Cesàro operators on Hardy spaces in the unit ball

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Abstract
This article establishes the boundedness of the generalized Cesàro operator on holomorphic Hardy spaces in the unit ball. The approach consists in writing the generalized Cesàro operator as a composition of certain integral operators.

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1. Introduction
Let \( B_n \) denote the unit ball in \( \mathbb{C}^n \) and \( d\sigma \) the normalized Lebesgue measure in the unit sphere \( S_n \). The Hardy space \( H^p(B_n) \) consists of all holomorphic functions \( f \in H(B_n) \), for which

\[
\|f\|_{H^p(B_n)} = \sup_{0 \leq r < 1} \left( \int_{S_n} |f(r \cdot \zeta)|^p \, d\sigma(\zeta) \right)^{1/p} < \infty.
\]

In the unit disc \( U \) in \( \mathbb{C} \), Stempark [1] introduced the Cesàro averaging operators \( \mathcal{C}^\eta \), defined by

\[
\mathcal{C}^\eta(f)(z) = \sum_{n=0}^\infty \left( \frac{1}{A_n} \sum_{j=0}^n A_n^{\eta-j} a_j \right) z^n,
\]

where \( f(z) = \sum_{n=0}^\infty a_n z^n \) is holomorphic in \( U \), \( \Re \eta > -1 \), and

\[
A_n^\eta = \frac{\Gamma(n + \eta + 1)}{\Gamma(n + 1) \Gamma(\eta + 1)}.
\]

When \( \eta = 0 \), \( \mathcal{C}^\eta \) is the classical Cesàro operators, which has been widely studied (see, e.g., [1–7]). Remarkably, \( \mathcal{C}^\eta \) is bounded on Hardy spaces in the unit disc \( U \) [1–3] for any \( \eta \) with \( \Re \eta > -1 \).

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In higher dimensions, Chang, Gilbert, Tie [8], Stević [9], and Hu [10] investigated the Cesàro averaging operators on the polydisc $\mathbb{U}^n$ and the unit ball $\mathbb{B}_n$, respectively. Let $f(z) = \sum_{|\alpha|=0}^{\infty} a_\alpha z^\alpha$ be a holomorphic function in $\mathbb{U}^n$ or $\mathbb{B}_n$ and define

$$
\mathcal{C}(f)(z) = \sum_{|\alpha|=0}^{\infty} \left( \sum_{\beta \leq \alpha} a_{\alpha-\beta} \prod_{j=1}^{n} A_{\beta_j}^{j+1} \right) z^\alpha,
$$

where $\vec{\gamma} = (\gamma_1, \ldots, \gamma_n)$ with $\Re \gamma_j > -1$, $\alpha$ and $\beta$ are multi-indices from $\mathbb{Z}_+^n$. The integral formula of $\mathcal{C}(f)$ is given by (see [9])

$$
\mathcal{C}(f)(z) = \prod_{j=1}^{n} (\gamma_j + 1) \int_{[0,1]^n} f(\tau \cdot z) \prod_{j=1}^{n} (1 - \tau_j z_j)^{\gamma_j+1} d\tau,
$$

where $\tau = (\tau_1, \ldots, \tau_n)$ and $d\tau = d\tau_1 \cdots d\tau_n$.

The boundedness of $\mathcal{C}(f)$ on the Hardy spaces in $\mathbb{U}^n$ has been established in [8,9]. It is natural to consider the corresponding result in the Hardy spaces in the unit ball. Along this line, Hu [10] obtained the weaker result that $\mathcal{C}(f)$ is bounded on certain weighted Bergman spaces in the unit ball when $\vec{\gamma} = 0$.

The purpose of this paper is to show that the Cesàro operator $\mathcal{C}(f)$ is bounded on all Hardy spaces in the unit ball $\mathbb{B}_n$ for any $\vec{\gamma}$ with $\Re \gamma_j > -1$.

Our technique is a new one: the function theory in the polydiscs is used to solve the problem in the unit ball.

In order to prove the main result, we will first extend the results in the polydisc by Chang, Gilbert, Tie [8] and Stević [9].

**Theorem 1.** Let $0 < p < \infty$, $\Re \gamma_j > -1$ and $0 \leq r_j < \rho_j \leq 1$, $j = 1, \ldots, n$. Then there exists a constant $C$ independent of $f$, $\rho$, and $r$, such that

$$
\int_{[0,2\pi]^n} |\mathcal{C}(f)(r \cdot e^{i\theta})|^p d\theta \leq C(p, \vec{\gamma}, n) \int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})|^p d\theta
$$

for any $f \in H(\rho \mathbb{U}^n)$. Here $\rho \mathbb{U}^n = \{z \in \mathbb{C}^n: |z_j| \leq \rho_j, \ j = 1, \ldots, n\}$.

When $\rho = 1$, Theorem 1 is exactly Theorem 3.9 in [8]. Note that one cannot obtain Theorem 1 from the case of $\rho = 1$ by replacing $f$ with $f_\rho$, where $f_\rho(z) = f(\rho z)$. The reason for this is that

$$
\mathcal{C}(f_\rho)(\frac{r}{\rho}) \neq \mathcal{C}(f)(r \cdot z).
$$

The bridge between the theory in the polydisc with that in the unit ball can be constructed by mathematical induction based on the following lemma.

**Lemma 2.** (See [11, p. 15].) Let $f$ be a function on $\mathbb{S}_n$, $n > 1$. Then the identity

$$
\int_{\mathbb{S}_n} f d\sigma = \frac{1}{2\pi} \int_{\mathbb{S}_{n-1}} d\nu(\zeta') \int_{\pi}^{-\pi} f(\zeta', e^{i\theta} \zeta_n) d\theta
$$

holds. Here $\zeta' = (\zeta_1, \ldots, \zeta_{n-1})$ and $d\nu(\zeta')$ is the normalized Lebesgue measure on $\mathbb{B}_{n-1}$.

The main result in this paper is the following theorem.

**Theorem 3.** Let $0 < p < +\infty$, $\gamma = (\gamma_1, \ldots, \gamma_n)$, $r = (r_1, \ldots, r_n)$ such that $\Re \gamma_j > -1$ and $0 \leq r_j < 1$, $j = 1, \ldots, n$. Then there exists a constant $C$ independent of $f$ and $r$ such that

$$
\int_{\mathbb{S}_n} |\mathcal{C}(f)(r \cdot \zeta)|^p d\sigma(\zeta) \leq C(p, \vec{\gamma}, n) \int_{\mathbb{S}_n} |f(r \cdot \zeta)|^p d\sigma(\zeta)
$$

for any $f \in H(\mathbb{B}_n)$. 
2. Cesàro operators in the polydisc

Chang, Gilbert, Tie [8], Stević [9] established the boundedness of Cesàro operators in the polydisc. Its generalized version in any polydisc \( \rho \mathbb{U}^n \) shall be needed in the proof of our main result in the unit ball. As pointed out in the introduction, one cannot obtain this general result simply by applying the known result to slice functions \( f_{\rho} \). However, we shall see that their arguments are valid in this general case.

In order to prove Theorem 1, we need some known lemmas.

**Lemma 4.** (See [12].) Let \( 0 < p < \infty \), and \( 0 \leq r_j < \rho_j \leq 1 \), \( j = 1, \ldots, n \). Then there exists a constant \( C \) independent of \( f \), \( r \), and \( \rho \), such that

\[
\int_{[0,2\pi]^n} \left( \sup_{0 \leq \tau < 1} |f(\tau \cdot r \cdot e^{i\theta})|^p \, d\theta \right)^{\frac{1}{p}} \prod_{j=1}^{n} (1 - \tau_j)^{-\frac{1}{a}} \, d\tau \leq C \int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})|^p \, d\theta.
\]

for all \( f \in H(\rho \mathbb{U}^n) \).

**Lemma 5.** (See [9].) Let \( 0 < p < +\infty, 1 < a < +\infty \) and \( 0 \leq r_j < 1 \), \( j = 1, \ldots, n \). Then there is a constant \( C \) depending only on \( p \) and \( n \) such that

\[
\int_{[0,1]^n} \left( \int_{[0,2\pi]^n} |f(\tau \cdot r \cdot e^{i\theta})|^p \, d\theta \right)^{\frac{1}{a}} \prod_{j=1}^{n} (1 - \tau_j)^{-\frac{1}{a}} \, d\tau \leq C \int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})|^p \, d\theta.
\]

**Lemma 6.** (See [2].) For \( \gamma > -1 \), there is a constant \( C = C(\gamma) \) such that

\[
\int_0^1 \frac{x^{\gamma+1}}{[x^2 + \varphi^2](x^2 + \theta^2)^{(\gamma+1)/2}} \, dx \leq C \frac{H^\gamma(\varphi/\theta)}{|\theta|}.
\]

Here

\[
H^\gamma(s) = \begin{cases} 
1 + |s|^\gamma, & \text{if } \gamma < 0, \\
\log(2 + 1/|s|), & \text{if } \gamma = 0, \\
1, & \text{if } \gamma > 0.
\end{cases}
\]

**Lemma 7.** (See [2].) Let \( \gamma_j > -1 \), \( j = 1, \ldots, n \), \( 1 < p < \infty \) and

\[
A_{\gamma, p} = 2^{1/p} \int_{[0,1]^n} \prod_{j=1}^{n} \frac{H^\gamma_j(s_j)}{|s_j + 1|^{1/p}} \, ds.
\]

Then \( A_{\gamma, p} < \infty \) and

\[
\int_{[-\pi,\pi]^n} \left( \int_{[0,1]^n} \prod_{j=1}^{n} H^\gamma_j(s_j) E_{s \varphi}(e^{i\theta}) \, ds \right)^p \, d\theta \leq A_{\gamma, p}^p \int_{[-\pi,\pi]^n} g^p(e^{i\theta}) \, d\theta
\]

for all measurable \( g \geq 0 \). Here

\[
E_{s \varphi}(e^{i\theta}) = \begin{cases} 
g(e^{i(s+1)\theta}), & \text{if } |s_j \theta_j| \leq \pi, \ j = 1, \ldots, n, \\
0, & \text{otherwise}.
\end{cases}
\]

**Proof of Theorem 1.** In what follows, for the sake of simplicity, we assume that \( \gamma_j > -1 \), \( j = 1, \ldots, n \). Let \( r = (r_1, \ldots, r_n) \) and \( \rho = (\rho_1, \ldots, \rho_n) \) be such that \( 0 \leq r_j < \rho_j \leq 1 \) for any \( j = 1, 2, \ldots, n \).

**Case 1.** \( 0 < p \leq 1 \).
Let $f \in H(\rho \cup ^0)$. From the integral formula

$$
\mathcal{E}^\gamma (f)(z) = \prod_{j=1}^n (\gamma_j + 1) \int_{[0,1]^p} \frac{f(\tau \cdot z)}{\prod_{j=1}^n (1 - \tau_j z \gamma_j + 1)} \prod_{j=1}^n (1 - \tau_j)^{\gamma_j} \, d\tau,
$$

we have

$$
M_p^\rho (\mathcal{E}^\gamma (f), r) \leq C \int_{[0,2\pi]^p} \left( \frac{\int_{[0,1]^p} \left| \frac{f(\tau \cdot r \cdot e^{i\theta})}{\prod_{j=1}^n (1 - \tau_j r_j e^{i\theta_j})^{\gamma_j + 1}} \right| \prod_{j=1}^n (1 - \tau_j)^{\gamma_j} \, d\tau}{\prod_{j=1}^n (1 - \tau_j r_j e^{i\theta_j})^{\gamma_j + 1}} \right)^p \, d\theta.
$$

We now apply the standard technique of Hardy–Littlewood [13,14]. Take $t_k = 1 - 2^{-k}, k \in \mathbb{N} \cup \{0\}$, and apply the inequality $\left( \sum |a_j|^p \right)^{1/p} \leq \sum |a_j|^p$ for $0 < p \leq 1$, to obtain

$$
M_p^\rho (\mathcal{E}^\gamma (f), r) \leq C \int_{[0,2\pi]^p} \left( \sum_{k_1,\ldots,k_n=1}^\infty \left( \int_{[0,1]^p} \left| \frac{f(\tau \cdot r \cdot e^{i\theta})}{\prod_{j=1}^n (1 - \tau_j r_j e^{i\theta_j})^{\gamma_j + 1}} \right| \prod_{j=1}^n (1 - \tau_j)^{\gamma_j} \, d\tau \right)^p \right) \, d\theta
$$

By Lemma 4, the preceding item can be further estimated

$$
\leq C \sum_{k_1,\ldots,k_n=1}^\infty \int_{[0,2\pi]^p} \left( \int_{[0,1]^p} \left| \frac{f(t_k \cdot r \cdot e^{i\theta})}{\prod_{j=1}^n (1 - t_k r_j e^{i\theta_j})^{\gamma_j + 1}} \right| \prod_{j=1}^n (1 - t_k r_j)^{\gamma_j} \, d\tau \right)^p \, d\theta
$$

Choose $a$ such that $(\gamma_j + 1)p a > 1$ for any $j = 1, \ldots, n$. We denote the exponent conjugate to $a$ by $a'$, i.e.,

$$
\frac{1}{a} + \frac{1}{a'} = 1.
$$

By Hölder’s inequality and the inequality in [15, p. 65]

$$
\int_{-\pi}^{\pi} \left| 1 - \rho e^{i\theta} \right|^{-a} \, d\theta \leq C(1 - \rho)^{1-a}, \quad 0 \leq \rho < 1, \quad a > 1,
$$

we have

$$
\int_{[0,2\pi]^p} \left( \int_{[0,1]^p} \left| \frac{f(\tau \cdot r \cdot e^{i\theta})}{\prod_{j=1}^n (1 - \tau_j r_j e^{i\theta_j})^{\gamma_j + 1}} \right| \prod_{j=1}^n (1 - \tau_j)^{\gamma_j} \, d\tau \right)^p \, d\theta
$$

$$
\leq \left( \int_{[0,2\pi]^p} \left| f(\tau \cdot r \cdot e^{i\theta}) \right|^{pa'} \, d\theta \right)^{1/p} \left( \int_{[0,2\pi]^p} \left( \int_{[0,1]^p} \frac{1}{\prod_{j=1}^n (1 - \tau_j r_j e^{i\theta_j})^{(\gamma_j + 1)p a} \, d\tau} \right)^{1/p} \, d\theta \right)^{1/p}.
$$
\[ \left( \int_{[0,2\pi]^n} |f(\mathbf{r} \cdot e^{i\theta})|^p \, d\theta \right)^{\frac{1}{n}} \prod_{j=1}^{n} (1 - \tau_j r_j)^{\frac{1}{2} - (\gamma_j + 1)p} \]

By Fubini's theorem, we have

\[ \leq C \left( \int_{[0,2\pi]^n} |f(\mathbf{r} \cdot e^{i\theta})|^p \, d\theta \right)^{\frac{1}{n}} \prod_{j=1}^{n} (1 - \tau_j r_j)^{\frac{1}{2} - (\gamma_j + 1)p} \]

Combining the above inequalities together, we finally obtain

\[ M_p^{B\gamma}(f, r) \leq C \int_{[0,1]^n} \left( \int_{[0,2\pi]^n} |f(\mathbf{r} \cdot e^{i\theta})|^p \, d\theta \right)^{\frac{1}{n}} \prod_{j=1}^{n} (1 - \tau_j r_j)^{\frac{1}{2} - (\gamma_j + 1)p} \, d\theta \prod_{j=1}^{n} (1 - \tau_j)^{-\frac{1}{2}} \, d\tau \leq CM_p^{B\gamma}(f, r). \]

The last inequality used Lemma 5.

**Case 2.** \( 1 < p < +\infty \).

Let \( f \in H(\rho B_n) \). Then for any \( \tau = (\tau_1, \ldots, \tau_n) \) with \( 0 < \tau_j < 1, \ j = 1, \ldots, n \),

\[ f(\mathbf{r} \cdot e^{i\theta}) = \frac{1}{(2\pi)^n} \int_{[-\pi,\pi]^n} f_r(e^{i\varphi}) \prod_{j=1}^{n} P(\tau_j, \varphi_j - \theta_j) \, d\varphi, \]

where \( P(s, \phi) \) is the Poisson kernel

\[ P(s, \phi) = \frac{1 - s^2}{1 - 2s \cos \phi + s^2}. \]

Then by Fubini's theorem,

\[ \mathcal{C}^{\gamma}(f)(\mathbf{r} \cdot e^{i\theta}) = C(\gamma) \int_{[0,1]^n} \frac{f(\mathbf{r} \cdot e^{i\theta})}{(2\pi)^n} \prod_{j=1}^{n} (1 - \tau_j r_j e^{i\theta_j})^{\gamma_j + 1} \prod_{j=1}^{n} (1 - \tau_j)^{\gamma_j} \, d\tau \]

\[ = C(\gamma) \int_{[0,1]^n} \frac{1}{(2\pi)^n} \left( \int_{[-\pi,\pi]^n} f_r(e^{i\varphi}) \prod_{j=1}^{n} P(\tau_j, \varphi_j - \theta_j) \, d\varphi \right) \prod_{j=1}^{n} \frac{(1 - \tau_j)^{\gamma_j}}{(1 - \tau_j r_j e^{i\theta_j})^{\gamma_j + 1}} \, d\tau \]

\[ = C(\gamma) \int_{[-\pi,\pi]^n} \left( \int_{[0,1]^n} \prod_{j=1}^{n} (1 - \tau_j)^{\gamma_j} P(\tau_j, \varphi_j) \frac{(1 - \tau_j r_j e^{i\theta_j})^{\gamma_j + 1}}{(1 - \tau_j r_j e^{i\theta_j})^{\gamma_j + 1}} \, d\tau \right) f_r(e^{i(\varphi + \theta)}) \, d\varphi. \]

By Lemma 6 and the inequality in [16, p. 96]

\[ P(s, \phi) \leq C \frac{1 - s}{(1 - s)^2 + \phi^2}, \quad \forall 0 < s < 1, \ |\phi| \leq \pi, \]

we have

\[ \int_{0}^{1} \left| \frac{(1 - \tau_j)^{\gamma_j} \, P(\tau_j, \varphi_j)}{(1 - \tau_j r_j e^{i\theta_j})^{\gamma_j + 1}} \right| \, d\tau_j \]

\[ \leq C \int_{0}^{1} \frac{(1 - \tau_j)^{\gamma_j + 1} (1 - \tau_j r_j e^{i\theta_j})^{\gamma_j + 1/2}}{((1 - \tau_j)^2 + \varphi_j^2)(1 - \tau_j r_j \cos \theta_j + \tau_j^2 r_j^2)^{\gamma_j + 1/2}} \frac{1}{(1 - \tau_j^2 r_j^2)^{\gamma_j + 1/2}} \, d\tau_j \]

\[ \leq C \int_{0}^{1} \frac{(1 - \tau_j)^{\gamma_j + 1}}{(1 - \tau_j^2 + \varphi_j^2)(1 - \tau_j r_j \cos \theta_j)^{\gamma_j + 1/2}} \frac{1}{(1 - \tau_j r_j)^{\gamma_j + 1/2}} \, d\tau_j \]
\[ \int_0^1 \frac{1 - \tau_j}{(1 - \tau_j)^2 + \varphi_j^2} \frac{1}{((1 - \tau_j)^2 + \theta_j^2)^{(\gamma_j + 1)/2}} d\tau_j = C \int_0^1 \frac{x^{\gamma_j + 1}}{(x^2 + \varphi_j^2)(x^2 + \theta_j^2)^{(\gamma_j + 1)/2}} dx \leq C \frac{\mathcal{H}_{\gamma_j}(\varphi_j/\theta_j)}{|\theta_j|}. \]

Hence,
\[
\left| \mathcal{C}_{\gamma_j}^\varphi (f \cdot e^{i\varphi}) \right| \leq C \int_{[-\pi,\pi]^n} \prod_{j=1}^n \mathcal{H}_{\gamma_j}(\varphi_j/\theta_j) |f_r(e^{i(\varphi_j + \theta)})| d\varphi = C \int_{\mathbb{S}^n_{\gamma_j}(\varphi_j/\theta_j)} \prod_{j=1}^n \mathcal{H}_{\gamma_j}(s_j) |f_r(e^{i(1+s)\theta})| ds \cdot
\]

Then, by Lemma 7, we get
\[
\int_{[0,2\pi]^n} \left| \mathcal{C}_{\gamma_j}^\varphi (f \cdot e^{i\varphi}) \right|^p d\theta \leq A_{\gamma,\varphi} \int_{[0,2\pi]^n} \left| f_r(e^{i\theta}) \right|^p d\theta,
\]
which completes the proof. \(\square\)

3. Cesàro operators in the unit ball

Now we consider the boundedness of the generalized Cesàro operator \(\mathcal{C}_{\gamma_j}^\varphi\) on Hardy spaces in the unit ball. We write \(\mathcal{C}_{\gamma_j}^\varphi\) as the composition of suitable integral operators.

Let \(f \in H(\mathbb{B}_n)\) and denote
\[ f_{z,j}(\zeta) = f(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n), \quad \zeta \in \mathbb{U}. \]

We introduce the operator
\[
\mathcal{C}_{\gamma_j}^\varphi (f)(z_1, \ldots, z_n) := (1 + \eta \int_0^1 f_{z,j}(\tau_j z_j) \left( 1 - \left( 1 - \tau_j \right)^{\gamma_j} \right) d\tau_j. \]

Lemma 8. Let \(f \in H(\mathbb{B}_n)\). For any \(1 \leq m \leq n\) and \(z \in \mathbb{B}_n\),
\[
\mathcal{C}_{\gamma_j}^m \circ \cdots \circ \mathcal{C}_{\gamma_1}^m (f)(z) = \prod_{j=1}^m (1 + \gamma_j) \int_{[0,1]^m} f(\tau_1 z_1, \ldots, \tau_m z_m, z_{m+1}, \ldots, z_n) \prod_{j=1}^m (1 - \tau_j)^{\gamma_j} d\tau_1 \cdots d\tau_m. \quad (3.1)
\]

In particular,
\[
\mathcal{C}_{\gamma_j}^n \circ \cdots \circ \mathcal{C}_{\gamma_1}^n (f)(z) = \mathcal{C}_{\gamma_j}^\varphi (f)(z), \quad \forall z \in \mathbb{B}_n. \quad (3.2)
\]

Proof. We prove formula (3.1) by induction on \(m\). It is clear that the formula holds when \(m = 1\). Assume that it holds when \(m = k < n\), i.e.,
\[
\mathcal{C}_{\gamma_j}^k \circ \cdots \circ \mathcal{C}_{\gamma_1}^k (f)(z) = \prod_{j=1}^k (1 + \gamma_j) \int_{[0,1]^k} f(\tau_1 z_1, \ldots, \tau_k z_k, z_{k+1}, \ldots, z_n) \prod_{j=1}^k (1 - \tau_j)^{\gamma_j} d\tau_1 \cdots d\tau_k,
\]
then
\[ C^\gamma_{k+1} \circ \cdots \circ C^\gamma_1 (f)(z) = \left( C^\gamma_k \circ \cdots \circ C^\gamma_1 (f) \right)(z) = C_1 \int_0^1 C^\gamma_k \circ \cdots \circ C^\gamma_1 (f)(z_1, \ldots, z_{k+1} z_{k+1}, z_{k+2}, \ldots, z_n) \frac{(1 - \tau_{k+1} z_{k+1})^{\gamma_{k+1}}}{(1 - \tau_{k+1} z_{k+1})^{\gamma_{k+1}+1}} d\tau_{k+1} \]

\[ = C_2 \int_0^1 \left( \int_0^{[0,1]^k} \frac{f(\tau_1 z_1, \ldots, \tau_k z_k, \tau_{k+1} z_{k+1}, z_{k+2}, \ldots, z_n) \prod_{j=1}^k (1 - \tau_j z_j)^{\gamma_j+1}}{(1 - \tau_{k+1} z_{k+1})^{\gamma_{k+1}+1}} d\tau_1 \ldots d\tau_k \right) \prod_{j=1}^{k+1} (1 - \tau_j z_j)^{\gamma_j+1} d\tau_{k+1} \]

Here $C_1 = (1 + \gamma_{k+1})$ and $C_2 = \prod_{j=1}^{k+1} (1 + \gamma_j)$.

Hence the formula holds for $m = k + 1 \leq n$. This completes the proof. □

**Lemma 9.** Let $0 < p < +\infty$, $0 \leq m \leq n - 1$, $\gamma = (\gamma_1, \ldots, \gamma_n)$, and $r = (r_1, \ldots, r_n)$ be such that $\Re \gamma_j > -1$ and $0 \leq r_j < 1$ for any $j = 1, \ldots, n$. Then there exists a constant $C$ independent of $m$, $f$, and $r$ such that

\[ \int_{\mathbb{B}_n} \left| C^\gamma_{m+1} \circ \cdots \circ C^\gamma_1 (f)(r \cdot \xi) \right|^p d\sigma(\xi) \leq C \int_{\mathbb{B}_n} \left| C^\gamma_m \circ \cdots \circ C^\gamma_1 (f)(r \cdot \xi) \right|^p d\sigma(\xi) \quad (3.3) \]

for all $f \in H(\mathbb{B}_n)$.

**Proof.** Fix $z_1, \ldots, z_m, z_{m+2}, \ldots, z_n$ and denote

\[ g_{z,m}(\xi) := C^\gamma_m \circ \cdots \circ C^\gamma_1 (f)(z_1, \ldots, z_m, \xi, z_{m+2}, \ldots, z_n), \quad \forall \xi \in \mathbb{U}. \]

Then Lemma 8 presents the integral formula for $g_{z,m}$

\[ g_{z,m}(\xi) = \prod_{j=1}^m (1 + \gamma_j) \int_{[0,1]^m} \frac{f(\tau_1 z_1, \ldots, \tau_m z_m, \xi, z_{m+2}, \ldots, z_n) \prod_{j=1}^m (1 - \tau_j z_j)^{\gamma_j+1}}{\prod_{j=1}^m (1 - \tau_j z_j)^{\gamma_j+1}} d\tau_1 \ldots d\tau_m. \]

Since $f \in H(\mathbb{B}_n)$, it follows that

\[ g_{z,m}(\xi) \in H\left( \sqrt{\left( 1 - \sum_{j=1, j \neq m+1}^n |z_j|^2 \right) \mathbb{U}} \right). \]

Recall the definition of the Cesàro operator in the unit disc

\[ C^\gamma_{m+1} (g_{z,m})(\xi) = (1 + \gamma_{m+1}) \int_0^1 \frac{g_{z,m}(\tau_{m+1} \xi) \prod_{j=1}^{m+1} (1 - \tau_{m+1} \xi)^{\gamma_{m+1}+1}}{\prod_{j=1}^{m+1} (1 - \tau_{m+1} \xi)^{\gamma_{m+1}+1}} d\tau_{m+1}. \quad (3.4) \]

Therefore,

\[ C^\gamma_{m+1} \circ C^\gamma_m \circ \cdots \circ C^\gamma_1 (f)(z) \]

\[ = C^\gamma_{m+1} \left( C^\gamma_m \circ \cdots \circ C^\gamma_1 (f) \right)(z) \]

\[ = (1 + \gamma_{m+1}) \int_0^1 \frac{C^\gamma_m \circ \cdots \circ C^\gamma_1 (f)(z_1, \ldots, z_m, \tau_{m+1} z_{m+1}, z_{m+2}, \ldots, z_n) \prod_{j=1}^{m+1} (1 - \tau_{m+1} z_{m+1})^{\gamma_{m+1}+1}}{\prod_{j=1}^{m+1} (1 - \tau_{m+1} z_{m+1})^{\gamma_{m+1}+1}} d\tau_{m+1}. \]
\[(1 + \gamma_{m+1}) \int_0^1 \frac{g_{z,m}(z_m+1)}{(1 - \tau_{m+1}z_m+1)^{\gamma_{m+1}}} (1 - \tau_{m+1})^{\gamma_{m+1}} d\tau_{m+1}
\]
\[= C_{\gamma_m} (g_{z,m})(z_m+1).\]

Note that for each \(z = r\xi\) with \(\xi \in \mathbb{S}^n\) and \(0 \leq r < 1\),
\[g_{z,m} \in H \left( \left| r_j \xi_j \right|^2 \right) \subseteq H \left( \left| \xi_{m+1} \right| \right).\]

Applying Lemma 2 and Theorem 1 for the case \(n = 1\) and \(\rho = \left| \xi_{m+1} \right|\), we obtain
\[\int_{\mathbb{S}^n} \left| C_{\gamma_{m+1}} \circ \cdots \circ C_{\gamma_1} (f) (r \cdot \xi) \right|^p d\sigma(\xi)
\]
\[= \int_{\mathbb{S}^n} \left| C_{\gamma_{m+1}} (g_{z,m}) (r_{m+1} \xi_{m+1}) \right|^p d\sigma(\xi)
\]
\[= \frac{1}{2\pi} \int_{\mathbb{B}^n-1} d\sigma(\xi_1, \ldots, \xi_m, \xi_{m+2}, \ldots, \xi_n) \int_{-\pi}^{\pi} \left| C_{\gamma_{m+1}} (g_{r \cdot \xi, m}) (r_{m+1} \xi_{m+1} e^{i\theta}) \right|^p d\theta
\]
\[\leq \frac{1}{2\pi} \int_{\mathbb{B}^n-1} d\sigma(\xi_1, \ldots, \xi_m, \xi_{m+2}, \ldots, \xi_n) \int_{-\pi}^{\pi} \left| g_{r \cdot \xi, m} (r_{m+1} \xi_{m+1} e^{i\theta}) \right|^p d\theta
\]
\[= C \int_{\mathbb{S}^n} \left( g_{r \cdot \xi, m} (r_{m+1} \xi_{m+1}) \right|^p d\sigma(\xi)
\]
\[= C \int_{\mathbb{S}^n} \left| C_{\gamma_{m+1}} \circ \cdots \circ C_{\gamma_1} (f) (r \cdot \xi) \right|^p d\sigma(\xi).
\]

This completes the proof. \(\square\)

By Lemmas 8 and 9, Theorem 3 can be immediately obtained.

Let \(0 < p, q < \infty\) and \(\mu\) be a positive measure on \([0, 1)\). We define the weighted Bergman space \(A^{p,q}_\mu (\mathbb{B}_n)\) to be the set of all holomorphic functions on \(\mathbb{B}_n\) such that
\[\|f\|_{A^{p,q}_\mu (\mathbb{B}_n)} = \left[ \int_{[0,1)} \left( \int_{\mathbb{S}^n} \left| f (r \cdot \xi) \right|^p d\sigma(\xi) \right)^{q/p} d\mu(r) \right]^{1/q} < \infty.
\]

Using Theorem 3, we have the following corollary.

**Corollary 10.** The generalized Cesàro operator is bounded on the space \(A^{p,q}_\mu (\mathbb{B}_n)\) for any \(0 < p, q < \infty\). More precisely, there exists a constant \(C\) independent of \(f\), such that
\[\|C^{\gamma} (f)\|_{A^{p,q}_\mu (\mathbb{B}_n)} \leq C \|f\|_{A^{p,q}_\mu (\mathbb{B}_n)}.
\]

**Note added in proof**

During the galley proof stage of this article, the authors learned that the case \(\rho = 1\) of Theorem 1 was proved with the same approach, mainly due to Andersen [2], in “The generalized Cesaro operator on the unit polydisc” by Der-Chen Chang and Stevo Stević (Taiwanese J. Math. 7 (2) (2003) 293–308).


References