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On Operator Ranges

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INTRODUCTION

The purpose of this paper is to survey a number of elegant but little-known results concerning the ranges of bounded linear operators in Hilbert space. There is reason to believe that the results and techniques of this theory will find increasing application, for instance in formulating and proving infinite-dimensional versions of finite-dimensional theorems. As an example of this, we mention the following extension of Burnside's theorem on irreducible matrix algebras, obtained recently by C. Foiaş [11]: if an algebra $\mathcal{Ol}$ of bounded operators on a Hilbert space $\mathscr{H}$ leaves invariant no operator range, then $\mathcal{Ol}$ is weakly dense in the algebra of all bounded operators on $\mathscr{H}$. In many such situations, where it is not (or is not known to be) enough to consider only closed subspaces, it may be hoped that operator ranges will suffice. This is because, as will be seen, operator ranges possess many special features that distinguish them from arbitrary linear subspaces; in general, they tend to resemble closed subspaces, but their behavior is much more pathological.

Most, but by no means all, of the results of this paper appear in the literature; most of the proofs are new. The basic sources for material on operator ranges are the exhaustive treatises of Dixmier [5, 6]. Other results are taken from Köthe [16, 17], von Neumann [20], and Mackey [18]. We also draw upon the (unpublished) lecture notes of R. G. Douglas and D. M. Topping on operator ranges, as well as a paper of A. Ramsay and K. Gustafson [12].

Section 1 opens with a basic result giving various characterizations of the notion of an operator range. Then it is proved that an operator range is a first category F_σ, and several examples are discussed that show, among other things, that these conditions are not sufficient. Section 2 is built on two recent theorems concerning inclusion and sums of operator ranges. From these follow many of Dixmier's results concerning the collection $\mathscr{L}$ of all operator ranges in a Hilbert space $\mathscr{H}$: for example, that it is a lattice generated by the closed subspaces, and that the only elements of $\mathscr{L}$ with complements in $\mathscr{L}$ are the closed subspaces. We also include Dixmier's classification of operator ranges in terms of the closed subspaces that they contain, and a new result that determines the class of the range of a positive operator by means of the Weyl limit points of its spectrum. In Section 3, ideas of Köthe are used to give a criterion for unitary equivalence (or similarity) of operator ranges. Then these ideas are presented in their original context—equivalence and congruence of operators. Finally, we give von Neumann's result that for any unbounded self-adjoint operator A, there is a unitary operator U such that the domains of A and U^*AU have only the zero vector in common.

In Section 4, we mention the basic facts about orthonormal bases in operator ranges. We also give an elementary proof of a characterization of the range of a contraction T by means of the defect operator $\sqrt{1 - TT^*}$. The remainder of the section is devoted to the infinite-dimensional extension of recent work of Anderson and Duffin [1] concerning the parallel sum $A : B$ of two positive operators. This sum is of interest here because it gives a canonical operator whose range is the intersection of two given ranges.

We conclude this introduction by establishing some notation and terminology. We are usually concerned with a complex Hilbert space $\mathscr{H}$, not necessarily separable. An *operator* on $\mathscr{H}$ will always be understood to be a bounded linear transformation from $\mathscr{H}$ into itself. The algebra of all bounded operators on $\mathscr{H}$ will be denoted by $\mathscr{B}(\mathscr{H})$. An operator is invertible if it has an inverse in $\mathscr{B}(\mathscr{H})$. If $T \in \mathscr{B}(\mathscr{H})$, then $\mathscr{R}(T)$ or $T\mathscr{H}$ denotes the generally nonclosed set of values of T, and $\mathscr{N}(T)$ denotes the null space of T. An *operator range* in $\mathscr{H}$ is a linear subspace of $\mathscr{H}$ that is the range of some bounded operator on $\mathscr{H}$. Dixmier calls an operator range a "Julia variety". Foiaş uses the term "semiclosed-subspace."

We shall denote operator ranges by $\mathscr{R}$, $\mathscr{S}$,... and closed subspaces by $\mathscr{M}$, $\mathscr{N}$,... .

1. Fundamental Properties

We begin with a number of characterizations of operator ranges.

THEOREM 1.1. *Let $\mathscr{R}$ be a vector subspace (not necessarily closed) of a Hilbert space $\mathscr{H}$. The following conditions are equivalent:*

(1) *$\mathscr{R}$ is the range of a bounded operator in $\mathscr{H}$.*

(2) *$\mathscr{R}$ is the range of a closed operator in $\mathscr{H}$.*

(3) *$\mathscr{R}$ is the domain of a closed operator in $\mathscr{H}$.*

(4) *There is an inner product $(\cdot, \cdot)'$ on $\mathscr{R}$ such that $(\mathscr{R}, \| \cdot \|')$ is a Hilbert space and $\| x \|' \geqslant \| x \|$ for all $x \in \mathscr{R}$.*

(5) *There is a sequence $\{\mathscr{H}_n \mid n \geqslant 0\}$ of closed mutually orthogonal subspaces of $\mathscr{H}$ such that*

$$\mathscr{R} = \left\{ \sum_{n=0}^{\infty} x_n \mid x_n \in \mathscr{H}_n \textit{ and } \sum_{n=0}^{\infty} (2^n \| x_n \|)^2 < \infty \right\}.$$

Proof. Obviously (1) implies (2). Assume that $\mathscr{R} = \mathscr{R}(T)$ for some T with closed graph. Let $\mathscr{D}$ be the domain of T, and put $T_1 = T \mid \mathscr{D} \cap \mathscr{N}(T)^{\perp}$. Then T_1 is closed, one-to-one, and has range $\mathscr{R}$, so that T_1^{-1} is closed with domain $\mathscr{R}$. Thus (2) implies (3).

If $\mathscr{R}$ is the domain of a closed operator T in $\mathscr{H}$, then $\mathscr{R}$ is complete in the inner product defined by

$$(x, y)' = (x, y) + (Tx, Ty) \qquad \text{for all} \quad x, y \in \mathscr{R},$$

and $\| x \|' \geqslant \| x \|$ for all $x \in \mathscr{R}$. Hence (3) implies (4). To show that (4) implies (1), note first that the inclusion map T from $(\mathscr{R}, \| \cdot \|')$ into $\mathscr{H}$ is bounded. If $T^* = U(TT^*)^{1/2}$ is the polar decomposition, then $U : \mathscr{H} \to (\mathscr{R}, \| \cdot \|')$ is a partial isometry with final space $\mathscr{R}(T^*)^- = \mathscr{N}(T)^{\perp} = \mathscr{R}$. Viewed as an operator from $\mathscr{H}$ into $\mathscr{H}$, U has range $\mathscr{R}$ and is bounded since $\| Ux \| \leqslant \| Ux \|' \leqslant \| x \|$ for all $x \in \mathscr{H}$.

Next we show that (1) implies (5). Recall that for any bounded operator T in $\mathscr{H}$, T and $A = (TT^*)^{1/2}$ have the same range (cf. Theorem 2.1). Let E be the spectral measure of A, let $E_n = E(2^{-n-1}\| A \|, 2^{-n}\| A \|]$, and let $\mathscr{H}_n = E_n\mathscr{H}$ for $n \geqslant 0$. Then

$$\mathscr{R}(A) = \left\{ \sum_{n=0}^{\infty} x_n \mid x_n \in \mathscr{H}_n \text{ and } \sum_{n=0}^{\infty} (2^n \| x_n \|)^2 < \infty \right\}.$$

For, if $x_n = E_n Ax$, then $Ax = \sum_{n=0}^{\infty} x_n$, $\| x_n \| \leqslant 2^{-n}\| A \| \| E_n x \|$, and therefore

$$\sum_{n=0}^{\infty} (2^n \| x_n \|)^2 \leqslant \| A \|^2 \| x \|^2 < \infty.$$

On the other hand, if $x_n \in \mathscr{H}_n$, then, because $A\mathscr{H}_n = \mathscr{H}_n$, there is $y_n \in \mathscr{H}_n$ with $Ay_n = x_n$, and we have $\| x_n \| = \| Ay_n \| \geqslant 2^{-n-1}\| A \| \| y_n \|$. Thus, if $x = \sum_{n=0}^{\infty} x_n$ with $\sum_{n=0}^{\infty} (2^n\| x_n \|)^2 < \infty$, we get $\sum_{n=0}^{\infty} \| y_n \|^2 < \infty$, $y = \sum_{n=0}^{\infty} y_n \in \mathscr{H}$, and $Ay = x$.

Finally (5) implies (1), for if P_n is the projection on $\mathscr{H}_n$ and $D = \sum_{n=0}^{\infty} 2^{-n}P_n$, then D is bounded and $\mathscr{R}(D) = \mathscr{R}$.

Remarks. (i) As the proof shows, the condition that $\mathscr{R}$ be the range of a bounded positive operator in $\mathscr{H}$ is equivalent to the conditions of the theorem. Using the Cayley transform, one sees that $\mathscr{R}$ is also the range of $U - I$, where U is a unitary operator.

(ii) The weight sequence $\{2^n\}$ used in (5) can be replaced by any increasing unbounded sequence $\{w_n\}$ of positive numbers.

(iii) Consider those vector subspaces of $\mathscr{H}$ that are the range of some bounded operator defined on a Banach space. The examples below show that there are subspaces of this type that do not satisfy (1)–(5). However, it is easy to see that the versions of (1)–(4) appropriate to subspaces of this kind are equivalent (even when $\mathscr{H}$ is replaced by a Banach space).

Next we discuss several special properties of operator ranges, and some examples. To begin with, the range of a bounded operator T in $\mathscr{H}$ is a Borel set and, in fact, an F_σ. For if $\mathscr{B}$ is any closed ball in $\mathscr{H}$, then $\mathscr{B}$ is weakly compact and convex, so that $T\mathscr{B}$ is weakly closed and convex, and is therefore norm closed. If $\mathscr{B}_n = \{x \mid \| x \| \leqslant n\}$, then $\mathscr{H} = \bigcup_{n=0}^{\infty} \mathscr{B}_n$, and consequently $\mathscr{R}(T) = \bigcup_{n=0}^{\infty} T\mathscr{B}_n$ is an F_σ. (This is valid for any bounded operator defined on a reflexive Banach space.)

Another necessary condition, due to Banach [2, p. 38], is the following: if T is a bounded linear transformation with domain a Banach space $\mathscr{X}$ and range contained in a Banach space $\mathscr{Y}$, then either $T\mathscr{X}$ is a first category subspace of $\mathscr{Y}$ or else $T\mathscr{X} = \mathscr{Y}$. This is proved by showing (as in some proofs of the open mapping theorem) that $T\mathscr{X} = \mathscr{Y}$ if $(T\mathscr{B}_n)^-$ has nonvoid interior for some n.

These conditions are not sufficient. For a simple example, consider in $\mathscr{H} = L^2[0, 1]$ the subspace $L^p[0, 1]$ $(2 < p < \infty)$. Since the inclusion map is continuous, the above results show that L^p is a first category F_σ in L^2. But it is not the range of any bounded operator in L^2, as the following remark shows: *if T is a continuous linear injection of a Banach space $\mathscr{X}$ in a Hilbert space $\mathscr{H}$, and if $T\mathscr{X}$ is the range of a bounded operator in $\mathscr{H}$, then $\mathscr{X}$ is isomorphic to a Hilbert space.* Since L^p is not isomorphic to a Hilbert space (see [2, p. 203] or [19], where it is shown that L^p has closed noncomplemented subspaces), it follows that L^p is not the range of any bounded operator in L^2.

To prove this remark, let A be a bounded operator in L^2 with range $T\mathscr{X}$, let $\mathscr{K} = \mathscr{N}(A)^\perp$, and let $A_1 = A \mid \mathscr{K}$. Then $A_1^{-1}T$ maps $\mathscr{X}$ onto the Hilbert space $\mathscr{K}$ and is one-to-one with closed graph. Hence, $A_1^{-1}T$ is bicontinuous, and $\mathscr{X}$ is isomorphic to a Hilbert space.

As further examples we mention the spaces $L^\infty[0, 1]$ and $C[0, 1]$ and the space $A[0, 1]$ of absolutely continuous functions f on $[0, 1]$ such that $f(0) = 0$. All are first category subspaces of L^2 by Banach's theorem (the inclusion maps of L^∞ and C in L^2 are continuous, and A is the range of the continuous linear map $V : L^1 \to L^2$ defined by $(Vf)(x) = \int_0^x f(t)\, dt$), but none is the range of a bounded operator in L^2 (again by the above remark, since L^∞, C, and L^1 are nonreflexive spaces).

We conclude this section with an observation (brought to our attention by C. Foiaş) that is occasionally useful. *Let T be a linear transformation with domain a Banach space and range contained in a Banach space. If the graph of T is the range of a bounded operator B defined on a Banach space, then T is bounded.* By passing to a quotient space B can be assumed to be an injection. Then the map

$$x \to B^{-1}\langle x, Tx\rangle$$

is closed, and the assertion follows from the usual closed graph theorem.

2. The Lattice of Operator Ranges

We begin with a recent and useful result of Douglas [8].

THEOREM 2.1. *Let A and B be bounded operators on $\mathscr{H}$. The following conditions are equivalent:*

(1) $\mathscr{R}(A) \subset \mathscr{R}(B)$.
(2) $AA^* \leqslant \lambda^2 BB^*$ *for some constant* $\lambda > 0$.
(3) $A = BC$ *for some bounded operator C on* $\mathscr{H}$.

Proof. Suppose that (1) holds, and let B_0 be the restriction of B to $\mathscr{N}(B)^\perp$. Then $B_0^{-1} : \mathscr{R}(B) \to \mathscr{N}(B)^\perp$ is a closed linear transformation. This implies that $C = B_0^{-1}A$ is a closed linear transformation from $\mathscr{H}$ into $\mathscr{N}(B)^\perp$. An application of the closed graph theorem shows that C is bounded. Since $BC = A$, (1) implies (3). That (3) implies (1) is trivial.

If (2) holds, then $\| A^*x \| \leqslant \lambda \| B^*x \|$ for all $x \in \mathscr{H}$. Therefore, the linear map $D : \mathscr{R}(B^*) \to \mathscr{R}(A^*)$ defined by $D(B^*x) = A^*x$ is bounded. Extend D to the closure of $\mathscr{R}(B^*)$ by continuity; then put $D = 0$ on $\mathscr{R}(B^*)^\perp = \mathscr{N}(B)$. This gives a bounded operator D on $\mathscr{H}$ such that $DB^* = A^*$, and so $A = BC$ with $C = D^*$. Thus (2) implies (3). Finally, the inequality

$$(AA^*x, x) = \| A^*x \|^2 = \| C^*B^*x \|^2 \leqslant \| C^* \|^2 \| B^*x \|^2 = \| C^* \|^2 (BB^*x, x)$$

shows that (3) implies (2).

Remark. As the proof shows, the operator C can be chosen so that $\mathscr{N}(C^*) \supset \mathscr{N}(B)$. This condition and the equation $C^*B^* = A^*$ uniquely determine C.

If $\mathscr{R}(A) = \mathscr{R}(B)$, then, by the proof that (1) implies (3), there are bounded operators C from $\mathscr{H}$ into $\mathscr{N}(B)^\perp$ and D from $\mathscr{H}$ into $\mathscr{N}(A)^\perp$ such that $A = BC$ and $B = AD$. Then DC is the identity operator on $\mathscr{N}(A)^\perp$, and CD is the identity operator on $\mathscr{N}(B)^\perp$. Therefore, C gives a one-to-one linear transformation from $\mathscr{N}(A)^\perp$ onto $\mathscr{N}(B)^\perp$. If we now suppose that A and B have the same nullity, then it is clear that we can redefine C on $\mathscr{N}(A)$ to obtain an invertible operator on $\mathscr{H}$:

COROLLARY 1. *Let A and B be bounded operators on $\mathscr{H}$. There exists an invertible operator C on $\mathscr{H}$ such that $A = BC$ if and only if A and B have the same range and nullity.*

In particular, it follows that two positive operators have the same range if and only if they differ by an invertible factor [5; Théorème 2.2]. This fact in turn shows that, if A is a positive operator, then $\mathscr{R}(A^{1/2}) = \mathscr{R}(A)$ if and only if A has closed range.

If A is a bounded operator and if y is a nonzero vector in $\mathscr{H}$, then $y \in \mathscr{R}(A)$ if and only if $\mathscr{R}(B) \subset \mathscr{R}(A)$, where B is the projection of $\mathscr{H}$ onto the one-dimensional subspace spanned by y. By applying (2) of Theorem 2.1, one gets a characterization of $\mathscr{R}(A)$ due to Shmuly'an [23]:

COROLLARY 2. *If $A \in \mathscr{B}(\mathscr{H})$, then $y \in \mathscr{R}(A)$ if and only if*

$$\sup_x \frac{|(x, y)|}{\| A^*x \|} < \infty.$$

It is perhaps worth noting that Shmuly'an's result can be formulated for a Banach space $\mathscr{X}$ as follows: If $A \in \mathscr{B}(\mathscr{X})$, then a vector $y \in \mathscr{X}^{**}$

belongs to the range of A^{**} if and only if $\sup_{f \in \mathscr{X}^*} |\langle y, f \rangle| / \| A^* f \| < \infty$. (Necessity is trivial; sufficiently is established by observing that the hypothesis insures that $A^*f \to \langle y, f \rangle$ is a bounded linear functional on $\mathscr{R}(A^*)$. Hence, by the Hahn–Banach Theorem there is a vector $x \in \mathscr{X}^{**}$ such that $\langle x, A^*f \rangle = \langle y, f \rangle$ for all $f \in \mathscr{X}^*$; therefore $y = A^{**}x$.)

It is an immediate consequence of Theorem 2.1 that the operators A and $(AA^*)^{1/2}$ have the same range. This fact leads to a simple proof of the following result due to T. Crimmins [unpublished].

THEOREM 2.2. *If $A, B \in \mathscr{B}(\mathscr{H})$, then*

$$\mathscr{R}(A) + \mathscr{R}(B) = \mathscr{R}(\sqrt{AA^* + BB^*}).$$

Proof. Let $T = \begin{pmatrix} A & -B \\ 0 & 0 \end{pmatrix}$ act on $\mathscr{H} \oplus \mathscr{H}$ in the usual way. Then

$$(\mathscr{R}(A) + \mathscr{R}(B)) \oplus \{0\} = \mathscr{R}(T) = \mathscr{R}((TT^*)^{1/2}) = \mathscr{R} \begin{pmatrix} (AA^* + BB^*)^{1/2} & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \mathscr{R}(\sqrt{AA^* + BB^*}) \oplus \{0\}.$$

COROLLARY 1. *Let A, B_1, $B_2 \in \mathscr{B}(\mathscr{H})$. The following conditions are equivalent*:

(1) $\mathscr{R}(A) \subset \mathscr{R}(B_1) + \mathscr{R}(B_2)$.

(2) $AA^* \leqslant \lambda^2(B_1B_1^* + B_2B_2^*)$ *for some constant* $\lambda > 0$.

(3) *There exist bounded operators X and Y such that*

$$A = B_1X + B_2Y.$$

Proof. Conditions (1) and (2) are equivalent by Theorems 2.1 and 2.2. Since it is obvious that (3) implies (1), it suffices to prove that (1) implies (3).

Now if $S = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix}$, then condition (1) implies $\mathscr{R}(S) \subset \mathscr{R}(T)$. Hence, $S = TX$ for some 2×2 matrix X by Theorem 2.1. This gives (3).

COROLLARY 2. *If $A, B \in \mathscr{B}(\mathscr{H})$, then there exist operators X, Y, Z in $\mathscr{B}(\mathscr{H})$ with $X \geqslant 0$, $Y \geqslant 0$ such that*

$$\mathscr{R}(A) \cap \mathscr{R}(B) = \mathscr{R}(AX) + \mathscr{R}(AZ^*) = \mathscr{R}(BZ) + \mathscr{R}(BY)$$
$$= \mathscr{R}((AXA^*)^{1/2}) = \mathscr{R}((BYB^*)^{1/2}).$$

Proof. Let $T = \begin{pmatrix} A & -B \\ 0 & 0 \end{pmatrix}$ and let $P = \begin{pmatrix} X & Z^* \\ Z & Y \end{pmatrix}$ be the orthogonal projection of $\mathscr{H} \oplus \mathscr{H}$ onto $\mathscr{N}(T)$. From $TP = 0$ follows $AX = BZ$ and $AZ^* = BY$; these in turn give

$$\mathscr{R}(A) \cap \mathscr{R}(B) \supset \mathscr{R}(AX) + \mathscr{R}(AZ^*).$$

Equality holds here, for if $w \in \mathscr{R}(A) \cap \mathscr{R}(B)$, then $w = Au = Bv$; so $\langle u, v \rangle \in \mathscr{N}(T)$. It follows that $u = Xu + Z^*v$, and therefore that

$$w = Au \in \mathscr{R}(AX) + \mathscr{R}(AZ^*).$$

Applying Theorem 2.2 and using the relation $X^2 + Z^*Z = X$ (from $P^2 = P$) gives $\mathscr{R}(A) \cap \mathscr{R}(B) = \mathscr{R}((AXA^*)^{1/2})$. The other two equations are obtained by a similar argument.

COROLLARY 3. *Let A and B be positive operators with closed range. Then $A + B$ has closed range if and only if $\mathscr{R}(A) + \mathscr{R}(B)$ is closed.*

Proof. The assertion is an immediate consequence of the theorem and the fact that a positive operator has closed range if and only if its square root has closed range.

Theorem 2.2 and Corollary 2 show that the collection of operator ranges is a lattice with respect to vector addition and set intersection. This was first observed by Dixmier [5] and Mackey [18]. The same two results in fact show that if $\mathfrak{A}$ is a von Neumann algebra, then the collection of ranges of operators in $\mathfrak{A}$ is a lattice. We do not know whether the same is true if $\mathfrak{A}$ is merely a C^*-algebra. (The operator X appearing in the proof of Corollary 2 need not belong to the C^*-algebra generated by A and B.)

The remaining results of this section are concerned with properties of the lattice $\mathscr{L}$ of all operator ranges in $\mathscr{H}$. The following result of Dixmier [5, Proposition 3.7] gives an intrinsic characterization of the closed subspaces in this lattice.

THEOREM 2.3. *An operator range $\mathscr{R}$ is complemented in the lattice $\mathscr{L}$ of all operator ranges if and only if $\mathscr{R}$ is closed.*

Proof. Suppose that A and B are bounded operators such that $\mathscr{R}(A) \cap \mathscr{R}(B) = 0$ and $\mathscr{R}(A) + \mathscr{R}(B)$ is closed. We assert that $\mathscr{R}(A)$ and $\mathscr{R}(B)$ are closed. To prove this, consider the transformation T from $\mathscr{N}(A)^\perp \oplus \mathscr{N}(B)^\perp$ into $\mathscr{H}$ given by the matrix $T = (A \quad B)$. It is easy to check that T is one-to-one with closed range. Hence T has a left inverse, say $\binom{X}{Z}$. Then XA is the identity on $\mathcal{N}(A)^{\perp}$, and ZB is the identity on $\mathcal{N}(B)^{\perp}$. These imply that A and B have closed ranges.

COROLLARY. *If $\mathcal{R}$ is a nonclosed operator range, then* $\dim \mathcal{R}^{-}/\mathcal{R} = \infty$.

Proof. Suppose that the quotient space $\mathcal{R}^{-}/\mathcal{R}$ has a finite basis $x_1 + \mathcal{R},..., x_n + \mathcal{R}$. If $\mathcal{N}$ is the subspace spanned by $x_1 ,..., x_n$, then $\mathcal{N} \cap \mathcal{R} = 0$, and $\mathcal{N} + \mathcal{R} = \mathcal{R}^{-}$ is closed. Hence $\mathcal{R}$ is closed.

Remark. With a stronger argument one can show that the algebraic dimension of the quotient $\mathcal{R}^{-}/\mathcal{R}$ is the cardinality of the continuum (see Corollary 2 of Theorem 3.6). Note also that the Corollary gives some information about operator ranges. Indeed, it is false without the assumption that $\mathcal{R}$ is an operator range. (Consider the null space of a discontinuous linear functional on $\mathcal{H}$.)

The next result, also due to Dixmier [5, Proposition 3.12], generalizes Theorem 2.3. We shall give a more direct proof.

THEOREM 2.4. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be operator ranges. Then $\mathcal{R}_1 + \mathcal{R}_2 = \mathcal{H}$ if and only if there are disjoint closed subspaces $\mathcal{M}_1 \subset \mathcal{R}_1$ and $\mathcal{M}_2 \subset \mathcal{R}_2$ such that $\mathcal{M}_1 + \mathcal{M}_2 = \mathcal{H}$.*

Proof. Let A and B be positive operators with $\mathcal{R}(A) = \mathcal{R}_1$, $\mathcal{R}(B) = \mathcal{R}_2$. Then $(A^2 + B^2)^{1/2}$ has range $\mathcal{R}_1 + \mathcal{R}_2 = \mathcal{H}$ by 2.2. Therefore, $(A^2 + B^2)^{1/2}$ is invertible. Hence, there is a positive number δ such that

$$\delta^2 \| x \|^2 \leqslant ((A^2 + B^2)x, x) = \| Ax \|^2 + \| Bx \|^2$$

for all $x \in \mathcal{H}$. Let E be the spectral measure of A, and choose a positive number $\epsilon < \delta$. If $x \in \mathcal{H}$, then

$$\begin{aligned} \| E[0, \epsilon]B^2E[0, \epsilon]x \| \, \| E[0, \epsilon]x \| &\geqslant (B^2E[0, \epsilon]x, E[0, \epsilon]x) \\ &= \| BE[0, \epsilon]x \|^2 \geqslant \delta^2 \| E[0, \epsilon]x \|^2 - \| AE[0, \epsilon]x \|^2 \\ &\geqslant (\delta^2 - \epsilon^2) \| E[0, \epsilon]x \|^2. \end{aligned}$$

Therefore,

$$\| B^2E[0, \epsilon]x \| \geqslant (\delta^2 - \epsilon^2) \| E[0, \epsilon]x \|$$

$$\| E[0, \epsilon]B^2E[0, \epsilon]x \| \geqslant (\delta^2 - \epsilon^2) \| E[0, \epsilon]x \|.$$

The first of these inequalities shows that $\mathscr{M}_2 = B^2\mathscr{R}(E[0, \epsilon])$ is closed. Also, if $\mathscr{M}_1$ is the range of $E[0, \epsilon]^\perp$, then

$$\mathscr{M}_1 + \mathscr{M}_2 = \mathscr{M}_1 \oplus E[0, \epsilon]\mathscr{M}_2 = \mathscr{M}_1 \oplus \mathscr{R}(E[0, \epsilon]B^2E[0, \epsilon])$$

is closed by the second inequality. Finally, the same two inequalities imply $\mathscr{M}_1^\perp \cap \mathscr{M}_2^\perp = 0$ and $\mathscr{M}_1 \cap \mathscr{M}_2 = 0$, respectively.

COROLLARY. *If $\mathscr{R}_2$ is the range of a compact operator and if $\mathscr{R}_1 + \mathscr{R}_2 = \mathscr{H}$, then $\mathscr{R}_1$ is a closed subspace of finite codimension in $\mathscr{H}$.*

Proof. With the notation as in the proof of the theorem, the subspace $\mathscr{M}_2$ is finite dimensional. Hence $\mathscr{R}(E[0, \epsilon])$ is finite dimensional (since B^2 is one-to-one on this space). This implies that A has closed range and finite-dimensional null space.

Remark. Dixmier [5, Proposition 3.11] also showed that two dense ranges that span $\mathscr{H}$ must have dense intersection.

If A is a bounded operator and if E is the spectral measure of $\sqrt{AA^*}$, then the spectral projections $E_\mu = E([\mu, \infty))$ satisfy $\mu^2 E_\mu \leqslant AA^*$. Hence $\mathscr{R}(E_\mu) \subset \mathscr{R}(A)$ for $\mu > 0$. This observation leads to a simple proof of a well-known theorem:

THEOREM 2.5. *Let V be a linear subspace of $\mathscr{H}$. These are equivalent:*

(1) *Any bounded operator A on $\mathscr{H}$ with $\mathscr{R}(A) \subset V$ is compact.*

(2) *V contains no closed infinite-dimensional subspace of $\mathscr{H}$.*

Proof. First, (1) implies (2) since a compact projection has finite rank. Now, assume (2), and let A be a bounded operator with $\mathscr{R}(A) \subset V$. Without loss of generality, we suppose that A is positive. The projections E_μ, $\mu > 0$, are compact by the assumption on V and the remark preceding Theorem 2.5. Since $\| AE_\mu - A \| \leqslant \mu$, it follows that A is also compact.

For a more elementary proof of the theorem, see [13].

The vector subspaces $C(0, 1)$ and $L^\infty(0, 1)$ of $L^2(0, 1)$ and the subspaces $\ell^p (p < 2)$ of ℓ^2 are examples of subspaces with the property indicated in Theorem 2.5. See [9], for example.

The following corollary is less well known:

COROLLARY. *Let A be a bounded operator. Then A is compact if and only if $\| Ax_n \| \to 0$ for every orthonormal sequence $\{x_n\}$ in $\mathscr{H}$.*

Proof. We may suppose that $A \geqslant 0$. If $\mathscr{M}$ is a closed subspace of $\mathscr{H}$ contained in $\mathscr{R}(A)$, then, by Theorem 2.1, there is a bounded operator C such that $P = AC = C^*A$, where P is the orthogonal projection of $\mathscr{H}$ onto $\mathscr{M}$. If $\{x_n\}$ is an orthonormal basis of $\mathscr{R}(P)$, then $x_n = Px_n = C^*Ax_n \to 0$. This means that $\mathscr{M}$ is finite dimensional. Hence, A is compact.

Remark. It follows easily from Theorem 2.5 and the Baire category theorem that an operator A is compact if and only if $\mathscr{R}(A)$ is a countable union of compact subsets of $\mathscr{H}$.

The next two results prove Mackey's theorem [18] that the lattice $\mathscr{L}$ of operator ranges is generated by the closed subspaces of $\mathscr{H}$. (See also [5, Théorèmes 5.1, 5.11].)

THEOREM 2.6. *Suppose that $\mathscr{H}$ is separable and let $\mathscr{R}$ be an operator range in $\mathscr{H}$ that contains a closed infinite-dimensional subspace of $\mathscr{H}$. Then there are disjoint closed subspaces $\mathscr{M}_1$, $\mathscr{M}_2$ such that $\mathscr{R} = \mathscr{M}_1 + \mathscr{M}_2$.*

Proof. Without loss of generality we may suppose that $\mathscr{R}$ is not closed. Let T be a closed linear transformation whose domain is $\mathscr{R}$, and let $\mathscr{N}$ be a closed infinite-dimensional subspace of $\mathscr{H}$ that is contained in $\mathscr{R}$. Let φ be an isomorphism from $(T\mathscr{R})^-$ onto $\mathscr{N}$. Then $\mathscr{N}' = \{x + \varphi Tx : x \in \mathscr{R} \cap \mathscr{N}^\perp\}$ is a closed subspace of $\mathscr{H}$. Also $\mathscr{R} = \mathscr{N} + \mathscr{N}'$. Therefore $\mathscr{M}_1 = \mathscr{N}'$ and $\mathscr{M}_2 = \mathscr{N} \ominus \mathscr{N} \cap \mathscr{N}'$ are disjoint closed subspaces whose sum is $\mathscr{R}$.

For a more transparent proof of the theorem, suppose that $\mathscr{R} = \mathscr{R}(A)$, where $A = \int_0^1 \lambda \, dE_\lambda$. Since $\mathscr{R}$ is not closed, we can choose $0 < a < 1$ such that the projections $E[0, a]$ and $E(a, 1]$ are both of infinite rank. If $S = \int_0^a \lambda \, dE_\lambda$, then $(2 - S^2)^{1/2}$ is invertible so that $B = S \oplus (2 - S^2)^{1/2}$ on $E[0, a]\mathscr{H} \oplus E(a, 1]\mathscr{H}$ has range $\mathscr{R}$. But $B^2 = S^2 \oplus (2 - S^2)$ is the sum of the projections

$$P = \begin{pmatrix} T & (T - T^2)^{1/2} \\ (T - T^2)^{1/2} & 1 - T \end{pmatrix}, \qquad Q = \begin{pmatrix} T & -(T - T^2)^{1/2} \\ -(T - T^2)^{1/2} & 1 - T \end{pmatrix},$$

where $T = \frac{1}{2}S^2$. Hence

$$\mathscr{R}(A) = \mathscr{R}(B) = \mathscr{R}((P + Q)^{1/2}) = \mathscr{R}(P) + \mathscr{R}(Q).$$

Remark. If $\mathscr{R}$ is not closed, then clearly $\mathscr{M}_1$ and $\mathscr{M}_2$ must both be infinite dimensional. Note also that the theorem is false if $\mathscr{H}$ is not separable. (For example, consider the operator $A = \sum_1^\infty 2^{-n} P_{\mathscr{K}_n}$, where dim $\mathscr{H}_n = c_n$ is a strictly increasing sequence of infinite cardinal numbers.)

For the following corollary, recall that bounded operators A and B are said to be *equivalent* if there are bounded invertible operators L and M such that $B = LAM$.

COROLLARY. *A positive operator A is equivalent to the sum of two projections of infinite rank if and only if A is not compact.*

Proof. If A is not compact, then, by Theorems 2.5 and 2.6, there are infinite rank projections P and Q such that $\mathscr{R}(A^{1/2}) = \mathscr{R}(P) + \mathscr{R}(Q)$. Hence, $A^{1/2}$ and $(P + Q)^{1/2}$ have the same range (Theorem 2.2). Therefore $A^{1/2} = (P + Q)^{1/2}C = C^*(P + Q)^{1/2}$ for some invertible operator C by Corollary 1 of Theorem 2.1. Hence $A = (A^{1/2})^*(A^{1/2}) = C^*(P + Q)C$ as required.

Conversely, if $A = S(P + Q)T$, where S and T are invertible and P and Q are projections of infinite rank, then $(P + Q)^{1/2}$ is not compact by Theorem 2.2 and 2.5. This implies that A is not compact.

Remark. The sum of two projections is characterized to within unitary equivalence in [3, 7, 10].

THEOREM 2.7. *Let $\mathscr{R}$ be an operator range that contains no closed infinite-dimensional subspace of $\mathscr{H}$. Then there are closed subspaces $\mathscr{M}_1, \ldots, \mathscr{M}_4$ such that $\mathscr{R} = (\mathscr{M}_1 + \mathscr{M}_2) \cap (\mathscr{M}_3 + \mathscr{M}_4)$.*

Proof. Let K be a positive compact operator whose range is $\mathscr{R}$, choose a closed subspace of infinite rank and deficiency that reduces K, and let $K = K_1 \oplus K_2$ be the corresponding decomposition of K. Then $\mathscr{R}(K) = \mathscr{R}(K_1 \oplus 1) \cap \mathscr{R}(1 \oplus K_2)$ is an intersection of two noncompact operator ranges. The proof is completed by an application of Theorem 2.6.

Theorems 2.6 and 2.7 show that every operator range is either the sum $\mathscr{M}_1 + \mathscr{M}_2$ of two disjoint closed subspaces or the intersection of two such sums. It is an easy exercise to verify that $\mathscr{M}_1 + \mathscr{M}_2$ is closed if and only if there is a constant $0 \leqslant c < 1$ such that $|(m_1, m_2)| \leqslant c\| m_1 \| \| m_2 \|$ for all $m_i \in \mathscr{M}_i$. This condition means that the angle between the subspaces is positive. An equivalent condition is that the idempotent $m_1 + m_2 \to m_1$ be bounded.

Theorem 2.5 provides a characterization of those operator ranges that are the range of a compact operator. Now if $\mathscr{R}$ is the range of a noncompact operator, then $\mathscr{R}$ contains a closed infinite-dimensional subspace $\mathscr{N}$. It may happen that any such subspace $\mathscr{N}$ is contained in another such subspace $\mathscr{N}'$ with $\dim(\mathscr{N}' \ominus \mathscr{N}) = \infty$. Ranges of this type are called *class* 2*a* ranges by Dixmier. On the other hand, a nonclosed range $\mathscr{R}$ is of *class* 2*b* if there is a closed infinite-dimensional subspace $\mathscr{N} \subset \mathscr{R}$ that is not itself of infinite deficiency in another closed subspace $\mathscr{N}' \subset \mathscr{R}$. Operator ranges are therefore of four types: closed subspaces, ranges of compact operators, and those of class 2a or 2b. The following theorem [5, Proposition 5.3] is an easy consequence of the definition and Theorem 2.5:

THEOREM 2.8. *An operator range $\mathscr{R}$ is of class 2b if and only if there exists a closed infinite-dimensional subspace $\mathscr{M}$ and a nonclosed compact operator range $\mathscr{R}_0 \subset \mathscr{M}^\perp$ such that $\mathscr{R} = \mathscr{M} + \mathscr{R}_0$.*

The next result (apparently new) characterizes those operators whose range is of class 2b. This in turn leads to a description of the range of a positive operator in terms of its spectrum.

Recall that a number λ is a *Weyl limit point* (see [21], for example) of the spectrum of a self-adjoint operator A if there is a sequence $\{x_n\}$ of unit vectors tending weakly to 0 such that $\|(A - \lambda)x_n\| \to 0$. If E is the spectral measure of A, then λ is a Weyl limit point if and only if $E(\sigma)$ has infinite rank for every neighborhood σ of λ. From this characterization it is easy to see that the Weyl limit points consist of accumulation points of $\sigma(A)$ together with those isolated points that are eigenvalues of infinite multiplicity.

THEOREM 2.9. *Let A be a positive operator with nonclosed range. Then $\mathscr{R}(A)$ is of class 2b if and only if $A = B \oplus K$, where K is compact and B is invertible with infinite rank.*

Proof. An operator of the asserted form clearly has range of class 2b by Theorem 2.8. Conversely, suppose $\mathscr{R}(A)$ is of class 2b. Then by Theorem 2.8 and Theorem 2.2 there is a projection P of infinite rank and a positive compact operator K such that $\mathscr{R}(A) = \mathscr{R}((P + K^2)^{1/2})$. Moreover, K can be chosen to have range contained in $\mathscr{R}(P)^\perp$, so that $PK = 0 = KP$. It follows from the uniqueness of the square root of a positive operator that $(P + K^2)^{1/2} = P + K$. Finally, Corollary 2 of Theorem 2.1 gives an invertible operator C such that $A = (P+K)C = C^*(P + K)$. The proof can now be completed by an appeal to Lemma 3.2. We shall continue with a direct proof however.

Choose a positive number δ such that $\| C^*x \| \geqslant \delta \| x \|$ for all $x \in \mathscr{H}$, and let λ be a positive Weyl limit point of the spectrum of A. Then there are unit vectors x_n tending weakly to 0 such that $\|(A - \lambda)x_n\| \to 0$. Therefore,

$$P^\perp A x_n = KCx_n \to 0,$$
$$(AP - \lambda)x_n = (A - \lambda)x_n - AP^\perp x_n = (A - \lambda)x_n - C^*Kx_n \to 0.$$

Hence, $\lambda P^\perp x_n = P^\perp A x_n - P^\perp(A - \lambda)x_n \to 0$. Since $\lambda \neq 0$, it follows that $\| Px_n \| \to 1$. Because $AP = C^*P$, this gives

$$\delta = \delta \lim_n \| Px_n \| \leqslant \lim_n \inf \| APx_n \| \leqslant \lim_n (\|(AP - \lambda)x_n\| + \| \lambda x_n \|) = \lambda.$$

Thus, there are no Weyl limit points of the spectrum of A in the open interval $(0, \delta)$. It follows easily that A is the direct sum of an invertible and a compact as required.

There is still another proof of Theorem 2.9 that is of interest because it is perhaps more natural than the one just given and because it will also be of use in proving Lemma 3.2. This goes as follows: Let $A = \int \lambda \, dE_\lambda$ be a positive operator whose range is of class 2b, and let $\mathscr{N}$ be a closed subspace contained in $\mathscr{R}(A)$ that is not of infinite deficiency in any other such closed subspace. There is a bounded operator D such that $P_{\mathscr{N}} = AD$. Choose $\delta > 0$ such that $\delta \| D \| < 1$, and fix a positive number $\epsilon \leqslant \delta$. If $x \in E[\epsilon, \delta]\mathscr{H}$ and $n \in \mathscr{N}$, then

$$|(x, n)| = |(x, P_{\mathscr{N}} n)| = |(D^*Ax, n)| \leqslant \delta \| D \| \| x \| \| n \|.$$

This implies that $E[\epsilon, \delta]\mathscr{H} + \mathscr{N}$ is closed and contained in the range of A (by the remark preceding Theorem 2.5). Therefore,

$$\dim E[\epsilon, \delta]\mathscr{H} = \dim((E[\epsilon, \delta]\mathscr{H} + \mathscr{N}) \ominus \mathscr{N}) < \infty$$

for $0 < \epsilon \leqslant \delta$. Hence, $K = \int_0^\delta \lambda \, dE_\lambda$ is compact. Finally, $B = \int_\delta^\infty \lambda \, dE_\lambda$ is clearly invertible and not of finite rank since A is not compact.

COROLLARY. *Let A be a positive operator with nonclosed range. Then*

(1) *$\mathscr{R}(A)$ contains no infinite-dimensional closed subspace if and only $\sigma(A)$ has no positive Weyl limit points.*

(2) *$\mathscr{R}(A)$ is of class 2a if and only if there is a sequence of positive Weyl limit points tending to 0.*

(3) *$\mathscr{R}(A)$ is of class 2b if and only if there are positive Weyl limit points, but these are bounded away from 0.*

It is clear from the foregoing that multiplication by the independent variable on $L^2(0, 1)$ defines an operator whose range is of class 2a. This fact is not easy to prove directly.

3. Equivalence of Operator Ranges

Operator ranges $\mathscr{R}$ and $\mathscr{S}$ in $\mathscr{H}$ are called *similar*, if there is an invertible operator $T \in \mathscr{B}(\mathscr{H})$ such that $\mathscr{S} = T\mathscr{R}$, and *unitarily equivalent*, if T can be taken to be unitary. These notions are identical, a result due to Dixmier [5].

Theorem 3.1. *Operator ranges are similar if and only if they are unitarily equivalent.*

Proof. Suppose $T\mathscr{R} = \mathscr{S}$ for operator ranges $\mathscr{R}$ and $\mathscr{S}$ and invertible T. Choose $A \in \mathscr{B}(\mathscr{H})$ with A self-adjoint and $\mathscr{R} = A\mathscr{H}$ (see the remark after 1.1). If $S = TA$, then $S\mathscr{H} = \mathscr{S}$ and $S^*\mathscr{H} = \mathscr{R}$. In the polar decomposition $S = (SS^*)^{1/2}U$, the partial isometry U has initial space $(S^*\mathscr{H})^-$ and final space $(S\mathscr{H})^-$; it is in fact true that $US^*\mathscr{H} = S\mathscr{H}$, i.e., $U\mathscr{R} = \mathscr{S}$. To complete the proof we need to know that $\mathscr{R}^\perp$ and $\mathscr{S}^\perp$ have the same dimension. But this follows from $T^*\mathscr{S}^\perp = \mathscr{R}^\perp$, and thus from the hypothesis $T\mathscr{R} = \mathscr{S}$.

As observed by Dixmier, Köthe's results on equivalence and congruence of operators [16, 17] can be interpreted to give a classification of operator ranges up to unitary equivalence. We begin our discussion of this with a lemma that contains the essence of one of Köthe's two key arguments, and that is of considerable interest in itself.

Lemma 3.2. *Let $A = \int_0^M \lambda \, dE_\lambda$ and $B = \int_0^N \lambda \, dF_\lambda$ be positive operators on $\mathscr{H}$ that have the same range. Then there is a constant $K \geqslant 1$ such that*

$$\dim E[\alpha, \beta]\mathscr{H} \leqslant \dim F[\alpha/K, K\beta]\mathscr{H}$$

and dually, whenever $0 < \alpha \leqslant \beta$.

Proof. Since A and B have the same range, Theorem 2.1 implies that there is an invertible operator C such that $A = BC$. Choose a constant $K > \sqrt{2} \max\{\| C \|, \| C^{-1} \|\}$. If $x \in E[\alpha, \infty]\mathscr{H}$ and $y \in F[0, \alpha/K]\mathscr{H}$, then, as in the proof of Theorem 2.9, $|(x, y)| \leqslant \| x \| \| y \| \| C \|/K$. Similarly, if

$x \in F[\beta K, \infty)\mathscr{H}$, $y \in E[0, \beta]\mathscr{H}$, then $|(x, y)| \leqslant \| x \| \| y \| \| C^{-1} \|/K$. Hence, if $x \in E[\alpha, \beta]\mathscr{H}$ and if

$$y = y_1 + y_2 \in F[0, \alpha/K)\mathscr{H} + F(K\beta, \infty)\mathscr{H} = (F[\alpha/K, \beta K]\mathscr{H})^{\perp},$$

then

$$|(x, y)| \leqslant |(x, y_1)| + |(x, y_2)| \leqslant \| x \| \| y \| \sqrt{2} \max\{\| C \|, \| C^{-1} \|\}/K.$$

It follows that the sum $E[\alpha, \beta]\mathscr{H} + (F[\alpha/K, \beta K]\mathscr{H})^{\perp}$ is direct and closed. The assertion of the lemma is an immediate consequence of this fact.

Remark. By means of a more intricate argument, it can be shown that if $K \geqslant \max\{\| C \|, \| C^{-1} \|\}$, then for $0 < \alpha \leqslant \beta$ the subspaces $E[\alpha, \beta]\mathscr{H}$ and $(F[\alpha/K, K\beta]\mathscr{H})^{\perp}$ intersect only at the zero vector. This is enough to prove Lemma 3.2.

The foregoing argument shows that by choosing a larger value of K one can also insure that the sum of these subspaces is closed.

The second of Köthe's arguments will be replaced by an application of the following well-known theorem of P. Hall [22]: if $\{S(e) \mid e \in \mathscr{E}\}$ is a system of finite sets with the property that for any positive integer p and any choice of elements $e_1, e_2, \ldots, e_p \in \mathscr{E}$, the union of the sets $S(e_1), S(e_2), \ldots, S(e_p)$ contains at least p elements, then there exists a one-to-one mapping φ such that $\varphi(e) \in S(e)$ for all $e \in \mathscr{E}$.

THEOREM 3.3. *Let $\mathscr{R}$ and $\mathscr{S}$ be operator ranges, determined as in* Theorem 1.1 *by systems $\{\mathscr{H}_n \mid n \geqslant 0\}$ and $\{\mathscr{K}_n \mid n \geqslant 0\}$ of closed pairwise orthogonal subspaces. Then $\mathscr{R}$ and $\mathscr{S}$ are unitarily equivalent if and only if*

(1) *the subspaces $(\Sigma \oplus \mathscr{H}_n)^{\perp}$ and $(\Sigma \oplus \mathscr{K}_n)^{\perp}$ have the same dimension, and*

(2) *there is an integer $k \geqslant 0$ such that for all integers $n \geqslant 0$ and $\ell \geqslant 0$*

$$\dim(\mathscr{H}_n \oplus \cdots \oplus \mathscr{H}_{n+l}) \leqslant \dim(\mathscr{K}_{n-k} \oplus \cdots \oplus \mathscr{K}_{n+l+k})$$

and dually. (*Here $\mathscr{H}_i = \mathscr{K}_i = 0$ for $i < 0$.*)

Proof. Assume that $\mathscr{R}$ and $\mathscr{S}$ are unitarily equivalent. Then so are $\mathscr{R}^{-}$ and $\mathscr{S}^{-}$, and since these spaces are respectively $\Sigma \oplus \mathscr{H}_n$ and $\Sigma \oplus \mathscr{K}_n$, (1) follows. In proving (2), there is no loss of generality in assuming that $\mathscr{R}$ and $\mathscr{S}$ are equal. If $A = \Sigma 2^{-n} P_n$ and $B = \Sigma 2^{-n} Q_n$, where P_n is the projection on $\mathscr{H}_n$ and Q_n is the projection on $\mathscr{K}_n$, then

A and B have the same range. If K is the positive constant provided by Lemma 3.2 and k is any positive integer with $2^k \geqslant K$, then (2) follows from the conclusion of that lemma.

Now assume that (1) and (2) are satisfied. Let $\mathscr{E}_n$ be an orthonormal basis for $\mathscr{H}_n$, $\mathscr{E} = \bigcup \mathscr{E}_n$, $\mathscr{F}_n$ an orthonormal basis for $\mathscr{K}_n$, and $\mathscr{F} = \bigcup \mathscr{F}_n$. To produce a unitary operator U with $U\mathscr{R} = \mathscr{S}$ it will clearly suffice to construct a bijection $\pi : \mathscr{E} \to \mathscr{F}$ with the property:

$$\exists k \text{ such that } e \in \mathscr{E}_i \text{ and } \pi(e) \in \mathscr{F}_j \text{ implies } | i - j | \leqslant k. \tag{*}$$

For this it will be sufficient in turn to construct injections $\varphi : \mathscr{E} \to \mathscr{F}$ and $\psi : \mathscr{F} \to \mathscr{E}$ that satisfy (*), for then, using the method of the usual proof of the Cantor–Bernstein Theorem, there will be a bijection $\pi : \mathscr{E} \to \mathscr{F}$, equal to φ on a part of $\mathscr{E}$ and ψ^{-1} on the rest, and therefore satisfying (*).

To construct φ, we need a slight extension of (2), namely, for any positive integer p and choice of indices $n_1 < n_2 < \cdots < n_p$,

$$\dim \sum_{i=1}^{p} \oplus \mathscr{H}_{n_i} \leqslant \dim \sum_{i=1}^{p} (\mathscr{K}_{n_i - k} \oplus \cdots \oplus \mathscr{K}_{n_i + k}). \tag{2'}$$

To prove this, note first that if the differences $n_2 - n_1, \ldots, n_p - n_{p-1}$ are all at most $2k$, we have

$$\begin{aligned}\dim \sum_{i=1}^{p} (\mathscr{K}_{n_i - k} \oplus \cdots \oplus \mathscr{K}_{n_i + k}) &= \dim(\mathscr{K}_{n_1 - k} \oplus \cdots \oplus \mathscr{K}_{n_p + k}) \\ &\geqslant \dim(\mathscr{H}_{n_1} \oplus \cdots \oplus \mathscr{H}_{n_p}) \\ &\geqslant \dim \sum_{i=1}^{p} \oplus \mathscr{H}_{n_i}\end{aligned}$$

by (2). Next, observe that if $n_q - n_{q-1} \geqslant 2k + 1$ for some $q \leqslant p$, then the corresponding pieces

$$\sum_{i<q} (\mathscr{K}_{n_i - k} \oplus \cdots \oplus \mathscr{K}_{n_i + k}) \qquad \text{and} \qquad \sum_{i \geqslant q} (\mathscr{K}_{n_i - k} \oplus \cdots \oplus \mathscr{K}_{n_i + k})$$

of the right hand side of (2′) are orthogonal. The general case therefore follows by adding the inequalities (2′) for each of the maximal segments of $n_1, \ldots, n_p$ with consecutive differences all at most $2k$.

We now wish to assign to each $e \in \mathscr{E}_n$, $n \geqslant 0$, a finite subset $S(e)$ of $\mathscr{F}_{n-k} \cup \cdots \cup \mathscr{F}_{n+k}$, in such a way that the hypotheses of Hall's Theorem are satisfied, for then the injection φ provided by that theorem will satisfy (*), and the proof will be complete.

(i) If $\mathscr{F}_{n-k}, \ldots, \mathscr{F}_{n+k}$ are all finite, put $S(e) = \mathscr{F}_{n-k} \cup \cdots \cup \mathscr{F}_{n+k}$ for all $e \in \mathscr{E}_n$.

(ii) If $\mathscr{F}_{n-k}, \ldots, \mathscr{F}_{n+k}$ are not all finite, one of them, say $\mathscr{F}_q$, is infinite and satisfies card $\mathscr{F}_q \geqslant$ card $\mathscr{E}_n$ (by (2) with $\ell = 0$); assign to the elements e of $\mathscr{E}_n$ distinct one-element subsets of $\mathscr{F}_q$, taking care not to reuse elements of $\mathscr{F}_q$ that have been used at any prior stage, and to leave a remainder of the same cardinality as $\mathscr{F}_q$.

It is obvious that this system satisfies the hypotheses of Hall's Theorem: for choices $e_1, \ldots, e_p$ all of type (i) by (2′), for choices $e_1, \ldots, e_p$ all of type (ii) by construction, and hence for arbitrary choices, since $S(e)$ and $S(e')$ are disjoint whenever e is of type (i) and e' of type (ii).

We remark that the weights $\{2^n\}$ employed in this theorem can be replaced by any geometric sequence $\{r^n\}$ with $r > 1$; thus two operator ranges, defined by orthogonal systems of subspaces using the same value of r, are unitarily equivalent if and only if (2) holds. Weights other than geometric sequences can be used, but then (2) must be modified in accordance with Lemma 3.2.

Next we present several results about equivalence and congruence. Bounded operators S and T are *equivalent* if there exist invertible operators L and M such that $T = LSM$, and *congruent* if there exists an invertible operator M such that $T = M^*SM$.

THEOREM 3.4. *Bounded operators S and T are equivalent if and only if they have the same nullity and unitarily equivalent (or similar) ranges.*

Proof. Suppose $T = LSM$ with L and M invertible. Then

$$\mathscr{N}(T) = \mathscr{N}(LSM) = \mathscr{N}(SM) = M^{-1}\mathscr{N}(S)$$

so that S and T have the same nullity. Also

$$T\mathscr{H} = LSM\mathscr{H} = L(S\mathscr{H})$$

and thus $\mathscr{R}(S)$ and $\mathscr{R}(T)$ are similar, hence unitarily equivalent by Theorem 3.1. Now suppose that S and T have the same nullity, and that $U\mathscr{R}(S) = \mathscr{R}(T)$ for some unitary U. Then US and T have the same range and nullity, and therefore $T = USM$ for some invertible operator M by Corollary 1 of Theorem 2.1.

Since $\mathcal{N}(N)$ is the orthogonal complement of $\mathcal{R}(N)$ when N is normal, we deduce:

COROLLARY. *Normal operators are equivalent if and only if their ranges are unitarily equivalent.*

Very little seems to be known about congruence, except in the case of self-adjoint operators, and there it is closely connected with equivalence. In fact:

THEOREM 3.5. *Let A and B be positive bounded operators. Then the following conditions are equivalent*:

(1) *A and B are equivalent.*

(2) *$A^{1/2}$ and $B^{1/2}$ are equivalent.*

(3) *A and B are congruent.*

Proof. If $A^{1/2}$ and $B^{1/2}$ are equivalent, then $A^{1/2} = UB^{1/2}M$ with U unitary and M invertible (by the proof of the preceding theorem.) Hence

$$A = (UB^{1/2}M)^*(UB^{1/2}M) = M^*BM.$$

Thus (2) implies (3). That (3) implies (1) is trivial.

The Corollary of Theorem 3.4 implies that positive operators A and B are equivalent if and only if there is a unitary operator U such that A and UBU^{-1} have the same range. Therefore, to prove that (1) implies (2) it will suffice to show that if $\mathcal{R}(A) = \mathcal{R}(B)$, then $\mathcal{R}(A^{1/2}) = \mathcal{R}(B^{1/2})$. This implication is an immediate consequence of condition (2) of Theorem 2.1 and monotonicity of the square root operation [14].

COROLLARY. *The ranges of positive operators A and B are unitarily equivalent if and only if the ranges of $A^{1/2}$ and $B^{1/2}$ are unitarily equivalent.*

We note that the condition that $A^{1/2}$ and $B^{1/2}$ be congruent could be added to the theorem. Also, the argument from (2) to (3) shows that for any equivalent operators S and T, S^*S and T^*T are congruent. The converse is true and follows from the polar decomposition, provided that S and T are one-to-one with dense range [15], or more generally, that the partial isometries in their polar decompositions can be taken to be unitary. Finally, the corollary of Theorem 3.5 is not valid with unitary equivalence replaced by equality. (See the example following Theorem 4.2.)

We conclude this section with a result of von Neumann [20] asserting that for any unbounded self-adjoint operator A, there is a unitary operator U with the property that the domains of A and U^*AU have only the zero vector in common. When stated in terms of operator ranges, this becomes:

THEOREM 3.6. *If $\mathscr{R}$ is a nonclosed operator range in a separable Hilbert space $\mathscr{H}$, there is a unitary operator U on $\mathscr{H}$ such that $\mathscr{R} \cap U\mathscr{R} = \{0\}$.*

Proof. The proof given by Dixmier [5] rests on the following statements:

(a) $\mathscr{S} \cap V\mathscr{S} = \{0\}$ for some unitary V and dense operator range $\mathscr{S}$ that contains a closed infinite-dimensional subspace.

(b) If $\mathscr{S}$ is as in (a) and $\mathscr{R}$ is any nonclosed range, then $W\mathscr{R} \subset \mathscr{S}$ for some unitary W.

Of course, then $U = W^{-1}VW$ will meet the requirements of the theorem. To prove (a), let $\{e_n\}$ be the usual exponential basis of $L^2[0, 2\pi]$, and let A be an operator that is diagonal in this basis: $Ae_n = a_n e_n$ for all n. If the eigenvalues $\{a_n\}$ decrease sufficiently rapidly, say $a_n = \exp(-n^2)$, then the range $\mathscr{S}$ of A consists of restrictions of entire functions to the interval $[0, 2\pi]$ on the real axis. If V is the unitary operator given by multiplication by the function φ equal to 1 in $[0, \pi)$ and -1 in $[\pi, 2\pi]$, it follows that $\mathscr{S} \cap V\mathscr{S} = \{0\}$. Now $\mathscr{S}$ does not contain an infinite-dimensional closed subspace (since A is compact), but this is remedied by forming a countably infinite direct sum of copies of A.

In the proof of (b) we will say that an operator range is of *type* J_S (Dixmier's notation) if it is dense and is determined, as in Theorem 1.1, by an orthogonal sequence of *infinite-dimensional* closed subspaces. Since any two such operator ranges are unitarily equivalent, (b) will be proved if it is shown (i) that any nonclosed operator range $\mathscr{R}$ is contained in a range of type J_S, and (ii) that any dense operator range $\mathscr{S}$ that contains an infinite-dimensional closed subspace contains a range of type J_S.

Let $\mathscr{R}$ be determined by the sequence $\{\mathscr{H}_n\}$. Because $\mathscr{R}$ is nonclosed, $\mathscr{H}_n \neq \{0\}$ for infinitely many n. For each $n \geqslant 0$, let $\mathscr{H}_n'$ be the span of infinitely many nonzero $\mathscr{H}_i$ with $i \geqslant n$; this is to be accomplished so as to exhaust the nonzero $\mathscr{H}_i$ without using any more than once. If $(\Sigma \oplus \mathscr{H}_i)^\perp$ is incorporated with (say) $\mathscr{H}_0'$, then the range $\mathscr{R}'$ determined by $\{\mathscr{H}_n'\}$ is of type J_S and contains $\mathscr{R}$.

Let $\mathscr{S}$ be determined by $\{\mathscr{K}_n\}$, and note that $\mathscr{K}_n$ is infinite dimensional for some n, say $n = p$. (Otherwise, the corresponding diagonal operator $D = \Sigma 2^{-n} Q_n$, where Q_n is the projection on $\mathscr{K}_n$, would be compact, and $\mathscr{S}$ could contain no infinite-dimensional closed subspace, by Theorem 2.5.) Let $\mathscr{K}_p = \Sigma \oplus \mathscr{L}_i$ with each $\mathscr{L}_i$ infinite dimensional, and put $\mathscr{K}_i' = \mathscr{K}_i \oplus \mathscr{L}_i$ for $i \neq p$ and $\mathscr{K}_p' = \mathscr{L}_p$. Then the range $\mathscr{S}'$ determined by $\{\mathscr{K}_n'\}$ is of type J_S and is contained in $\mathscr{S}$.

COROLLARY 1. *If $\mathscr{R}$ is a nonclosed operator range in a separable Hilbert space, then there is a continuous unitary group $\{U_t\}_{-\infty<t<\infty}$ such that $U_s\mathscr{R} \cap U_t\mathscr{R} = 0$ for $s \neq t$.*

Proof. This result appears without proof in [5]. To prove it, it suffices to modify the argument used to prove Theorem 3.6 as follows: Let $W\mathscr{R} \subset \mathscr{S}$, and let $U_t = W^{-1}V_tW$, where V_t is the unitary operator given by multiplication by the function φ_t equal to e^{itx} in $[0, \pi)$ and e^{-itx} in $[\pi, 2\pi]$. To prove that $V_s\mathscr{S} \cap V_t\mathscr{S} = 0$ for $s \neq t$, it suffices to observe that if $V_t\mathscr{S} \cap \mathscr{S} \neq 0$, then there is a nonzero entire function g such that $e^{2itx}g(x) = g(x)$ for all $x \in [0, 2\pi]$. This implies that $e^{2itx} = 1$ for all x in some interval and hence that $t = 0$.

COROLLARY 2. *For any nonclosed operator range $\mathscr{R}$ in a separable space $\mathscr{H}$, the algebraic dimension of $\mathscr{H}/\mathscr{R}$ is the power of the continuum.*

As another application of Theorem 3.6 we mention the following [5, Proposition 9.7]:

COROLLARY 3. *Let T be a bounded operator on a separable space $\mathscr{H}$, and suppose that T has proper dense range. There exists an orthonormal basis $\{e_n\}$ of $\mathscr{H}$ such that the vectors $Te_n, Te_{n+1}, \ldots$ span a dense linear subspace for each $n \geqslant 1$.*

Proof. Let $\mathscr{S}$ be a dense operator range that is disjoint from $\mathscr{R}(T^*)$, and let $\{e_k\}$ be an orthonormal basis of $\mathscr{H}$ that is contained in $\mathscr{S}$. If x is orthogonal to Te_k for $k \geqslant n$, then $T^*x \in \mathscr{S} \cap \mathscr{R}(T^*) = \{0\}$; hence $x \in \mathscr{N}(T^*) = \mathscr{R}(T)^\perp = \{0\}$.

Remark. The proof of Theorem 1.1 shows that if A is a positive operator whose spectrum contains an interval of the form $(0, \epsilon)$, then $\mathscr{R}(A)$ is of type J_S. This fact, the polar decomposition, and the Corollary of Theorem 3.4 show that *if S and T are one-to-one with dense ranges, and if the spectra of SS^* and TT^* contain an interval* $(0, \epsilon)$, *then S and T are equivalent* [5, Proposition 10.1].

4. Further Results

In this final section, we shall discuss three miscellaneous topics. The first of these is a characterization of the range of a contraction T by means of the defect operator $D_* = (1 - TT^*)^{1/2}$. This result is implicit in the work of de Branges and Rovnyak [4, p. 23]. It seems to have been first proven by R. G. Douglas by an ingenious use of the spectral theorem. We shall give an elementary proof of this result.

The second and major topic of discussion in this section is concerned with parallel sums of positive operators. Here, we use the techniques of Section 2 to prove extensions of a number of results previously known only for matrices [1].

Finally, we shall make a few comments about orthonormal bases in operator ranges.

Theorem 4.1. *Let T be a contraction on $\mathscr{H}$ and let $D_* = (1 - TT^*)^{1/2}$. Then $z \in \mathscr{R}(T)$ if and only if* $\sup_y(\| z + D_* y \|^2 - \| y \|^2) < \infty$.

Proof. Let $D = (1 - T^*T)^{1/2}$. Then $TD = D_*T$. The necessity is proven by the following computation:

$$\begin{aligned}\| Tx + D_* y \|^2 - \| y \|^2 &= \| Tx \|^2 + 2 \operatorname{Re}(Tx, D_* y) + \| D_* y \|^2 - \| y \|^2 \\ &= \| Tx \|^2 + 2 \operatorname{Re}(Tx, D_* y) + \| y \|^2 - \| T^* y \|^2 - \| y \|^2 \\ &= \| x \|^2 - \| Dx \|^2 + 2 \operatorname{Re}(Dx, T^* y) - \| T^* y \|^2 \\ &= \| x \|^2 - \| Dx - T^* y \|^2 \leqslant \| x \|^2.\end{aligned}$$

Thus, the supremum in question is $\leqslant \| x \|^2$ and equality holds if and only if $Dx \in \mathscr{R}(T^*)^-$. (This condition is easily seen to be equivalent to $x \in \mathscr{N}(T)^\perp$. Thus the square root of the supremum is exactly the minimum of the norms of the preimages of z under T.)

Conversely, suppose that $\| z + D_* y \|^2 \leqslant c^2 + \| y \|^2$ for all $y \in \mathscr{H}$. Expanding the inner product we get

$$2 \operatorname{Re}(z, D_* y) \leqslant c_0^2 + \| T^* y \|^2,$$

where $c_0^2 = c^2 - \| z \|^2$ is a nonnegative constant. Now, replace y by $te^{i\theta}y$ where t is real and θ is a suitably chosen real number. By minimizing the resulting quadratic in t we are lead to the condition

$$|(z, D_* y)| \leqslant c_0 \| T^* y \| \qquad (y \in \mathscr{H}).$$

This shows that the map $T^*y \to (D_* y, z)$ is a bounded linear functional; therefore by the Riesz representation theorem there is a vector x in $\mathscr{H}$ such that $(D_* y, z) = (T^* y, x)$ for all $y \in \mathscr{H}$. This implies $Tx = D_* z$. Hence $z = (1 - TT^*)z + TT^*z = D_* Tx + TT^*z = TDx + TT^*z$. Thus, z belongs to the range of T as asserted.

Crimmins' formula (Theorem 2.2) identifies a canonical operator whose range is the sum of given ranges. Recently, Anderson and Duffin [1] found a less explicit but analogous formula for the intersection of operator ranges when the underlying Hilbert space is finite dimensional. We proceed to describe their result and indicate its infinite-dimensional generalization.

Suppose temporarily then that $\mathscr{H}$ is finite-dimensional. To each $A \in \mathscr{B}(\mathscr{H})$ we can associate a uniquely determined inverse transformation A^+ from $\mathscr{R}(A)$ onto $\mathscr{N}(A)^\perp$. If A and B are positive operators, then $\mathscr{R}(A + B)$ includes the ranges of A and B so that the composition $A(A + B)^+B$ makes sense. This operator is denoted by $A : B$ and called the *parallel sum* of A and B. (The definition is motivated by consideration of how parallel resistances add in electrical circuits.) Anderson and Duffin show that the set of positive matrices is a commutative semigroup under parallel addition. They also show that $\mathscr{R}(A : B) = \mathscr{R}(A) \cap \mathscr{R}(B)$.

If $\mathscr{H}$ is infinite dimensional, the definition of A^+ is still meaningful and defines a linear transformation in $\mathscr{H}$ that, in general, cannot be extended to a bounded operator on $\mathscr{H}$. However, the formula defining $A : B$ no longer makes sense, and so the parallel sum must be redefined.

Now if A and B are positive operators on $\mathscr{H}$, then

$$\mathscr{R}(A^{1/2}) + \mathscr{R}(B^{1/2}) = \mathscr{R}((A + B)^{1/2}).$$

Therefore, by Theorem 2.1, there are uniquely determined operators C, D on $\mathscr{H}$ such that

$$A^{1/2} = (A + B)^{1/2}C, \qquad \mathscr{N}(C^*) \supset \mathscr{N}((A + B)^{1/2})$$

$$B^{1/2} = (A + B)^{1/2}D, \qquad \mathscr{N}(D^*) \supset \mathscr{N}((A + B)^{1/2}).$$

In fact, one has $C = (\sqrt{A+B})^{+}\sqrt{A}$ and $D = (\sqrt{A+B})^{+}\sqrt{B}$. We define the *parallel sum* of A and B to be the bounded operator $A : B = A^{1/2}C^{*}DB^{1/2}$.

We shall show that the new definition of $A : B$ produces a positive operator. Before doing this, however, we show that $A : B = A(A+B)^{+}B$ *whenever* $X = (A+B)^{+}B$ *is everywhere defined, that is, whenever* $\mathscr{R}(B) \subset \mathscr{R}(A+B)$. Indeed, X must then be bounded and

$$X^{*}(A+B) = B = B^{1/2}B^{1/2} = B^{1/2}D^{*}(A+B)^{1/2}.$$

Therefore, $X^{*}(A+B)^{1/2} = B^{1/2}D^{*}$, since both vanish on $\mathscr{R}((A+B)^{1/2})^{\perp}$. Hence, $A : B = A^{1/2}C^{*}DB^{1/2} = A^{1/2}C^{*}(A+B)^{1/2}X = AX$ as asserted. In particular, this formula for $A : B$ is valid when $A + B$ has closed range, for then $\mathscr{R}(B) \subset \mathscr{R}(B^{1/2}) \subset \mathscr{R}((A+B)^{1/2}) = \mathscr{R}(A+B)$.

THEOREM 4.2. *Suppose that A and B are bounded positive operators on $\mathscr{H}$. Then*

(1) *$A : B$ is bounded and positive.*

(2) $A : B = B : A$.

(3) $\mathscr{R}((A : B)^{1/2}) = \mathscr{R}(A^{1/2}) \cap \mathscr{R}(B^{1/2})$.

(4) $\mathscr{R}(A) \cap \mathscr{R}(B) \subset \mathscr{R}(A : B)$.

Proof. As in the proof of Corollary 2 of Theorem 2.2, let T be the operator on $\mathscr{H} \oplus \mathscr{H}$ with matrix $\begin{pmatrix} A^{\frac{1}{2}} & -B^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}$, and let $P = \begin{pmatrix} X & Z^{*} \\ Z & Y \end{pmatrix}$ be the projection onto $\mathscr{N}(T)$. Then

$$A^{1/2}X = B^{1/2}Z, \qquad A^{1/2}Z^{*} = B^{1/2}Y,$$
$$\mathscr{R}(A^{1/2}) \cap \mathscr{R}(B^{1/2}) = \mathscr{R}((A^{1/2}XA^{1/2})^{1/2}).$$

Next, let $V = \begin{pmatrix} C_0 & -D_0 \\ E & F \end{pmatrix}$ be the partial isometry satisfying the conditions

$$T = (TT^{*})^{1/2}V, \qquad \mathscr{N}(V^{*}) = \mathscr{N}(T^{*}) = \mathscr{N}(A+B)^{1/2} \oplus \mathscr{H}.$$

A simple calculation shows that $E = F = 0$ and that

$$\mathscr{N}(C_0^{*}) \cap \mathscr{N}(D_0^{*}) = \mathscr{N}(A+B)^{1/2}.$$

These relations and the uniqueness of the operators C, D introduced above imply that $C_0 = C$, $D_0 = D$. Therefore $V = \begin{pmatrix} C & -D \\ 0 & 0 \end{pmatrix}$.

Now V^*V is the projection onto $\mathscr{N}(T)^\perp$; hence $1 - V^*V = P$. By comparing the entries of these two matrices one gets

$$X = 1 - C^*C, \qquad Z = D^*C, \qquad Y = 1 - D^*D.$$

Therefore,

$$\begin{aligned} A : B &= A^{1/2}C^*DB^{1/2} = A^{1/2}Z^*B^{1/2} \\ &= A^{1/2}XA^{1/2} = B^{1/2}ZA^{1/2} = B^{1/2}D^*CA^{1/2} = B : A. \end{aligned}$$

These last equations complete the proof of (1), (2), and (3). To prove (4), we use the fact that VV^* is the projection onto $\mathscr{N}(V^*)^\perp = \mathscr{N}(T^*)^\perp$. This implies that CC^*+DD^* is the projection of $\mathscr{H}$ onto $\mathscr{N}((A+B)^{1/2})^\perp$. Therefore,

$$A^{1/2}C^* + B^{1/2}D^* = (A + B)^{1/2}(CC^* + DD^*) = (A + B)^{1/2}.$$

Now, suppose $z = Ax = By \in \mathscr{R}(A) \cap \mathscr{R}(B)$. Then,

$$(A + B)^{1/2}CA^{1/2}x = z = (A + B)^{1/2}DB^{1/2}y;$$

hence, $CA^{1/2}x = DB^{1/2}y$. Therefore,

$$\begin{aligned} z = A^{1/2}A^{1/2}x &= (A + B)^{1/2}CA^{1/2}x = (A^{1/2}C^* + B^{1/2}D^*)CA^{1/2}x \\ &= A^{1/2}C^*DB^{1/2}y + B^{1/2}D^*CA^{1/2}x = (A : B)(x + y) \end{aligned}$$

and this completes the proof of (4).

Remark. As we have previously noted, a positive operator and its square root have the same range when either range is closed. Hence, in the finite-dimensional case, assertion (3) of Theorem 4.2 is true without square roots. In general, however, the inclusion in (4) is proper as was pointed out to us by W. N. Anderson. His remark suggested the following example:

Example. There exist positive operators A and B with dense ranges such that $\mathscr{R}(A^{1/2}) = \mathscr{R}(B^{1/2})$ and $\mathscr{R}(A) \cap \mathscr{R}(B) = 0$.

To see this, let A_1 and A_2 be positive operators with disjoint dense ranges (Theorem 3.6). Then, $A = A_1 + A_2$ and $B = 2A_1 + A_2$ have disjoint dense ranges and $A \leqslant B \leqslant 2A$ so that $\mathscr{R}(A^{1/2}) = \mathscr{R}(B^{1/2})$ by Theorem 2.1.

The next two results were also obtained by Anderson and Duffin in the finite-dimensional case.

THEOREM 4.3. *If P and Q are projections, then $2(P:Q) = P \wedge Q$ is the projection onto $\mathscr{R}(P) \cap \mathscr{R}(Q)$.*

Proof. Since $(P:Q)^{1/2}$ has closed range by Theorem 4.2, it follows that $\mathscr{R}(P:Q) = \mathscr{R}(P) \cap \mathscr{R}(Q) = \mathscr{R}(P \wedge Q)$. Also, if $z = Px = Qy \in \mathscr{R}(P) \cap \mathscr{R}(Q)$, then, as in the proof of (4) of Theorem 4,

$$\begin{aligned} z = Px &= P^{1/2}C^*DQ^{1/2}y + Q^{1/2}D^*CP^{1/2}x \\ &= P^{1/2}C^*DQ^{1/2}z + Q^{1/2}D^*CP^{1/2}z = 2(P:Q)z. \end{aligned}$$

This shows that $2(P:Q)$ and $P \wedge Q$ agree on their common range. Since both operators are self-adjoint, they are identical.

Remark. It is easy to exhibit rank one projections P_n, P such that $\| P_n - P \| \to 0$ and $P_n \wedge P = 0$. Theorem 4.3 therefore implies that the map $A \to A : B$ is not continuous in the weak, strong, or uniform operator topologies.

THEOREM 4.4. *If $0 \leqslant A_1 \leqslant A_2$, then $A_1 : B \leqslant A_2 : B$ for all positive operators B.*

Proof. By Theorem 2.1 there is a contraction E such that

$$(A_1 + B)^{1/2} = (A_2 + B)^{1/2}E \qquad \text{and} \qquad \mathscr{N}(E^*) \supset \mathscr{N}((A+B)^{1/2}).$$

Let D_1 and D_2 satisfy $B^{1/2} = (A_1 + B)^{1/2}D_1 = (A_2 + B)^{1/2}D_2$ as in the definition of the parallel sums $A_1 : B$, $A_2 : B$. The proof of Theorem 4.2 shows that $A_1 : B = B^{1/2}(1 - D_1{}^*D_1)B^{1/2}$ and similarly for $A_2 : B$. Hence it suffices to show that $D_1{}^*D_1 \geqslant D_2{}^*D_2$.

Now $(A_2 + B)^{1/2}D_2 = (A_1 + B)^{1/2}D_1 = (A_2 + B)^{1/2}ED_1$. Taking adjoints one sees that $D_1{}^*E^* - D_2{}^*$ vanishes on the range of $(A_2 + B)^{1/2}$. Since this difference is trivially 0 on the orthogonal complement, we conclude that $D_2 = D_1{}^*E^*$. Hence, $D_2{}^*D_2 = D_1{}^*E^*ED_1 \leqslant D_1{}^*D_1$ as required.

COROLLARY. *If A and B are nonzero positive operators, then $\| A : B \| \leqslant \| A \| : \| B \| = (\| A \|^{-1} + \| B \|^{-1})^{-1}$. Equality holds if A and B are projections.*

If A_1, A_2, A_3 are positive operators, then Theorem 4.2 shows that the square roots of the parallel sums $(A_1 : A_2) : A_3$ and $A_1 : (A_2 : A_3)$ have the same range. From this it is not difficult to see that *parallel* *summation is an associative operation on the set of positive operators with closed range*. Associativity of the operation in general however seems difficult to resolve without a more useful definition of $A : B$. In what sense is $A : B$ the "minimal" positive operator whose square root has range equal to the intersection of the ranges of $A^{1/2}$ and $B^{1/2}$?

Finally we mention a few facts about orthonormal bases in operator ranges. To begin with, note that (by the usual proof for complete spaces) any two maximal orthonormal sets in an inner-product space have the same cardinality. Thus the notion of dimension is well defined for arbitrary linear manifolds in a Hilbert space, and, in particular, for operator ranges.

Let $\mathscr{R}$ be a linear subspace in a Hilbert space; we ask whether $\dim \mathscr{R} = \dim \mathscr{R}^-$. That this is true when $\mathscr{R}$ is an operator range is clear from (5) of Theorem 1.1. In general, it is false, as the following example (due to P. R. Halmos) shows. Let $\mathscr{K}$ and $\mathscr{L}$ be Hilbert spaces with $\dim \mathscr{K} = \aleph_0$ and $\dim \mathscr{L} = c$ (the power of the continuum), let $\mathscr{E}$ and $\mathscr{F}$ be orthonormal bases of $\mathscr{K}$ and $\mathscr{L}$, respectively, and let $\mathscr{B}$ be a Hamel (algebraic) basis of $\mathscr{K}$ with $\mathscr{B} \supset \mathscr{E}$. Since card $(\mathscr{B}\backslash\mathscr{E}) = c$, there is a linear transformation $T : \mathscr{K} \to \mathscr{L}$ that vanishes on $\mathscr{E}$ and is a bijection of $\mathscr{B}\backslash\mathscr{E}$ on $\mathscr{F}$. Then the orthonormal set

$$\{\langle e, 0\rangle : e \in \mathscr{E}\}$$

is maximal in the graph $\mathscr{G}(T)$ of T, and

$$\mathscr{G}(T)^- = \mathscr{K} \oplus \mathscr{L}.$$

Hence $\dim \mathscr{G}(T) = \aleph_0$ and $\dim \mathscr{G}(T)^- = c$.

These facts may be reformulated as follows: any operator range $\mathscr{R}$ contains an orthonormal basis for $\mathscr{R}^-$, but this fails for arbitrary linear subspaces. Or again, for an infinite-dimensional operator range $\mathscr{R}$, the minimum cardinality of a dense subset of $\mathscr{R}$ is $\dim \mathscr{R}$, but this is false for arbitrary linear subspaces.

Even though an operator range $\mathscr{R}$ and its closure have the same dimension, it is not true that *every* maximal orthonormal subset of $\mathscr{R}$ is also maximal in $\mathscr{R}^-$. In fact, if $\mathscr{D}$ and $\mathscr{D}'$ are dense linear manifolds such that $\mathscr{D} \subset \mathscr{D}'$, then there is an orthonormal subset $\mathscr{E}$ of $\mathscr{D}$ that is maximal in $\mathscr{D}$ but not maximal in $\mathscr{D}'$. (Let $\mathscr{E}$ be a maximal orthonormal subset of $\mathscr{D} \cap \{x_0\}^\perp$, where $x_0 \in \mathscr{D}'$, $x_0 \notin \mathscr{D}$.) In particular, if A is a positive operator with nonclosed range, then there is a maximal orthonormal subset $\mathscr{E}$ of $\mathscr{R}(A)$ that is not maximal in $\mathscr{R}(A^{1/2})$.

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