On the Combinatorial Antipodal-Point Lemmas

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Tucker's Lemma is a combinatorial result which can be used to prove many antipodal-point theorems for the $n$-sphere, such as the Borsuk–Ulam and the Lusternik–Schnirelmann, among others (cf. Lefschetz [5]). This lemma originally appeared in [6] stated for a rectangular subdivision of a square. Tucker later generalized this to barycentrically derived subdivisions of a particular complex representing the $n$-cube, [5]. Fan, in [3], has proven a generalization of Tucker's Lemma for octahedral decompositions of the $n$-cube. Then Baker, in [1], established this result for the cubical decomposition of the $n$-cube. Below we shall give a more spherical interpretation of this Lemma, which we shall establish using a combination of the parity arguments of the other combinatorial proofs and a geometric search technique similar to that found in the author's [2].

Tucker's Lemma, as stated for the plane, is as follows: Given an array of $N^2$ elements, in $N$ rows and $N$ columns ($N > 1$), and given the four labels $1, -1, 2, -2$, let each element of the array be assigned one of the four labels in such a way that each pair of antipodal elements on the boundary of the array is assigned a pair of labels whose sum is zero; then there is at least one pair of adjoining elements of the array that have labels whose sum is zero. Elements can be adjoining in a row, column or diagonal with slope 1.

We will first generalize this statement to apply to labellings of particular triangulations of the unit disk.

Definition. An antipodal triangulation of the unit disk $D$ is any triangulation in which the points on the boundary occur in antipodal pairs.

This does not imply that there need be any central symmetry to the triangulation except on the boundary. To label this triangulation we assign to each vertex one of the numbers $\pm 1, \pm 2$ in such a way that antipodal boundary vertices receive labels that sum to 0.
For example:

\[\begin{array}{c}
\includegraphics[width=0.5\textwidth]{example-diagram}
\end{array}\]

**Theorem.** In every labelled antipodal triangulation of the disk there is at least one pair of connected vertices whose labels sum to 0.

**Proof.** Let us assume that no pair of adjacent vertices have labels which sum to 0. First we prove that the number of edges labelled \((1, -2)\) on the boundary is even. If it is not zero, start with one such edge. This edge is part of a triangle whose third vertex is also a 1 or a \(-2\). If it is a \(-2\) then we drop our former vertex labelled \(-2\) and consider the new edge \((1, -2)\). This edge is part of exactly one new triangle. The third vertex of this new triangle is again either a 1 or a \(-2\). As before we drop whichever old vertex duplicates the label of our new vertex and thereby obtain a new edge labelled \((1, -2)\). This process must terminate since the triangulation is finite and no triangle can be entered more than once. This search can only end at an edge on the circumference labelled \((1, -2)\). In this way we pair off all the edges labelled \((1, -2)\) on the circumference. This process is illustrated below.

\[\begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram1}
\end{array}\]

This proves that the total number of edges on the circumference labelled \((1, -2)\) is even.

Let us now select any vertex on the boundary \(A\) and its antipodal vertex \(A'\). Let us choose one of the two possible semiperimeters from \(A\) to \(A'\) and call
it \( S \). Let the other semi-perimeter be called \( S' \). Each edge along \( S' \) labelled \((1, -2)\) corresponds to an edge along \( S \) labelled \((-1, 2)\). Therefore the total number of border edges labelled \((1, -2)\) is the sum of the number of edges along \( S \) labelled \((1, -2)\) plus the number of edges along \( S \) labelled \((-1, 2)\). We have already shown that this total is even.

If we look at the labels along \( S \) from \( A \) to \( A' \) we see that the only edges which reverse the sign of the label from positive to negative or vice versa, are those edges labelled \((1, -2)\) and \((-1, 2)\) since no edges have labels \((1, -1)\) or \((2, -2)\). Since \( A \) and \( A' \) are of different signs the total number of reversals must be odd.

This contradicts our previous finding and thereby proves the theorem.

We shall now show how this generalized statement and method of proof can be applied to shorten the proof of Fan's \( n \)-dimensional version of Tucker's Lemma, which we will illustrate only in the case \( n = 3 \).

Let us define an antipodal triangulation of the 3-disk, \( D^3 \), as any triangulation in which the boundary vertices and triangles occur in antipodal pairs.

**Theorem.** If the vertices of an antipodal triangulation of \( D^3 \) are assigned the labels \( \pm 1, \pm 2, \) and \( \pm 3 \) in such a way that antipodal vertices receive labels which sum to 0, then some pair of adjacent vertices receives labels which sum to 0.

**Proof.** Let us assume that no pair of adjacent vertices has labels which sum to 0. Let us start with a triangle on the boundary with vertices labelled \((1, 2, 3)\). This triangle is the face of a three-simplex whose new vertex must duplicate the label of one of the old vertices. Dropping the old vertex we have a new triangular face which belongs to exactly one new 3-simplex. As with the previous search technique this will end in another triangular face on the boundary labelled \((1, 2, 3)\). This procedure applies equally well to any boundary face labelled \((a, b, c)\), where \(|a|, |b| \) and \(|c| \) are distinct. In general we have shown that the number of boundary triangles labelled \((a, b, c)\) is even.

Let us now consider any vertex \( A \) on the boundary. Let us call its antipodal vertex \( A' \). Let \( S \) be a simple path from \( A \) to \( A' \) along boundary edges which does not contain a pair of antipodal vertices. The existence of such a path for at least one choice of \( A \) is not hard to demonstrate. (If there is a path from \( A \) to \( A' \) which includes the antipodal vertices \( B \) and \( B' \) then select \( B \) as our chosen vertex instead of \( A \).) Let \( S' \) be the path from \( A' \) to \( A \) antipodal to \( S \). The sphere, which is the boundary of \( D^3 \), is cut into two hemi-spheres by the paths \( S \) and \( S' \). Call one of these \( H \) and the other \( H' \). Every triangular face labelled \((a, b, c)\) in \( H \) is antipodal to a face labelled \((-a, -b, -c)\) in \( H' \). For \(|a|, |b| \) and \(|c| \) distinct the number of faces labelled \((a, b, c)\) in \( H \) plus the number of faces labelled \((-a, -b, -c)\) in \( H \) is equal to the
total number of faces labelled \((a, b, c)\) on the boundary and is therefore even.

Let the symbol \(S(a, b)\) denote the number of edges along \(S\) labelled \((a, b)\). There are only six possibilities for edges with labels of opposite signs. They are \((1, -2), (1, -3), (-1, 2), (-1, 3), (2, -3)\) and \((-2, 3)\). As we go from \(A\) to \(A'\) along \(S\) the total number of sign changes must be odd since \(A\) and \(A'\) have opposite sign. Therefore,

\[
S(1, -2) + S(-1, 2) + S(1, -3) + S(-1, 3) + S(2, -3) + S(-2, 3) \quad (**)
\]

is odd.

Let the symbol \(H(a, b, c)\) denote the number of faces on \(H\) with labels \(a, b\) and \(c\). We have shown above that

\[
H(a, b, c) + H(-a, -b, -c) \quad (***)
\]

is even when \(|a|, |b|, |c|\) are distinct.

Let us start with an edge along \((S \cup S')\) labelled \((1, -2)\) and construct a chain of triangles going through \(H\) in the same manner as in the two-dimensional case (dropping a repeated vertex). This chain can terminate in two different ways. Either it reaches another edge on \((S \cup S')\) labelled \((1, -2)\) or else the path meets a triangle in \(H\) labelled \((1, -2, \pm 3)\). If we start with a triangle in \(H\) labelled \((1, -2, \pm 3)\) and begin a chain by dropping the vertex \(\pm 3\) this chain would terminate either at another triangle with vertices labelled \((1, -2, \pm 3)\) or else it would run into an edge labelled \((1, -2)\) on \((S \cup S')\). This shows that some pairs of edges \((1, -2)\) pair off and some triangles \((1, -2, \pm 3)\) pair off while the remaining edges pair off with the remaining triangles as in the diagram below. This proves that the parity of \(H(1, -2, 3) + H(1, -2, -3)\) is the same as the parity of \(S(1, -2) + S'(1, -2)\), which in turn is the same as the parity of \(S(1, -2) + S(-1, 2)\).

Therefore,

\[
S(1, -2) + S(-1, 2) \equiv H(1, -2, 3) + H(1, -2, -3) \quad (\text{mod } 2).
\]
Similarly,

\[ S(2, -3) + S(-2, 3) = H(2, -3, 1) + H(2, 3, 1) \pmod{2} \]

and

\[ S(-1, 3) + S(1, -3) = H(-1, 3, 2) + H(1, -3, -2) \pmod{2}. \]

Adding up these three congruences we find that (*) says that the left-hand side is odd, while (**) says that the right-hand side is the sum of three even numbers. This contradiction proves the theorem.

This theorem includes Tucker’s Lemma as a special case and implies the continuous topological theorems in a natural fashion.

The proof given above differs from the existing proofs of similar theorems in that it is geometric and constructive. The search technique shows that the reason there are an odd number of simplicies labelled \((1, -2, 3,...)\) on the boundary of the \(n\)-disk is geometrically related to the fact that there are an odd number of sign changes from \(A\) to \(A'\) along \(S\). The fact that this number is also even follows from the natural pairing arising from the chains. This method is constructive in that it will locate an edge whose vertices sum to 0 in dimension \(n\) if we already know the location of all relevant simplices in dimension \(n - 1\).

REFERENCES