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Eigenvalues and perfect matchings

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Abstract

We give sufficient conditions for existence of a perfect matching in a graph in terms of the eigenvalues of the Laplacian matrix. We also show that a distance-regular graph of degree k is k-edge-connected.

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1. Introduction

A *matching* of a graph Γ is a set of mutually disjoint edges. A matching is *perfect* if every vertex of Γ is incident with an edge of the matching. A perfect matching is also called a *1-factor*. A set of vertices of Γ that is incident with all edges of Γ is called a *vertex cover* of Γ. For perfect matchings in bipartite graphs, Frobenius [6] gave the following characterization.

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Theorem 1.1 (Frobenius). *A bipartite graph with v vertices has a perfect matching if and only if each vertex cover has size at least $v/2$.*

Similar results are due to König [10] and Hall [8]. The characterization of Frobenius implies that the adjacency matrix of a bipartite graph with no perfect matching must be singular. So a bipartite graph with only nonzero adjacency eigenvalues has a perfect matching. For regular bipartite graphs (of positive degree), Frobenius' characterization implies that there is always a perfect matching [9]. Because a graph with adjacency eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_v$ is regular if and only if $v\lambda_1 = \sum_{i=1}^{v} \lambda_i^2$, the latter condition can also be interpreted as a sufficient eigenvalue condition for the presence of a perfect matching in a bipartite graph.

For perfect matchings in arbitrary graphs we have the following characterization by Tutte [11]. (An *odd component* is a component with an odd number of vertices.)

Theorem 1.2 (Tutte). *A graph $\Gamma = (V, E)$ has no perfect matching if and only if there exists a subset $S \subset V$, such that the subgraph of Γ induced by $V \backslash S$ has more than $|S|$ odd components.*

This characterization generalizes Theorem 1.1 (in a bipartite graph with no perfect matching take S to be a vertex cover of size less than $v/2$), but by no means it implies that the two mentioned sufficient conditions for complete matchings in bipartite graphs also work in general. Indeed, Fig. 1 gives counterexamples for both cases. The first graph has a nonsingular adjacency matrix and yet no perfect matching, and the second one is regular with no perfect matching. In this note we look for sufficient conditions for perfect matchings in terms of the eigenvalues of the Laplacian matrix L. We recall that L is related to the adjacency matrix A by $L = D - A$, where D is the diagonal matrix of the vertex degrees. The Laplacian matrix L is positive semi-definite with row sum 0. Its eigenvalues will be denoted by $0 = \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_v$. The eigenvalue μ_2 is often called the *algebraic connectivity*; $\mu_2 = 0$ if and only if the graph is disconnected, and $\mu_2 \leqslant \kappa_v$, where κ_v is the vertex connectivity of the graph. For k-regular graphs $D = kI$, and hence the adjacency eigenvalues $k = \lambda_1 \geqslant \cdots \geqslant \lambda_v$ are related to the Laplacian eigenvalues by $\mu_i = k - \lambda_i$ ($i = 1, \ldots, v$). For these and other properties of the Laplacian matrix, see for example Section 2.5 of [3].

We shall show that Tutte's result implies that a graph on v vertices with v even, whose Laplacian eigenvalues satisfy $2\mu_2 \geqslant \mu_v$, has a perfect matching. For regular graphs with v even, we prove that $\mu_2 \geqslant 1$ is already sufficient for existence of a perfect matching.

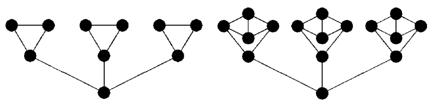

Fig. 1. Graphs without a perfect matching.

2. Arbitrary graphs

In this section we will use an inequality for disconnected vertex sets in graphs, due to the second author (see [7–Lemma 6.1]). Two disjoint vertex sets A and B in a graph are called *disconnected* if there are no edges between A and B.

Lemma 2.1. *If A and B are disconnected vertex sets of a graph with v vertices and Laplacian eigenvalues* $0 = \mu_1 \leqslant \cdots \leqslant \mu_v$, *then*

$$\frac{|A| \cdot |B|}{(v - |A|)(v - |B|)} \leqslant \left(\frac{\mu_v - \mu_2}{\mu_v + \mu_2}\right)^2.$$

Another tool is the following elementary lemma.

Lemma 2.2. *Let* $x_1 \ldots x_n$ *be n positive integers such that* $\sum_{i=1}^{n} x_i = k \leqslant 2n - 1$. *Then for every integer* ℓ, *satisfying* $0 \leqslant \ell \leqslant k$, *there exists an* $I \subset \{1, \ldots, n\}$ *such that* $\sum_{i \in I} x_i = \ell$.

Proof. Induction on n. The case $n = 1$ is trivial. If $n \geqslant 2$, assume $x_1 \geqslant \cdots \geqslant x_n$. Then $n - 1 \leqslant k - x_1 \leqslant 2(n - 1) - 1$ and we apply the induction hypothesis to $\sum_{i=2}^{n} x_i = k - x_1$ with the same ℓ if $\ell \leqslant n - 1$, and $\ell - x_1$ otherwise. $\square$

Theorem 2.3. *Let* Γ *be a graph with v vertices, and Laplacian eigenvalues* $0 = \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_v$. *If v is even and* $\mu_v \leqslant 2\mu_2$, *then* Γ *has a perfect matching.*

Proof. Assume $\Gamma = (V, E)$ has no perfect matching. By Tutte's theorem there exists a set $S \subset V$ of size s (say), such that the subgraph Γ' of Γ induced by $V \backslash S$ has $q > s$ odd components. But since v is even, $s + q$ is even, hence $q \geqslant s + 2$.

First assume $v \leqslant 3s + 3$. Then Γ' has at most $2s + 3$ vertices and at least $s + 2$ components. By Lemma 2.2, Γ' and hence Γ, has a pair of disconnected vertex sets A and B with $|A| = \left\lfloor \frac{1}{2}(v - s) \right\rfloor$ and $|B| = \left\lceil \frac{1}{2}(v - s) \right\rceil$. Now Lemma 2.1 implies

$$\left(\frac{\mu_v - \mu_2}{\mu_v + \mu_2}\right)^2 \geqslant \frac{|A| \cdot |B|}{vs + |A| \cdot |B|} = \frac{(v - s)^2 - \epsilon}{(v + s)^2 - \epsilon},$$

where $\epsilon = 0$ if $v - s$ is even and $\epsilon = 1$ if $v - s$ is odd. Using $v \geqslant 2s + 2$ we obtain

$$\frac{\mu_v - \mu_2}{\mu_v + \mu_2} > \frac{v - s - 1}{v + s} \geqslant \frac{s + 1}{3s + 2} > \frac{1}{3}.$$

Hence $2\mu_2 < \mu_v$.

Next assume $v \geqslant 3s + 4$. Now Γ', and hence Γ, has a pair of disconnected vertex sets A and B with $|A| + |B| = v - s$ and $\min\{|A|, |B|\} \geqslant s + 1$, so $|A| \cdot |B| \geqslant (s + 1)(v - 2s - 1) > vs - 2s^2$. Now Lemma 2.1 implies

$$\left(\frac{\mu_v - \mu_2}{\mu_v + \mu_2}\right)^2 \geqslant \frac{|A| \cdot |B|}{vs + |A| \cdot |B|} \geqslant \frac{vs - 2s^2}{2vs - 2s^2} = \frac{1}{2} - \frac{s}{2v - 2s} > \frac{1}{4},$$

by use of $v \geqslant 3s + 4$. So

$$\frac{\mu_v - \mu_2}{\mu_v + \mu_2} > \frac{1}{2} > \frac{1}{3},$$

hence $2\mu_2 < \mu_v$. □

The complete bipartite graphs $K_{m,n}$ with $m \leqslant n$ have Laplacian eigenvalues $\mu_2 = m$ and $\mu_v = v = m + n$. This shows that $2\mu_2$ can get arbitrarily close to μ_v for graphs with v even and no perfect matching.

3. Regular graphs

For regular graphs the condition of the previous section can be improved.

Theorem 3.1. *A connected k-regular graph on v vertices with adjacency eigenvalues $k = \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_v$ and v even which satisfies*

$$\lambda_3 \leqslant \begin{cases} k - 1 + \frac{3}{k+1} & \text{if } k \text{ is even,} \\ k - 1 + \frac{3}{k+2} & \text{if } k \text{ is odd,} \end{cases}$$

has a perfect matching.

Proof. Let $\Gamma = (V, E)$ be a k-regular graph with $v = |V|$ even and no perfect matching. By Tutte's theorem there exists a set $S \subset V$ of size s such that $V \backslash S$ induces a subgraph with $q \geqslant s + 2$ odd components $\Gamma_1, \Gamma_2, \ldots, \Gamma_q$ (say). Let t_i denote the number of edges in Γ between S and Γ_i. Then clearly $\sum_{i=1}^{q} t_i \leqslant ks$, $s \geqslant 1$, and $t_i \geqslant 1$ (since Γ is connected). Hence $t_i < k$ and $n_i > 1$ for at least three values of i, say $i = 1, 2$ and 3. Let ℓ_i denote the largest adjacency eigenvalue of Γ_i, and assume $\ell_1 \geqslant \ell_2 \geqslant \ell_3$. Then eigenvalue interlacing (see for example [5–p. 19]) applied to the subgraph induced by the union of Γ_1, Γ_2 and Γ_3 gives $\ell_i \leqslant \lambda_i$ for $i = 1, 2, 3$.

Consider Γ_3 with n_3 vertices and e_3 edges (say). Then $2e_3 = kn_3 - t_3 \leqslant n_3(n_3 - 1)$. We saw that $t_3 < k$ and $n_3 > 1$, hence $k < n_3$. Moreover, the average degree $\bar{d}_3$ of Γ_3 equals $2e_3/n_3 = k - t_3/n_3$. If k is even, t_3 must be even and hence $t_3 \leqslant k - 2$. If k is odd, $k < n_3$ implies $k \leqslant n_3 - 2$ (remember that n_3 is odd). Hence

$$\bar{d}_3 \geqslant \begin{cases} k - \frac{k-2}{k+1} & \text{if } k \text{ is even,} \\ k - \frac{k-1}{k+2} & \text{if } k \text{ is odd.} \end{cases}$$

Note that $t_3 < n_3$ implies that Γ_3 cannot be regular. Next we use the fact (see [5–p. 84]) that the largest adjacency eigenvalue of a graph is bounded from below by the average degree with equality if and only if the graph is regular. Thus $\bar{d}_3 < \ell_3$. We saw that $\ell_3 \leqslant \lambda_3$, which finishes the proof. □

The example from the introduction of a 3-regular connected graph with an even number of vertices, but no perfect matching can easily be extended to arbitrary $k \geqslant 3$ (see [4]). We give the description for k even. For odd k the construction is similar but slightly more complicated. Let Γ' be the complete graph K_{k+1} from which a matching of size $(k-2)/2$ is deleted. Take k disjoint copies of Γ'. Add $k-2$ new vertices and connect each of these vertices to a vertex of degree $k-1$ in each Γ'. This gives a connected k-regular graph with k^2+2k-2 vertices and no perfect matching. It is not very difficult to see that both the second and third largest eigenvalue of this example are equal to the largest eigenvalue of Γ', which equals $(k-2+\sqrt{k^2+12})/2 = k-1+\frac{3}{k+1}+\mathcal{O}(k^{-2})$. This shows that for v and k even there exist graphs with no perfect matching, for which λ_3 gets arbitrarily close to the value of Theorem 3.1.

From the above it is clear that v even and $\lambda_2 \leqslant k-1$ implies existence of a perfect matching. In terms of the Laplacian matrix this translates into:

Corollary 3.2. *A regular graph with an even number of vertices and algebraic connectivity at least* 1 *has a perfect matching.*

But we can say more. The Laplacian matrix of a disjoint union of $v/2$ edges has eigenvalues 0 and 2. This implies that deletion of the edges of a perfect matching of a graph Γ reduces the eigenvalues of the Laplacian matrix of Γ with at most 2. Hence:

Corollary 3.3. *A regular graph with an even number of vertices and algebraic connectivity μ_2 has at least $\lfloor(\mu_2+1)/2\rfloor$ disjoint perfect matchings.*

Corollary 3.4. *A regular graph with an even number of vertices and diameter at most* 3 *has a perfect matching.*

Proof. In the proof of Theorem 3.1, we saw that $t_i < n_i$ for $i = 1$, 2 and 3. Hence there exist vertices x and y in Γ_1 and Γ_2 respectively, that are not adjacent to a vertex of S. Therefore the distance between x and y is at least 4. □

4. Distance-regular graphs

The research for this note was motivated by the question: ‘Do all distance-regular graphs on an even number of vertices have a perfect matching?’ (For necessary information on distance-regular graphs we refer to [1].)

We know of no distance-regular graph that does not satisfy the condition of Theorem 3.1, but it is not immediately clear that that condition is fulfilled for all distance-regular graphs. An example that comes close is the Biggs-Smith graph, where $k = 3$

and $\lambda_2 = \lambda_3 = (1+\sqrt{17})/2 \approx 2.562$. It may be possible to classify all distance-regular graphs with second largest eigenvalue larger than $k-1$.

But in the distance-regular case we can prove a stronger result and have the existence of perfect matchings as an immediate corollary.

Theorem 4.1. *A distance-regular graph of degree k is k-edge-connected, that is, cannot be disconnected by removing fewer than k edges. Moreover, if $k > 2$ then the only disconnecting sets of k edges are the sets of k edges on a single vertex.*

We conjecture that in fact distance-regular graphs are k-vertex-connected. This is known for strongly regular graphs [2].

Corollary 4.2. *A distance-regular graph with an even number of vertices has a perfect matching.*

Proof. It is known (see [4]), that a k-regular graph with edge connectivity at least $k-1$ on an even number of vertices has a perfect matching. (The result is also clear from the proof of Theorem 3.1.) $\square$

Let Γ be a distance-regular graph of degree $k > 2$. As usual, let us write v for the total number of vertices, d for the diameter and k_i for the number of vertices at distance i from a given vertex. Also, for two vertices x, y at distance i, let c_i, a_i, b_i be the number of vertices adjacent to y, at distance $i-1, i, i+1$ (respectively) from x. Write $\lambda := a_1$ and $\mu := c_2$ and $k := k_1$. Finally, let $\Gamma_i(x)$ be the set of vertices of Γ at distance i from the vertex x.

First a lemma giving a lower bound for the size of connected components.

Lemma 4.3

(i) *Let S be a disconnecting set of vertices of Γ, and let A be the vertex set of a component of the complement of S in Γ. Fix a vertex $a \in A$ and let $s_i := |S \cap \Gamma_i(a)|$. Then $|A \cap \Gamma_i(a)| \geqslant \big(1 - \sum_{j=1}^{i} \frac{s_j}{k_j}\big)k_i$, so that*

$$|A| \geqslant v - \sum_i \frac{s_i}{k_i}(k_i + \cdots + k_d).$$

(ii) *Let T be a disconnecting set of edges of Γ, and let A be the vertex set of a component of Γ minus T. Fix a vertex $a \in A$ and let t_i be the number of edges in T that join $\Gamma_{i-1}(a)$ and $\Gamma_i(a)$. Then $|A \cap \Gamma_i(a)| \geqslant \big(1 - \sum_{j=1}^{i} \frac{t_j}{c_j k_j}\big)k_i$, so that*

$$|A| \geqslant v - \sum_i \frac{t_i}{c_i k_i}(k_i + \cdots + k_d).$$

Proof. Because (i) and (ii) are similar, we will only prove (ii). The case $i = 0$ is trivial. Suppose $i \geqslant 1$, let e_i denote the number of edges in Γ between $A \cap \Gamma_{i-1}(a)$

and $A \cap \Gamma_i(a)$, and put $m_i := |A \cap \Gamma_i(a)|$. Then $m_i c_i \geqslant e_i \geqslant m_{i-1}b_{i-1} - t_i$. Since $b_{i-1}/c_i = k_i/k_{i-1}$, this gives $m_i \geqslant m_{i-1}k_i/k_{i-1} - t_i/c_i$, and the inequality follows by induction. □

For applications of this lemma, it is useful to note that $(k_i + \cdots + k_d)/k_i \geqslant (k_{i+1} + \cdots + k_d)/k_{i+1}$ (indeed, $b_0 \geqslant \cdots \geqslant b_{d-1}$ and $c_1 \leqslant \cdots \leqslant c_d$ implies $k_i/k_0 \geqslant \cdots \geqslant k_d/k_{d-i}$), so that, for example, part (ii) implies that

$$|A| > v\left(1 - \frac{|T|}{\mu k_2}\right)$$

when T is a disconnecting set of edges none of which is incident with a.

Proof of Theorem 4.1. Suppose we remove (at most) k edges from Γ, disconnecting this graph, where none of the components consists of a single vertex. If B is the vertex set of a component such that each vertex of B is incident with at least one of the removed edges, then $|B| \leqslant k$, and each vertex of B is on at least $k - (|B| - 1)$ removed edges with other vertex outside B, so that $|B|(k - |B| + 1) \leqslant k$. Since by assumption $|B| > 1$ it follows that $|B| = k$ and the subgraph induced by B is complete. The local graph of Γ at a vertex of B is regular and contains a $(k-1)$-clique, so is a k-clique, and also Γ itself is complete, contradiction. This means that every component has at least $k + 1$ vertices, and contains a vertex b not incident with any of the removed edges.

If $b_1 > 1$, then by the above remark we have for any component with vertex set A and any vertex $a \in A$ not incident with one of the removed edges

$$|A| > v\left(1 - \frac{k}{\mu k_2}\right) = v\left(1 - \frac{1}{b_1}\right) \geqslant \frac{v}{2},$$

so that there is at most one component, contradiction.

Otherwise $b_1 = 1$, that is, $k = \lambda + 2$. We may assume that $d \geqslant 3$, since the case $d = 2$ was treated in [2]. Then $c_i \leqslant b_j$ for $i + j \leqslant d$ implies $\mu = 1$. Now the local graph of Γ is a disjoint union of $(\lambda + 1)$-cliques, so $\lambda = 0$ and $k = 2$, contrary to the assumption. □

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