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Common fixed point theorems for a weaker Meir–Keeler type function in cone metric spaces☆

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ABSTRACT

In this work, we define a weaker Meir–Keeler type function $\psi : \operatorname{int} P \cup \{0\} \to \operatorname{int} P \cup \{0\}$ in a cone metric space, and under this weaker Meir–Keeler type function, we show the common fixed point theorems of four single-valued functions in cone metric spaces.

1. Introduction and preliminaries

Huang and Zhang [1] have introduced the concept of the cone metric space, replacing the set of real numbers by an ordered Banach space, and they showed some fixed point theorems of contractive type mappings on cone metric spaces. Later, many authors generalized their results. In this work, we first recall the Meir–Keeler type function, and define a weaker Meir–Keeler type function $\psi : \operatorname{int} P \cup \{0\} \to \operatorname{int} P \cup \{0\}$ in a cone metric space. Under this weaker Meir–Keeler type function, we show the common fixed point theorems of four single-valued functions in cone metric spaces. We recall some definitions of the cone metric spaces and some of the properties [1], as follows:

Definition 1 (*[1]*). Let E be a real Banach space and P a subset of E. P is called a cone if and only if:

(i) P is nonempty, closed, and $p \neq \{0\}$,
(ii) $a, b \in \Re$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$,
(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$, a partial ordering $\leq$ with respect to P is defined by $x \leq y$ if and only if $y - x \in P$ for all $x, y \in E$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P.

The cone P is called normal if there exists a real number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|.$$

The least positive number K satisfying above is called the normal constant of P.

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The cone P is called regular if every increasing sequence which is bounded from above is convergent, that is, if $\{x_n\}$ is a sequence such that

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y,$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

In the following we always suppose that E is a real Banach space with cone P in E with $\operatorname{int} P \neq \phi$, and $\leq$ is a partial ordering with respect to P.

Definition 2 (*[1]*). Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(i) $0 < d(x, y)$, for all $x, y \in X, x \neq y$,
(ii) $d(x, y) = 0$ if and only if $x = y$,
(iii) $d(x, y) = d(y, x)$, and
(iv) $d(x, y) + d(y, z) \geq d(x, z)$, for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 3 (*[1]*). Let (X, d) be a cone metric space, and let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$, there is $n_0 \in \mathcal{N}$ such that

$$d(x_n, x) \ll c, \quad \text{for all } n > n_0,$$

then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x.

Definition 4 (*[1]*). Let (X, d) be a cone metric space, and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is a Cauchy sequence if for any $c \in E$ with $0 \ll c$, there is N such that

$$d(x_n, x_m) \ll c, \quad \text{for all } n, m > N.$$

Definition 5 (*[1]*). Let (X, d) be a cone metric space. If every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Remark 1 (*[1]*). If P is a normal cone, then $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$. Further, in this case, $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $m, n \to \infty$.

Let (X, d) be a cone metric space, $T : X \to X$ and $x_0 \in X$. Then T is continuous at x_0 if for any sequence $\{x_n\}$ in X with $d(x_n, x_0) \to 0$, we have $d(Tx_n, Tx_0) \to 0$.

Definition 6. Let (X, d) be a cone metric space, and let $S, F : X \to X$ be two single-valued functions. We say that S and F are compatible if

$$\lim_{n\to\infty} d(SFx_n, FSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} d(Fx_n, Sx_n) = 0$.

In particular, $d(SFx, FSx) = 0$ if $d(Fx, Sx) = 0$ on taking $x_n = x$ for all n.

Recall the notion of the Meir–Keeler type function. A function $\psi : \Re^+ \to \Re^+$ is said to be a Meir–Keeler type function (see [2]) if for each $\eta \in \Re^+$, there exists $\delta = \delta(\eta) > 0$ such that for $t \in \Re^+$ with $\eta \leq t < \eta + \delta$, we have $\psi(t) < \eta$. We now define a new weaker Meir–Keeler type function, as follows:

Definition 7. Let (X, d) be a cone metric space with cone P, and let $\psi : \operatorname{int} P \cup \{0\} \to \operatorname{int} P \cup \{0\}$. Then the function ψ is called a weaker Meir–Keeler type function if for each η, $0 \ll \eta$, there exists δ, $0 \ll \delta$ such that for $t \in \operatorname{int} P$ with $\eta \leq t \ll \delta + \eta$, there exists $n_0 \in \mathcal{N}$ such that $\psi^{n_0}(t) \ll \eta$.

2. Main results

In the sequel, we let the function $\psi : \operatorname{int} P \cup \{0\} \to \operatorname{int} P \cup \{0\}$ satisfy the following conditions:

(i) ψ is a weaker Meir–Keeler type function;
(ii) for each $t \in \operatorname{int} P$, we have $0 \ll \psi(t) \ll t$;
(iii) for $t_n \in \operatorname{int} P$, if $\lim_{n\to\infty} t_n = \gamma \gg 0$, then $\lim_{n\to\infty} \psi(t_n) \ll \gamma$;
(iv) $\{\psi^n(t)\}_{n\in\mathcal{N}}$ is nonincreasing.

Moreover, we call this mapping a ψ-mapping.

Theorem 1. *Let (X, d) be a complete cone metric space with regular cone P such that $d(x, y) \in \operatorname{int} P$ for all $x, y \in X$ with $x \neq y$, and let $F, G, S, T : X \to X$ be four single-valued functions with $SX \subset GX$ and $TX \subset FX$ such that for all $x, y \in X$,*

$$d(Sx, Ty) \leq \psi\left(\max\left\{d(Fx, Gy), d(Fx, Sx), d(Gy, Ty), \frac{1}{2}[d(Fx, Ty) + d(Gy, Sx)]\right\}\right).$$

If S and F are compatible, T and G are compatible, and if either F or G is continuous, then S, T, F and G have a unique common fixed point z in X.

Proof. Given $x_0 \in X$, define the sequence $\{x_n\}$ recursively as follows:

$$Gx_{2n+1} = Sx_{2n} = z_{2n}, \qquad Fx_{2n+2} = Tx_{2n+1} = z_{2n+1}.$$

Since

$$\begin{aligned} d(z_{2n}, z_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leq \psi\left(\max\left\{d(Fx_{2n}, Gx_{2n+1}), d(Fx_{2n}, Sx_{2n}), d(Gx_{2n+1}, Tx_{2n+1}), \frac{1}{2}[d(Fx_{2n}, Tx_{2n+1}) + d(Gx_{2n+1}, Sx_{2n})]\right\}\right) \\ &\leq \psi\left(\max\left\{d(z_{2n-1}, z_{2n}), d(z_{2n-1}, z_{2n}), d(z_{2n}, z_{2n+1}), \frac{1}{2}[d(z_{2n-1}, z_{2n}) + d(z_{2n}, z_{2n+1})]\right\}\right), \end{aligned}$$

we hence have

$$d(z_{2n}, z_{2n+1}) \ll d(z_{2n-1}, z_{2n}).$$

Similarly,

$$d(z_{2n+1}, z_{2n+2}) \ll d(z_{2n}, z_{2n+1}).$$

Generally, we have

$$\begin{aligned} &d(z_n, z_{n+1}) \ll d(z_{n-1}, z_n), \quad \text{and} \\ &d(z_n, z_{n+1}) \leq \psi(d(z_{n-1}, z_n)) \leq \cdots \leq \psi^n(d(z_0, z_1)). \end{aligned}$$

Since $\{\psi^n(d(z_0, z_1))\}_{n\in\mathcal{N}}$ is nonincreasing, it must converge to some η, $0 \leq \eta$. We claim that $\eta = 0$. On the contrary, assume that $0 \ll \eta$. Then by the definition of the weaker Meir–Keeler type function, there exists δ, $0 \ll \delta$ such that for $0 \ll d(z_0, z_1)$ with $\eta \leq d(z_0, z_1) \ll \delta + \eta$, there exists $n_0 \in \mathcal{N}$ such that $\psi^{n_0}(d(z_0, z_1)) \ll \eta$. Since $\lim_{n\to\infty} \psi^n(d(z_0, z_1)) = \eta$, there exists $m_0 \in \mathcal{N}$ such that $\eta \leq \psi^m d(z_0, z_1) \ll \delta + \eta$, for all $m \geq m_0$. Thus, we conclude that $\psi^{m_0+n_0}(d(z_0, z_1)) \ll \eta$. So we get a contradiction. So $\lim_{n\to\infty} \psi^n(d(z_0, z_1)) = 0$, and so $\lim_{n\to\infty} d(z_n, z_{n+1}) = 0$.

Next, we let $c_m = d(z_m, z_{m+1})$, and we claim that the following result holds:

for each $\gamma > 0$, there is $n_0(\gamma) \in N$ such that for all $m, n \geq n_0(\gamma)$,

$$d(z_m, z_n) < \gamma. \qquad (*)$$

We shall prove $(*)$ by contradiction. Suppose that $(*)$ is false. Then there exists some $\gamma > 0$ such that for all $k \in N$, there are $m_k, n_k \in N$ with $m_k > n_k \geq k$ satisfying:

(i) m_k is even and n_k is odd,
(ii) $d(z_{m_k}, z_{n_k}) \geq \gamma$, and
(iii) m_k is the smallest even number such that the conditions (i), (ii) hold (see [3]).

Since $c_m \searrow 0$, by (ii), we have $\lim_{k\to\infty} d(z_{m_k}, z_{n_k}) = \gamma$, and

$$\begin{aligned} \gamma &\leq d(z_{m_k}, z_{n_k}) = d(Sx_{m_k}, Tx_{n_k}) \\ &\leq \psi\left(\max\left\{d(Fx_{m_k}, Gx_{n_k}), d(Fx_{m_k}, Sx_{m_k}), d(Gx_{n_k}, Tx_{n_k}), \frac{1}{2}[d(Fx_{m_k}, Tx_{n_k}) + d(Gx_{n_k}, Sx_{m_k})]\right\}\right) \\ &\leq \psi\left(\max\left\{d(z_{m_k-1}, z_{n_k-1}), c_{m_k} - 1, c_{n_k} - 1, \frac{1}{2}[d(z_{m_k-1}, z_{n_k}) + d(z_{n_k-1}, z_{n_k}) + d(z_{n_k-1}, z_{m_k})]\right\}\right) \\ &\leq \psi\left(\max\left\{c_{m_k-1} + d(z_{m_k}, z_{n_k}) + c_{n_k-1}, c_{m_k} - 1, c_{n_k} - 1, \frac{1}{2}[c_{m_k-1} + d(z_{m_k}, z_{n_k}) + c_{n_k-1} + d(z_{m_k}, z_{n_k})]\right\}\right) \\ &\leq \psi(c_{m_k-1} + d(z_{m_k}, z_{n_k}) + c_{n_k-1}). \end{aligned}$$

Since $\lim_{k\to\infty} c_{m_k-1} + d(z_{m_k}, z_{n_k}) + c_{n_k-1} = \gamma$, and by the condition (iii) of the ψ-mapping, we have $\gamma \leq \lim_{k\to\infty} \psi(c_{m_k-1} + c_{n_k-1} + d(z_{m_k}, z_{n_k})) < \gamma$, a contradiction. It follows from $(*)$ that the sequence $\{z_n\}$ must be a Cauchy sequence; hence $\{z_n\}$ converges to some $z \in X$. So, $d(Fx_{2n}, z) \to 0$, $d(Gx_{2n+1}, z) \to 0$, $d(Sx_{2n}, z) \to 0$ and $d(Tx_{2n+1}, z) \to 0$ as $n \to \infty$.

Assume F is continuous. Then we have

$$d(F^2x_{2n}, Fz) \to 0 \quad \text{and} \quad d(FSx_{2n}, Fz) \to 0 \quad \text{as } n \to \infty.$$

Since S and F are compatible and $d(Sx_{2n}, Fx_{2n}) \to 0$ as $n \to \infty$, we have

$$d(SFx_{2n}, Fz) \leq d(SFx_{2n}, FSx_{2n}) + d(FSx_{2n}, Fz),$$

and so,

$$d(SFx_{2n}, Fz) \to 0, \quad \text{as } n \to \infty.$$

For any $n \in N$,

$$d(SFx_{2n}, Tx_{2n+1}) \le \psi\left(\max\left\{d(F^2x_{2n}, Gx_{2n+1}), d(F^2x_{2n}, SFx_{2n}), d(Gx_{2n+1}, Tx_{2n+1}), \frac{1}{2}[d(F^2x_{2n}, Tx_{2n+1}) + d(Gx_{2n+1}, SFx_{2n})]\right\}\right).$$

(1) If

$$\max\left\{d(F^2x_{2n}, Gx_{2n+1}), d(F^2x_{2n}, SFx_{2n}), d(Gx_{2n+1}, Tx_{2n+1}), \frac{1}{2}[d(F^2x_{2n}, Tx_{2n+1}) + d(Gx_{2n+1}, SFx_{2n})]\right\} = d(F^2x_{2n}, Gx_{2n+1}),$$

then we have $\lim_{n\to\infty} d(F^2x_{2n}, Gx_{2n+1}) = d(Fz, z)$, and

$$\lim_{n\to\infty} d(SFx_{2n}, Tx_{2n+1}) \le \lim_{n\to\infty} \psi(d(F^2x_{2n}, Gx_{2n+1})) < d(Fz, z),$$

that is, $d(Fz, z) < d(Fz, z)$, which implies that $Fz = z$.

(2) If

$$\max\left\{d(F^2x_{2n}, Gx_{2n+1}), d(F^2x_{2n}, SFx_{2n}), d(Gx_{2n+1}, Tx_{2n+1}), \frac{1}{2}[d(F^2x_{2n}, Tx_{2n+1}) + d(Gx_{2n+1}, SFx_{2n})]\right\} = d(F^2x_{2n}, SFx_{2n}),$$

then we have $\lim_{n\to\infty} d(F^2x_{2n}, SFx_{2n}) = 0$, and

$$\lim_{n\to\infty} d(SFx_{2n}, Tx_{2n+1}) \le 0,$$

which implies that $Fz = z$.

(3) If

$$\max\left\{d(F^2x_{2n}, Gx_{2n+1}), d(F^2x_{2n}, SFx_{2n}), d(Gx_{2n+1}, Tx_{2n+1}), \frac{1}{2}[d(F^2x_{2n}, Tx_{2n+1}) + d(Gx_{2n+1}, SFx_{2n})]\right\} = d(Gx_{2n+1}, Tx_{2n+1}),$$

then we have $\lim_{n\to\infty} d(Gx_{2n+1}, Tx_{2n+1}) = 0$, and

$$\lim_{n\to\infty} d(SFx_{2n}, Tx_{2n+1}) \le 0,$$

which implies that $Fz = z$.

(4) If

$$\max\left\{d(F^2x_{2n}, Gx_{2n+1}), d(F^2x_{2n}, SFx_{2n}), d(Gx_{2n+1}, Tx_{2n+1}), \frac{1}{2}[d(F^2x_{2n}, Tx_{2n+1}) + d(Gx_{2n+1}, SFx_{2n})]\right\} = \frac{1}{2}[d(F^2x_{2n}, Tx_{2n+1}) + d(Gx_{2n+1}, SFx_{2n})],$$

then we have $\lim_{n\to\infty} \frac{1}{2}[d(F^2x_{2n}, Tx_{2n+1}) + d(Gx_{2n+1}, SFx_{2n})] = d(Fz, z) + d(Fz, z)$, and

$$\lim_{n\to\infty} d(SFx_{2n}, Tx_{2n+1}) \le \lim_{n\to\infty} \psi(d(F^2x_{2n}, Gx_{2n+1})) < d(Fz, z),$$

that is, $d(Fz, z) < d(Fz, z)$, which implies that $Fz = z$.

Follow (1)–(4), we get $Fz = z$.

For any $n \in N$,

$$d(Sz, Tx_{2n+1}) \le \psi\left(\max\left\{d(Fz, Gx_{2n+1}), d(Fz, Sz), d(Gx_{2n+1}, Tx_{2n+1}), \frac{1}{2}[d(Fz, Tx_{2n+1}) + d(Gx_{2n+1}, Sz)]\right\}\right).$$

(5) If

$$\max\left\{d(Fz, Gx_{2n+1}), d(Fz, Sz), d(Gx_{2n+1}, Tx_{2n+1}), \frac{1}{2}[d(Fz, Tx_{2n+1}) + d(Gx_{2n+1}, Sz)]\right\} = d(Fz, Gx_{2n+1}),$$

then we have $\lim_{n\to\infty} d(Fz, Gx_{2n+1}) = d(Fz, z) = 0$, and

$$\lim_{n\to\infty} d(Sz, Tx_{2n+1}) = d(Sz, z) \le \lim_{n\to\infty} \psi(d(F^2x_{2n}, Gx_{2n+1})) < 0,$$

which implies that $Sz = z$.

(6) If

$$\max\left\{d(Fz, Gx_{2n+1}), d(Fz, Sz), d(Gx_{2n+1}, Tx_{2n+1}), \frac{1}{2}[d(Fz, Tx_{2n+1}) + d(Gx_{2n+1}, Sz)]\right\} = d(Fz, Sz),$$

then we have

$$\lim_{n\to\infty} d(Sz, Tx_{2n+1}) = d(Sz, z) \le \lim_{n\to\infty} \psi(d(Fz, Sz)) < d(Fz, Sz),$$

a contradiction, which implies that $Sz = z$.

(7) If

$$\max\left\{d(Fz, Gx_{2n+1}), d(Fz, Sz), d(Gx_{2n+1}, Tx_{2n+1}), \frac{1}{2}[d(Fz, Tx_{2n+1}) + d(Gx_{2n+1}, Sz)]\right\} = d(Gx_{2n+1}, Tx_{2n+1}),$$

then we have $\lim_{n\to\infty} d(Fz, Gx_{2n+1}) = 0$, and

$$\lim_{n\to\infty} d(Sz, Tx_{2n+1}) = d(Sz, z) \le \lim_{n\to\infty} \psi(d(Fz, Gx_{2n+1})) < 0,$$

which implies that $Sz = z$.

(8) If

$$\max\left\{d(Fz, Gx_{2n+1}), d(Fz, Sz), d(Gx_{2n+1}, Tx_{2n+1}), \frac{1}{2}[d(Fz, Tx_{2n+1}) + d(Gx_{2n+1}, Sz)]\right\}$$
$$= \frac{1}{2}[d(Fz, Tx_{2n+1}) + d(Gx_{2n+1}, Sz)],$$

then we have $\lim_{n\to\infty} \frac{1}{2}[d(Fz, Tx_{2n+1}) + d(Gx_{2n+1}, Sz)] = \frac{1}{2}d(Sz, z)$, and

$$\lim_{n\to\infty} d(Sz, Tx_{2n+1}) = d(Sz, z) \le \lim_{n\to\infty} \psi\left(\frac{1}{2}[d(Fz, Tx_{2n+1}) + d(Gx_{2n+1}, Sz)]\right) < \frac{1}{2}d(Sz, z),$$

a contradiction, which implies that $Sz = z$.

Following (5)–(8), we get $Sz = z$.

Select $z' \in X$ such that $Gz' = z = Sz$. Then $TGz' = Tz$, and

$$\begin{aligned} d(z, Tz') &= d(Sz, Tz') \\ &\le \psi\left(\max\left\{d(Fz, Gz'), d(fz, Sz), d(Gz', Tz'), \frac{1}{2}[d(Fz, Tz') + d(Gz', Sz)]\right\}\right) \\ &\le \psi(\max\{0, 0, d(z, Tz'), d(z, Tz')\}), \end{aligned}$$

which implies that $Tz' = z$ and so $GTz' = Gz$.

Since T and G are compatible and $d(Tz', Gz') = 0$, we get $d(Tz, Gz) = d(TGz', GTz') = 0$, which implies $Tz = Gz$. Since

$$\begin{aligned} d(z, Tz) &= d(Sz, Tz) \\ &\le \psi\left(\max\left\{d(Fz, gz), d(Fz, Sz), d(Gz, Tz), \frac{1}{2}[d(Fz, Tz) + d(Gz, Sz)]\right\}\right) \\ &\le \psi\left(\max\left\{d(z, Tz), 0, 0, \frac{1}{2}[d(z, Tz) + d(z, Tz)]\right\}\right) \\ &= \psi(d(z, Tz)), \end{aligned}$$

we have $d(z, Tz) = 0$, and so $Tz = z$.

Hence z is a common fixed point of S, T, F and G with $Sz = Tz = z = Fz = Gz$.

Let y be a common fixed point of S, T, F and G. We have

$$\begin{aligned} d(y, z) &\le d(Sy, Tz) \\ &\le \psi\left(\max\left\{d(Fy, Gz), d(Fy, Sy), d(Gz, Tz), \frac{1}{2}[d(Fy, Tz) + d(Gz, Sy)]\right\}\right) \\ &= \psi(d(y, z)). \end{aligned}$$

This implies $y = z$. Hence z is the unique common fixed point of S, T, F and G.

Similarly, we can prove the continuity of G. □

For the case $F = G = I$ (identity mapping), we have the following corollary.

Corollary 1. *Let (X, d) be a complete cone metric space with regular cone P such that $d(x, y) \in \operatorname{int} P$ for all $x, y \in X$ with $x \neq y$, and let $S, T : X \to X$ be two single-valued functions such that for all $x, y \in X$,*

$$d(Sx, Ty) \leq \psi\left(\max\left\{d(x, y), d(x, Sx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Sx)]\right\}\right).$$

Then S and T have a unique common fixed point z in X.

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