac{\varepsilon_{2n-1}}{2^{2n-1}} \begin{pmatrix} 1 - \Re a_{2n} & -\Im a_{2n} \\ -\Im a_{2n} & 1 + \Re a_{2n} \end{pmatrix}, \quad n \geqslant 1, \tag{8}$$

being a_n *the Schur parameters related to* $\mathrm{d}\mu$ *and* ε_n *given in* (5).

Proof. From the definition of ϕ_n^* we have that

$$f_n^{(1)}(z) = 2^{-n}(z^{-n+1}\phi_{2n-1}(z) + z^{n-1}\bar{\phi}_{2n-1}(z^{-1})),$$

$$f_n^{(2)}(z) = -\mathrm{i}2^{-n}(z^{-n+1}\phi_{2n-1}(z) - z^{n-1}\bar{\phi}_{2n-1}(z^{-1})),$$

and, thus, $\bar{f}_n^{(k)}(z^{-1}) = f_n^{(k)}(z)$.

If the decomposition given in (i) exists, it must be unique. If we suppose two such decompositions

$$f_n^{(k)}(z) = f_n^{(k1)}(x) + y f_n^{(k2)}(x) = g_n^{(k1)}(x) + y g_n^{(k2)}(x), \quad f_n^{(kj)}, g_n^{(kj)} \in \mathscr{P},$$

then

$$f_n^{(k)}(z^{-1}) = f_n^{(k1)}(x) - y f_n^{(k2)}(x) = g_n^{(k1)}(x) - y g_n^{(k2)}(x),$$

and above equalities give $f_n^{(kj)} = g_n^{(kj)}$ for $k = 1, 2$.

To see that this decomposition exist, let us write $\phi_n(z) = \sum_{k=0}^{n} \alpha_k z^k$, $\alpha_k \in \mathbb{C}$, with $\alpha_n = 1$. Then,

$$\begin{aligned} f_n^{(1)}(z) &= 2^{-n} z^{-n} \sum_{k=0}^{2n-1} (\alpha_k z^{k+1} + \bar{\alpha}_k z^{2n-k-1}) \\ &= T_n(x) + \sum_{j=0}^{n-1} 2^{j-n} \Re(\alpha_{n-j-1} + \alpha_{n+j-1}) T_j(x) \\ &\quad + y \sum_{j=0}^{n-2} 2^{j+1-n} \Im(\alpha_{n-j-2} + \alpha_{n+j}) U_j(x), \end{aligned} \tag{9}$$

where $T_j(x) = 2^{-j}(z^j + z^{-j})$ and $U_j(x) = -\mathrm{i} y^{-1} 2^{-j-1}(z^{j+1} - z^{-j-1})$ are, respectively, the jth Tchebychev monic polynomials of first and second kind (see for example [5] or [24]). This proves (i) for $f_n^{(1)}$. The proof for $f_n^{(2)}$ is similar.

Notice that $f_n^{(k)} \in \Lambda^{\#}_{-n,n}$ for $i=1,2$. In fact, $f_n^{(1)}(z)=2^{-n}(z^n+z^{-n})+\cdots$ and $f_n^{(2)}(z)=-\mathrm{i}2^{-n}(z^n-z^{-n})+\cdots$, where the dots mean terms belonging to $\Lambda_{-n+1,n-1}$. Thus, it is obvious that $\mathscr{B}_n$ is a basis of $\Lambda_{-n,n}$. The block-diagonal structure (7) of the matrix of $\langle\cdot,\cdot\rangle_{\mathrm{d}\mu}$ with respect to $\mathscr{B}$,

$$\begin{pmatrix} \langle 1,1\rangle_{\mathrm{d}\mu} & \langle 1,f_1^{(1)}\rangle_{\mathrm{d}\mu} & \langle 1,f_1^{(2)}\rangle_{\mathrm{d}\mu} & \cdots \\ \langle f_1^{(1)},1\rangle_{\mathrm{d}\mu} & \langle f_1^{(1)},f_1^{(1)}\rangle_{\mathrm{d}\mu} & \langle f_1^{(1)},f_1^{(2)}\rangle_{\mathrm{d}\mu} & \cdots \\ \langle f_1^{(2)},1\rangle_{\mathrm{d}\mu} & \langle f_1^{(2)},f_1^{(1)}\rangle_{\mathrm{d}\mu} & \langle f_1^{(2)},f_1^{(2)}\rangle_{\mathrm{d}\mu} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is just a direct consequence of the orthogonality relations for ϕ_n and ϕ_n^*, that is, the only conditions $\langle \phi_n, z^k\rangle_{\mathrm{d}\mu} = \langle \phi_n^*, z^{k+1}\rangle_{\mathrm{d}\mu} = 0$ for $0 \leqslant k \leqslant n-1$ imply $\langle 1, f_n^{(k)}\rangle_{\mathrm{d}\mu} = \langle f_n^{(k)}, f_m^{(j)}\rangle_{\mathrm{d}\mu} = 0$ for $n \neq m$ and $k,j=1,2$.

Finally, expression (8) for the matrix

$$C_n = \begin{pmatrix} \langle f_n^{(1)}, f_n^{(1)}\rangle_{\mathrm{d}\mu} & \langle f_n^{(1)}, f_n^{(2)}\rangle_{\mathrm{d}\mu} \\ \langle f_n^{(2)}, f_n^{(1)}\rangle_{\mathrm{d}\mu} & \langle f_n^{(2)}, f_n^{(2)}\rangle_{\mathrm{d}\mu} \end{pmatrix}$$

follows straightforward from the relations $\langle \phi_n^*, \phi_n^*\rangle_{\mathrm{d}\mu} = \langle \phi_n, \phi_n\rangle_{\mathrm{d}\mu} = \varepsilon_n$ and $\langle z\phi_n, \phi_n^*\rangle_{\mathrm{d}\mu} = -\varepsilon_n a_{n+1}$, the last one obtained from the recurrence formula (1). □

Notice that property (i) implies that SOF are real on $\mathbb{T}$. The incomplete orthogonality expressed in (ii) of previous proposition is the origin of the name "semi-orthogonal functions" given to the functions $f_n^{(k)}$. Notice that, when the measure $\mathrm{d}\mu$ is symmetric, the monic OP ϕ_n have real coefficients and it follows from previous proof that $f_n^{(12)} = f_n^{(21)} = 0$, $n \geqslant 1$. Moreover, in this case the Schur parameters are real and, then, the SOF are indeed strictly orthogonal.

Before continuing, it is useful to introduce a new notation.

Definition 2. The vector semi-orthogonal functions (VSOF) associated to the measure $\mathrm{d}\mu$ are the functions $\mathbf{f}_n:\mathbb{C}\backslash\{0\} \to \mathbb{C}^2$, $n \geqslant 0$, defined by

$$\mathbf{f}_n(z) = \begin{pmatrix} f_n^{(1)}(z) \\ f_n^{(2)}(z) \end{pmatrix},$$

where $f_0^{(1)}(z) = 1, f_0^{(2)}(z) = 0$ and $f_n^{(k)}(z)$, $n \geqslant 1$, $k = 1,2$, are the SOF related to $\mathrm{d}\mu$.

Remark 1. Proposition 1(i), that is given for $n \geqslant 1$, holds for $n=0$ too, being $f_0^{(11)}(x)=1, f_0^{(12)}(x) = f_0^{(21)}(x) = f_0^{(22)}(x) = 0$. We will refer to $(f_n^{(k)})_{n\geqslant 0, k=1,2}$, as the complete family of SOF associated to $\mathrm{d}\mu$.

Remark 2. The fact that $\mathscr{B}_n$ is a basis of $\Lambda_{-n,n}$ ensures that $(\mathbf{f}_m)_{m=0}^n$ is a set of generators for the modulus $\Lambda^2_{-n,n}$ over the ring $\mathbb{C}^{(2,2)}$ of 2×2 complex matrices. Hence, $(\mathbf{f}_m)_{m\geqslant 0}$ is a set of generators for Λ^2. Although $(\mathbf{f}_n)_{n\geqslant 0}$ is not a basis, it is not difficult to see that in the decomposition of an arbitrary element of Λ^2 as a linear combination of $(\mathbf{f}_n)_{n\geqslant 0}$, all the matrix coefficients are univocally determined excepting the one related to $\mathbf{f}_0$, whose first column is determined whereas the second one is arbitrary.

Definition 3. Given an arbitrary measure $\mathrm{d}\mu$ on $\mathbb{T}$ we define the sesquilinear functional $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\mathrm{d}\mu}$:$\Lambda^2\times\Lambda^2 \to \mathbb{C}^{(2,2)}$ in the following way

$$\langle\!\langle \mathbf{f},\mathbf{g} \rangle\!\rangle_{\mathrm{d}\mu} = \int_0^{2\pi} \mathbf{f}(\mathrm{e}^{\mathrm{i}\theta})\mathbf{g}(\mathrm{e}^{\mathrm{i}\theta})^*\,\mathrm{d}\mu(\theta), \quad \mathbf{f},\mathbf{g}\in\Lambda^2,$$

where, for an arbitrary matrix A, we write $A^* = \bar{A}^{\mathrm{T}}$ and the symbol T denotes the operation of transposition.

Remark 3. If $\mathbf{f} = \binom{f^{(1)}}{f^{(2)}}, \mathbf{g} = \binom{g^{(1)}}{g^{(2)}}$ with $f^{(k)}, g^{(k)} \in \Lambda$ for $k = 1,2$, then

$$\langle\!\langle \mathbf{f},\mathbf{g} \rangle\!\rangle_{\mathrm{d}\mu} = \begin{pmatrix} \langle f^{(1)}, g^{(1)}\rangle_{\mathrm{d}\mu} & \langle f^{(1)}, g^{(2)}\rangle_{\mathrm{d}\mu} \\ \langle f^{(2)}, g^{(1)}\rangle_{\mathrm{d}\mu} & \langle f^{(2)}, g^{(2)}\rangle_{\mathrm{d}\mu} \end{pmatrix}.$$

Notice that for all $\mathbf{f},\mathbf{g}\in\Lambda^2$ it is $\langle\!\langle z\mathbf{f},z\mathbf{g} \rangle\!\rangle_{\mathrm{d}\mu} = \langle\!\langle \mathbf{f},\mathbf{g} \rangle\!\rangle_{\mathrm{d}\mu}$ and $\langle\!\langle \mathbf{g},\mathbf{f} \rangle\!\rangle_{\mathrm{d}\mu} = \langle\!\langle \mathbf{f},\mathbf{g} \rangle\!\rangle^*_{\mathrm{d}\mu}$.

The orthogonality properties of SOF, translated to the language of VSOF, give the following result.

Proposition 2. *The VSOF associated to* $\mathrm{d}\mu$ *are orthogonal with respect to* $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\mathrm{d}\mu}$. *More precisely,*

$$\langle\!\langle \mathbf{f}_n,\mathbf{f}_m \rangle\!\rangle_{\mathrm{d}\mu} = C_n\delta_{n,m}, \quad n,m\geqslant 0,$$

where C_n *is given in* (8) *for* $n\geqslant 1$ *and* $C_0 = \varepsilon_0 C$ *with* $C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Definition 4. The Schur matrices associated to the measure $\mathrm{d}\mu$ are the following real symmetric traceless matrices:

$$H_n = \begin{pmatrix} \Re a_n & \Im a_n \\ \Im a_n & -\Re a_n \end{pmatrix}, \quad n\geqslant 0,$$

where a_n are the Schur parameters related to $\mathrm{d}\mu$.

Remark 4. Notice that we can write

$$C_n = \frac{\varepsilon_{2n-1}}{2^{2n-1}}(I - H_{2n}), \quad n\geqslant 1. \tag{10}$$

For $n \geqslant 1$, the condition $|a_n| < 1$, which is equivalent to $|\det H_n| < 1$, ensures that C_n is positive definite and, therefore, nonsingular. With above notation,

$$C_n^{-1} = \frac{2^{2n-1}}{\varepsilon_{2n}}(I + H_{2n}), \quad n \geqslant 1, \tag{11}$$

where we have used (4).

Bearing in mind that $(\mathbf{f}_m)_{m=0}^{n}$ is a set of generators for $\Lambda_{-n,n}$, we get from Proposition 2 the following consequence.

Corollary 1. *The VSOF associated to* $\mathrm{d}\mu$ *satisfy for* $n \geqslant 1$

$$\langle\!\langle \mathbf{f}_n, \mathbf{f} \rangle\!\rangle_{\mathrm{d}\mu} = 0, \quad \forall \mathbf{f} \in \Lambda^2_{-n+1,n-1}.$$

2. Recurrence relation for semi-orthogonal functions

The VSOF associated to a measure form a set of orthogonal vector Laurent polynomials, where the orthogonality is respect to some sesquilinear functional related to the measure. The natural question that arises is if, analogously to OP, they satisfy a three-term recurrence relation. The answer to this question is given in the following proposition.

Proposition 3. *The VSOF associated to* $\mathrm{d}\mu$ *satisfy the recurrence relation*

$$z\mathbf{f}_n(z) = (I + \mathrm{i}J)\mathbf{f}_{n+1}(z) + \mathbf{L}_n \mathbf{f}_n(z) + \mathbf{M}_n \mathbf{f}_{n-1}(z), \quad n \geqslant 1,$$

where I *is the* 2×2 *identity matrix and*

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\mathbf{L}_n = \frac{1}{2}\{(I - H_{2n})H_{2n-1}(I + \mathrm{i}J) - (I + \mathrm{i}J)H_{2n+1}(I + H_{2n})\}, \quad n \geqslant 1,$$

$$\mathbf{M}_n = \frac{1}{4}\det(I - H_{2n-1})(I - H_{2n})(I - \mathrm{i}J)(I + H_{2n-2}), \quad n \geqslant 1,$$

being H_n *the Schur matrices related to* $\mathrm{d}\mu$.

Proof. As it is usual, we begin by decomposing $z\mathbf{f}_n$ with respect to the set of generators $(\mathbf{f}_n)_{n\geqslant 0}$. Since $z\mathbf{f}_n \in \Lambda^2_{-n+1,n+1} \subset \Lambda^2_{-n-1,n+1}$, it is obvious that

$$z\mathbf{f}_n(z) = \sum_{k=0}^{n+1} A_k^{(n)} \mathbf{f}_k(z), \quad A_k^{(n)} \in \mathbb{C}^{(2,2)}, \quad n \geqslant 1.$$

From Remark 2 we know that the matrix coefficients $A_k^{(n)}$ are univocally determined for $k \geqslant 1$, while for $A_0^{(n)}$ only the product $A_0^{(n)}\mathbf{f}_0$ is fixed, that is, the first column of $A_0^{(n)}$ is determined whereas the second one is arbitrary.

Now, by projecting above decomposition of $z\mathbf{f}_n$ over $\mathbf{f}_j$, with $j \leqslant n+1$, we find that $\langle\!\langle z\mathbf{f}_n, \mathbf{f}_j \rangle\!\rangle_{\mathrm{d}\mu} = A_j^{(n)} C_j$.

When $j = 0$ and $n \geqslant 2$ we get $A_0^{(n)} C_0 = \langle\!\langle \mathbf{f}_n, z^{-1}\mathbf{f}_0 \rangle\!\rangle_{\mathrm{d}\mu} = 0$ due to Corollary 1. Therefore, $A_0^{(n)}\mathbf{f}_0 = A_0^{(n)} C \mathbf{f}_0 = 0$.

If $1 \leqslant j \leqslant n-2$, then $A_j^{(n)} C_j = \langle\!\langle \mathbf{f}_n, z^{-1}\mathbf{f}_j \rangle\!\rangle_{\mathrm{d}\mu} = 0$ again by means of Corollary 1. Now, the non-singularity of C_j for $j \geqslant 1$ forces $A_j^{(n)} = 0$.

Hence, we can write

$$z\mathbf{f}_n(z) = A_{n+1}^{(n)}\mathbf{f}_{n+1}(z) + A_n^{(n)}\mathbf{f}_n(z) + A_{n-1}^{(n)}\mathbf{f}_{n-1}(z), \quad n \geqslant 1, \tag{12}$$

where the only indetermination is in the second column of $A_0^{(1)}$ that can be arbitrarily choosen. Notice that the coefficients $A_{n-1}^{(n)}$ and $A_n^{(n-1)}$ are related for $n \geqslant 1$ by $A_{n-1}^{(n)} C_{n-1} = \langle\!\langle z\mathbf{f}_n, \mathbf{f}_{n-1} \rangle\!\rangle_{\mathrm{d}\mu} = \langle\!\langle z^{-1}\mathbf{f}_{n-1}, \mathbf{f}_n \rangle\!\rangle_{\mathrm{d}\mu}^* = \langle\!\langle z\mathbf{f}_{n-1}, \mathbf{f}_n \rangle\!\rangle_{\mathrm{d}\mu}^{\mathrm{T}} = C_n (A_n^{(n-1)})^{\mathrm{T}}$, where we have used the fact that VSOF, like SOF, are real on $\mathbb{T}$. Thus,

$$\begin{aligned} A_0^{(1)}\mathbf{f}_0 &= \varepsilon_0^{-1} A_0^{(1)} C_0 \mathbf{f}_0 = \varepsilon_0^{-1} C_1 (A_1^{(0)})^{\mathrm{T}} \mathbf{f}_0, \\ A_{n-1}^{(n)} &= C_n (A_n^{(n-1)})^{\mathrm{T}} C_{n-1}^{-1}. \end{aligned} \tag{13}$$

This means that we only need to calculate $A_{n+1}^{(n)}$ and $A_n^{(n)}$. To this end we introduce the following elements of Λ^2

$$\mathbf{g}_0(z) = \mathbf{f}_0(z)$$

$$\mathbf{g}_n(z) = \begin{pmatrix} z^n \\ z^{-n} \end{pmatrix}, \quad n \geqslant 1,$$

so that, $(\mathbf{g}_n)_{n\geqslant 0}$ is a set of generators for the module Λ^2 with decomposition properties similar to those above described for $(\mathbf{f}_n)_{n\geqslant 0}$. Thus, we can decompose both sides of (12) in $(\mathbf{g}_n)_{n\geqslant 0}$ and, then, equal coefficients of $\mathbf{g}_n$ for $n \geqslant 1$.

We begin with the decomposition of $\mathbf{f}_n$ for $n \geqslant 1$. If $Q = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$, then

$$\begin{aligned} \mathbf{f}_n(z) &= 2^{-n} Q \begin{pmatrix} z^{1-n}\phi_{2n-1}(z) \\ z^{-n}\phi_{2n-1}^*(z) \end{pmatrix} \\ &= 2^{-n} Q \left\{ \mathbf{g}_n(z) + \begin{pmatrix} b_{2n-1} & a_{2n-1} \\ \bar{a}_{2n-1} & \bar{b}_{2n-1} \end{pmatrix} \mathbf{g}_{n-1}(z) + \cdots \right\}, \end{aligned} \tag{14}$$

where the dots mean terms belonging to $\Lambda^2_{-n+2,n-2}$ for $n \geqslant 2$ and no terms for $n = 1$. Besides, we need the decomposition of $z\mathbf{f}_n$ for $n \geqslant 1$

$$z\mathbf{f}_n(z) = 2^{-n}Q\begin{pmatrix} z^{2-n}\phi_{2n-1}(z) \\ z^{1-n}\phi^*_{2n-1}(z) \end{pmatrix}$$
$$= 2^{-n}Q\left\{C_0\mathbf{g}_{n+1}(z) + \begin{pmatrix} b_{2n-1} & a_{2n-1} \\ \bar{a}_{2n-1} & \bar{b}_{2n-1} \end{pmatrix} C_0\mathbf{g}_n(z) + \cdots\right\}, \tag{15}$$

where now the dots mean terms belonging to $\Lambda^2_{-n+1,n-1}$.

Introducing (14) and (15) into (12) and equaling coefficients of $\mathbf{g}_{n+1}$ and $\mathbf{g}_n$, lead to

$$QC_0 = \frac{1}{2}A^{(n)}_{n+1}Q,$$

$$Q\begin{pmatrix} b_{2n-1} & a_{2n-1} \\ \bar{a}_{2n-1} & \bar{b}_{2n-1} \end{pmatrix} C_0 = \frac{1}{2}A^{(n)}_{n+1}Q\begin{pmatrix} b_{2n+1} & a_{2n+1} \\ \bar{a}_{2n+1} & \bar{b}_{2n+1} \end{pmatrix} + A^{(n)}_n Q,$$

which have the solutions

$$A^{(n)}_{n+1} = 2QC_0Q^{-1},$$

$$A^{(n)}_n = Q\begin{pmatrix} b_{2n-1} - b_{2n+1} & -a_{2n+1} \\ \bar{a}_{2n-1} & 0 \end{pmatrix} Q^{-1}$$
$$= Q\left\{\bar{a}_{2n-1}\begin{pmatrix} -a_{2n} & 0 \\ 1 & 0 \end{pmatrix} - a_{2n+1}\begin{pmatrix} \bar{a}_{2n} & 1 \\ 0 & 0 \end{pmatrix}\right\} Q^{-1}$$
$$= Q\left\{\bar{a}_{2n-1}\begin{pmatrix} 1 & -a_{2n} \\ -\bar{a}_{2n} & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - a_{2n+1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & a_{2n} \\ \bar{a}_{2n} & 1 \end{pmatrix}\right\} Q^{-1}.$$

Here we have used relation (2) between a_n and b_n. At this point it is useful to notice that

$$QC_0Q^{-1} = \frac{1}{2}(I + \mathrm{i}J), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$Q\begin{pmatrix} 1 & \pm a_n \\ \pm\bar{a}_n & 1 \end{pmatrix} Q^{-1} = I \pm H_n, \quad n \geqslant 0,$$

$$\bar{a}_n Q\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} Q^{-1} = \frac{1}{2}H_n(I + \mathrm{i}J), \quad n \geqslant 0,$$

$$a_n Q\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Q^{-1} = \frac{1}{2}(I + \mathrm{i}J)H_n, \quad n \geqslant 0,$$

where H_n are the Schur matrices related to $\mathrm{d}\mu$. Thus, we finally find

$$\begin{aligned} A_{n+1}^{(n)} &= I + \mathrm{i}J, \quad n \geqslant 1, \\ A_n^{(n)} &= \frac{1}{2}\{(I - H_{2n})H_{2n-1}(I + \mathrm{i}J) - (I + \mathrm{i}J)H_{2n+1}(I + H_{2n})\}, \quad n \geqslant 1. \end{aligned} \tag{16}$$

Now, by using (4), (10), (11), (13) and (16), we see that

$$A_0^{(1)}\mathbf{f}_0 = \frac{1}{2}(1 - |a_1|^2)(I - H_2)(I - \mathrm{i}J)\mathbf{f}_0,$$

$$A_{n-1}^{(n)} = \frac{1}{4}(1 - |a_{2n-1}|^2)(I - H_{2n})(I - \mathrm{i}J)(I + H_{2n-2}), \quad n \geqslant 2.$$

It is possible to choose arbitrarily the second column of $A_0^{(1)}$, so we can fix it to be null, that is

$$A_0^{(1)} = \frac{1}{2}(1 - |a_1|^2)(I - H_2)(I - \mathrm{i}J)C = \frac{1}{4}(1 - |a_1|^2)(I - H_2)(I - \mathrm{i}J)(I + H_0).$$

Hence, all the coefficients $A_{n-1}^{(n)}$ can be given by

$$A_{n-1}^{(n)} = \frac{1}{4}\det(I - H_{2n-1})(I - H_{2n})(I - \mathrm{i}J)(I + H_{2n-2}), \quad n \geqslant 1.$$

Since $\mathbf{L}_n = A_n^{(n)}$ and $\mathbf{M}_n = A_{n-1}^{(n)}$ the proposition is proved. $\square$

3. Semi-orthogonal functions and matrix measures

Now we are going to translate previous results on Λ^2 to the modulus $\mathscr{P}^{(2,2)}$ of 2×2 matrices with coefficients in $\mathscr{P}$. To do that we will associate a sequence of matrix polynomials to any family of SOF.

Definition 5. The matrix polynomials $\mathbf{F}_n \in \mathscr{P}^{(2,2)}$, $n \geqslant 0$, associated to the measure $\mathrm{d}\mu$, are defined by

$$\mathbf{F}_n(x) = \begin{pmatrix} f_n^{(11)}(x) & f_n^{(12)}(x) \\ f_n^{(21)}(x) & f_n^{(22)}(x) \end{pmatrix},$$

where $f_n^{(k)}(z) = f_n^{(k1)}(x) + y f_n^{(k2)}(x)$, $n \geqslant 0$, $k = 1, 2$, is the unique decomposition given in Proposition 1 and Remark 1 for the complete family of SOF related to $\mathrm{d}\mu$.

Remark 5. The VSOF and the matrix polynomials associated to $\mathrm{d}\mu$ are related by

$$\mathbf{f}_n(z) = \mathbf{F}_n(x)\begin{pmatrix} 1 \\ y \end{pmatrix}, \quad n \geqslant 0, \tag{17}$$

and, thus,

$$(\mathbf{f}_n(z) \quad \mathbf{f}_n(z^{-1})) = \mathbf{F}_n(x) \begin{pmatrix} 1 & 1 \\ y & -y \end{pmatrix}, \quad n \geqslant 0,$$

from which we get

$$\mathbf{F}_n(x) = \frac{1}{2} (\mathbf{f}_n(z) \quad \mathbf{f}_n(z^{-1})) \begin{pmatrix} 1 & y^{-1} \\ 1 & -y^{-1} \end{pmatrix}, \quad n \geqslant 0, \tag{18}$$

for $x \neq \pm 1$.

It is natural to expect for the above matrix polynomials to inherit some orthogonality properties from the ones satisfied by the corresponding SOF. To see this we introduce the following matrix measure.

Definition 6. Given an arbitrary measure $d\mu$ on $\mathbb{T}$ the matrix measure $d\Omega$ associated to $d\mu$ is the following 2×2 symmetric matrix measure on $[-1, 1]$:

$$d\Omega(x) = \frac{1}{2} \begin{pmatrix} d\rho(x) & \sqrt{1-x^2}\, d\sigma(x) \\ \sqrt{1-x^2} d\sigma(x) & (1-x^2)\, d\rho(x) \end{pmatrix}, \tag{19}$$

where $d\rho$ and $d\sigma$ are scalar measures on $[-1, 1]$ given by

$$d\rho(x) = d\nu_1(x) + d\nu_2(x),$$

$$d\sigma(x) = d\nu_1(x) - d\nu_2(x),$$

and $d\nu_1$, $d\nu_2$ are the projected measures of $d\mu$ defined in (6).

Remark 6. Notice that a matrix measure $d\Omega$ with form (19), being $d\rho$ and $d\sigma$ arbitrary scalar measures on $[-1, 1]$, is always associated to some measure $d\mu$ on $\mathbb{T}$. The related measure $d\mu$ is positive iff $d\nu_1$, $d\nu_2$ so are, which holds iff $|d\sigma| \leqslant d\rho$ (this implies that $d\rho$ is positive and that $\mathrm{supp}(d\sigma) \subset \mathrm{supp}(d\rho)$). Therefore, when $d\mu$ is positive it has an infinite support iff $d\rho$ so does.

Now, the results in Proposition 1 for SOF have the following consequences for the corresponding matrix polynomials.

Proposition 4. *The matrix polynomials* $(\mathbf{F}_n)_{n \geqslant 0}$ *and the matrix measure* $d\Omega$ *associated to* $d\mu$ *satisfy*:

(i) $\deg \mathbf{F}_n = n$. *More precisely*, $\mathbf{F}_0(x) = C$, *with the matrix* C *as in Proposition* 2, *and*

$$\mathbf{F}_{n+1}(x) = Cx^{n+1} + \begin{pmatrix} \eta_n & 0 \\ \gamma_n & 1 \end{pmatrix} x^n + \cdots, \quad n \geqslant 0,$$

where the dots mean terms with degree less than n and

$$\eta_n = \frac{1}{2}\,\Re(a_{2n+1}+b_{2n+1}), \qquad \gamma_n = \frac{1}{2}\,\Im(a_{2n+1}+b_{2n+1}),$$

being a_n the Schur parameters related to $\mathrm{d}\mu$ *and b_n given in* (3),

(ii) $\int_{-1}^{1} \mathbf{F}_n(x)\,\mathrm{d}\Omega(x)\,\mathbf{F}_m^{\mathrm{T}}(x) = \frac{1}{2}\,C_n\delta_{n,m},\ \ n,m \geqslant 0.$

Proof. Result (i) follows straightforward from Proposition 1(i) and Definition 5, wherefrom we see that γ_n is the leading coefficient of $f_{n+1}^{(21)}(x)$, while η_n is the coefficient of x^n in $f_{n+1}^{(11)}(x)$. The expression (9) for $f_n^{(1)}$ shows that $\eta_n = \frac{1}{2}\,\Re(a_{2n+1}+b_{2n+1})$ and a similar expression for $f_n^{(2)}$ gives $\gamma_n = \frac{1}{2}\,\Im(a_{2n+1}+b_{2n+1})$.

To prove (ii) it is enough to notice that, using (17), we get from Definition 3

$$\begin{aligned}\langle\!\langle \mathbf{f}_n, \mathbf{f}_m \rangle\!\rangle_{\mathrm{d}\mu} &= \int_0^{2\pi} \mathbf{f}_n(\mathrm{e}^{\mathrm{i}\theta})\mathbf{f}_m^{\mathrm{T}}(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\mu(\theta)\\ &= \int_{-1}^{1} \mathbf{F}_n(x) \begin{pmatrix} 1 & \sqrt{1-x^2} \\ \sqrt{1-x^2} & 1-x^2 \end{pmatrix} \mathbf{F}_m^{\mathrm{T}}(x)\,\mathrm{d}v_1(x)\\ &\quad + \int_{-1}^{1} \mathbf{F}_n(x) \begin{pmatrix} 1 & -\sqrt{1-x^2} \\ -\sqrt{1-x^2} & 1-x^2 \end{pmatrix} \mathbf{F}_m^{\mathrm{T}}(x)\,\mathrm{d}v_2(x),\end{aligned}$$

with the positive choice for the square root. Taking into account Definition 6 we see that

$$\langle\!\langle \mathbf{f}_n, \mathbf{f}_m \rangle\!\rangle_{\mathrm{d}\mu} = 2\int_{-1}^{1} \mathbf{F}_n(x)\,\mathrm{d}\Omega(x)\mathbf{F}_m^{\mathrm{T}}(x),$$

and Proposition 2 gives (ii). □

Remark 7. From Proposition 4(i) we see that

$$(I-C)\mathbf{F}_{n+1}(x) + \mathbf{F}_n(x) = \Gamma_n x^n + \cdots, \qquad \Gamma_n = \begin{pmatrix} 1 & 0 \\ \gamma_n & 1 \end{pmatrix}, \qquad n \geqslant 0, \tag{20}$$

where, again, the dots mean terms with degree less than n. So, it is obvious that every element of $\mathscr{P}_n^{(2,2)}$ is a linear combination of $(\mathbf{F}_m)_{m=0}^{n+1}$, and, therefore, $(\mathbf{F}_n)_{n\geqslant 0}$ is a set of generators for $\mathscr{P}^{(2,2)}$.

Unfortunately, in spite of Proposition 4(ii), we cannot say that $(\mathbf{F}_n)_{n\geqslant 0}$ is a sequence of left orthogonal matrix polynomials with respect to $\mathrm{d}\Omega$. A sequence $(\mathbf{P}_n)_{n\geqslant 0}$ of 2×2 real matrix polynomials is called a sequence of left orthogonal matrix polynomials (LOMP) with respect to $\mathrm{d}\Omega$ if it satisfies [6,17,23]:

(I) $\deg \mathbf{P}_n = n$, and the leading coefficient of $\mathbf{P}_n$ is nonsingular.

(II) $\int_{-1}^{1} \mathbf{P}_n(x)\,\mathrm{d}\Omega(x)x^k = 0$ for $0 \leqslant k < n$ and $\int_{-1}^{1} \mathbf{P}_n(x)\,\mathrm{d}\Omega(x)x^n$ is nonsingular.

However, the leading coefficient C of $\mathbf{F}_n$ is singular. Even more, although

$$\int_{-1}^{1} \mathbf{F}_n(x)\,\mathrm{d}\Omega(x)x^k = 0, \quad 0 \leqslant k \leqslant n-2, \tag{21}$$

since $\mathbf{F}_n$ is orthogonal to $\operatorname{span}\{\mathbf{F}_0, \mathbf{F}_1, \ldots, \mathbf{F}_{n-1}\} \supset \mathscr{P}^{(2,2)}_{n-2}$, we have from (20) that

$$\int_{-1}^{1} \mathbf{F}_n(x)\,\mathrm{d}\Omega(x)x^{n-1} = \frac{1}{2}C_n(I-C)^{\mathrm{T}}(\Gamma_{n-1}^{-1})^{\mathrm{T}} = \frac{1}{2}C_n(I-C), \quad n \geqslant 1. \tag{22}$$

In other words, all what we can say is that $\mathbf{F}_n$ is what we could call a left quasi-orthogonal matrix polynomial of order n with respect to $\mathrm{d}\Omega$, that is, a nonnull matrix polynomial with $\deg \mathbf{F}_n \leqslant n$ and left orthogonal to $I, Ix, Ix^2, \ldots, Ix^{n-2}$ with respect to the measure $\mathrm{d}\Omega$ (see for example [5] or [9] for introducing the analogous conception in the scalar case).

Notice that, when the measure $\mathrm{d}\mu$ is symmetric, both $\mathbf{F}_n$ and $\mathrm{d}\Omega$ are diagonal. Then, our quasi-orthogonal matrix polynomials provide two sequences of scalar OP on $[-1,1]$, that is, we recover Szegő's result.

The complex recurrence formula for VSOF provides two real recurrence relations for the corresponding matrix polynomials.

Proposition 5. *The matrix polynomials associated to* $\mathrm{d}\mu$ *satisfy the recurrence relations*

$$\begin{aligned} x\mathbf{F}_n(x) &= \mathbf{F}_{n+1}(x) + L_n\mathbf{F}_n(x) + M_n\mathbf{F}_{n-1}(x), \quad n \geqslant 1, \\ \mathbf{F}_n(x)Y(x) &= J\mathbf{F}_{n+1}(x) + \tilde{L}_n\mathbf{F}_n(x) + \tilde{M}_n\mathbf{F}_{n-1}(x), \quad n \geqslant 1, \end{aligned} \tag{23}$$

where

$$Y(x) = \begin{pmatrix} 0 & 1 \\ 1-x^2 & 0 \end{pmatrix},$$

$$M_n = \frac{1}{4}\det(I - H_{2n-1})(I - H_{2n})(I + H_{2n-2}),$$

$$\tilde{M}_n = -\frac{1}{4}\det(I - H_{2n-1})(I - H_{2n})J(I + H_{2n-2}),$$

$$L_n = \frac{1}{2}\{(I - H_{2n})H_{2n-1} - H_{2n+1}(I + H_{2n})\},$$

$$\tilde{L}_n = \frac{1}{2}\{(I - H_{2n})H_{2n-1}J - JH_{2n+1}(I + H_{2n})\}, \tag{24}$$

and H_n *are the Schur matrices related to* $\mathrm{d}\mu$.

Proof. VSOF, like SOF, satisfy $\bar{\mathbf{f}}_n(z^{-1}) = \mathbf{f}_n(z)$. Therefore, from the recurrence relation in Proposition 3 we get

$$z^{-1}\mathbf{f}_n(z) = (I - \mathrm{i}J)\mathbf{f}_{n+1}(z) + \bar{\mathbf{L}}_n\mathbf{f}_n(z) + \bar{\mathbf{M}}_n\mathbf{f}_{n-1}(z).$$

Appropriate linear combinations of this new recurrence relation and the original one give

$$\begin{aligned} x\mathbf{f}_n(z) &= \mathbf{f}_{n+1}(z) + \Re\mathbf{L}_n\mathbf{f}_n(z) + \Re\mathbf{M}_n\mathbf{f}_{n-1}(z), \\ y\mathbf{f}_n(z) &= J\mathbf{f}_{n+1}(z) + \Im\mathbf{L}_n\mathbf{f}_n(z) + \Im\mathbf{M}_n\mathbf{f}_{n-1}(z). \end{aligned} \tag{25}$$

From (17) we get

$$y\mathbf{f}_n(z) = \mathbf{F}_n(x)Y(x)\begin{pmatrix} 1 \\ y \end{pmatrix}, \quad Y(x) = \begin{pmatrix} 0 & 1 \\ 1-x^2 & 0 \end{pmatrix}. \tag{26}$$

Introducing (17) and (26) in (25) we find two relations for $\mathbf{F}_n$ with the form

$$A(x)\begin{pmatrix} 1 \\ y \end{pmatrix} = B(x)\begin{pmatrix} 1 \\ y \end{pmatrix}, \quad A(x), B(x) \in \mathscr{P}^{(2,2)},$$

that are true if $x=(z+z^{-1})/2$, $y=(z-z^{-1})/2\mathrm{i}$ for any $z \neq 0$. Evaluating in z and z^{-1} we obtain

$$A(x)\begin{pmatrix} 1 & 1 \\ y & -y \end{pmatrix} = B(x)\begin{pmatrix} 1 & 1 \\ y & -y \end{pmatrix}$$

and, therefore, it must be $A(x)=B(x)$. Taking into account the expressions for $\mathbf{M}_n$ and $\mathbf{L}_n$ given in Proposition 3, we see that the two equalities that we find in this way are exactly the desired recurrence relations for $\mathbf{F}_n$. $\square$

From Proposition 4 we see that

$$\mathbf{F}_0(x) = C,$$

$$\mathbf{F}_1(x) = (xI - I + H_1)C + I. \tag{27}$$

Therefore, starting from the Schur matrices H_n associated to $\mathrm{d}\mu$, the first recurrence relation in (23), together with the expressions (27) for the two first matrix polynomials, let us obtain the complete sequence of matrix polynomials associated to $\mathrm{d}\mu$. Conversely, suppose that we have an arbitrary sequence $(H_n)_{n\geqslant 0}$ of 2×2 real symmetric traceless matrices with $H_0 = 2C - I$ and $|\det H_n| < 1$ for $n \geqslant 1$. If a sequence $(\mathbf{F}_n)_{n\geqslant 0}$ of matrix polynomials satisfies a recurrence relation like the first one in (23) with M_n and L_n given by (24) and the initial conditions (27), then the matrix polynomials $(\mathbf{F}_n)_{n\geqslant 0}$ are associated to some measure on $\mathbb{T}$ (and, therefore, they are left quasi-orthogonal with respect some matrix measure on $[-1,1]$). To see this, just notice that the matrix sequence $(H_n)_{n\geqslant 0}$ provides, through the relation $a_n = H_n^{(11)} + \mathrm{i}H_n^{(12)}$, a complex sequence $(a_n)_{n\geqslant 0}$ such that $a_0 = 1$ and $|a_n| < 1$ for $n \geqslant 1$ ($A^{(kj)}$ denotes the (k,j)th element of the matrix A). Now, it is well known that this conditions for a_n ensure that the complex polynomials $(\phi_n)_{n\geqslant 0}$ defined by (1) form a sequence of monic OP with respect to some positive measure $\mathrm{d}\mu$ on $\mathbb{T}$. We have shown that the measure $\mathrm{d}\mu$ generates an associated matrix measure $\mathrm{d}\Omega$ on $[-1,1]$ and that the OP $(\phi_n)_n$ let us construct a sequence of left quasi-orthogonal matrix polynomials with respect to $\mathrm{d}\Omega$ satisfying (23), (24), and (27). This sequence must be $(\mathbf{F}_n)_{n\geqslant 0}$.

4. Semiorthogonal functions and left orthogonal matrix polynomials

We have discovered that the generalization of Szegő's method leads in general, not to a sequence of scalar OP, neither a sequence of LOMP, but to a sequence $(\mathbf{F}_n)_{n\geqslant 0}$ of left quasi-orthogonal matrix polynomials with respect to the matrix measure $\mathrm{d}\Omega$. However, if $\mathrm{d}\Omega$ is a positive matrix measure then there exists a sequence of LOMP with respect to $\mathrm{d}\Omega$ iff [6],

$$\int_{-1}^{1} p^{\mathrm{T}}(x)\,\mathrm{d}\Omega(x)\,p(x) \neq 0, \quad \forall p \in \mathbb{C}^{(2,1)}[x]\backslash\{0\}. \tag{28}$$

As we see in the following proposition, when a matrix measure is associated to a measure on $\mathbb{T}$, it is possible to give simple conditions equivalent to (28).

Proposition 6. *Let* $\mathrm{d}\mu$ *be an arbitrary measure on* $\mathbb{T}$ *and let* $\mathrm{d}\Omega$ *be the matrix measure associated to* $\mathrm{d}\mu$. *Then,* $\mathrm{d}\Omega$ *is positive iff* $\mathrm{d}\mu$ *is positive. Moreover, when* $\mathrm{d}\mu$ *is positive the following statements are equivalent*:

(i) *There exists a sequence of LOMP with respect to* $\mathrm{d}\Omega$.
(ii) *There exists a sequence of OP with respect to* $\mathrm{d}\mu$.
(iii) $\mathrm{supp}(\mathrm{d}\mu)$ *is infinite*.
(iv) $\mathrm{supp}(\mathrm{d}\Omega)$ *is infinite*.

Proof. Let us suppose that $\mathrm{d}\mu$ is positive. Then $|\mathrm{d}\sigma| \leqslant \mathrm{d}\rho$ and, thus, $\mathrm{d}\rho$ is positive (see Remark 6). In order to prove the positivity of $\mathrm{d}\Omega$ we have just to see that $\int_a^b \mathrm{d}\Omega(x)$ is a nonnegative definite matrix for all $a,b \in [-1,1]$, which is equivalent to say that its trace and determinant are both nonnegative. Since $\mathrm{d}\rho$ is positive,

$$\mathrm{tr}\int_a^b \mathrm{d}\Omega(x) = \int_a^b (2-x^2)\mathrm{d}\rho(x) \geqslant 0.$$

Moreover, taking into account that $|\mathrm{d}\sigma| \leqslant \mathrm{d}\rho$ we get that

$$\det\int_a^b \mathrm{d}\Omega(x) = \int_a^b \mathrm{d}\rho(x)\int_a^b (1-x^2)\,\mathrm{d}\rho(x) - \left(\int_a^b \sqrt{1-x^2}\mathrm{d}\sigma(x)\right)^2$$
$$\geqslant \int_a^b \mathrm{d}\rho(x)\int_a^b (1-x^2)\mathrm{d}\rho(x) - \left(\int_a^b \sqrt{1-x^2}\mathrm{d}\rho(x)\right)^2 \geqslant 0,$$

where we have used the Schwarz's inequality. Thus, if $\mathrm{d}\mu$ is positive then $\mathrm{d}\Omega$ is positive too.

To see the converse first notice that if $p \in \mathbb{C}^{(2,1)}[x]$ then $f(z)=(1,y)p(x)$ is a Laurent polynomial. Even more, for every $f \in \Lambda$ there is a unique decomposition $f(z)=(1,y)p(x)$, $p \in \mathbb{C}^{(2,1)}[x]$. This decomposition holds iff $p^{\mathrm{T}}=(p_1,p_2)$ with $p_1(x)=(f(z)+f(z^{-1}))/2$ and $p_2(x)=(f(z)-f(z^{-1}))/2y$. Now, using the Tchebychev polynomials of first and second kind we see that $p_1,p_2 \in \mathbb{C}[x]$. This provides an isomorphism between Λ and $\mathbb{C}^{(2,1)}[x]$. Let us consider an arbitrary $f \in \Lambda$ and the corresponding $p \in \mathbb{C}^{(2,1)}[x]$. Then,

$$\int_0^{2\pi} |f(\mathrm{e}^{\mathrm{i}\theta})|^2\mathrm{d}\mu(\theta) = 2\int_{-1}^{1} p^{\mathrm{T}}(x)\,\mathrm{d}\Omega(x)\,p(x).$$

If $d\Omega$ is a positive matrix measure then $\int_{-1}^{1} p^{T}(x)\, d\Omega(x)\, p(x) \geqslant 0$ for all $p \in \mathbb{C}^{(2,1)}[x]$. Therefore, $\int_0^{2\pi} |f(e^{i\theta})|^2 d\mu(\theta) \geqslant 0$ for all $f \in \Lambda$ and, thus, $d\mu$ is positive too.

Now, assume that $d\mu$ is positive. The equivalence between (ii) and (iii) is known. From (28) and above results we see that the statement (i) means that $\int_0^{2\pi} |f(e^{i\theta})|^2 d\mu(\theta) \neq 0$ for all $f \in \Lambda \backslash \{0\}$, which holds iff $d\mu$ has an infinite support. So it is proved that (i) is equivalent to (iii). The equivalence between (iii) and (iv) is just a consequence of the following facts that are true for any positive measure $d\mu$ (see Remark 6): $\operatorname{supp}(d\rho)$ is infinite iff $\operatorname{supp}(d\mu)$ so is; $\operatorname{supp}(d\sigma) \subset \operatorname{supp}(d\rho)$ and, hence, $\operatorname{supp}(d\Omega) = \operatorname{supp}(d\rho)$.

In what follows we will suppose again that $d\mu$ is a positive measure on $\mathbb{T}$ with infinitely many points in the support. Then, the next proposition gives a sequence of LOMP with respect to the related matrix measure in terms of the associated matrix polynomials.

Proposition 7. *Let* $d\Omega$ *and* $(\mathbf{F}_n)_{n\geqslant 0}$ *be the matrix measure and the matrix polynomials associated to* $d\mu$, *respectively. Then, the matrix polynomials* $(\mathbf{P}_n)_{n\geqslant 0}$ *given by*

$$\mathbf{P}_n(x) = \alpha_n \mathbf{F}_{n+1}(x) + \beta_n \mathbf{F}_n(x), \quad n \geqslant 0,$$

$$\alpha_n = I - C, \quad \beta_n = \begin{pmatrix} 1 & r_n \\ 0 & 0 \end{pmatrix},$$

$$r_n = \Im a_{2n}/(1 + \Re a_{2n}),$$

define a sequence of LOMP with respect $d\Omega$, *where* a_n *are the Schur parameters related to* $d\mu$. *Moreover*,

$$\mathbf{F}_n(x) = \tilde{\alpha}_n \mathbf{P}_n(x) + \tilde{\beta}_n \mathbf{P}_{n-1}(x), \quad n \geqslant 0,$$

$$\tilde{\alpha}_n = C, \quad \tilde{\beta}_n = \begin{pmatrix} 0 & -r_n \\ 0 & 1 \end{pmatrix},$$

with the convention $\mathbf{P}_{-1} = 0$.

Proof. Let $(\mathbf{P}_n)_{n\geqslant 0}$ be an arbitrary sequence of LOMP with respect to $d\Omega$. From the algebraic and orthogonality properties of $\mathbf{F}_n$ and $\mathbf{P}_n$ we have that, for $n \geqslant 0$

$$\begin{aligned} \mathbf{P}_n(x) &= \alpha_n \mathbf{F}_{n+1}(x) + \beta_n \mathbf{F}_n(x), \quad \alpha_n, \beta_n \in \mathbb{R}^{(2,2)}, \\ \mathbf{F}_n(x) &= \tilde{\alpha}_n \mathbf{P}_n(x) + \tilde{\beta}_n \mathbf{P}_{n-1}(x), \quad \tilde{\alpha}_n, \tilde{\beta}_n \in \mathbb{R}^{(2,2)}, \end{aligned} \tag{29}$$

where $\mathbf{P}_{-1} = 0$ and the matrix coefficients must satisfy the relations

$$\begin{aligned} \alpha_n \tilde{\alpha}_{n+1} &= \beta_n \tilde{\beta}_n = 0, \\ \alpha_n \tilde{\beta}_{n+1} + \beta_n \tilde{\alpha}_n &= I. \end{aligned} \tag{30}$$

Now, we proceed to determine α_n, β_n, $\tilde{\alpha}_n$, $\tilde{\beta}_n$ by imposing on $\mathbf{P}_n$ the conditions (I) and (II) given after Remark 7.

From (29) and Proposition 4(i) we get

$$\mathbf{P}_n(x) = \alpha_n C x^{n+1} + \left\{ \alpha_n \begin{pmatrix} \eta_n & 0 \\ \gamma_n & 1 \end{pmatrix} + \beta_n C \right\} x^n + \cdots, \quad n \geqslant 0,$$

where the dots mean terms with degree less than n. Therefore, (I) is equivalent to

$$\begin{aligned} &\alpha_n C = 0, \quad n \geqslant 0, \\ &\alpha_n \begin{pmatrix} \eta_n & 0 \\ \gamma_n & 1 \end{pmatrix} + \beta_n C \quad \text{nonsingular}, \ n \geqslant 0. \end{aligned} \tag{31}$$

The quasi-orthogonality of $\mathbf{F}_n$ implies that, for all $\alpha_n, \beta_n \in \mathbb{R}^{(2,2)}$, the matrix polynomial $\mathbf{P}_n$ given in (29) is orthogonal to Ix^k, $0 \leqslant k \leqslant n-2$, with respect to $\mathrm{d}\Omega$. For Ix^{n-1}, we can use (21) and (22) to obtain

$$\int_{-1}^{1} \mathbf{P}_n(x) \mathrm{d}\Omega(x) x^{n-1} = \frac{1}{2} \beta_n C_n (I - C), \quad n \geqslant 1.$$

Since this integral must vanish, with the aid of (20) we find

$$\int_{-1}^{1} \mathbf{P}_n(x)\, \mathrm{d}\Omega(x) x^n = \frac{1}{2} \{\alpha_n C_{n+1}(I - C) + \beta_n C_n\} (\Gamma_n^{-1})^{\mathrm{T}}, \quad n \geqslant 0.$$

Hence, (II) is equivalent to

$$\begin{aligned} &\beta_n C_n (I - C) = 0, \quad n \geqslant 1, \\ &\alpha_n C_{n+1}(I - C) + \beta_n C_n \quad \text{nonsingular}, \ n \geqslant 0. \end{aligned} \tag{32}$$

If $V_n \in \mathbb{R}^{(2,2)}$ is nonsingular for all $n \geqslant 0$, then

$$\alpha_n = V_n (I - C), \quad n \geqslant 0,$$

$$\beta_0 = V_0 C,$$

$$\beta_n = V_n C C_n^{-1}, \quad n \geqslant 1,$$

are solutions of (31) and (32). The expressions given in the proposition for α_n and β_n correspond to the choice $V_0 = I$ and

$$V_n = \begin{pmatrix} \frac{1}{(C_n^{-1})^{(11)}} & 0 \\ 0 & 1 \end{pmatrix}, \quad n \geqslant 1.$$

The relations (30) give then $\tilde{\alpha}_n$ and $\tilde{\beta}_n$. $\square$

Remark 8. Taking into account Propositions 4 and 7, we get that the leading coefficient of $\mathbf{P}_n$ is Γ_n (see (20)) and

$$\int_{-1}^{1} \mathbf{P}_n(x)\mathrm{d}\Omega(x)\mathbf{P}_n(x) = 2^{-2n} \begin{pmatrix} \varepsilon_{2n}(1+\Re a_{2n})^{-1} & 0 \\ 0 & \frac{1}{4}\varepsilon_{2n+1}(1+\Re a_{2n+2}) \end{pmatrix},$$

where ε_n is given in (5). Therefore, for the matrix measure $\mathrm{d}\Omega$, the monic LOMP are $\tilde{\mathbf{P}}_n = \Gamma_n^{-1}\mathbf{P}_n$ while $\hat{\mathbf{P}}_n = W_n\mathbf{P}_n$ are left orthonormal polynomial (LONP), being

$$W_n = 2^n \begin{pmatrix} \kappa_{2n}(1+\Re a_{2n})^{1/2} & 0 \\ 0 & 2\kappa_{2n+1}(1+\Re a_{2n+2})^{-1/2} \end{pmatrix}. \tag{33}$$

with $\kappa_n = \varepsilon_n^{-1/2}$.

Given a matrix measure, LOMP are determined up to multiplication on the left by a nonsingular constant matrix. Therefore, monic LOMP are unique but LONP are defined up to multiplication on the left by an orthogonal constant matrix. Thus, LONP can be fixed if we ask for their coefficients to be symmetric and positive definite. We will refer to the standard LONP when this choice is made. As for the measure $\mathrm{d}\Omega$, we see that $(\hat{\mathbf{P}}_n)_{n\geqslant 0}$ is not the sequence of standard LONP because the corresponding leading coefficients $W_n\Gamma_n$ are not symmetric. The following proposition gives the standard LONP in this case.

Proposition 8. *Let $\mathrm{d}\Omega$ be the matrix measure associated to $\mathrm{d}\mu$ and let $(\mathbf{P}_n)_{n\geqslant 0}$ be the corresponding LOMP defined in Proposition 7. Then, the sequence of standard LONP $(\mathbf{Q}_n)_{n\geqslant 0}$ with respect to $\mathrm{d}\Omega$ is given by $\mathbf{Q}_n = \Xi_n^T W_n \mathbf{P}_n$, where*

$$\Xi_n = \frac{1}{\sqrt{\det \Theta_n}}\Theta_n, \quad \Theta_n = K_n - JK_nJ = K_n + \operatorname{adj} K_n^{\mathrm{T}},$$

being $K_n = W_n\Gamma_n$ with Γ_n and W_n defined in (20) and (33), respectively.

Proof. Since $\hat{\mathbf{P}}_n = W_n\mathbf{P}_n$ is a LONP, $\mathbf{Q}_n$ will be a LONP too iff Ξ_n is an orthogonal matrix. To see that Ξ_n is indeed orthogonal, first notice that $JAJ = -\operatorname{adj} A^{\mathrm{T}}$ for all $A \in \mathbb{R}^{(2,2)}$. Therefore, the nonnegative definite matrix $(A-JAJ)^{\mathrm{T}}(A-JAJ) = A^{\mathrm{T}}A + \operatorname{adj}(A^{\mathrm{T}}A) + A^{\mathrm{T}}(\operatorname{adj} A^{\mathrm{T}}) + (\operatorname{adj} A)A = (\operatorname{tr}(A^{\mathrm{T}}A) + 2(\det A))I$ is a multiple of the identity. Taking determinants in above expression we see that this multiple is the nonnegative factor $\det(A + \operatorname{adj} A^{\mathrm{T}})$. For $A = K_n$, this factor can not vanish because $\det K_n = \det W_n > 0$. Thus, $\Xi_n^{\mathrm{T}}\Xi_n = I$.

Now, the leading coefficient of $\hat{\mathbf{P}}_n$ is $K_n = W_n\Gamma_n$. So, the leading coefficient of $\mathbf{Q}_n$ is $\Xi_n^{\mathrm{T}}K_n = K_n^{\mathrm{T}}K_n + (\det K_n)I$, which is symmetric and positive definite because $\det K_n > 0$. Hence, $(\mathbf{Q}_n)_{n\geqslant 0}$ are the standard LONP. □

In the next section we will deal with the strong asymptotics of LOMP with respect to a matrix measure associated to a measure on $\mathbb{T}$. As usual, we will take the standard LONP as a reference to express the asymptotic behaviour.

5. Asymptotics of semi-orthogonal functions and matrix OP

Once we have shown above connection between scalar and matrix OP, it is natural to take advantage of known properties for OP on $\mathbb{T}$ to develop new results about the more unfamiliar world of matrix OP. As an example we present here the implications of the asymptotics of OP on $\mathbb{T}$ when Szegő's condition,

$$\int_0^{2\pi} \log \mu'(\theta)\, \mathrm{d}\theta > -\infty, \tag{34}$$

for the measure $\mathrm{d}\mu$ holds (as it is usual, μ' denotes the Radon–Nikodym derivative of the absolutely continuous part of $\mathrm{d}\mu$ with respect to the Lebesgue measure $\mathrm{d}\theta$). It can be proved [9,24] that Szegő's condition is equivalent to $(a_n)_{n\geqslant 0} \in \ell^2$, which, in sight of (5), means that $\varepsilon = \lim_n \varepsilon_n > 0$ (or, in other words, $\kappa = \lim_n \kappa_n < \infty$). Thus, a necessary condition for (34) is $\lim_n a_n = 0$.

When Szegő's condition holds, asymptotic properties of OP on $\mathbb{T}$ are given in terms of the function

$$D(\mathrm{d}\mu; z) = \exp\left(\frac{1}{4\pi}\int_0^{2\pi} \log \mu'(\theta) \frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z}\, \mathrm{d}\theta\right), \quad |z| \neq 1,$$

which, for $|z| < 1$, is known as Szegő's function for the measure $\mathrm{d}\mu$, and satisfies the following remarkable property [9,24]:

$$\lim_{r\to 1^-} D(\mathrm{d}\mu; r\mathrm{e}^{\mathrm{i}\theta})\overline{D(\mathrm{d}\mu; r\mathrm{e}^{\mathrm{i}\theta})} = \mu'(\theta), \quad \text{a.e.}$$

Notice that, for $\mathrm{d}\tilde{\mu}$, the symmetric measure of $\mathrm{d}\mu$,

$$D(\mathrm{d}\tilde{\mu}; z) = \overline{D(\mathrm{d}\mu; \bar{z})} = D(\mathrm{d}\mu; z^{-1})^{-1}.$$

It can be proved [9,24] that, under (34),

$$\kappa = \frac{1}{\sqrt{2\pi}} D(\mathrm{d}\mu; 0)^{-1} \tag{35}$$

and the orthonormal polynomials satisfy

$$\lim_n \varphi_n(z) = 0, \quad |z| < 1,$$

$$\lim_n \varphi_n^*(z) = \frac{1}{\sqrt{2\pi}} D(\mathrm{d}\mu; z)^{-1}, \quad |z| < 1,$$

where the convergence is uniform on compact sets. Therefore,

$$\varepsilon = 2\pi D(\mathrm{d}\mu; 0)^2$$

and for the monic OP we get

$$\begin{aligned} &\lim_n \phi_n(z) = 0, \quad |z| < 1, \\ &\lim_n \phi_n^*(z) = D(\mathrm{d}\mu; 0) D(\mathrm{d}\mu; z)^{-1}, \quad |z| < 1. \end{aligned} \tag{36}$$

From above results the following asymptotics of VSOF follows straightforward:

$$\begin{aligned}
&\lim_n 2^n z^n \mathbf{f}_n(z) = \begin{pmatrix} 1 \\ i \end{pmatrix} D(\mathrm{d}\mu;0)D(\mathrm{d}\mu;z)^{-1}, \quad 0<|z|<1,\\
&\lim_n 2^n z^{-n} \mathbf{f}_n(z) = \begin{pmatrix} 1 \\ -i \end{pmatrix} D(\mathrm{d}\mu;0)D(\mathrm{d}\mu;z), \quad |z|>1,
\end{aligned} \tag{37}$$

where, again, the convergence is uniform on compact sets.

As for matrix OP on $[-1,1]$, some general results are known, but they are not so good as previous ones. More precisely, let us suppose that a positive matrix measure $\mathrm{d}\omega$ on $[-1,1]$ satisfies the Szegő's matrix condition

$$\int_{-1}^{1} \log\det\omega'(x)\,\frac{\mathrm{d}x}{\sqrt{1-x^2}} > -\infty,$$

where ω' is the Radon–Nikodym derivative of the absolutely continuous part of $\mathrm{d}\omega$ with respect to the Lebesgue scalar measure $\mathrm{d}\theta$. If $(\mathbf{q}_n)_{n\geqslant 0}$ is the sequence of standard LONP with respect to $\mathrm{d}\omega$ then, for any other sequence of LONP $(\mathbf{p}_n)_{n\geqslant 0}$ such that $\mathbf{p}_n = \xi_n \mathbf{q}_n$ with $(\xi_n)_{n\geqslant 0}$ convergent, we have that

$$\lim_n z^n \mathbf{p}_n(x) = \frac{1}{\sqrt{2\pi}}\mathbf{D}(\mathrm{d}\omega;z)^{-1}, \quad |z|<1, \tag{38}$$

where the convergence is uniform on compact sets and $\mathbf{D}(\mathrm{d}\omega;z)$ is certain matrix-valued analytic function on $|z|<1$ without zeros there [2]. The Szegő's matrix function $\mathbf{D}(\mathrm{d}\omega;z)$ is uniquely determined by ω' and satisfies the boundary condition

$$\lim_{r\to 1^-} \mathbf{D}(\mathrm{d}\omega;r\mathrm{e}^{\mathrm{i}\theta})\mathbf{D}(\mathrm{d}\omega;r\mathrm{e}^{\mathrm{i}\theta})^* = \omega'(\cos\theta)|\sin\theta|, \quad \text{a.e.}$$

Unfortunately, an explicit expression for $\mathbf{D}(\mathrm{d}\omega;z)$ in terms of ω' is not available. On that score, all what we can state is that [2]

$$\mathbf{D}(\mathrm{d}\omega;z) = \overset{\curvearrowright}{\int_0^{2\pi}} \exp\left(\mathbf{M}(\theta)\frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z}\,\mathrm{d}\theta\right)\zeta, \quad |z|<1,$$

where ζ is a constant orthogonal matrix factor depending on $\lim_n \xi_n$, and $\mathbf{M}(\theta)$ is a Hermitian matrix-valued integrable function on $[0,2\pi)$ such that

$$\mathrm{tr}\mathbf{M}(\theta) = \log\det\{\omega'(\cos\theta)|\sin\theta|\}, \quad \theta\in[0,2\pi).$$

The symbol $\overset{\curvearrowright}{\int_0^{2\pi}}$ means the multiplicative integral

$$\overset{\curvearrowright}{\int_0^{2\pi}} \exp(F(\theta))\,\mathrm{d}\theta = \lim_n \prod_{k=1}^{n} \exp(F(t_k))(\theta_k-\theta_{k-1}),$$

where $t_k \in [\theta_{k-1}, \theta_k)$ and the limit is taken in the usual sense over the partitions $0=\theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{n-1} < \theta_n = 2\pi$ of the interval $[0, 2\pi)$.

When the matrix measure is associated to a measure on $\mathbb{T}$ above results can be improved by translating the better known asymptotics of OP on $\mathbb{T}$ to matrix OP on $[-1,1]$. As a consequence we can obtain in this case an explicit expression for the Szegő's matrix function.

Theorem 1. *Let*

$$\mathrm{d}\Omega(x) = \frac{1}{2}\begin{pmatrix} \mathrm{d}\rho(x) & \sqrt{1-x^2}\,\mathrm{d}\sigma(x) \\ \sqrt{1-x^2}\,\mathrm{d}\sigma(x) & (1-x^2)\mathrm{d}\rho(x) \end{pmatrix}$$

be a positive matrix measure on $[-1,1]$ ($\mathrm{d}\rho$ *and* $\mathrm{d}\sigma$ *are scalar measures on* $[-1,1]$) *that satisfies the condition*

$$\int_{-1}^{1} \log\det \Omega'(x)\, \frac{\mathrm{d}x}{\sqrt{1-x^2}} > -\infty.$$

Let $\mathscr{R}(z), \mathscr{I}(z), \gamma$ *be*

$$\mathscr{R}(z) = \frac{1-z^2}{4\pi}\int_0^{\pi} \frac{\log\det\Omega'(x)}{\sqrt{1-x^2}}\frac{\mathrm{d}x}{1-2xz+z^2}, \quad |z| \neq 1,$$

$$\mathscr{I}(z) = -\frac{z}{2\pi}\int_{-1}^{1} \log\left\{\frac{\rho'(x)+\sigma'(x)}{\rho'(x)-\sigma'(x)}\right\}\frac{\mathrm{d}x}{1-2xz+z^2}, \quad |z| \neq 1,$$

$$\gamma = -\frac{1}{4\pi}\int_{-1}^{1} \log\left\{\frac{\rho'(x)+\sigma'(x)}{\rho'(x)-\sigma'(x)}\right\}\mathrm{d}x,$$

and let $(\mathbf{Q}_n)_{n\geqslant 0}$ *be the sequence of standard LONP with respect to* $\mathrm{d}\Omega$. *Then, for* $x \in \mathbb{C}\setminus[-1,1]$, *if we write* $z = x + \sqrt{x^2-1}$ *with the choice of* $\sqrt{x^2-1}$ *such that* $|z| < 1$, *we have that*

$$\lim_n z^n \mathbf{Q}_n(x) = \frac{1}{\sqrt{2\pi}}\mathbf{D}(\mathrm{d}\Omega; z)^{-1}$$

where

$$\mathbf{D}(\mathrm{d}\Omega; z) = \frac{1}{\sqrt{9+4\gamma^2}}\begin{pmatrix} 1 & 0 \\ 0 & -\sqrt{x^2-1} \end{pmatrix}\exp(I\mathscr{R}(z) + J\mathscr{I}(z))\begin{pmatrix} 3 & -2\gamma \\ 2\gamma z & 3z \end{pmatrix},$$

being the convergence uniform on compact sets.

Proof. Notice first that Szegő's matrix condition implies that $\mathrm{d}\Omega$ has an infinite support. Thus, from Proposition 6 we see that there exist LOMP with respect to $\mathrm{d}\Omega$ and that the matrix measure $\mathrm{d}\Omega$ is associated to some positive measure $\mathrm{d}\mu$ on $\mathbb{T}$ with an infinite support (therefore, there exist OP with respect to $\mathrm{d}\mu$).

The expression that gives the Szegő condition for $\mathrm{d}\mu$ can be rewritten in the following way:

$$\begin{aligned}\int_0^{2\pi} \log \mu'(\theta)\,\mathrm{d}\theta &= \int_{-1}^{1} [\log(v_1'(x)\sqrt{1-x^2}) + \log(v_2'(x)\sqrt{1-x^2})]\frac{\mathrm{d}x}{\sqrt{1-x^2}} \\ &= \int_{-1}^{1} \log\left\{\frac{1}{4}(\rho'(x)^2 - \sigma'(x)^2)(1-x^2)\right\}\frac{\mathrm{d}x}{\sqrt{1-x^2}} \\ &= \int_{-1}^{1} \log\det \Omega'(x)\,\frac{\mathrm{d}x}{\sqrt{1-x^2}}.\end{aligned}$$

Thus, Szegő's matrix condition for $\mathrm{d}\Omega$ is equivalent to Szegő's condition for $\mathrm{d}\mu$. Hence, under the assumptions of the theorem, the Szegő's function $D(\mathrm{d}\mu;z)$ governs the asymptotic behaviour of the VSOF related to $\mathrm{d}\mu$ in the way shown in (37).

Then, from (18) we find for the quasi-orthogonal matrix polynomials associated to $\mathrm{d}\mu$ that

$$\lim_n 2^n z^{-n}\mathbf{F}_n(x) = D(\mathrm{d}\mu;0)\mathbb{D}(\mathrm{d}\mu;z)\begin{pmatrix}1 & 0\\ 0 & \mathrm{i}y\end{pmatrix}^{-1}, \quad |z|>1,$$

where

$$\mathbb{D}(\mathrm{d}\mu;z) = \begin{pmatrix} D_s(\mathrm{d}\mu;z) & \mathrm{i}D_a(\mathrm{d}\mu;z)\\ -\mathrm{i}D_a(\mathrm{d}\mu;z) & D_s(\mathrm{d}\mu;z)\end{pmatrix} = ID_s(z) + \mathrm{i}JD_a(z),$$

and $D_s(\mathrm{d}\mu;z)$, $D_a(\mathrm{d}\mu;z)$ are what we could call the symmetric and antisymmetric part of $D(\mathrm{d}\mu;z)$, that is,

$$\begin{aligned}D_s(z) &= \frac{1}{2}(D(\mathrm{d}\mu;z) + D(\mathrm{d}\tilde{\mu};z)),\\ D_a(z) &= \frac{1}{2}(D(\mathrm{d}\mu;z) - D(\mathrm{d}\tilde{\mu};z)).\end{aligned}$$

Notice that, when $\mathrm{d}\mu$ is symmetric, $D_s(\mathrm{d}\mu;z)=D(\mathrm{d}\mu;z)$ and $D_a(\mathrm{d}\mu;z)=0$. Hence, the matrix $\mathbb{D}(\mathrm{d}\mu;z)$ is diagonal. This is natural, because in this case the quasi-orthogonal polynomials are diagonal.

Let us define

$$\begin{aligned}\mathscr{R}(z) &= \frac{1}{4\pi}\int_0^{2\pi} \log\mu'(\theta)\Re_\theta\left(\frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z}\right)\mathrm{d}\theta,\\ \mathscr{I}(z) &= \frac{1}{4\pi}\int_0^{2\pi} \log\mu'(\theta)\Im_\theta\left(\frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z}\right)\mathrm{d}\theta,\end{aligned}$$

where $\Re_\theta$ and $\Im_\theta$ are real and imaginary part operators with conjugation acting only on $\mathrm{e}^{\mathrm{i}\theta}$-dependence. Then, $D(\mathrm{d}\mu;z) = \exp(\mathscr{R}(z) + \mathrm{i}\mathscr{I}(z))$ and

$$\mathbb{D}(\mathrm{d}\mu;z) = \exp\mathscr{R}(z)\begin{pmatrix}\cos\mathscr{I}(z) & -\sin\mathscr{I}(z)\\ \sin\mathscr{I}(z) & \cos\mathscr{I}(z)\end{pmatrix} = \exp(I\mathscr{R}(z) - J\mathscr{I}(z)).$$

Notice that

$$\mathbb{D}(\mathrm{d}\tilde{\mu};z) = \exp(I\mathscr{R}(z) + J\mathscr{I}(z)) = \mathbb{D}(\mathrm{d}\mu;z)^{\mathrm{T}} = \overline{\mathbb{D}(\mathrm{d}\mu;\bar{z})} = \mathbb{D}(\mathrm{d}\mu;z^{-1})^{-1}.$$

The asymptotic behaviour of the LOMP given in Proposition 7 can now be deduced. Since $\lim_n a_n = 0$ we get

$$\lim_n 2^n z^{-n} \mathbf{P}_n(x) = D(\mathrm{d}\mu;0) \begin{pmatrix} 1 & 0 \\ 0 & z/2 \end{pmatrix} \mathbb{D}(\mathrm{d}\mu;z) \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i}y \end{pmatrix}^{-1}, \quad |z| > 1.$$

As for the standard LONP, Proposition 6 together with (20), (33) and (35) give

$$\lim_n z^{-n} \mathbf{Q}_n(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{9+4\gamma^2}} \begin{pmatrix} 3 & 2\gamma z \\ -2\gamma & 3z \end{pmatrix} \mathbb{D}(\mathrm{d}\mu;z) \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i}y \end{pmatrix}^{-1}, \quad |z| > 1,$$

where $\gamma = \lim_n \gamma_n$. This limit exists because, from the expression for γ_n given in Proposition 4(i) and relation (3), we see that

$$\gamma = \frac{1}{2} \Im \left(\lim_n b_n \right) = \frac{1}{2} \Im \left(\sum_{k=1}^{\infty} a_k \bar{a}_{k-1} \right),$$

and the convergence of $\sum_{k=0}^{\infty} a_{k+1}\bar{a}_k$ follows from the convergence of $\sum_{k=0}^{\infty} |a_k|^2$ and the Schwarz's inequality.

By comparing with (38) we see that, in our case, the asymptotics of $(\mathbf{Q}_n)_{n\geqslant 0}$ is governed by the Szegő's matrix function

$$\mathbf{D}(\mathrm{d}\Omega;z) = \frac{1}{\sqrt{9+4\gamma^2}} \begin{pmatrix} 1 & 0 \\ 0 & -\mathrm{i}y \end{pmatrix} \mathbb{D}(\mathrm{d}\mu;z)^{\mathrm{T}} \begin{pmatrix} 3 & -2\gamma \\ 2\gamma z & 3z \end{pmatrix}, \quad |z| < 1.$$

To complete the proof it only remains to see that $\mathscr{R}(z)$, $\mathscr{I}(z)$ and γ are given by the expressions that appear in the theorem. We can rewrite $\mathscr{R}(z)$ and $\mathscr{I}(z)$ in terms of the matrix measure $\mathrm{d}\Omega$ in the following way:

$$\begin{aligned}
\mathscr{R}(z) &= \frac{1}{4\pi} \int_0^{\pi} \log\{v_1'(\cos\theta) v_2'(\cos\theta)\sin^2\theta\} \Re_\theta \left(\frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z} \right) \mathrm{d}\theta \\
&= \frac{1}{4\pi} \int_0^{\pi} \log\det \Omega'(\cos\theta) \Re_\theta \left(\frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z} \right) \mathrm{d}\theta, \\
\mathscr{I}(z) &= \frac{1}{4\pi} \int_0^{\pi} \log\{v_1'(\cos\theta)/v_2'(\cos\theta)\} \Im_\theta \left(\frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z} \right) \mathrm{d}\theta \\
&= \frac{1}{4\pi} \int_0^{\pi} \log \left\{ \frac{\rho'(\cos\theta)+\sigma'(\cos\theta)}{\rho'(\cos\theta)-\sigma'(\cos\theta)} \right\} \Im_\theta \left(\frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z} \right) \mathrm{d}\theta.
\end{aligned}$$

Now, the change of variables $x = \cos\theta$ gives the desired expressions for $\mathscr{R}(z)$ and $\mathscr{I}(z)$.

Besides, we can give an expression for $\gamma = \frac{1}{2}\Im(\lim_n b_n)$ in terms of $\mathrm{d}\Omega$. Notice first that $b_n = \overline{\phi_n^{*\prime}(0)}$. Since $(\phi_n^*)_{n\geqslant 0}$ is a sequence of analytic functions in the complex plane that converges uniformly on compact subsets of $|z| < 1$, we can write $\lim_n \phi_n^{*\prime}(z) = (\lim_n \phi_n^*)'(z)$ for $|z| < 1$ (see [22]). Thus, from (36) we get

$$\lim_n b_n = -\overline{(\log D)'(\mathrm{d}\mu;0)} = -\frac{1}{2\pi} \int_0^{2\pi} \log \mu'(\theta) \mathrm{e}^{\mathrm{i}\theta}\, \mathrm{d}\theta.$$

Therefore,

$$\begin{aligned}\gamma &= -\frac{1}{4\pi}\int_0^{2\pi}\log\mu'(\theta)\sin\theta\,\mathrm{d}\theta\\ &= -\frac{1}{4\pi}\int_0^{\pi}\log\{v_1'(\cos\theta)/v_2'(\cos\theta)\}\sin\theta\,\mathrm{d}\theta\\ &= -\frac{1}{4\pi}\int_{-1}^{1}\log\left\{\frac{\rho'(x)+\sigma'(x)}{\rho'(x)-\sigma'(x)}\right\}\mathrm{d}x,\end{aligned}$$

which completes the proof. □

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