Positive solutions for third-order three-point nonhomogeneous boundary value problems

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A B S T R A C T
In this work, by employing the Guo–Krasnosel’skii fixed point theorem and Schauder’s fixed point theorem, we study the existence and nonexistence of positive solutions to the third-order three-point nonhomogeneous boundary value problem

\[ u'''(t) + a(t)f(u(t)) = 0, \quad 0 < t < 1, \]
\[ u(0) = u'(0) = 0, \quad u'(1) - \alpha u'\eta = \lambda, \]

where \( \eta \in (0, 1) \), \( \alpha \in [0, 1/\eta) \) are constants and \( \lambda \in (0, \infty) \) is a parameter.

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1. Introduction

We are interested in the existence or nonexistence of positive solutions for the nonlinear third-order three-point boundary value problem (BVP for short)

\[ u'''(t) + a(t)f(u(t)) = 0, \quad 0 < t < 1, \]  \( (1.1) \)
\[ u(0) = u'(0) = 0, \quad u'(1) - \alpha u'(\eta) = \lambda, \]  \( (1.2) \)

where \( \eta \in (0, 1) \), \( \alpha \in [0, 1/\eta) \) are constants and \( \lambda \in (0, \infty) \) is a parameter. Here, by a positive solution we mean a function \( u'(t) \) which is positive on \((0, 1)\) and satisfies differential equation \((1.1)\) and the boundary conditions \((1.2)\). It is assumed throughout that

\[ (H) \quad a \in C((0, 1), [0, \infty)) \quad \text{and} \quad 0 < \int_0^1 (1-s)a(s)ds < \infty, \quad f \in C([0, \infty), [0, \infty)). \]

Third-order differential equations arise in a variety of different areas of applied mathematics and physics. Recently, the three-point boundary value problems of third-order differential equations have received much attention. One may see Anderson [1,2], Anderson and Davis [3], Bai [4], Boucherif and Al-Malki [5], Graef and Yang [7], Grossinho and Minhós [8], Sun [15], Yao [16] and Yu et al. [17], and the references therein for related results. In a recent paper, by using the Guo–Krasnosel’skii fixed point theorem, Guo et al. in [10] established existence results for at least one positive solution for BVP \((1.1), (1.2)\) when \( \lambda = 0 \), \( \alpha \in (1, 1/\eta) \) and the nonlinearity \( f \) is either superlinear or sublinear.

However, to the author’s knowledge, fewer results on three-point nonhomogeneous boundary value problems of second-order ordinary differential equations can be found in the literature. It is worth mentioning that Chen [6] and Ma [14] studied the existence of positive solutions of three-point nonhomogeneous boundary value problems of second-order ordinary differential equations.

Inspired and motivated by the works mentioned above, in this work we will consider the existence or nonexistence of positive solutions to BVP (1.1), (1.2). We shall first give a new form of the solution, and then determine the properties of the Green’s function for associated linear boundary value problems; finally, by employing the Guo–Krasnosel’skii fixed point theorem and Schauder’s fixed point theorem, some sufficient conditions guaranteeing the existence or nonexistence of a positive solution if the nonlinearity \( f \) is either superlinear or sublinear are established for the above boundary value problem. The results obtained extend and complement some known results.

The rest of the article is organized as follows. In Section 2, we present some preliminaries and the Guo–Krasnosel’skii fixed point theorem that will be used in Section 3. The main results and proofs will be given in Section 3.

2. Preliminary lemmas

In this section, we present some notation and preliminary lemmas that will be used in the proofs of the main results. We also state in this section a fixed point theorem due to Guo and Krasnosel’skii.

**Lemma 2.1.** Let \( x \in C^1[0, 1] := \{x \in C[0, 1], \ x(t) \geq 0, \ t \in [0, 1]\}; \) then the unique solution of the following BVP:

\[
\begin{align*}
\frac{d^3 u}{dt^3} + a(t)f(x(t)) &= 0, \quad 0 < t < 1, \\
\frac{du}{dt}(0) &= u'(0) = 0, \quad u(1) - au'(\eta) = \lambda, \\
\end{align*}
\]

is given by

\[
\begin{align*}
u(t) &= \int_0^1 G(t, s)a(s)f(x(s))ds + \frac{\alpha t^2}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s)a(s)f(x(s))ds + \frac{\lambda t^2}{2(1 - \alpha \eta)},
\end{align*}
\]

where

\[
G(t, s) = \begin{cases} 
2(t - s^2) & s \leq t, \\
\frac{1}{2} & s \geq t,
\end{cases}
\]

and

\[
G_1(t, s) := \frac{\partial G(t, s)}{\partial t} = \begin{cases} 
(1 - t)s & s \leq t, \\
(1 - s)t & s \geq t.
\end{cases}
\]

**Proof.** In fact, if \( u(t) \) is a solution of the BVP (2.1), (2.2), then we may suppose that

\[
u(t) = -\frac{1}{2} \int_0^t (t - s)^2 a(s)f(x(s))ds + At^2 + Bt + C.
\]

By the boundary conditions (2.2), we get \( B = C = 0 \) and

\[
A = \frac{1}{2(1 - \alpha \eta)} \int_0^1 (1 - s)a(s)f(x(s))ds - \frac{\alpha}{2(1 - \alpha \eta)} \int_0^\eta (\eta - s)a(s)f(x(s))ds + \frac{\lambda}{2(1 - \alpha \eta)}.
\]

Therefore, BVP (2.1), (2.2) has a unique solution

\[
\begin{align*}
u(t) &= -\frac{1}{2} \int_0^t (t - s)^2 a(s)f(x(s))ds + \frac{\alpha t^2}{2(1 - \alpha \eta)} \int_0^1 (1 - s)a(s)f(x(s))ds \\
&\quad - \frac{\alpha t^2}{2(1 - \alpha \eta)} \int_0^\eta (\eta - s)a(s)f(x(s))ds + \frac{\lambda t^2}{2(1 - \alpha \eta)} \\
&= -\frac{1}{2} \int_0^t (t - s)^2 a(s)f(x(s))ds + \frac{\alpha t^2}{2(1 - \alpha \eta)} \int_0^\eta a(s)f(x(s))ds \\
&\quad + \frac{\alpha t^2}{2(1 - \alpha \eta)} \int_0^1 a(s)f(x(s))ds - \frac{\alpha t^2}{2(1 - \alpha \eta)} \int_0^\eta a(s)f(x(s))ds + \frac{\lambda t^2}{2(1 - \alpha \eta)} \\
&= \frac{1}{2} \int_0^t (2t - s^2)a(s)f(x(s))ds + \frac{1}{2} \int_0^1 (1 - s)^2 a(s)f(x(s))ds \\
&\quad + \frac{\alpha t^2}{2(1 - \alpha \eta)} \left( \int_0^\eta a(s)f(x(s))ds + \int_0^1 \eta(1 - s)a(s)f(x(s))ds \right) + \frac{\lambda t^2}{2(1 - \alpha \eta)} \\
&= \int_0^1 G(t, s)a(s)f(x(s))ds + \frac{\alpha t^2}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s)a(s)f(x(s))ds + \frac{\lambda t^2}{2(1 - \alpha \eta)}.
\end{align*}
\]

The proof is complete. \( \square \)
We need some properties of functions $G(t, s)$ and $G_t(t, s)$ in order to discuss the existence of positive solutions.

**Lemma 2.2.** For all $(t, s) \in [0, 1] \times [0, 1]$, we have

$$0 \leq G_t(t, s) \leq (1 - s)s.$$

**Proof.** The conclusion is obvious. The proof is complete. \(\Box\)

**Lemma 2.3.** For all $(t, s) \in [\tau, 1] \times [0, 1]$, we have

$$\gamma G(1, s) \leq G(t, s) \leq G(1, s) = \frac{1}{2} (1 - s)s,$$

where $\gamma = t^2$, and $\tau$ satisfies $\int_{\tau}^{1} (1 - s)sa(s) \, ds > 0$.

**Proof.** For all $t, s \in [0, 1]$, if $s \leq t$, it follows from (2.4) that

$$G(t, s) = \frac{1}{2} (2t - t^2 - s) = \frac{1}{2} [1 - s - (1 - t)^2]s \leq \frac{1}{2} (1 - s)s = G(1, s),$$

and

$$G(t, s) = \frac{1}{2} (2t - t^2 - s) = \frac{1}{2} t^2 (1 - s) + \frac{1}{2} (1 - t)(t - s + (1 - s)t) \geq \frac{1}{2} t^2 (1 - s)s = t^2 G(1, s).$$

If $t \leq s$, then from (2.4) we have

$$\frac{1}{2} t^2 (1 - s)s \leq G(t, s) = \frac{1}{2} t^2 (1 - s) \leq \frac{1}{2} (1 - s)s = G(1, s).$$

Thus

$$t^2 G(1, s) \leq G(t, s) \leq \frac{1}{2} (1 - s)s, \quad \forall (t, s) \in [0, 1] \times [0, 1].$$

Therefore,

$$t^2 G(1, s) \leq G(t, s) \leq G(1, s), \quad \forall (t, s) \in [\tau, 1] \times [0, 1].$$

The proof is complete. \(\Box\)

**Lemma 2.4.** If $x \in C^+[0, 1]$, then the unique solution $u(t)$ of the BVP (2.1), (2.2) is nonnegative and satisfies

$$\min_{t \in [\tau, 1]} u(t) \geq \gamma \| u \|.$$

**Proof.** Let $x \in C^+[0, 1]$; it is obvious that $u(t)$ is nonnegative. For any $t \in [0, 1]$, by (2.3) and Lemma 2.3, it follows that

$$u(t) = \int_{0}^{1} G(t, s)a(s)f(x(s)) \, ds + \frac{\alpha t^2}{2(1 - \alpha \eta)} \int_{0}^{1} G_1(\eta, s)a(s)f(x(s)) \, ds + \frac{\lambda t^2}{2(1 - \alpha \eta)} \int_{0}^{1} G_t(\eta, s)a(s)f(x(s)) \, ds \leq \int_{0}^{1} G(1, s)a(s)f(x(s)) \, ds + \frac{\alpha}{2(1 - \alpha \eta)} \int_{0}^{1} G_1(\eta, s)a(s)f(x(s)) \, ds + \frac{\lambda}{2(1 - \alpha \eta)}$$

and thus,

$$\| u \| \leq \int_{0}^{1} G(1, s)a(s)f(x(s)) \, ds + \frac{\alpha}{2(1 - \alpha \eta)} \int_{0}^{1} G_1(\eta, s)a(s)f(x(s)) \, ds + \frac{\lambda}{2(1 - \alpha \eta)}.$$

On the other hand, (2.3) and Lemma 2.3 imply that, for any $t \in [\tau, 1]$,

$$u(t) = \int_{0}^{1} G(t, s)a(s)f(x(s)) \, ds + \frac{\alpha t^2}{2(1 - \alpha \eta)} \int_{0}^{1} G_1(\eta, s)a(s)f(x(s)) \, ds + \frac{\lambda t^2}{2(1 - \alpha \eta)} \int_{0}^{1} G_t(\eta, s)a(s)f(x(s)) \, ds \geq \gamma \int_{0}^{1} G(1, s)a(s)f(x(s)) \, ds + \frac{\alpha}{2(1 - \alpha \eta)} \int_{0}^{1} G_1(\eta, s)a(s)f(x(s)) \, ds + \frac{\lambda}{2(1 - \alpha \eta)}$$

Therefore,

$$\min_{t \in [\tau, 1]} u(t) \geq \gamma \| u \|.$$

This completes the proof. \(\Box\)
Define the cone \( K \) by
\[
K = \left\{ u \in C^+[0, 1] : \min_{t \in [r, 1]} u(t) \geq \gamma \| u \| \right\}.
\]

Define an operator \( T \) by
\[
Tu(t) = \int_0^1 G(t, s)a(s)f(u(s))ds + \frac{\alpha t^2}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s)a(s)f(u(s))ds + \frac{\lambda t^2}{2(1 - \alpha \eta)}.
\]

By Lemma 2.1, BVP (1.1), (1.2) has a positive solution \( u = u(t) \) if and only if \( u \) is a fixed point of \( T \).

**Lemma 2.5.** The operator defined in (2.5) is completely continuous and satisfies \( T(K) \subseteq K \).

**Proof.** By Lemma 2.4, we know that \( T(K) \subseteq K \). The operator \( T \) is completely continuous by an application of the Ascoli–Arzela theorem. The proof is complete. \( \square \)

To establish the existence or nonexistence of positive solutions of BVP (1.1), (1.2), we will employ the following Guo–Krasnosel'skii fixed point theorem [9,13].

**Theorem 2.1.** Let \( E \) be a Banach space and \( K \subseteq E \) a cone in \( E \). Assume \( \Omega_1 \) and \( \Omega_2 \) are open subsets of \( E \) with \( 0 \in \Omega_1 \) and \( \overline{\Omega}_1 \subseteq \Omega_2 \). Let \( T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow K \) be a completely continuous operator. In addition suppose either
(A) \( \| Tu \| \leq \| u \|, \forall u \in K \cap \partial \Omega_1 \) and \( \| Tu \| \geq \| u \|, \forall u \in K \cap \partial \Omega_2 \) or
(B) \( \| Tu \| \leq \| u \|, \forall u \in K \cap \partial \Omega_1 \) and \( \| Tu \| \leq \| u \|, \forall u \in K \cap \partial \Omega_2 \).
holds. Then \( T \) has a fixed point in \( K \cap (\overline{\Omega}_2 \setminus \Omega_1) \).

3. Main results

In this section, we discuss the existence or nonexistence of a positive solution of BVP (1.1), (1.2). Throughout this section, we shall use the following notation:
\[
\begin{align*}
\Lambda_1 &= \left( \int_0^1 (1 - s)sa(s)ds + \frac{\alpha}{1 - \alpha \eta} \int_0^1 G_1(\eta, s)a(s)ds \right)^{-1}, \\
\Lambda_2 &= \left( \gamma \int_\tau^1 (1 - s)sa(s)ds + \frac{\alpha \gamma}{1 - \alpha \eta} \int_\tau^1 G_1(\eta, s)a(s)ds \right)^{-1}.
\end{align*}
\]

It is obvious that \( \Lambda_2 > \Lambda_1 > 0 \). Also we define
\[
f_0 = \lim_{r \to 0^+} \frac{f(r)}{r}, \quad f_\infty = \lim_{r \to \infty} \frac{f(r)}{r}.
\]

**Theorem 3.1.** Suppose that \( f \) is superlinear, i.e.
\[
f_0 = 0, \quad f_\infty = \infty.
\]

Then BVP (1.1), (1.2) has at least one positive solution for \( \lambda \) small enough and has no positive solution for \( \lambda \) large enough.

**Proof.** We divide the proof into two steps.

**Step 1.** We prove that BVP (1.1), (1.2) has at least one positive solution for sufficiently small \( \lambda > 0 \).

Since \( f_0 = 0 \), for \( \Lambda_1 > 0 \), there exists \( R_1 > 0 \) such that \( \frac{\Lambda_1}{r} \leq \Lambda_1 \), \( r \in (0, R_1] \). Therefore,
\[
f(r) \leq \Lambda_1 r, \quad \text{for } r \in [0, R_1].
\]

Set \( \Omega_1 = \{ u \in C[0, 1] : \| u \| < R_1 \} \) and let \( \lambda \) satisfy
\[
0 < \lambda \leq (1 - \alpha \eta)R_1.
\]

Then, for any \( u \in K \cap \partial \Omega_1 \), it follows from Lemmas 2.2, 2.5, (3.1) and (3.2) that
\[
Tu(t) = \int_0^1 G(t, s)a(s)f(u(s))ds + \frac{\alpha t^2}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s)a(s)f(u(s))ds + \frac{\lambda t^2}{2(1 - \alpha \eta)}
\leq \frac{1}{2} \int_0^1 (1 - s)sa(s)u(s)ds + \frac{\alpha}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s)a(s)u(s)ds + \frac{\lambda}{2(1 - \alpha \eta)}R_1
\leq \frac{1}{2} \Lambda_1 \left( \int_0^1 (1 - s)sa(s)u(s)ds + \frac{\alpha}{1 - \alpha \eta} \int_0^1 G_1(\eta, s)a(s)u(s)ds \right) + \frac{\alpha (1 - \alpha \eta)R_1}{2(1 - \alpha \eta)}
\leq \frac{1}{2} \Lambda_1 \left( \int_0^1 (1 - s)sa(s)ds + \frac{\alpha}{1 - \alpha \eta} \int_0^1 G_1(\eta, s)a(s)ds \right) \| u \| + \frac{1}{2} R_1
= \frac{1}{2} R_1 + \frac{1}{2} R_1 = \| u \|.
\]
Lemma 2.4: We know that the operator \( T \) has no positive solution for \( f \) in \( (1.1) \). Hence, we can conclude that for \( r \in [\gamma R_2, \infty) \), we have
\[
f(r) \geq 2A_2r, \\
\text{for } r \in [\gamma R_2, \infty).
\]

Set \( \Omega_2 = \{ u \in C(0, 1) : \| u \| < R \} \). For any \( u \in K \cap \partial \Omega_2 \), by Lemma 2.4 one has \( \min_{r \in [\gamma R_2, \infty)} u(s) \geq \gamma \| u \| = \gamma R_2 \). Thus, from (2.5) and (3.4) we can conclude that
\[
Tu(1) = \int_0^1 G(1, s)a(s)f(u(s))ds + \frac{\alpha}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s)a(s)f(u(s))ds + \frac{\lambda}{2(1 - \alpha \eta)} \geq 2A_2 \int_0^1 (1 - s)a(s)4A_2u(s)ds + \frac{\alpha}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s)a(s)f(u(s))ds
\]
which implies that
\[
\| Tu \| \geq \| u \|, \\
\text{for } u \in K \cap \partial \Omega_2.
\]
Therefore, by (3.3), (3.5) and the first part of Theorem 2.1 we know that the operator \( T \) has at least one fixed point \( u^* \in K \cap (\overline{\Omega_2} \setminus \Omega) \), which is a positive solution of BVP (1.1), (1.2).

Step 2: We verify that BVP (1.1), (1.2) has no positive solution for \( \lambda \) large enough. Otherwise, there exist 0 < \( \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \), with \( \lim_{n \to \infty} \lambda_n = +\infty \), such that for any positive integer \( n \), the BVP
\[
\begin{align*}
&u''(t) + a(t)f(u(t)) = 0, \quad 0 < t < 1, \\
u(0) = u'(0) = 0, \quad u'(1) - \alpha u'(\eta) = \lambda_n,
\end{align*}
\]
has a positive solution \( u_n(t) \). By (2.5), we have
\[
u_n(1) = \int_0^1 G(1, s)a(s)f(u_n(s))ds + \frac{\alpha}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s)a(s)f(u_n(s))ds + \frac{\lambda_n}{2(1 - \alpha \eta)} \geq \frac{\lambda_n}{2(1 - \alpha \eta)} \to \infty, \quad (n \to \infty).
\]
Thus
\[
\| u_n \| \to \infty, \quad (n \to \infty).
\]
Since \( f_{\infty} = \infty \), for \( 4A_2 > 0 \), there exists \( \hat{R} > 0 \) such that \( \frac{\lambda_n}{r} \geq 4A_2 \), for \( r \in [\gamma \hat{R}, \infty) \), which implies that
\[
f(r) \geq 4A_2r, \\
\text{for } r \in [\gamma \hat{R}, \infty).
\]

Let \( n \) be large enough that \( \| u_n \| \geq \hat{R} \). Then
\[
\| u_n \| \geq u_n(1)
\]
\[
\begin{align*}
&= \int_0^1 G(1, s)a(s)f(u_n(s))ds + \frac{\alpha}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s)a(s)f(u_n(s))ds + \frac{\lambda_n}{2(1 - \alpha \eta)} \\
&\geq \frac{1}{2} \int_0^1 (1 - s)a(s)f(u_n(s))ds + \frac{\alpha}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s)a(s)f(u_n(s))ds \\
&\geq \frac{1}{2} \int_0^1 (1 - s)a(s)4A_2u_n(s)ds + \frac{\alpha}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s)a(s)4A_2u_n(s)ds \\
&\geq 2A_2 \left( \gamma \int_0^1 (1 - s)a(s)ds + \frac{\alpha \gamma}{1 - \alpha \eta} \int_0^1 G_1(\eta, s)a(s)ds \right) \| u_n \| \\
&= 2\| u_n \|,
\end{align*}
\]
which is a contradiction. The proof is complete. \( \Box \)

Moreover, if the function \( f \) is nondecreasing, we have the following result.
Theorem 3.2. Suppose that \( f \) is superlinear. If \( f \) is nondecreasing, then there exists a positive constant \( \lambda^* \) such that BVP (1.1), (1.2) has at least one positive solution for \( \lambda \in (0, \lambda^*) \) and has no positive solution for \( \lambda \in (\lambda^*, \infty) \).

Proof. Let \( \Sigma = \{ \lambda \mid \) BVP (1.1), (1.2) has at least one positive solution \} and \( \lambda^* = \sup \Sigma \); it follows from Theorem 3.1 that \( 0 < \lambda^* < \infty \). From the definition of \( \lambda^* \), we know that for any \( \lambda \in (0, \lambda^*) \), there is a \( \lambda_0 > \lambda \) such that BVP

\[
\begin{align*}
    u''(t) + a(t)f(u(t)) &= 0, \quad 0 < t < 1, \\
    u(0) &= u'(0) = 0, \\
    u'(1) - a\mu'(\eta) &= \lambda_0,
\end{align*}
\]

has a positive solution \( u_0(t) \). Now we prove that for any \( \lambda \in (0, \lambda_0) \), BVP (1.1), (1.2) has a positive solution.

In fact, let

\[
    K(u_0) = \{ u \in K \mid u(t) \leq u_0(t), \quad t \in [0, 1] \}.
\]

For any \( \lambda \in (0, \lambda_0) \), \( u \in K(u_0) \), it follows from (2.5) and the monotonicity of \( f \) that we have that

\[
\begin{align*}
    Tu(t) &= \int_0^1 G(t, s)a(s)f(u(s))ds + \frac{\alpha t^2}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s)a(s)f(u(s))ds + \frac{\lambda t^2}{2(1 - \alpha \eta)} \\
    &= \int_0^1 G(t, s)a(s)f(u_0(s))ds + \frac{\alpha t^2}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s)a(s)f(u_0(s))ds + \frac{\lambda t^2}{2(1 - \alpha \eta)} \\
    &= u_0(t).
\end{align*}
\]

Thus \( T(K(u_0)) \subseteq K(u_0) \). By Schauder’s fixed point theorem we know that there exists a fixed point \( u \in K(u_0) \), which is a positive solution of BVP (1.1), (1.2). The proof is complete. \( \square \)

Now we consider the case when \( f \) is sublinear.

Theorem 3.3. Suppose that \( f \) is sublinear, i.e.

\[
    f_0 = \infty, \quad f_\infty = 0.
\]

Then the BVP (1.1), (1.2) has at least one positive solution for any \( \lambda \in (0, \infty) \).

Proof. Since \( f_0 = \infty \), there exists \( R_1 > 0 \) such that \( f(r) \geq 2A_2r \), for \( r \in [0, R_1] \). So, for any \( u \in K \) with \( \| u \| = R_1 \) and any \( \lambda > 0 \), we have

\[
\begin{align*}
    Tu(1) &= \int_0^1 G(1, s)a(s)f(u(s))ds + \frac{\alpha}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s)a(s)f(u(s))ds + \frac{\lambda}{2(1 - \alpha \eta)} \\
    &\geq \frac{1}{2} \int_0^1 (1 - s)a(s)f(u(s))ds + \frac{\alpha}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s)a(s)f(u(s))ds \\
    &\geq \frac{1}{2} \int_0^1 (1 - s)a(s)2A_2u(s)ds + \frac{\alpha}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s)a(s)2A_2u(s)ds \\
    &\geq A_2 \left( \gamma \int_0^1 (1 - s)a(s)ds + \frac{\alpha \gamma}{1 - \alpha \eta} \int_0^1 G_1(\eta, s)a(s)ds \right) \| u \| \\
    &= \| u \|,
\end{align*}
\]

and consequently, \( \| Tu \| \geq \| u \| \). So, if we set \( \Omega_1 = \{ u \in K : \| u \| < R_1 \} \), then

\[
    \| Tu \| \geq \| u \|, \quad \text{for } u \in K \cap \partial \Omega_1.
\]

Next we construct the set \( \Omega_2 \). We consider two cases: \( f \) is bounded or \( f \) is unbounded.

Case (i) Suppose that \( f \) is bounded, say \( f(r) \leq M \) for all \( r \in [0, \infty) \). In this case we choose

\[
    R_2 \geq \max \left\{ 2R_1, \frac{M}{A_1}, \frac{\lambda}{1 - \alpha \eta} \right\},
\]

and then for \( u \in K \) with \( \| u \| = R_2 \), we have

\[
\begin{align*}
    Tu(t) &= \int_0^1 G(t, s)a(s)f(u(s))ds + \frac{\alpha t^2}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s)a(s)f(u(s))ds + \frac{\lambda t^2}{2(1 - \alpha \eta)} \\
    &\leq \frac{M}{2A_1} \int_0^1 (1 - s)a(s)ds + \frac{M \alpha}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s)a(s)ds + \frac{\lambda t^2}{2(1 - \alpha \eta)} \\
    &\leq \frac{M}{2A_1} + \frac{1}{2} R_2 \leq \frac{1}{2} R_2 + \frac{1}{2} R_2 = \| u \|.
\end{align*}
\]

So, \( \| Tu \| \leq \| u \| \).
Case (ii) When \( f \) is unbounded. Now, since \( f_\infty = 0 \), there exists \( R_0 \) such that

\[
f(r) \leq A_1 r, \quad \text{for } r \in [R_0, \infty).
\]

(3.7)

Let \( R_2 \geq \max\{2R_1, R_0, \frac{\lambda}{\alpha \eta_1}\} \) and be such that

\[
f(r) \leq f(R_2), \quad \text{for } r \in [0, R_2].
\]

(We are able to do this since \( f \) is unbounded.) For \( u \in K \) with \( \|u\| = R_2 \), from (2.5) and (3.7), we have

\[
Tu(t) = \int_0^1 G(t, s)a(s)f(u(s))ds + \frac{\alpha t^2}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s)a(s)f(u(s))ds + \frac{\lambda t^2}{2(1 - \alpha \eta)}
\]

\[
\leq \frac{1}{2} \int_0^1 (1 - s)sa(s)f(R_2)ds + \frac{\alpha}{2(1 - \alpha \eta)} \int_0^1 G_1(\eta, s)a(s)f(R_2)ds + \frac{\lambda}{2(1 - \alpha \eta)}
\]

\[
\leq \frac{1}{2} A_1 \left( \int_0^1 (1 - s)sa(s)ds + \frac{\alpha}{1 - \alpha \eta} \int_0^1 G_1(\eta, s)a(s)ds \right) R_2 + \frac{1}{2} R_2
\]

\[
= \frac{1}{2} R_2 + \frac{1}{2} R_2 = \|u\|.
\]

Thus, \( \|Tu\| \leq \|u\| \).

Therefore, in either case we may put \( \Omega_2 = \{u \in K : \|u\| < R_2\} \); then

\[
\|Tu\| \leq \|u\|, \quad \text{for } u \in K \cap \partial \Omega_2.
\]

(3.8)

So, it follows from (3.6), (3.8) and the second part of the Theorem 2.1 that \( \Gamma \) has a fixed point \( u^* \in K \cap (\overline{\Omega_2} \setminus \Omega_1) \). Then \( u^* \) is a positive solution of BVP (1.1), (1.2). The proof is complete. \( \square \)

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References