ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT®

J. Math. Pures Appl. 83 (2004) 417–432

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES

www.elsevier.com/locate/matpur

Real interpolation and closed operator ideals ☆

Fernando Cobos [a,*], Luz M. Fernández-Cabrera [b], Antonio Manzano [c], Antón Martínez [d]

[a] *Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain*

[b] *Sección Departamental de Matemática Aplicada, Escuela de Estadística, Universidad Complutense de Madrid, 28040 Madrid, Spain*

[c] *Departamento de Matemáticas y Computación, Escuela Politécnica Superior, Universidad de Burgos, 09001 Burgos, Spain*

[d] *Departamento de Matemática Aplicada, E.T.S. Ingenieros Industriales, Universidad de Vigo, 36200 Vigo, Spain*

Received 12 November 2003

Abstract

We investigate the behaviour by general J- and K-methods of certain closed operator ideals. In particular, the results apply to weakly compact operators, Rosenthal operators and Banach–Saks operators.

Résumé

Nous étudions le comportement de certains ideaux d'opérateurs par les J- et K-méthodes. En particulier, les résultats sont appliqués aux opérateurs faiblement compacts, aux opérateurs de Rosenthal et aux opérateurs de Banach–Saks.

MSC: 46B70; 47B10

Keywords: Real interpolation; Weakly compact operators; Rosenthal operators; Banach–Saks operators

☆ Authors have been supported in part by Ministerio de Ciencia y Tecnología (BFM2001-1424).

* Corresponding author.

E-mail addresses: cobos@mat.ucm.es (F. Cobos), luz_fernandez-c@mat.ucm.es (L.M. Fernández-Cabrera), amanzano@ubu.es (A. Manzano), antonmar@uvigo.es (A. Martínez).

0021-7824/$ – see front matter

doi:10.1016/j.matpur.2004.01.002

1. Introduction

In 1974, Davis, Figiel, Johnson and Pelczyński [12] established their celebrated result on the factorization property of weakly compact operators through reflexive spaces. The proof given in [12] has a clear interpolation flavour. This motivated the investigation on the behaviour of weak compactness under interpolation, as well as it started the research on the factorization property for operator ideals $\mathcal{I}$, that is to say, to study whether or not every operator T of the ideal $\mathcal{I}$ can be factorized through a Banach space E whose identity operator I_E belongs to $\mathcal{I}$. Relevant contributions on these problems are due to Beauzamy [1] and Heinrich [16] (other related results can be found in [20] and [28]; a quantitative version of the results of Beauzamy and Heinrich were established by the present authors in [10] and [8]). In both cases they deal with the classical real method $(A_0, A_1)_{\theta,q}$.

But the real method is not enough to describe all interpolation spaces with respect to many important couples. For example, applying this method to (L_1, L_∞) we only obtain L_p and $L_{p,q}$ spaces, while Lorentz spaces, Marcinkiewicz spaces and the majority of symmetric spaces are interpolation spaces with respect to (L_1, L_∞) (see [2] and [18]). However, as a famous result of Calderón [5] and Mitjagin [23] says, any interpolation space with respect to the couple (L_1, L_∞) is K-monotone, and so (see [4] or [25]) it can be obtained by the general K-method, that is, extending the definition of the classical real method by replacing the usual weighted L_q norm by a more general lattice norm.

The general K-method has been studied widely, as well as the general J-method. We only mention here the monograph by Peetre [26], by Brudnyĭ and Krugljak [4], the paper by Cwikel and Peetre [11] and by Nilsson [24] and [25]. In many cases J-spaces arise as dual of K-spaces, but not always.

The behaviour of weakly compact operators by the general K-method has been investigated by Aizenstein and Brudnyĭ [4, Section 4.6], and by Mastylo [22]. Other related results have been obtained by Mastylo in [21], this time dealing with the Rosenthal property of K-spaces. To our knowledge, there is no known corresponding versions of any of these results for J-spaces.

In this paper we develop a new approach to these results that allows us to establish the J-versions at the same time, as well as to extend the results to other closed operator ideals. In particular, we cover the cases of Rosenthal operators and Banach–Saks operators. The new approach is based on ideas of Heinrich [16] and our previous results in [6] and [9].

The organization of the paper is as follows. In Section 2 we recall the definitions of general J- and K-method in the discrete form presented in [24], and we establish some preliminary results. In Section 3 we show that if $\mathcal{I}$ is the ideal of weakly compact operators, Rosenthal operators or Banach–Saks operators, then $\mathcal{I}$ satisfies a certain property, the so-called Σ_Γ-condition, relative to vector valued sequence spaces generated by Γ. Here Γ is the sequence space that we are using to define the J- or K-method. For this we assume that the identity operator I_Γ on Γ belongs to the ideal $\mathcal{I}$. Finally, in Section 4, we establish the interpolation theorems by using the Σ_Γ-condition. We also discuss the limit case when I_Γ does not belong to $\mathcal{I}$, uncovering an inaccuracy in [28, Theorem 1].

2. Preliminaries

By a *Banach couple* $\bar{A} = (A_0, A_1)$ we mean two Banach spaces A_j, $j = 0, 1$, which are continuously embedded in some Hausdorff topological vector space. For each $t > 0$ we put:

$$K(t,a) = K\big(t,a;\bar{A}\big) = \inf\big\{\|a_0\|_{A_0} + t\|a_1\|_{A_1}\colon a = a_0 + a_1,\ a_j \in A_j\big\}, \quad a \in A_0 + A_1,$$

and

$$J(t,a) = J\big(t,a;\bar{A}\big) = \max\big\{\|a\|_{A_0}, t\|a\|_{A_1}\big\}, \quad a \in A_0 \cap A_1.$$

Then $\{K(t,\cdot)\}_{t>0}$ (respectively, $\{J(t,\cdot)\}_{t>0}$) is a family of norms on $A_0 + A_1$ (respectively, $A_0 \cap A_1$), and any two of which are equivalent.

A Banach space A is said to be an *intermediate space* with respect to the couple $\bar{A}$ if $A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1$. Here $\hookrightarrow$ means continuous inclusion. The "position" of A within the couple $\bar{A}$ can be described by using the functions:

$$\psi_A(t) = \psi_A\big(t,\bar{A}\big) = \sup\big\{K(t,a)\colon \|a\|_A = 1\big\},$$
$$\rho_A(t) = \rho_A\big(t,\bar{A}\big) = \inf\big\{J(t,a)\colon a \in A_0 \cap A_1, \|a\|_A = 1\big\} \quad \text{(see [6])}.$$

A function $\varphi : (0,\infty) \to (0,\infty)$ is said to be *quasiconcave* if $\varphi(s) \leqslant \max\{1, s/t\}\varphi(t)$ for all $s, t > 0$. Functions ψ_A, ρ_A are examples of this kind of functions. Note that if φ is quasiconcave then $\varphi^*(t) = 1/\varphi(1/t)$ has also this property. If a quasiconcave function φ satisfies that

$$\min\{1, 1/t\}\varphi(t) \to 0 \quad \text{as } t \to 0 \text{ or as } t \to \infty,$$

then we write $\varphi \in \mathcal{P}_0$.

Let $\overline{B} = (B_0, B_1)$ be another Banach couple. We write $T \in \mathcal{L}(\bar{A}, \overline{B})$ and also $T : \bar{A} \to \overline{B}$ to mean that T is a linear operator from $A_0 + A_1$ into $B_0 + B_1$ whose restriction to each A_j defines a bounded operator from A_j into B_j for $j = 0, 1$. We set:

$$\|T\|_{\bar{A},\overline{B}} = \max\big\{\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}\big\}.$$

If the couple $\bar{A}$ (respectively, $\overline{B}$) reduces to a single Banach space, i.e., if $A_0 = A_1 = A$ (respectively, $B_0 = B_1 = B$), then we write $T \in \mathcal{L}(A, \overline{B})$ (respectively, $T \in \mathcal{L}(\bar{A}, B)$).

An *interpolation method* is a procedure $\mathcal{F}$ that associates to each Banach couple $\bar{A}$ an intermediate space $\mathcal{F}(\bar{A})$ in such a way that given any other Banach couple $\overline{B}$ and any $T \in \mathcal{L}(\bar{A}, \overline{B})$, the restriction of T to $\mathcal{F}(\bar{A})$ gives a bounded operator from $\mathcal{F}(\bar{A})$ into $\mathcal{F}(\overline{B})$.

By the closed graph theorem, for any couples $\bar{A}$, $\overline{B}$ there is a positive constant C such that for all $T \in \mathcal{L}(\bar{A}, \overline{B})$ it holds:

$$\|T\|_{\mathcal{F}(\bar{A}),\mathcal{F}(\overline{B})} \leqslant C \max\big\{\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}\big\}. \tag{2.1}$$

If $C = 1$ in (2.1) for all couples $\bar{A}$, $\overline{B}$, then the method $\mathcal{F}$ is called *exact*.

For $t > 0$, let $t\mathbb{R}$ be $\mathbb{R}$ with the norm $\|\lambda\|_{t\mathbb{R}} = t|\lambda|$. If $\mathcal{F}$ is an exact interpolation method, the *characteristic function* $\varphi_{\mathcal{F}}$ of $\mathcal{F}$ is defined by:

$$\mathcal{F}\big(\mathbb{R}, (1/t)\mathbb{R}\big) = \big(1/\varphi_{\mathcal{F}}(t)\big)\mathbb{R} \quad \text{(see [17]).}$$

The function $\varphi_{\mathcal{F}}$ is quasiconcave. The following result shows the connection between $\varphi_{\mathcal{F}}$, $\psi_{\mathcal{F}(\bar{A})}$ and $\rho_{\mathcal{F}(\bar{A})}$.

Lemma 2.1. *Let $\mathcal{F}$ be an exact interpolation method. For any Banach couple $\bar{A}$, it holds*:

$$\psi_{\mathcal{F}(\bar{A})}(t) \leqslant \varphi_{\mathcal{F}}(t) \leqslant \rho_{\mathcal{F}(\bar{A})}(t) \quad \textit{for all } t > 0.$$

Proof. Let $a \in \mathcal{F}(\bar{A})$ and $t > 0$. By the Hahn–Banach theorem, there is

$$f : \bar{A} \to \big(\mathbb{R}, (1/t)\mathbb{R}\big)$$

such that $f(a) = K(t, a)$ and $\|f\|_{\bar{A},(\mathbb{R},(1/t)\mathbb{R})} \leqslant 1$. Hence

$$K(t, a)/\varphi_{\mathcal{F}}(t) = \big\|f(a)\big\|_{\mathcal{F}(\mathbb{R},(1/t)\mathbb{R})} \leqslant \|a\|_{\mathcal{F}(\bar{A})}$$

and so $\psi_{\mathcal{F}(\bar{A})}(t) \leqslant \varphi_{\mathcal{F}}(t)$.

On the other hand, given any $a \in A_0 \cap A_1$ and any $t > 0$, the operator $T\lambda = \lambda a$ satisfies that

$$T : \big(\mathbb{R}, (1/t)\mathbb{R}\big) \to \bar{A} \quad \text{with} \quad \|T\|_{(\mathbb{R},(1/t)\mathbb{R}),\bar{A}} \leqslant J(t, a).$$

It follows that $\|a\|_{\mathcal{F}(\bar{A})} \leqslant J(t, a)/\varphi_{\mathcal{F}}(t)$. This implies that $\varphi_{\mathcal{F}}(t) \leqslant \rho_{\mathcal{F}(\bar{A})}(t)$. $\square$

We are interested in sufficient conditions on $\mathcal{F}$ such that $T : \mathcal{F}(\bar{A}) \to \mathcal{F}(\overline{B})$ inherits a certain property that $T : A_0 \cap A_1 \to B_0 + B_1$ has. For this reason, we review now some concepts from operator theory (see [13] and [27]). As usual, $\mathcal{L}(E, F)$ designates the collection of all bounded linear operators from the Banach space E into the Banach space F, endowed with the operator norm. We put U_E for the closed unit ball of E, and E^* for the dual space of E.

An *operator ideal* $\mathcal{I}$ is a method of ascribing to each pair (E, F) of Banach spaces a linear subspace $\mathcal{I}(E, F)$ of $\mathcal{L}(E, F)$ such that

(i) $\mathcal{I}(E, F)$ contains the finite rank operators; and
(ii) for all Banach spaces E, F, X, Y, whenever $R \in \mathcal{L}(X, E)$, $T \in \mathcal{I}(E, F)$, $S \in \mathcal{L}(F, Y)$, then the composed operator $STR \in \mathcal{I}(X, Y)$.

The ideal $\mathcal{I}$ is said to be *closed* if $\mathcal{I}(E, F)$ is a closed subspace of $\mathcal{L}(E, F)$ for all Banach spaces E and F. Other properties that an ideal may have are surjectivity and injectivity. The ideal $\mathcal{I}$ is said to be *surjective* if for every $T \in \mathcal{L}(E, F)$ it follows from

$TQ_E \in \mathcal{I}(\ell_1(U_E), F)$ that $T \in \mathcal{I}(E, F)$. Here $\ell_1(U_E)$ stands for the Banach space of all absolutely summable families of scalars with U_E as index set and where $Q_E : \ell_1(U_E) \to E$ is the surjection $Q_E\{\lambda_x\} = \sum_{x \in U_E} \lambda_x x$. The ideal is said to be *injective* if whenever $T \in \mathcal{L}(E, F)$ and in addition $J_F T \in \mathcal{I}(E, \ell_\infty(U_{F^*}))$, then it follows that $T \in \mathcal{I}(E, F)$. Now $\ell_\infty(U_{F^*})$ is the Banach space of all bounded families of scalars indexed by the elements of U_{F^*}, and $J_F : F \to \ell_\infty(U_{F^*})$ is the isometric embedding defined by $J_F y = (\langle f, y\rangle)_{f \in U_{F^*}}$. Further details can be found in [27]. Compact operators and weakly compact operators are examples of closed injective and surjective operator ideals. Other examples will be given in the next section.

The following results are immediate consequences of Lemma 2.1 and [9, Corollaries 3.5 and 3.6].

Lemma 2.2. *Let $\mathcal{I}$ be an injective closed operator ideal and let $\mathcal{F}$ be an exact interpolation method with $\varphi^*_{\mathcal{F}} \in \mathcal{P}_0$. Suppose that A is a Banach space and $\overline{B} = (B_0, B_1)$ is a Banach couple. If $T \in \mathcal{L}(A, \overline{B})$ with $T \in \mathcal{I}(A, B_0 + B_1)$ then $T \in \mathcal{I}(A, \mathcal{F}(\overline{B}))$.*

Lemma 2.3. *Let $\mathcal{I}$ be a surjective closed operator ideal and let $\mathcal{F}$ be an exact interpolation method with $\varphi_{\mathcal{F}} \in \mathcal{P}_0$. Suppose that $\bar{A} = (A_0, A_1)$ is a Banach couple and let B be a Banach space. If $T \in \mathcal{L}(\bar{A}, B)$ with $T \in \mathcal{I}(A_0 \cap A_1, B)$ then $T \in \mathcal{I}(\mathcal{F}(\bar{A}), B)$.*

Peetre's J- and K-methods are important examples of exact interpolation methods. Next we recall the discrete version of the general form of these methods (see [24,11] and [4]).

Let Γ be a Banach space of real valued sequences with $\mathbb{Z}$ as index set. We say that Γ is a $\mathbb{Z}$*-lattice* if Γ contains all sequences with only finitely many non-zero coordinates, and moreover Γ satisfies that whenever $|\xi_m| \leqslant |\mu_m|$ for each $m \in \mathbb{Z}$ and $\{\mu_m\} \in \Gamma$, then $\{\xi_m\} \in \Gamma$ and $\|\{\xi_m\}\|_\Gamma \leqslant \|\{\mu_m\}\|_\Gamma$.

The associated space Γ' of Γ consists of all sequences $\{\eta_m\}$ for which

$$\big\|\{\eta_m\}\big\|_{\Gamma'} = \sup\left\{\sum_{m=-\infty}^{\infty} |\eta_m \xi_m| : \ \big\|\{\xi_m\}\big\|_\Gamma \leqslant 1\right\} < \infty.$$

The space Γ' is also a $\mathbb{Z}$-lattice.

We say that Γ is *K-non-trivial* if

$$\big\{\min(1, 2^m)\big\} \in \Gamma. \tag{2.2}$$

The $\mathbb{Z}$-lattice Γ is called *J-non-trivial* if

$$\sup\left\{\sum_{m=-\infty}^{\infty} \min(1, 2^{-m})|\xi_m| : \ \|\xi\|_\Gamma \leqslant 1\right\} < \infty. \tag{2.3}$$

Let Γ be a K-non-trivial $\mathbb{Z}$-lattice. Given any Banach couple $\bar{A} = (A_0, A_1)$, the *K-space* $\bar{A}_{\Gamma;K} = (A_0, A_1)_{\Gamma;K}$ is formed by all $a \in A_0 + A_1$ such that $\{K(2^m, a)\} \in \Gamma$. We put $\|a\|_{\bar{A}_{\Gamma;K}} = \|\{K(2^m, a)\}\|_\Gamma$.

If Γ is J-non-trivial, the *J-space* $\bar{A}_{\Gamma;J} = (A_0, A_1)_{\Gamma;J}$ is defined as the collection of all sums $a = \sum_{m=-\infty}^{\infty} u_m$ (convergence in $A_0 + A_1$) where $\{u_m\} \subseteq A_0 \cap A_1$ and $\{J(2^m, u_m)\} \in \Gamma$. We set

$$\|a\|_{\bar{A}_{\Gamma;J}} = \inf\left\{ \left\| \{J(2^m, u_m)\} \right\|_{\Gamma} \colon a = \sum_{m=-\infty}^{\infty} u_m \right\}.$$

It is easy to check that J- and K-methods are exact interpolation methods. Conditions (2.2) and (2.3) are essential to get meaningful definitions (see [24] and [4]).

The classical real method $(A_0, A_1)_{\theta,q}$ coincide with the K- and J-methods generated by $\Gamma = \ell_q(2^{-\theta m})$, the space ℓ_q with the weight $\{2^{-\theta m}\}$,

$$(A_0, A_1)_{\ell_q(2^{-\theta m});K} = (A_0, A_1)_{\ell_q(2^{-\theta m});J} = (A_0, A_1)_{\theta,q} \quad \text{(see [3,4,30])}.$$

Here $0 < \theta < 1$ and $1 \leqslant q \leqslant \infty$. In a more general way, if f is a function parameter and $\Gamma = \ell_q(1/f(2^m))$ then

$$(A_0, A_1)_{\ell_q(1/f(2^m));K} = (A_0, A_1)_{\ell_q(1/f(2^m));J} = (A_0, A_1)_{f,q},$$

where $(A_0, A_1)_{f,q}$ is the real method with a function parameter (see [26,17,15]).

If Γ is any $\mathbb{Z}$-lattice satisfying (2.2) and (2.3), then $\bar{A}_{\Gamma;K} \hookrightarrow \bar{A}_{\Gamma;J}$. But it is not true in general that $\bar{A}_{\Gamma;K}$ coincides with $\bar{A}_{\Gamma;J}$. It is shown in [24], Lemma 2.5, that a necessary and sufficient condition for equality is that the Calderón transform

$$\Omega\{\xi_m\} = \left\{ \sum_{k=-\infty}^{\infty} \min\bigl(1, 2^{m-k}\bigr)|\xi_k| \right\}_{m\in\mathbb{Z}}$$

is bounded on Γ.

It is easy to see that the characteristic function φ_K of the K-method is:

$$\varphi_K(t) = \left\| \{\min(1, 2^m/t)\} \right\|_{\Gamma}^{-1}, \quad t > 0.$$

Next we determine the characteristic function of the J-method.

Lemma 2.4. *Let Γ be a J-non-trivial $\mathbb{Z}$-lattice. The fundamental function of the J-method defined by Γ is*:

$$\varphi_J(t) = \sup\left\{ \sum_{m=-\infty}^{\infty} \min(1, t/2^m)|\xi_m| \colon \left\|\{\xi_m\}\right\|_{\Gamma} \leqslant 1 \right\}, \quad t > 0.$$

Proof. Write

$$\eta(t) = \sup\left\{ \sum_{m=-\infty}^{\infty} \min(1, t/2^m)|\xi_m| \colon \left\|\{\xi_m\}\right\|_{\Gamma} \leqslant 1 \right\}.$$

For any $\lambda \in \mathbb{R}$ and any J-representation $\lambda = \sum_{m=-\infty}^{\infty} \lambda_m$, we have:

$$
\begin{aligned}
|\lambda| = K\big(t, \lambda; \mathbb{R}, (1/t)\mathbb{R}\big) &\leqslant \sum_{m=-\infty}^{\infty} K(t, \lambda_m) \\
&\leqslant \big\|\big\{J(2^m, \lambda_m)\big\}\big\|_\Gamma \sum_{m=-\infty}^{\infty} \min(1, t/2^m)\frac{J(2^m, \lambda_m)}{\|\{J(2^m, \lambda_m)\}\|_\Gamma} \\
&\leqslant \big\|\big\{J(2^m, \lambda_m)\big\}\big\|_\Gamma \eta(t).
\end{aligned}
$$

Therefore, $\varphi_J(t) \leqslant \eta(t)$ for any $t > 0$.

Conversely, given any $\varepsilon > 0$ there exists $\{\xi_m\} \in U_\Gamma$ such that $\eta(t) - \varepsilon \leqslant \sum_{m=-\infty}^{\infty} \min(1, t/2^m)|\xi_m|$. Put $C = \sum_{m=-\infty}^{\infty} \min(1, t/2^m)|\xi_m|$. Any $\lambda \in \mathbb{R}$ can be represented as $\lambda = \sum_{m=-\infty}^{\infty} \min(1, t/2^m)|\xi_m|\lambda/C$. Since

$$
\begin{aligned}
&J\big(2^m, \min(1, t/2^m)|\xi_m|\lambda/C; \mathbb{R}, (1/t)\mathbb{R}\big) \\
&\quad = \min(1, t/2^m)\max(1, 2^m/t)|\xi_m\lambda|/C = |\xi_m\lambda|/C,
\end{aligned}
$$

it follows that $(\eta(t) - \varepsilon)\|\lambda\|_{(\mathbb{R},(1/t)\mathbb{R})_{\Gamma;J}} \leqslant C\|\{\xi_m\}\|_\Gamma|\lambda|/C \leqslant |\lambda|$. This implies that $\eta(t) \leqslant \varphi_J(t)$ for all $t > 0$, and completes the proof. □

The next result shows that the behaviour at 0 and ∞ of the functions φ_K and φ_J can be controlled by the norms of shift operators on Γ. For $k \in \mathbb{Z}$, the *shift operator* τ_k is defined by $\tau_k\{\xi_m\}_{m\in\mathbb{Z}} = \{\xi_{m+k}\}_{m\in\mathbb{Z}}$.

Lemma 2.5. *Let Γ be a $\mathbb{Z}$-lattice such that*

$$2^{-n}\|\tau_n\|_{\Gamma,\Gamma} \to 0 \quad \textit{and} \quad \|\tau_{-n}\|_{\Gamma,\Gamma} \to 0 \quad \textit{as } n \to \infty.$$

The following holds:

(a) *if Γ is K-non-trivial, then $\varphi_K \in \mathcal{P}_0$.*
(b) *if Γ is J-non-trivial, then $\varphi_J^* \in \mathcal{P}_0$.*

Proof. Write $C = \|\{\min(1, 2^m)\}\|_\Gamma$. The norm of $(\mathbb{R}, \mathbb{R})_{\Gamma;K}$ is $C|\cdot|$. Interpolating the identity operator $I \in \mathcal{L}((\mathbb{R}, (1/t)\mathbb{R}), (\mathbb{R}, \mathbb{R}))$, we get:

$$\varphi_K(t) = C^{-1}\|I\|_{(\mathbb{R},(1/t)\mathbb{R})_{\Gamma;K},(\mathbb{R},\mathbb{R})_{\Gamma;K}}, \quad t > 0.$$

Since $\|I\|_{\mathbb{R},\mathbb{R}} = 1$ and $\|I\|_{(1/t)\mathbb{R},\mathbb{R}} = t$, using [7, Lemma 2.6/(ii)], we derive that $\lim_{t\to 0}\varphi_K(t) = 0$. On the other hand, working with $(1/t)I$ and using [7, Lemma 2.6/(i)], we obtain that $\lim_{t\to\infty}\varphi_K(t)/t = 0$.

The proof for the J-method is similar, but reversing $(\mathbb{R}, (1/t)\mathbb{R})$ and $(\mathbb{R}, \mathbb{R})$. □

3. The Σ_Γ-condition

Let Γ be a $\mathbb{Z}$-lattice. Given any sequence of Banach spaces $\{E_m\}$, the vector valued space $\Gamma(E_m)$ is defined by:

$$\Gamma(E_m) = \left\{x = \{x_m\}\colon\ x_m \in E_m \text{ and } \|x\|_{\Gamma(E_m)} = \left\|\left\{\|x_m\|_{E_m}\right\}\right\|_\Gamma < \infty\right\}.$$

We denote by $Q_k : \Gamma(E_m) \to E_k$ the projection $Q_k\{x_m\} = x_k$, and by $P_r : E_r \to \Gamma(E_m)$ the embedding $P_r x = \{\delta_m^r x\}$ where δ_m^r is the Kronecker delta. If the sequence $\{E_m\}$ reduces to a single Banach space, i.e., $E_m = E$ for all $m \in \mathbb{Z}$, then we write $\Gamma(E)$ instead of $\Gamma(E_m)$.

Definition 3.1. We say that an operator ideal $\mathcal{I}$ satisfies the *Σ_Γ-condition* if for any sequences of Banach spaces $\{E_m\}, \{F_m\}$ and for any operator $T \in \mathcal{L}(\Gamma(E_m), \Gamma(F_m))$, it follows from $Q_k T P_r \in \mathcal{I}(E_r, F_k)$ for any $r, k \in \mathbb{Z}$ that $T \in \mathcal{I}(\Gamma(E_m), \Gamma(F_m))$.

For the special case $\Gamma = \ell_q$, this condition was investigated by Heinrich in [16]. To say that $\mathcal{I}$ satisfies the Σ_Γ-condition means that the operator $T \in \mathcal{L}(\Gamma(E_m), \Gamma(F_m))$ belongs to $\mathcal{I}$ if and only if all elements of its matrix representation belong to $\mathcal{I}$. Such an ideal must be closed as the following result shows.

Lemma 3.2. *Let Γ be a $\mathbb{Z}$-lattice. Each operator ideal $\mathcal{I}$ which satisfies the Σ_Γ-condition is closed.*

Proof. Take any Banach space E, F and any sequence of operators $\{T_n\}_{n\in\mathbb{N}} \subseteq \mathcal{I}(E, F)$ with $\sum_{n=1}^\infty \|T_n\|_{E,F} < \infty$. We should prove that the operator $T = \sum_{n=1}^\infty T_n$ belongs to $\mathcal{I}(E, F)$. We may assume that $\|T_n\|_{E,F} > 0$ for each $n \in \mathbb{N}$.

Since $\left\{\|T_n\|_{E,F}^{1/2}\right\} \in \ell_2$ and $\ell_2 = \Gamma^{1/2}(\Gamma')^{1/2}$ (see [19]), we can find sequences $\alpha = \{\alpha_m\} \in \Gamma$, $\beta = \{\beta_m\} \in \Gamma'$ with non-negative coordinates, such that $\|T_n\|_{E,F} = \alpha_n\beta_n$ for all $n \in \mathbb{N}$ and $\alpha_m = \beta_m = 0$ for all $m \in \mathbb{Z} - \mathbb{N}$. Each operator T_n can be factorized as $T_n = S_n R_n$ where $R_n = \beta_n^{-1} T_n$ and $S_n = \beta_n I_F$. Put $R_m = S_m = 0$ for $m \in \mathbb{Z} - \mathbb{N}$.

The operator $R : E \to \Gamma(F)$ defined by $Rx = \{R_m x\}_{m\in\mathbb{Z}}$ is bounded because $\|Rx\|_{\Gamma(F)} \leqslant \|\alpha\|_\Gamma \|x\|_E$. We claim that $R \in \mathcal{I}(E, \Gamma(F))$. Indeed, the space E can be realized as a vector valued space $\Gamma(E_m)$ if we choose, for example, $E_0 = E$ and $E_m = 0$ for all $m \neq 0$. Since for any $r, k \in \mathbb{Z}$,

$$Q_k R P_r = \begin{cases} 0 & \text{if } r \neq 0, \\ R_k & \text{if } r = 0 \end{cases}$$

belongs to $\mathcal{I}(E, F)$, the Σ_Γ-condition implies that $R \in \mathcal{I}(E, \Gamma(F))$.

Let $S : \Gamma(F) \to F$ be the operator defined by $S\{z_m\} = \sum_{m=-\infty}^\infty S_m z_m = \sum_{n=1}^\infty \beta_n z_n$. We have:

$$\left\|S\{z_m\}\right\|_F \leqslant \sum_{m=-\infty}^\infty \beta_m \|z_m\|_F \leqslant \left\|\{\beta_m\}\right\|_{\Gamma'} \left\|\left\{\|z_m\|_F\right\}\right\|_\Gamma = \|\beta\|_{\Gamma'} \left\|\{z_m\}\right\|_{\Gamma(F)},$$

so S is bounded. Since $T = SR$, we conclude that $T \in \mathcal{I}(E, F)$. $\square$

Clearly, if $\mathcal{I}$ satisfies the Σ_Γ-condition then the identity operator I_Γ on Γ must belong to $\mathcal{I}$.

As in the case $\Gamma = \ell_q$ (see [16]), the following lemma will be useful later on to check if an ideal satisfies the Σ_Γ-condition.

Lemma 3.3. *Let Γ be a $\mathbb{Z}$-lattice. An operator ideal $\mathcal{I}$ satisfies the Σ_Γ-condition provided the following holds for any Banach spaces E, F, G_m $(m \in \mathbb{Z})$:*

If $T_1 \in \mathcal{L}(E, \Gamma(G_m))$, $T_2 \in \mathcal{L}(\Gamma(G_m), F))$ and $T_2 P_s Q_s T_1 \in \mathcal{I}(E, F)$ for all $s \in \mathbb{Z}$, then $T_2 T_1 \in \mathcal{I}(E, F)$.

Proof. Let $\{E_m\}, \{F_m\}$ be arbitrary sequences of Banach spaces and let $T \in \mathcal{L}(\Gamma(E_m), \Gamma(F_m))$ such that $Q_k T P_r \in \mathcal{I}(E_r, F_k)$ for any $r, k \in \mathbb{Z}$. Fix any $r \in \mathbb{Z}$ and put $T_1 = T P_r \in \mathcal{L}(E_r, \Gamma(F_m))$ and $T_2 = I_{\Gamma(F_m)} \in \mathcal{L}(\Gamma(F_m), \Gamma(F_m))$. For any $s \in \mathbb{Z}$, we have $T_2 P_s Q_s T_1 = P_s(Q_s T P_r)$, so $T_2 P_s Q_s T_1 \in \mathcal{I}(E_r, \Gamma(F_m))$. Assumption on $\mathcal{I}$ implies that $T P_r \in \mathcal{I}(E_r, \Gamma(F_m))$.

Now take $S_1 = I_{\Gamma(E_m)} \in \mathcal{L}(\Gamma(E_m), \Gamma(E_m))$ and $S_2 = T \in \mathcal{L}(\Gamma(E_m), \Gamma(F_m))$. For any $r \in \mathbb{Z}$, we get $S_2 P_r Q_r S_1 = (T P_r) Q_r \in \mathcal{I}(\Gamma(E_m), \Gamma(F_m))$. Hence, using again the assumption on $\mathcal{I}$, we derive that $T \in \mathcal{I}(\Gamma(E_m), \Gamma(F_m))$. $\square$

Next we give an example of an ideal which satisfies Σ_Γ-condition. We recall that a $\mathbb{Z}$-lattice Γ is said to be *regular* if for any $\{\xi_n\}_{n \in \mathbb{N}} \subseteq \Gamma$ with $\xi_n \downarrow 0$ it follows that $\|\xi_n\|_\Gamma \to 0$ as $n \to \infty$.

Theorem 3.4. *The ideal $\mathcal{W}$ of weakly compact operators satisfies the Σ_Γ-condition for any reflexive $\mathbb{Z}$-lattice Γ.*

Proof. It is well known that if Γ is reflexive then it is regular. Hence, $\Gamma^* = \Gamma'$ and for any sequence $\{E_m\}$ of Banach spaces it holds:

$$\Gamma(E_m)^{**} = \Gamma'\big(E_m^*\big)^* = \Gamma\big(E_m^{**}\big) \quad \text{(see [18] or [21]).} \tag{3.1}$$

Let $\{F_m\}_{m \in \mathbb{Z}}$ be another sequence of Banach spaces and let:
$T \in \mathcal{L}(\Gamma(E_m), \Gamma(F_m))$ such that $Q_k T P_r \in \mathcal{W}(E_r, F_k)$ for any $r, k \in \mathbb{Z}$. According to Gantmacher's theorem and (3.1), to check that T is weakly compact we should show that

$$T^{**} x^{**} \in \Gamma(F_m) \quad \text{for all } x^{**} \in \Gamma\big(E_m^{**}\big). \tag{3.2}$$

But $x^{**} = \lim_{s \to \infty} \sum_{r=-s}^{s} P_r Q_r x^{**}$, so it suffices to establish (3.2) when x^{**} has finitely many non-zero coordinates. Say,

$$x^{**} = \{\ldots, 0, 0, x_{-s}^{**}, x_{-s+1}^{**}, \ldots, x_{s-1}^{**}, x_s^{**}, 0, 0, \ldots\}.$$

We have:

$$T^{**}x^{**} = \sum_{r=-s}^{s} T^{**}P_r x_r^{**} = \lim_{n\to\infty} \sum_{k=-n}^{n} \sum_{r=-s}^{s} P_k Q_k T^{**} P_r x_r^{**}$$
$$= \lim_{n\to\infty} \sum_{k=-n}^{n} \sum_{r=-s}^{s} P_k (Q_k T P_r)^{**} x_r^{**}.$$

Using weak compactness of operators $Q_k T P_r$, we get that

$$\sum_{k=-n}^{n} \sum_{r=-s}^{s} P_k (Q_k T P_r)^{**} x_r^{**} \in \Gamma(F_m).$$

This implies that $T^{**}x^{**} \in \Gamma(F_m)$ because $\Gamma(F_m)$ is a closed subspace of

$$\Gamma(F_m^{**}) = \Gamma(F_m)^{**}. \qquad \square$$

In order to show other example, we recall that an operator $T \in \mathcal{L}(E, F)$ is said to be a *Rosenthal operator* if for every bounded sequence $\{x_n\} \subseteq E$, the sequence $\{Tx_n\}$ admits a weak Cauchy subsequence. By Rosenthal's theorem [29], the former condition is equivalent to the fact that no subspace of $T(E)$ is isomorphic to ℓ_1. In other words, $T(E)$ does not contain a copy of ℓ_1. Rosenthal operators form an injective and surjective operator ideal.

Theorem 3.5. *The ideal $\mathcal{R}$ of Rosenthal operators satisfies the Σ_Γ-condition for any $\mathbb{Z}$-lattice Γ which does not contain a copy of ℓ_1.*

Proof. According to Lemma 3.3, it is enough to show that for any Banach spaces E, F, G_m ($m \in \mathbb{Z}$) and any operators $T_1 \in \mathcal{L}(E, \Gamma(G_m))$, $T_2 \in \mathcal{L}(\Gamma(G_m), F)$ such that $T_2 P_s Q_s T_1 \in \mathcal{R}(E, F)$ for any $s \in \mathbb{Z}$, it holds $T_2 T_1 \in \mathcal{R}(E, F)$. For this aim, take any bounded sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ and let $M = \sup_{n\in\mathbb{N}}\{\|x_n\|_E\}$. Using that $T_2 P_s Q_s T_1 \in \mathcal{R}(E, F)$, we can find a subsequence $\{\widetilde{x}_n\}$ of $\{x_n\}$ such that $\{\sum_{s=-N}^{N} T_2 P_s Q_s T_1 \widetilde{x}_n\}_{n\in\mathbb{N}}$ is a weak Cauchy sequence for any $N \in \mathbb{N}$. Let us check that $\{T_2 T_1 \widetilde{x}_n\}_{n\in\mathbb{N}}$ is a weak Cauchy sequence.

Since Γ does not contain a copy of ℓ_1, according to [31, Theorem 117.3], Γ is regular. Hence, $\Gamma(G_m)^* = \Gamma'(G_m^*)$. Using again that $\ell_1 \not\subset \Gamma$, it follows from [31, Theorem 117.2], that Γ' is also regular. Whence, given any $f \in F^*$ and any $\varepsilon > 0$, we can find $N \in \mathbb{N}$ with

$$\left\| T_2^* f - \sum_{s=-N}^{N} P_s Q_s T_2^* f \right\|_{\Gamma'(G_m^*)} \leqslant \frac{\varepsilon}{4M\|T_1\|_{E,\Gamma(G_m)}}.$$

Now, since $\{\sum_{s=-N}^{N} T_2 P_s Q_s T_1 \widetilde{x}_n\}_{n\in\mathbb{N}}$ is a weak Cauchy sequence, there exits $n_0 \in \mathbb{N}$ such that

$$\left| \left\langle \sum_{s=-N}^{N} T_2 P_s Q_s T_1 (\widetilde{x}_n - \widetilde{x}_k), f \right\rangle \right| \leqslant \frac{\varepsilon}{2} \quad \text{for all } n, k \geqslant n_0.$$

Consequently, for any $n, k \geqslant n_0$, we obtain:

$$\begin{aligned}
\left|\left\langle T_2 T_1(\widetilde{x}_n - \widetilde{x}_k), f\right\rangle\right| &\leqslant \left|\left\langle T_1(\widetilde{x}_n - \widetilde{x}_k), T_2^* f - \sum_{s=-N}^{N} P_s Q_s T_2^* f\right\rangle\right| \\
&\quad + \left|\left\langle T_1(\widetilde{x}_n - \widetilde{x}_k), \sum_{s=-N}^{N} P_s Q_s T_2^* f\right\rangle\right| \\
&\leqslant 2M \|T_1\|_{E,\Gamma(G_m)} \left\| T_2^* f - \sum_{s=-N}^{N} P_s Q_s T_2^* f \right\|_{\Gamma'(G_m^*)} \\
&\quad + \left|\left\langle \sum_{s=-N}^{N} T_2 P_s Q_s T_1(\widetilde{x}_n - \widetilde{x}_k), f\right\rangle\right| \\
&\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This completes the proof. $\square$

The next example refers to Banach–Saks operators. Recall that an operator $T \in \mathcal{L}(E, F)$ is called a *Banach–Saks operator* if it maps bounded sequences into sequences possessing Cesaro convergent subsequences. A Banach space E is said to have the *Banach–Saks property* if the identity operator I_E is Banach–Saks.

Theorem 3.6. *The ideal $\mathcal{BS}$ of Banach–Saks operators satisfies the Σ_Γ-condition for any $\mathbb{Z}$-lattice Γ with the Banach–Saks property.*

Proof. We follow the main lines of the proof for the case $\Gamma = \ell_q$ established by Heinrich in [16]. Take any Banach spaces E, F, G_m $(m \in \mathbb{Z})$ and any operators $T_1 \in \mathcal{L}(E, \Gamma(G_m))$, $T_2 \in \mathcal{L}(\Gamma(G_m), F))$ with $T_2 P_s Q_s T_1 \in \mathcal{BS}(E, F)$ for any $s \in \mathbb{Z}$. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ be any bounded sequence. Using that $T_2 P_s Q_s T_1 \in \mathcal{BS}(E, F)$ and applying a result of Erdős and Magidor [14], for each $s \in \mathbb{N}$ we can find a subsequence $\{x_n'\}$ of $\{x_n\}$ such that all subsequences of $\{T_2 P_s Q_s T_1 x_n'\}$ are Cesaro convergent. It follows that $\{x_n\}$ has a subsequence $\{\widetilde{x}_n\}$ such that $\{T_2 P_s Q_s T_1 \widetilde{x}_n\}$ is Cesaro convergent for all s simultaneously.

Let $\xi_n = \{\|Q_m T_1 \widetilde{x}_n\|_{G_m}\}_{m\in\mathbb{Z}}$. We have $\|\xi_n\|_\Gamma = \|T_1 \widetilde{x}_n\|_{\Gamma(G_m)}$, so the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ is bounded in Γ. Since Γ has the Banach–Saks property, we may assume that $\{\widetilde{x}_n\}$ has been chosen in such a way that $\{\xi_n\}$ is Cesaro convergent in Γ. Let $\mu = \{\mu_m\}_{m\in\mathbb{Z}}$ be its limit. Banach–Saks property of Γ implies also that Γ is regular, and so $\|\{\gamma_m^N \mu_m\}\|_\Gamma \to 0$ as $N \to \infty$, where:

$$\gamma_m^N = \begin{cases} 0 & \text{if } |m| \leqslant N, \\ 1 & \text{if } |m| > N. \end{cases}$$

Combining this fact with the Cesaro convergence of $\{\xi_n\}$ to μ, we derive that for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n > N$, it holds:

$$\left\| \left\{ \gamma_m^N \frac{1}{n} \sum_{j=1}^{n} \| Q_m T_1 \widetilde{x}_j \|_{G_m} \right\} \right\|_{\Gamma} \leqslant \frac{\varepsilon}{\|T_2\|_{\Gamma(G_m),F}}.$$

Whence,

$$\left\| \frac{1}{n} \sum_{j=1}^{n} T_2 \sum_{|m|>N} P_m Q_m T_1 \widetilde{x}_j \right\|_F \leqslant \varepsilon.$$

Since

$$\frac{1}{n} \sum_{j=1}^{n} T_2 T_1 \widetilde{x}_j = \frac{1}{n} \sum_{j=1}^{n} T_2 \sum_{|m|\leqslant N} P_m Q_m T_1 \widetilde{x}_j + \frac{1}{n} \sum_{j=1}^{n} T_2 \sum_{|m|>N} P_m Q_m T_1 \widetilde{x}_j$$

and $\{\sum_{|m|\leqslant N} T_2 P_m Q_m T_1 \widetilde{x}_j\}_{j\in\mathbb{N}}$ is Cesaro convergent, it follows that $\{\frac{1}{n}\sum_{j=1}^{n} T_2 T_1 \widetilde{x}_j\}_{n\in\mathbb{N}}$ is a Cauchy sequence and therefore it is convergent.

The proof is complete. □

Banach–Saks operators form also an injective and surjective operator ideal.

4. Real interpolation and operator ideals

In this section we establish interpolation results for general couples by using the Σ_Γ-condition.

Theorem 4.1. *Let Γ be a K-non-trivial $\mathbb{Z}$-lattice with $\varphi_K \in \mathcal{P}_0$, and let $\mathcal{I}$ be an injective and surjective operator ideal which satisfies the Σ_Γ-condition. Suppose $\bar{A} = (A_0, A_1)$, $\overline{B} = (B_0, B_1)$ are Banach couples and let $T \in \mathcal{L}(\bar{A}, \overline{B})$. Then*

$$T \in \mathcal{I}\big(\bar{A}_{\Gamma;K}, \overline{B}_{\Gamma;K}\big) \quad \textit{if and only if} \quad T \in \mathcal{I}(A_0 \cap A_1, B_0 + B_1).$$

Proof. Assume that $T \in \mathcal{I}(A_0 \cap A_1, B_0 + B_1)$. Since $\mathcal{I}$ is surjective and $\varphi_K \in \mathcal{P}_0$, applying Lemma 2.3 we get that $T \in \mathcal{I}(\bar{A}_{\Gamma;K}, B_0 + B_1)$. Let F_m be the space $B_0 + B_1$ endowed with the norm $K(2^m, \cdot)$, $m \in \mathbb{Z}$, and let $\widehat{T} : \bar{A}_{\Gamma;K} \to \Gamma(F_m)$ be the operator defined by $\widehat{T}x = \{\ldots, Tx, Tx, Tx, \ldots\}$. For each $m \in \mathbb{Z}$, $Q_m \widehat{T} = T$ belongs to $\mathcal{I}(\bar{A}_{\Gamma;K}, F_m)$. Then, the Σ_Γ-condition implies that $\widehat{T} \in \mathcal{I}(\bar{A}_{\Gamma;K}, \Gamma(F_m))$. Now we consider the isometric embedding $j : \overline{B}_{\Gamma;K} \to \Gamma(F_m)$ given by $j(y) = \{\ldots, y, y, y, \ldots\}$. Since $jT = \widehat{T}$, using the injectivity of $\mathcal{I}$, we conclude that $T \in \mathcal{I}(\bar{A}_{\Gamma;K}, \overline{B}_{\Gamma;K})$.

Obviously, if $T \in \mathcal{I}(\bar{A}_{\Gamma;K}, \overline{B}_{\Gamma;K})$ then $T \in \mathcal{I}(A_0 \cap A_1, B_0 + B_1)$. □

The result for the J-method reads:

Theorem 4.2. *Let Γ be a J-non-trivial $\mathbb{Z}$-lattice with $\varphi_J^* \in \mathcal{P}_0$, and let $\mathcal{I}$ be an injective and surjective operator ideal which satisfies the Σ_Γ-condition. Suppose $\bar{A} = (A_0, A_1)$, $\overline{B} = (B_0, B_1)$ are Banach couples and let $T \in \mathcal{L}(\bar{A}, \overline{B})$. Then*

$$T \in \mathcal{I}\big(\bar{A}_{\Gamma;J}, \overline{B}_{\Gamma;J}\big) \quad \textit{if and only if} \quad T \in \mathcal{I}(A_0 \cap A_1, B_0 + B_1).$$

Proof. We only need to prove that $T \in \mathcal{I}(A_0 \cap A_1, B_0 + B_1)$ implies $T \in \mathcal{I}(\bar{A}_{\Gamma;J}, \overline{B}_{\Gamma;J})$ because the converse implication is clear. If $T \in \mathcal{I}(A_0 \cap A_1, B_0 + B_1)$, since $\mathcal{I}$ is injective and $\varphi_J^* \in \mathcal{P}_0$, Lemma 2.2 yields that $T \in \mathcal{I}(A_0 \cap A_1, \overline{B}_{\Gamma;J})$. Let G_m be the space $A_0 \cap A_1$ with the norm $J(2^m, \cdot)$, $m \in \mathbb{Z}$, and let $\widetilde{T} : \Gamma(G_m) \to \overline{B}_{\Gamma;J}$ be the operator defined by $\widetilde{T}\{u_m\} = T(\sum_{m=-\infty}^{\infty} u_m)$. For each $m \in \mathbb{Z}$, $\widetilde{T} P_m = T$ belongs to $\mathcal{I}(G_m, \overline{B}_{\Gamma;J})$. Whence, according to the Σ_Γ-condition, $\widetilde{T} \in \mathcal{I}(\Gamma(G_m), \overline{B}_{\Gamma;J})$. Let $\pi : \Gamma(G_m) \to \bar{A}_{\Gamma;J}$ be the metric surjection given by $\pi\{u_m\} = \sum_{m=-\infty}^{\infty} u_m$. Using that $\mathcal{I}$ is surjective, it follows from $\widetilde{T} = T\pi$ that $T \in \mathcal{I}(\bar{A}_{\Gamma;J}, \overline{B}_{\Gamma;J})$. □

Remark 4.3. Lemma 2.5 gives a sufficient condition for $\varphi_K \in \mathcal{P}_0$ and $\varphi_J^* \in \mathcal{P}_0$ in terms of shift operators on Γ.

Combining Theorems 4.1 and 4.2 with the results of Section 3 we derive:

Corollary 4.4. *Let Γ be a reflexive $\mathbb{Z}$-lattice. Let $\bar{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be Banach couples, and let $T \in \mathcal{L}(\bar{A}, \overline{B})$ such that $T : A_0 \cap A_1 \to B_0 + B_1$ is weakly compact.*

(i) *If Γ is K-non-trivial with $\varphi_K \in \mathcal{P}_0$, then $T : \bar{A}_{\Gamma;K} \to \overline{B}_{\Gamma;K}$ is weakly compact.*
(ii) *If Γ is J-non-trivial with $\varphi_J^* \in \mathcal{P}_0$, then $T : \bar{A}_{\Gamma;J} \to \overline{B}_{\Gamma;J}$ is weakly compact.*

Corollary 4.5. *Let Γ be a $\mathbb{Z}$-lattice which does not contain a copy of ℓ_1. Let $\bar{A} = (A_0, A_1)$, $\overline{B} = (B_0, B_1)$ be Banach couples, and let $T \in \mathcal{L}(\bar{A}, \overline{B})$ such that $T : A_0 \cap A_1 \to B_0 + B_1$ is Rosenthal.*

(i) *If Γ is K-non-trivial with $\varphi_K \in \mathcal{P}_0$, then $T : \bar{A}_{\Gamma;K} \to \overline{B}_{\Gamma;K}$ is a Rosenthal operator.*
(ii) *If Γ is J-non-trivial with $\varphi_J^* \in \mathcal{P}_0$, then $T : \bar{A}_{\Gamma;J} \to \overline{B}_{\Gamma;J}$ is a Rosenthal operator.*

Corollary 4.6. *Let Γ be a $\mathbb{Z}$-lattice with the Banach–Saks property. Let $\bar{A} = (A_0, A_1)$, $\overline{B} = (B_0, B_1)$ be Banach couples, and let $T \in \mathcal{L}(\bar{A}, \overline{B})$ such that $T : A_0 \cap A_1 \to B_0 + B_1$ is Banach–Saks.*

(i) *If Γ is K-non-trivial with $\varphi_K \in \mathcal{P}_0$, then $T : \bar{A}_{\Gamma;K} \to \overline{B}_{\Gamma;K}$ is a Banach–Saks operator.*
(ii) *If Γ is J-non-trivial with $\varphi_J^* \in \mathcal{P}_0$, then $T : \bar{A}_{\Gamma;J} \to \overline{B}_{\Gamma;J}$ is a Banach–Saks operator.*

All results for the J-method are new. The results for the K-method comprise several known theorems. To be precise:

(a) Corollary 4.4/(i) is a result of Aizenstein and Brudnyĭ [4, Theorem 4.6.8] (see also the paper by Mastylo [22, Corollary 11]). Writing down the special case $\Gamma = \ell_q(2^{-\theta m})$ with $1 < q < \infty$, $0 < \theta < 1$, we recover results of Heinrich [16] and Maligranda and Quevedo [20, Theorem 1]. In particular, if $\Gamma = \ell_q(2^{-\theta m})$, $\bar{A} = \overline{B}$ and T is taken to be the identity operator, we get a well-known result of Beauzamy [1, Proposition II. 2.3], on reflexivity of spaces $(A_0, A_1)_{\theta,q}$.
(b) Corollary 4.5/(i) for $\bar{A} = \overline{B}$ and $T = I$ is due to Mastylo [21, Theorem 3.3]. The special case $\Gamma = \ell_q(2^{-\theta m})$ with $1 < q < \infty$, $0 < \theta < 1$, and $T = I$ is a result of Beauzamy [1, Proposition II. 3.3], on copies of ℓ_1 in $(A_0, A_1)_{\theta,q}$.
(c) If we write down Corollary 4.6/(i) for $\bar{A} = \overline{B}$, $T = I$ and $\Gamma = \ell_q(2^{-\theta m})$ with $1 < q < \infty$, $0 < \theta < 1$, we recover a result of Heinrich [16, Corollary 2.5/(i)].

In the last three corollaries, we are assuming that the identity operator on Γ belongs to $\mathcal{I}$ to get that $\mathcal{I}$ satisfies the Σ_Γ-condition. The corollaries fail in general if $I_\Gamma \notin \mathcal{I}$. Easy counterexamples can be constructed, taking even $\bar{A} = \overline{B}$ and $T = I$. However, we show next that under a very restrictive condition on the couple, a positive result still holds when $I_\Gamma \notin \mathcal{I}$.

Proposition 4.7. *Let Γ be a K- and J-non-trivial $\mathbb{Z}$-lattice, and let $\mathcal{I}$ be an injective operator ideal. Suppose $\bar{A} = (A_0, A_1)$ is a Banach couples. If the embedding*

$$i : A_0 \cap A_1 \to A_0 + A_1$$

belongs to $\mathcal{I}$ and its range is closed, then the identity operators $I_{\bar{A}_{\Gamma;K}}$, $I_{\bar{A}_{\Gamma;J}}$ belong to $\mathcal{I}$.

Proof.
As we pointed out in Section 2

$$A_0 \cap A_1 \hookrightarrow \bar{A}_{\Gamma;K} \hookrightarrow \bar{A}_{\Gamma;J} \hookrightarrow A_0 + A_1.$$

Moreover, $A_0 \cap A_1$ is dense in $\bar{A}_{\Gamma;J}$ for the norm of $A_0 + A_1$. Therefore, if

$$i : A_0 \cap A_1 \to A_0 + A_1$$

has closed range, we obtain that

$$A_0 \cap A_1 = \bar{A}_{\Gamma;K} = \bar{A}_{\Gamma;J} = \overline{A_0 \cap A_1}^{A_0+A_1}$$

with equivalent norms. Now using that the embedding $i : A_0 \cap A_1 \to A_0 + A_1$ belongs to $\mathcal{I}$ and that $\mathcal{I}$ is injective, we conclude that identity operators $I_{\bar{A}_{\Gamma;K}}$, $I_{\bar{A}_{\Gamma;J}}$ belong to $\mathcal{I}$. □

Given any operator ideal $\mathcal{I}$ and any $\mathbb{Z}$-lattice Γ it is clear that $I_{\bar{A}_{\Gamma;K}} \in \mathcal{I}(\bar{A}_{\Gamma;K}, \bar{A}_{\Gamma;K})$ implies that the embedding $i : A_0 \cap A_1 \to A_0 + A_1$ belongs to $\mathcal{I}$. But it is false in general that the embedding has closed range. Even if we ask, in addition, that $\mathcal{I}$ is injective and $I_\Gamma \notin \mathcal{I}(\Gamma, \Gamma)$. Next we show it by means of an example.

Counterexample 4.8. Let $0 < \theta < 1$ and put:

$$\Gamma = \left\{ \{\xi_m\}_{m\in\mathbb{Z}} \colon \left\| \{\xi_m\} \right\|_\Gamma = \left(\sum_{m=-1}^{-\infty} \left(2^{-\theta m} |\xi_m| \right)^2 \right)^{1/2} + \sum_{m=0}^{\infty} 2^{-\theta m} |\xi_m| < \infty \right\}.$$

It is easy to check that Γ is a K-non-trivial $\mathbb{Z}$-lattice. Take any ordered Banach couple $\bar{A} = (A_0, A_1)$, that is , a pair $A_1 \hookrightarrow A_0$, with the embedding having norm 1. We claim that

$$(A_0, A_1)_{\Gamma;K} = (A_0, A_1)_{\theta,2} \quad \text{(equivalent norms)}.$$

Indeed, we have:

$$K(t,a) = \|a\|_{A_0} \quad \text{for } t \geqslant 1, \qquad t\|a\|_{A_0} \leqslant K(t,a) \quad \text{for } 0 < t \leqslant 1.$$

Whence,

$$\begin{aligned} \|a\|_{\bar{A}_{\Gamma;K}} &= \frac{1}{1-2^{-\theta}} \|a\|_{A_0} + \left(\sum_{m=-1}^{-\infty} \left(2^{-\theta m} K(2^m, a) \right)^2 \right)^{1/2} \\ &\leqslant \left[\frac{2^{(1-\theta)}(1-2^{-(1-\theta)2})^{1/2}}{1-2^{-\theta}} + 1 \right] \left(\sum_{m=-1}^{-\infty} \left(2^{-\theta m} K(2^m, a) \right)^2 \right)^{1/2} \\ &\sim \|a\|_{(A_0,A_1)_{\theta,2}}, \end{aligned}$$

where $\sim$ means equivalence with constants which do not depend on a. Now choose $\theta = 1/2$, $\mathcal{I} = \mathcal{W}$ and $\bar{A} = (\ell_\infty, \ell_1)$. The embedding $i : \ell_1 \to \ell_\infty$ is weakly compact, the $\mathbb{Z}$-lattice Γ is not reflexive, the interpolation space $\bar{A}_{\Gamma;K} = (\ell_\infty, \ell_1)_{1/2,2} = \ell_2$ is reflexive, but the embedding $i : \ell_1 \to \ell_\infty$ does not have closed range.

The counterexample uncover an inaccuracy in [28]: Theorem 1/(b) is not true in general.

References

[1] B. Beauzamy, Espaces d'interpolation réels: topologie et géométrie, in: Lecture Notes in Math., vol. 666, Springer-Verlag, Berlin/New York, 1978.

[2] C. Bennett, R. Sharpley, Interpolation of Operators, Academic Press, New York, 1988.

[3] J. Bergh, J. Löfström, Interpolation Spaces. An Introduction, Springer-Verlag, Berlin/New York, 1976.

[4] Y. Brudnyĭ, N. Krugljak, Interpolation Functors and Interpolation Spaces, vol. 1, North-Holland, Amsterdam, 1991.

[5] A.P. Calderón, Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz, Studia Math. 26 (1966) 273–299.

[6] F. Cobos, M. Cwikel, P. Matos, Best possible compactness results of Lions–Peetre type, Proc. Edinburgh Math. Soc. 44 (2001) 153–172.

[7] F. Cobos, L.M. Fernández-Cabrera, A. Martínez, Compact operators between K- and J-spaces, Studia Math., in press.

[8] F. Cobos, A. Manzano, A. Martínez, Interpolation theory and measures related to operator ideals, Quart. J. Math. 50 (1999) 401–416.
[9] F. Cobos, A. Manzano, A. Martínez, P. Matos, On interpolation of strictly singular operators, strictly cosingular operators and related operator ideals, Proc. Roy. Soc. Edinburgh Sect. A 130 (2000) 971–989.
[10] F. Cobos, A. Martínez, Extreme estimates for interpolated operators by the real method, J. London Math. Soc. 60 (1999) 860–870.
[11] M. Cwikel, J. Peetre, Abstract K and J spaces, J. Math. Pures Appl. 60 (1981) 1–50.
[12] W.J. Davis, T. Figiel, W.B. Johnson, A. Pelczyński, Factoring weakly compact operators, J. Funct. Anal. 17 (1974) 311–327.
[13] J. Diestel, H. Jarchow, A. Tonge, Absolutely Summing Operators, in: Cambridge Stud. Adv. Math., vol. 43, Cambridge Univ. Press, Cambridge, UK, 1995.
[14] P. Erdős, M. Magidor, A note on regular methods of summability and the Banach–Saks property, Proc. Amer. Math. Soc. 59 (1976) 232–234.
[15] J. Gustavsson, A function parameter in connection with interpolation of Banach spaces, Math. Scand. 42 (1978) 289–305.
[16] S. Heinrich, Closed operator ideals and interpolation, J. Funct. Anal. 35 (1980) 397–411.
[17] S. Janson, Minimal and maximal methods of interpolation, J. Funct. Anal. 44 (1981) 50–73.
[18] S.G. Kreĭn, Ju.I. Petunin, E.M. Semenov, Interpolation of Linear Operators, Amer. Math. Soc., Providence, RI, 1982.
[19] G.Ya. Lozanovskii, On some Banach lattices, Siberian Math. J. 10 (1969) 419–431.
[20] L. Maligranda, A. Quevedo, Interpolation of weakly compact operators, Arch. Math. 55 (1990) 280–284.
[21] M. Mastylo, Interpolation spaces not containing ℓ^1, J. Math. Pures Appl. 68 (1989) 153–162.
[22] M. Mastylo, On interpolation of weakly compact operators, Hokkaido Math. J. 22 (1993) 105–114.
[23] B.S. Mitjagin, An interpolation theorem for modular spaces, Mat. Sbornik 66 (1965) 472–482.
[24] P. Nilsson, Reiteration theorems for real interpolation and approximation spaces, Ann. Mat. Pura Appl. 132 (1982) 291–330.
[25] P. Nilsson, Interpolation of Calderón and Ovchinnikov pairs, Ann. Mat. Pura Appl. 134 (1983) 201–232.
[26] J. Peetre, A theory of interpolation of normed spaces, Notes Mat. 39 (1968) 1–86. Lecture Notes, Brasilia, 1963.
[27] A. Pietsch, Operator Ideals, North-Holland, Amsterdam, 1980.
[28] A. Quevedo, Some remarks on the reflexivity of the real interpolation space, J. Math. Anal. Appl. 162 (1991) 189–195.
[29] H.P. Rosenthal, A characterization of Banach spaces containing ℓ_1, Proc. Natl. Acad. Sci. USA 71 (1974) 2411–2413.
[30] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, North-Holland, Amsterdam, 1978.
[31] A.C. Zaanen, Riesz Spaces II, North-Holland, Amsterdam, 1983.