Absolute mean square exponential stability of Lur’e stochastic distributed parameter control systems

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Abstract
In this work the absolute mean square exponential stability of Lur’e stochastic distributed parameter control systems has been addressed. Delay-dependent sufficient conditions for the stochastic stability in Hilbert spaces are established in terms of linear operator inequalities (LOIs). Finally, the stochastic wave equation illustrates our result.

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1. Introduction
The absolute stability problem, which was initiated in [1], has been extensively investigated by many authors (see [2,3] and the references therein). Delay-dependent robust absolute stability criteria for uncertain Lur’e control systems of neutral type, in terms of linear matrix inequalities (LMIs), have been dealt with in [2] without employing any model transformation and bounding technique for cross terms, which is further under consideration in [3] using the discretized Lyapunov functional approach. So far, however, the overwhelming majority of absolute stability results are only available for systems governed by ordinary differential equations (ODEs) rather than by partial differential equations (PDEs), not to mention, by stochastic partial differential equations (SPDEs). Motivated by the fact that distributed parameter systems described by SPDEs are more common in general, there is a realistic need to discuss the absolute stability problem of such systems. Very recently, the linear operator inequalities (LOIs) approach [4] has provided new insights into the control theory of distributed parameter systems. In this paper, the concept of absolute stochastic stability will be extended to the infinite-dimensional case where delay-dependent sufficient conditions for absolute stochastic stability in Hilbert spaces are established in terms of linear operator inequalities (LOIs).

2. Preliminaries
Consider the mean square stability of Lur’e stochastic control systems in a Hilbert space $H$ described by

$$
\Sigma_0 : \begin{cases}
    dx(t) = [Ax(t) + Bx(t - h) + Ew(t)]dt + [Cx(t) + Dx(t - h)]d\omega(t) \\
    z(t) = Mx(t) + Nx(t - h) \\
    w(t) = -\varphi(t, z(t)) \\
    x(t) = \phi(t), \text{ for } \forall t \in [-h, 0]
\end{cases}
$$

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where \(x(t), z(t) \in H\) are the states, \(w(t) \in U\) is the control input, \(h\) denotes positive constant delay, \(\phi \in C([-h, 0], H)\) is the given initial state, \(\varphi(t, z(t)) : R \times H \rightarrow H\) is an abstract nonlinear function satisfying the following sector condition:

\[
\langle \varphi(t, z(t)) - K_1 z(t), \varphi(t, z(t)) - K_2 z(t) \rangle \leq 0
\]

(2)

\(\omega(t)\) is a zero-mean real scalar Wiener process on probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with assumption that \(\mathbb{E}[d\omega(t)] = 0, \mathbb{E}[d\omega^2(t)] = dt\).

Without loss of generality, it is assumed that

(i) Operator \(A\) generates a \(C_0\)-semigroup \(T(t)\).

(ii) Operators \(B, C, D, E, M\) and \(N\) are all linear and bounded.

(iii) Operators \(K_1\) and \(K_2\) are linear and may be \textit{unbounded}.

In what follows, we will have a position to define the concept of absolute stochastic exponential stability in Hilbert spaces.

**Definition 2.1.** The system \(\Sigma_0\) governed by (1) is said to be absolutely stochastically exponentially stable in the sector \([K_1, K_2]\) if the zero solution is globally uniformly stochastically exponentially stable for each abstract nonlinear function \(\varphi(t, z(t))\) satisfying (2).

As a key tool to developing our result in this paper, the following lemma will be introduced.

**Lemma 2.1** (Wirtinger’s Inequality and its Generalization [4]). Let \(z \in W^{1,2}([a, b], \mathbb{R})\) be a scalar function with \(z(a) = z(b) = 0\). Then

\[
\int_a^b z^2(\xi) d\xi \leq \frac{(b - a)^2}{\pi^2} \int_a^b \left[ \frac{dz(\xi)}{d\xi} \right]^2 d\xi.
\]

If additionally \(z \in W^{2,2}([a, b], \mathbb{R})\), then

\[
\int_a^b \left[ \frac{dz(\xi)}{d\xi} \right]^2 d\xi \leq \frac{(b - a)^2}{\pi^2} \int_a^b \left[ \frac{d^2z(\xi)}{d\xi^2} \right]^2 d\xi.
\]

### 3. Absolute stochastic exponential stability in a Hilbert space

In this section, the delay-dependent absolute stochastic mean square exponential stability condition for system \(\Sigma_0\) is presented in terms of linear operator inequalities (LOIs).

In the case when the nonlinear function \(\varphi(t, z(t))\) belongs to the sector \([0, K]\), i.e.,

\[
\langle \varphi(t, z(t)), \varphi(t, z(t)) - Kz(t) \rangle \leq 0,
\]

the following result is given.

**Theorem 3.1.** Given a positive constant delay \(h > 0\), let there exist positive constants \(\varepsilon > 0, \beta > 0\), a linear positive definite operator \(Q_1 : D(A) \rightarrow H\) and a nonnegative definite operator \(Q_2 \in L(H)\) with the following inequalities:

\[
\alpha \langle x, x \rangle \leq \langle x, Q_1 x \rangle \leq \gamma_{Q_1} \langle x, x \rangle + \langle Ax, Ax \rangle
\]

(4)

\[
\langle x, Q_2 x \rangle \leq \gamma_{Q_2} \langle x, x \rangle
\]

(5)

for some positive constants \(\alpha, \gamma_{Q_1}, \gamma_{Q_2}\) such that the LOI

\[
\mathcal{Z} := \begin{bmatrix}
Q_1 (A + \beta I) + (A + \beta I)^* Q_1 + C^* Q_1 C + Q_2 & Q_1 B + C^* Q_1 D & Q_1 E - \varepsilon M^* K^* \\
Q_1 B + C^* Q_1 D & -\varepsilon N^* K^* - 2\varepsilon I & -2\varepsilon I \\
C^* Q_1 D & -2\varepsilon I & -2\varepsilon I
\end{bmatrix} < 0
\]

(6)

holds. Then system \(\Sigma_0\) is absolutely stochastically mean square exponentially stable in the sector \([0, K]\) in a Hilbert space.

**Proof.** Choose the following positive semi-definite Lyapunov–Krasovskii functional in Hilbert spaces:

\[
V(t, x_t) = \langle x(t), Q_1 x(t) \rangle + \int_0^t e^{2\beta \theta} \langle x(t + \theta), Q_2 x(t + \theta) \rangle d\theta.
\]

(7)

The stochastic differential \(dV(t, x_t)\), by the Itô formula, can be computed as

\[
dV(t, x_t) = \mathcal{L}V(t, x_t) dt + 2 \langle x(t), Q_1 (Cx(t) + Dx(t - h)) \rangle d\omega(t)
\]

(8)
where

\[ \mathcal{L}V(t, x(t)) = 2 \langle x(t), Q_1(Ax(t) + Bx(t - h) + Ew(t)) \rangle + \langle x(t), Q_2x(t) \rangle \]

\[ - e^{-2\beta h} \langle x(t - h), Q_2x(t - h) \rangle - 2\beta \int_{t-h}^{t} e^{2\beta(t-s)} \langle x(s), Q_2x(s) \rangle \, ds \]

\[ + \langle (Cx(t) + Dx(t - h)), Q_1(Cx(t) + Dx(t - h)) \rangle. \]

It follows from (7) and (9) that

\[ \mathcal{L}V(t, x(t)) + 2\beta V(t, x(t)) = \langle \eta(t), \Pi \eta(t) \rangle \]

where \( \eta(t) := \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}, \Pi := \begin{bmatrix} Q_1(A + \beta I) + (A + \beta I)^*Q_1 + C^*Q_1C + Q_2 & Q_1B + C^*Q_1D \\
* & -e^{-2\beta h}Q_2 + D^*Q_1D \end{bmatrix} \).

In view of Dynkin’s formula [5] and inequality (11), we have that

\[ \langle \eta(t), \Pi \eta(t) \rangle - 2\varepsilon \langle w(t), w(t) \rangle - 2\varepsilon \langle w(t), K(Mx(t) + Nx(t-h)) \rangle < 0 \]

holds for \( \forall \eta(t) \neq 0 \), which implies that the inequality \( \langle \eta(t), \Pi \eta(t) \rangle < 0 \) holds for \( \forall \eta(t) \neq 0 \) satisfying (3) and hence from the equality (10) it follows that

\[ \mathcal{L}V(t, x(t)) + 2\beta V(t, x(t)) < 0 \]

for \( \forall \eta(t) \neq 0 \) satisfying (3).

In view of Dynkin’s formula [5] and inequality (11), we have that

\[ \alpha \mathbb{E} \left\{ \| x(t, \phi) \|^2 \right\} \leq \mathbb{E} \left\{ V(t, x(t)) \right\} \leq e^{-2\beta t} V(\phi) \leq e^{-2\beta t} \left[ \gamma_{q_1} \left( \| \phi(0) \|^2 + \| A\phi(0) \|^2 \right) + h\gamma_{q_2} \| \phi \|^2 \right], \hspace{1cm} t \geq 0 \]

which brings us to the conclusion

\[ \mathbb{E} \left\{ \| x(t, \phi) \|^2 \right\} \leq \frac{1}{\sqrt{\alpha}} e^{-\beta t} \gamma_{q_1} \left( \| \phi(0) \|^2 + \| A\phi(0) \|^2 \right) + h\gamma_{q_2} \| \phi \|^2, \hspace{1cm} t \geq 0. \]

And hence from Definition 2.1, the proof is completed. \( \square \)

Under the more general circumstances that the nonlinear function \( \varphi(t, z(t)) \) satisfies the sector condition (2), using the loop transformation technique [6], we come to the conclusion that the absolute stochastic exponential stability of system \( \Sigma_0 \) in the sector \([K_1, K_2] \) is equivalent to that of the following system:

\[ \Sigma_1 : \quad \begin{cases} dx(t) = [(A - EK_3M)x(t) + (B - EK_1N)x(t - h) + E\tilde{w}(t)]dt + [Cx(t) + Dx(t - h)]d\omega(t) \\
\tilde{w}(t) = -\tilde{\varphi}(t, z(t)) \\
x(t) = \phi(t), \quad \text{for} \forall t \in [-h, 0] \end{cases} \]

in the sector \([0, K_2 - K_1] \), where the abstract nonlinear function \( \tilde{\varphi}(t, z(t)) \) satisfies

\[ \langle \tilde{\varphi}(t, z(t)), \tilde{\varphi}(t, z(t)) \rangle - (K_2 - K_1)z(t) \rangle \leq 0. \]

Applying Theorem 3.1 to system (14) and (15) yields immediately our main result as follows.

**Theorem 3.2.** Given a positive constant delay \( h > 0 \), let there exist positive constants \( \varepsilon > 0, \beta > 0 \), a linear positive definite operator \( Q_1 : \mathcal{D}(A) \rightarrow H \) and nonnegative definite operator \( Q_2 \in \mathcal{L}(H) \) with the following inequalities:

\[ \alpha \langle x, x \rangle \leq \langle x, Q_1x \rangle \leq \gamma_{q_1} [\langle x, x \rangle + \langle Ax, Ax \rangle] \]

\[ \langle x, Q_2x \rangle \leq \gamma_{q_2} \langle x, x \rangle \]

for some positive constants \( \alpha, \gamma_{q_1}, \gamma_{q_2} \) such that the LOI

\[ \begin{bmatrix} Q_1(A - EK_1M + \beta I) + (A - EK_1M + \beta I)^*Q_1 + C^*Q_1C + Q_2 & Q_1(B - EK_1N) + C^*Q_1D \\
* & -e^{-2\beta h}Q_2 + D^*Q_1D \end{bmatrix} \]

holds. Then system \( \Sigma_0 \) is absolutely stochastically mean square exponentially stable in the sector \([K_1, K_2] \) in a Hilbert space.
4. Application to stochastic wave equations

Consider the stochastic wave equation

\[ \text{d}z_t(\xi, t) = (a \nabla^2 z(\xi, t) - \mu_0 z_t(\xi, t) - \mu_1 z_t(\xi, t - h) - a_0 z(\xi, t) - a_1 z(\xi, t - h) + c_1 w_1(\xi, t) + c_2 w_2(\xi, t)) \, \text{d}t 
+ (bz(\xi, t) - b_0 z(\xi, t) - b_1 z(\xi, t - h) - b_2 z(\xi, t - h)) \, \text{d}\omega(t) \]  
(19)

with Neumann boundary condition

\[ z^{(i)}_t(0, t) = z^{(i)}(\pi, t) = 0 \quad (i = 0, 1) \]  
(20)

where \( \nabla^2 \) denotes the Laplace operator, i.e., \( \nabla^2 := \frac{\partial^2}{\partial r^2} \), constant parameters \( a > 0, \mu_0 > 0, \) and \( \xi \in [0, \pi], t \geq 0. \)

The boundary-value problem (19) and (20) can be rewritten as Eq. (1) in a Hilbert space

\[ \mathcal{H} = \left\{ z \in W^{2,2} (0, \pi), \mathbb{R} \, | \, \text{s.t. boundary condition (20)} \right\} \]

with operators \( A = \begin{bmatrix} a \nabla^2 - a_0 & -a_1 \\ -b_0 & -b_1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ -b_2 & -b_2 \end{bmatrix}, C = \begin{bmatrix} 0 & -b_0 \\ \mu_0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, E = \begin{bmatrix} 0 & 0 \\ c_1 & c_2 \end{bmatrix}, M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \)

and with the state \( x(t) := \begin{bmatrix} z(\xi, t) \\ \dot{z}(\xi, t) \end{bmatrix} \) and control \( u(t) := \begin{bmatrix} u_1(\xi, t) \\ u_2(\xi, t) \end{bmatrix} \).

**Theorem 4.1.** Given the scalars \( \beta > 0, a > 0, \mu_0 > 0, \mu_1, a_0, a_1, b_0, b_1, b_2, c_1, c_2, m_1, m_2, n_1, n_2, k_{11}, k_{12}, k_{13}, k_{12}, k_{22}, k_{23} \) and \( p_{11}, p_{12}, p_{22} \), let there exist a positive constant \( \varepsilon > 0 \), a symmetric positive definite matrix \( Q_m = \begin{bmatrix} q_{01} + (a + h_1) q_{03} & q_{02} \\ q_{02} & q_{03} \end{bmatrix} > 0 \) (where \( q_{03} > 0 \)) and a nonnegative definite matrix \( Q_2 \geq 0 \) such that the following LMI holds:

\[ \begin{bmatrix} \Pi & Q_m (B - E K_{1m} N) + C^T Q_m D \\ * & -e^{2\beta t} Q_2 + D^T Q_m D \end{bmatrix} \begin{bmatrix} Q_m E - \varepsilon M^T (K_{2m} - K_{1m})^T \\ * \end{bmatrix} \leq 0 \]  
(22)

where

\[ \Pi := Q_m \begin{bmatrix} 1 - a_0 - a - h_1 - h_3 & \beta - \mu_0 - h_2 \\ \beta - \mu_0 - h_1 & 1 - a_0 - a - h_1 - h_3 \end{bmatrix} + \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix} Q_m \]  
\[ k_{1m} := \begin{bmatrix} c_1 k_1 & c_2 p_{12} \end{bmatrix}, \quad k_{2m} := \begin{bmatrix} k_{21} \\ k_{22} \end{bmatrix}. \]

Then boundary-value problem (19) and (20) is absolutely stochastically mean square exponentially stable in the sector \([K_1, K_2]\).

**Proof.** In the case of stochastic wave equation (19), consider the Lyapunov–Krasovskii functional \( V \) taken in (7) where

\[ Q_1 = \begin{bmatrix} q_{01} - a q_{03} \nabla^2 + h_3 q_{03} \frac{\partial^4}{\partial r^4} & q_{02} \\ q_{02} & q_{03} \end{bmatrix}, \quad Q_2 \geq 0. \]  
(23)

The proof is given in the following steps.

Step 1. Integrating by parts and utilizing the Wirtinger’s inequality given in **Lemma 2.1**, direct computation can obtain that

\[ \langle x, Q_1 x \rangle = -a q_{03} \int_0^\pi z \cdot \nabla^2 z d\xi + h_3 q_{03} \int_0^\pi z \cdot \frac{\partial^4}{\partial r^4} z d\xi + \langle x, \begin{bmatrix} q_{01} & q_{02} \\ q_{02} & q_{03} \end{bmatrix} x \rangle \]
\[ \geq (a + h_1) q_{03} \int_0^\pi z^2 d\xi + \langle x, \begin{bmatrix} q_{01} & q_{02} \\ q_{02} & q_{03} \end{bmatrix} x \rangle \]
\[ = \langle x, Q_3 x \rangle > 0 \quad \text{for } x \neq 0 \]  
(24)

which, together with the self-adjointness of operator \( Q_1 \), implies that operator \( Q_1 \) is positive definite.
Step 2. In view of the Wirtinger’s inequality given in Lemma 2.1 and the inequality (21), we have that
\[
\langle x, (A - EK_1M + \beta I)^\ast Q_1 + Q_1(A - EK_1M + \beta I)x \rangle \\
= \langle x, \left( \begin{array}{cc}
\beta & 1 \\
\beta & 1
\end{array} \right) \left( \begin{array}{cc}
q_{01} - aq_{03} & \frac{\partial^4}{\partial \xi^4} q_{02} \\
q_{02} & q_{03}
\end{array} \right) \left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right) \left( \begin{array}{cc}
0 & aq_{03} \frac{\partial^4}{\partial \xi^4} q_{02} \\
\frac{\partial^4}{\partial \xi^4} q_{02} & q_{03}
\end{array} \right) \left( \begin{array}{c}
\beta - \mu_0 - h_2 \\
\beta - \mu_0 - h_2
\end{array} \right) \right) x \\
\leq \langle x, Q_m \left( \begin{array}{ccc}
-a_0 & -a - h_1 & -h_3 \\
-1 & -1 & -1
\end{array} \right) \left( \begin{array}{cc}
\beta & 1 \\
\beta & 1
\end{array} \right) \left( \begin{array}{cc}
0 & 0 \\
0 & 0
\end{array} \right) \left( \begin{array}{cc}
\beta & 1 \\
\beta & 1
\end{array} \right) \left( \begin{array}{c}
\beta - \mu_0 - h_2 \\
\beta - \mu_0 - h_2
\end{array} \right) x \rangle.
\] (25)

From the above analysis it follows that if LMIs (21) and (22) hold, then LOI (18) is satisfied, and hence by Theorem 3.2, the proof is completed. \( \square \)

**Remark 4.1.** Utilizing Theorem 4.1 for the stochastic wave equation (19) with coefficients \( a = 30, \mu_0 = 20, \mu_1 = -0.2, a_0 = 0.14, a_1 = -0.12, b = 1.2, b_0 = 1.5, b_1 = 0.2, b_2 = -0.1, c_1 = 1.3, c_2 = 0.2, M = \left[ \begin{array}{cc}
1 & 0 \\
0 & 2
\end{array} \right], N = \left[ \begin{array}{cc}
0 & 0 \\
1.2 & 0.3
\end{array} \right], K_1 = \left[ \begin{array}{cc}
1 & 0 \\
0 & 0.2
\end{array} \right], K_2 = \left[ \begin{array}{cc}
2 & 0 \\
0 & 0.9
\end{array} \right], p_{11} = 1, h_1 = 1.3, h_2 = 0.08 \) and \( h_3 = 0.2 \) yields that system (19) and (20) is absolutely stochastically mean square exponentially stable in the sector \( \{ K_1, K_2 \} \) with decay rate \( \beta = 1.3 \) and maximum delay \( h_{\text{max}} = 1.5420 \) where operators \( K_1 = \left[ \begin{array}{cc}
0 & \frac{\partial^4}{\partial \xi^4} \\
\frac{\partial^4}{\partial \xi^4} & 0
\end{array} \right] + K_1m \) and \( K_2 = \left[ \begin{array}{cc}
0 & 0 \\
0 & 0
\end{array} \right] + K_2m. \)

5. Conclusions

The present work has addressed the absolute mean square exponential stability problem of Lur’e stochastic distributed parameter control systems for which delay-dependent sufficient conditions are established in terms of linear operator inequalities (LOIs). Finally, the stochastic wave equation is given to illustrate our result.

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**References**


