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Lie–Rinehart cohomology and integrable connections on modules of rank one

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ABSTRACT

Let k be an algebraically closed field of characteristic 0, let R be a commutative k-algebra, and let M be a torsion free R-module of rank one with a connection ∇. We consider the Lie–Rinehart cohomology with values in $\mathrm{End}_R(M)$ with its induced connection, and give an interpretation of this cohomology in terms of the integrable connections on M. When R is an isolated singularity of dimension $d \geqslant 2$, we relate the Lie–Rinehart cohomology to the topological cohomology of the link of the singularity, and when R is a quasi-homogenous hypersurface of dimension two, we give a complete computation of the cohomology.

1. Introduction

Rinehart introduced Lie–Rinehart cohomology in Rinehart [16] as a generalization of de Rham cohomology. Later, Lie–Rinehart cohomology has been considered by several authors, see for instance Huebschmann [11] and Casas, Ladra and Pirashvili [3]. In singularity theory, Huang, Luk and Yau [10] studied the so-called punctured de Rham cohomology, and although it is not mentioned in their paper, it turns out that this cohomology coincides with the Lie–Rinehart cohomology.

The purpose of this paper is to study the Lie Rinehart cohomology when R is a representative of an isolated singularity, and to interpret this cohomology in terms of integrable connections on R-modules of rank one. The emphasis is on explicit results and examples.

Let k be an algebraically closed field of characteristic 0, let R be a commutative k-algebra and let M be a torsion free R-module of rank one with a (not necessarily integrable) connection $\nabla : \mathrm{Der}_k(R) \to \mathrm{End}_k(M)$. We consider the Lie–Rinehart cohomology $\mathrm{H}^n_{\mathrm{Rin}}(\mathrm{Der}_k(R), \mathrm{End}_R(M))$ where $\mathrm{End}_R(M)$ has the

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(integrable) connection induced by ∇. We give the following interpretation of this cohomology for $n = 1, 2$:

Theorem A. *Let R be a complete reduced local k-algebra and let M be a torsion free R-module of rank one. Then we have*:

(1) *There is a canonical obstruction class* $\mathrm{ic}(M) \in \mathrm{H}^2_{\mathrm{Rin}}(\mathrm{Der}_k(R), \mathrm{End}_R(M))$ *which vanishes if and only if there is an integrable connection on M.*
(2) *If* $\mathrm{ic}(M)$ *vanishes, then* $\mathrm{H}^1_{\mathrm{Rin}}(\mathrm{Der}_k(R), \mathrm{End}_R(M))$ *is the moduli space of integrable connections on M up to equivalence.*

Using a spectral sequence, we relate $\mathrm{H}^*_{\mathrm{Rin}}(\mathrm{Der}_k(R), \mathrm{End}_R(M))$ to the topological cohomology of the link of the singularity:

Theorem B. *Let R be a finitely generated Cohen–Macaulay domain over $\mathbb{C}$ of dimension $d \geqslant 2$ with a unique isolated singularity $x \in X = \mathrm{Spec}(R)$. Then $\mathrm{End}_R(M) \cong R$ and there is a natural exact sequence*

$$0 \to \mathrm{H}^1_{\mathrm{Rin}}\big(\mathrm{Der}(R), R\big) \to \mathrm{H}^1(U_{\mathrm{an}}, \mathbb{C}) \to E_2^{0,1} \to \mathrm{H}^2_{\mathrm{Rin}}\big(\mathrm{Der}(R), R\big) \to \mathrm{H}^2(U_{\mathrm{an}}, \mathbb{C}),$$

where $E_2^{0,1} = \ker(\mathrm{H}^1(U, \mathcal{O}_X) \to \mathrm{H}^1(U, \Omega^1_X))$ *and* $U = X \setminus \{x\}$. *Moreover, if* $d \geqslant 3$, *then* $E_2^{0,1} = 0$.

We give an example to show that $\mathrm{H}^1_{\mathrm{Rin}}(\mathrm{Der}(R), R)$ and $\mathrm{H}^1(U_{\mathrm{an}}, \mathbb{C})$ are not in general isomorphic in dimension $d = 2$. To further clarify the case $d = 2$, we show the following result:

Theorem C. *Let $R = k[x_1, x_2, x_3]/(f)$ be an integral quasi-homogenous surface singularity. The grading on R induces a grading on $\mathrm{H}^*_{\mathrm{Rin}}(\mathrm{Der}_k(R), R)$, and*

$$\begin{aligned}
&\mathrm{H}^0_{\mathrm{Rin}}\big(\mathrm{Der}_k(R), R\big) = \mathrm{H}^0_{\mathrm{Rin}}\big(\mathrm{Der}_k(R), R\big)_0 \cong R_0 = k, \\
&\mathrm{H}^i_{\mathrm{Rin}}\big(\mathrm{Der}_k(R), R\big) = \mathrm{H}^i_{\mathrm{Rin}}\big(\mathrm{Der}_k(R), R\big)_0 \cong R_{d-d_1-d_2-d_3} \quad \text{for } i = 1, 2, \\
&\mathrm{H}^i_{\mathrm{Rin}}\big(\mathrm{Der}_k(R), R\big) = 0 \quad \text{for } i \geqslant 3,
\end{aligned}$$

as graded k-vector space, where $d_i = \deg x_i$ for $i = 1, 2, 3$ and $d = \deg f \geqslant 2$.

In particular, we have that $\mathrm{H}^i_{\mathrm{Rin}}(\mathrm{Der}_k(R), R) \cong R_{d-1-1-1} = R_{d-3}$ when R is the cone over a plane curve, so that

$$\dim_{\mathbb{C}} \mathrm{H}^i_{\mathrm{Rin}}\big(\mathrm{Der}_k(R), R\big) = \frac{(d-1)(d-2)}{2}$$

is the genus of the curve $V(f)$ in $\mathbb{P}^2$ for $i = 1, 2$.

Our Theorem C is related to the work of Huang, Luk and Yau [10] on the punctured local de Rham cohomology $H^i_h(V, x)$ of a germ of a complex analytic space. When (V, x) is a hypersurface singularity of dimension $d \geqslant 2$ with local ring R, $H^i_h(V, x)$ coincides with $\mathrm{H}^i_{\mathrm{Rin}}(\mathrm{Der}_{\mathbb{C}}(R), R)$ for $i \geqslant 1$.

We also prove that if R is a curve, then any connection on a torsion free R-module (of any rank) is integrable. Moreover, if $R = k[\Gamma]$ is an affine monomial curve and M is a graded torsion free R-module of rank one with a connection ∇, then $\mathrm{H}^i_{\mathrm{Rin}}(\mathrm{Der}_k(R), M, \nabla) = 0$ for $i \geqslant 1$.

2. Connections and cohomology

Let k be an algebraically closed field of characteristic 0, and let R be a commutative k-algebra. Assume that R is a reduced noetherian k-algebra, and that M is a rank one torsion free finitely generated R-module. In this section, we relate the set of integrable connections on M to Lie–Rinehart cohomology.

Let $S \subset R$ be the set of regular elements, and let $Q = Q(R) = S^{-1}R$ be the total ring of fractions. Then by assumption, $M \to M \otimes_R Q$ is injective and $M \otimes_R Q$ is a free Q-module of rank one. Moreover, fixing an embedding $M \subseteq Q$, we identify $\overline{R} := \mathrm{End}_R(M)$ as $\{\psi \in Q\colon\ \psi M \subseteq M\}$. We view the R-algebra $\overline{R}$ as a *commutative* extension of R with $R \subseteq \overline{R} \subseteq Q$.

Let g be a Lie–Rinehart algebra and assume that there is a g-connection ∇ on M, see Eriksen and Gustavsen [6] for definitions and basic properties. Then there is an induced g-connection $\overline{\nabla}$ on $\overline{R} = \mathrm{End}_R(M)$, given by $\overline{\nabla}_D(\phi) = \nabla_D\,\phi - \phi\nabla_D$ for $D \in \mathrm{g}$ and $\phi \in \mathrm{End}_R(M)$.

Proposition 2.1. *Let ∇ be a (not necessarily integrable) g-connection on M.*

(1) *The induced g-connection $\overline{\nabla} : \mathrm{g} \to \mathrm{End}_k(\overline{R})$ is given by $\overline{\nabla}_D(\phi) = D(\phi)$ for $\phi \in \overline{R}$, where ϕ is identified with an element in Q and D is extended to Q. In particular, $\overline{\nabla}$ is an integrable connection that is independent of ∇.*
(2) *If R is normal, then $\overline{R} = R$, and $\overline{\nabla}$ is the action $\tau : \mathrm{g} \to \mathrm{Der}_k(R)$.*

Proof. We first prove (1). An element $D \in \mathrm{g}$ has a lifting to a derivation on Q (which we also denote by D). For any connection ∇ on M, we may consider ∇_D and D as maps from $M \subseteq Q$ into Q. Then $\nabla_D - D$ is in $\mathrm{Hom}_R(M, Q)$, so that $\nabla_D = D + \psi_D$ for some $\psi_D \in Q$. A calculation shows that

$$\begin{aligned}\overline{\nabla}_D(\phi) &= (\nabla_D\phi - \phi\nabla_D) = (D + \psi_D)\phi - \phi(D + \psi_D) \\ &= D\phi - \phi D = D(\phi),\end{aligned}$$

where we consider ϕ as an element in Q. In other words, g acts on $\overline{R}$ through $\overline{\nabla}$ by extending the action of g to Q.

To prove (2), note that $\overline{R} \subseteq Q$ is integral over R since $\overline{R}$ is a finitely generated R-module. By assumption, R is normal, hence $R = \overline{R}$. □

With assumptions as above and with $\overline{\nabla}$ as in the proposition, we will consider the Lie–Rinehart cohomology groups $\mathrm{H}^n_{\mathrm{Rin}}(\mathrm{g}, \overline{R}) = \mathrm{H}^n_{\mathrm{Rin}}(\mathrm{g}, \overline{R}, \overline{\nabla})$ of $(\overline{R}, \overline{\nabla})$, see Huebschmann [11], Casas, Ladra and Pirashvili [3] and Maakestad [14] for definitions and properties. We shall give an interpretation of these cohomology groups for $n = 1$ and $n = 2$:

Proposition 2.2. *If M admits a g-connection, then there is a canonical class*

$$\mathrm{ic}(M) \in \mathrm{H}^2_{\mathrm{Rin}}(\mathrm{g}, \overline{R})$$

called the integrability class, such that $\mathrm{ic}(M) = 0$ if and only if M admits an integrable g-connection.

Proof. Let ∇ be a g-connection on M and let $\overline{\nabla}$ be the induced g-connection on $\overline{R} = \mathrm{End}_R(M)$. It follows from the Bianchi identity

$$\begin{aligned}\big(d^2(K_\nabla)\big)(D_1 \wedge D_2 \wedge D_3) = {}& \overline{\nabla}_{D_1} K_\nabla(D_2 \wedge D_3) - \overline{\nabla}_{D_2} K_\nabla(D_1 \wedge D_3) \\ &+ \overline{\nabla}_{D_3} K_\nabla(D_1 \wedge D_2) - K_\nabla\big([D_1, D_2] \wedge D_3\big) \\ &+ K_\nabla\big([D_1, D_3] \wedge D_2\big) - K_\nabla\big([D_2, D_3] \wedge D_1\big) = 0\end{aligned}$$

that K_∇ is a 2-cocycle in the Rinehart complex $\mathrm{C}^*_{\mathrm{Rin}}(\mathrm{g}, \overline{R})$. We define $\mathrm{ic}(M) = [K_\nabla] \in \mathrm{H}^2_{\mathrm{Rin}}(\mathrm{g}, \overline{R})$, and see that $\mathrm{ic}(M) = 0$ if and only if $K_\nabla = d^1(\tau)$ for some potential $\tau \in \mathrm{C}^1_{\mathrm{Rin}}(\mathrm{g}, \overline{R})$. A calculation shows that this condition holds if and only if $\nabla - \tau$ is an integrable g-connection on M, since $K_{\nabla'} = K_\nabla + d^1(P)$ when $\nabla' = \nabla + P$. □

Definition 2.3. Let ∇ and ∇' be two g-connections on M. We will say that ∇ and ∇' are equivalent if there is an R-linear automorphism φ of M such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\nabla_D} & M \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ M & \xrightarrow{\nabla'_D} & M \end{array}$$

commutes for all $D \in \mathrm{g}$.

One may consider the category of modules with g-connections, see Section 1 in Eriksen and Gustavsen [6]. Then $\nabla \sim \nabla'$ if and only if (M, ∇) and (M, ∇') are isomorphic in this category.

Theorem 2.4. *Assume that* $(R, \mathfrak{m})$ *is a reduced complete local noetherian* k*-algebra with residue field* k, *and let* (M, ∇) *be a rank one torsion free finitely generated* R*-module with an integrable* g*-connection. Then there is a bijective correspondence between* $\mathrm{H}^1_{\mathrm{Rin}}(\mathrm{g}, \overline{R})$ *and the set of equivalence classes of integrable* g*-connections on* M.

Proof. Let $\tau \in \mathrm{Hom}_R(\mathrm{g}, \overline{R})$. We claim that $\nabla - \tau$ is an integrable g-connection if and only if τ is a 1-cocycle in $\mathrm{C}^*_{\mathrm{Rin}}(\mathrm{g}, \overline{R})$. By definition, $d^1(\tau)(D_1 \wedge D_2) = \overline{\nabla}_{D_1}\tau(D_2) - \overline{\nabla}_{D_2}\tau(D_1) - \tau([D_1, D_2])$ and therefore

$$K_{\nabla-\tau}(D_1 \wedge D_2) = K_\nabla(D_1 \wedge D_2) + d^1(\tau)(D_1 \wedge D_2) + \big[\tau(D_1), \tau(D_2)\big].$$

Since $\overline{R}$ is commutative, $[\tau(D_1), \tau(D_2)] = 0$ and this proves the claim.

The correspondence between $H^1(\mathrm{g}, \overline{R})$ and equivalence classes of integrable g-connections is induced by $\tau \mapsto \nabla - \tau$. We must show that $\tau \in \operatorname{im} d^0$ if and only if ∇ and $\nabla - \tau$ are equivalent.

Assume $\tau = d^0(\phi)$, where $\phi \in \mathrm{End}_R(M) \cong \mathrm{C}^0_{\mathrm{Rin}}(\mathrm{g}, \overline{R})$. This means that $\tau(D) = \overline{\nabla}_D(\phi)$ for all $D \in \mathrm{g}$. We claim that there exists an automorphism $\psi \in \overline{R}$ such that $\psi\nabla_D = (\nabla_D - \tau(D))\psi$ for all $D \in \mathrm{g}$. In fact, we have $\tau(D) = D(\phi)$ and $\nabla_D\psi - \psi\nabla_D = D(\psi)$ by Proposition 2.1, so $\psi\nabla_D = (\nabla_D - \tau(D))\psi$ if and only if $D(\psi) = D(\phi)\psi$. Since $\overline{R}$ is a finitely generated R-module, $\overline{R}/\mathfrak{m}\overline{R}$ is an artinian ring. We have that $J \cap R = \mathfrak{m}$ where J is the Jacobson radical in $\overline{R}$, and it follows that $J^n \subseteq \mathfrak{m}\overline{R}$ for some n. Thus $\overline{R}$ is complete in the J-adic topology. It follows that $\overline{R}$ is a product of complete local rings with residue field k. We have $k^r \subset \overline{R}$ and $\overline{R}/J \cong k^r$ for some r. If e is any idempotent and D a k-linear derivation, it follows that $D(e) = 0$. Hence $d^0(k^r) = 0$ and we may assume that $\tau = D(\phi)$ for $\phi \in J$. It follows that $\psi = \exp(\phi)$ is in $\overline{R}$ and $\psi\nabla_D = (\nabla_D - \tau(D))\psi$.

Conversely, assume that there is an automorphism $\psi \in \overline{R}$ such that $\psi\nabla_D = (\nabla_D - \tau(D))\psi$ for all $D \in \mathrm{g}$. Since ψ is a unit, we can take $\phi = \log(\psi)$, and by an argument similar to the one above $\phi \in \overline{R}$. This implies $\tau = d^0(\phi)$. □

3. The curve case

In this section we assume that R is reduced noetherian k-algebra of dimension one, and consider in some detail the case when R is a monomial curve.

Proposition 3.1. *Let $\mathrm{g} \subseteq \mathrm{Der}_k(R)$ be a Lie–Rinehart algebra and let M be a torsion free R-module. Then any* g*-connection on M is integrable.*

Proof. If $\dim R = 1$, one has that $\mathrm{Hom}_R(\wedge^2 \mathrm{g}, \mathrm{End}_R(M)) = 0$ when $\mathrm{g} \subset \mathrm{Der}_k(R)$ since $\mathrm{End}_R(M)$ is torsion free and $\wedge^2 \mathrm{g} \subseteq \wedge^2 \mathrm{Der}_k(R)$ is a torsion module. If there exists a connection ∇ on M, the curvature, which is an R-linear map $K_\nabla : \mathrm{g} \wedge \mathrm{g} \to \mathrm{End}_R(M)$, is necessarily zero. Thus the connection is automatically integrable. □

Let $R = k[\Gamma]$ be a monomial curve singularity given by a numerical semigroup $\Gamma \subseteq \mathbb{N}_0$, where $H = \mathbb{N}_0 \setminus \Gamma$ is a finite set. We consider a finitely generated graded torsion free R-module M of rank one. Up to graded isomorphism and a shift, we may assume that M has the form $M = k[\Lambda]$, where Λ is a set such that $\Gamma \subseteq \Lambda \subseteq \mathbb{N}_0$ and $\Gamma + \Lambda \subseteq \Lambda$. Let $\Gamma^{(1)} = \{w \in H\colon\ w + (\Gamma \setminus \{0\}) \subseteq \Gamma\}$ and let E be the Euler derivation. Then the set $\{E\} \cup \{t^w E\colon\ w \in \Gamma^{(1)}\}$ is a minimal generating set for $\mathrm{g} = \mathrm{Der}_k(R)$ as a left R-module, see Eriksen [4].

Theorem 3.2. *Let $R = k[\Gamma]$ be a monomial curve singularity given by a numerical semigroup $\Gamma \subseteq \mathbb{N}_0$, where $\mathbb{N}_0 \setminus \Gamma$ is a finite set, and let M be a finitely generated graded torsion free R-module of rank one. If ∇ is a connection on M, then*

$$\mathrm{H}^i_{\mathrm{Rin}}\big(\mathrm{Der}_k(R), M, \nabla\big) = 0 \quad \textit{for } i \geqslant 1.$$

Proof. Let $\mathrm{g} = \mathrm{Der}_k(R)$. Consider the set $S = \{\lambda \in \Lambda\colon\ \lambda + \Gamma^{(1)} \nsubseteq \Lambda\}$ and let l be the cardinality of S. There are three possibilities:

(1) $l = 0$: $\nabla_E = E - c$ defines an integrable connection on M for all $c \in k$.
(2) $l = 1$: $\nabla_E = E - c$ defines an integrable connection on M iff $c = \lambda_0$ is the unique element in S.
(3) $l \geqslant 2$: there are no connections on M

Assume $l \leqslant 1$ and consider the connection given by $\nabla_E = E - c$, where $c \in k$ if $l = 0$ and $c = \lambda_0$ if $l = 1$. Let $r : \mathrm{Hom}_R(\mathrm{g}, M) \to M$ be given by $\phi \mapsto \phi(E)$. Then the composition

$$M \xrightarrow{d^0} \mathrm{Hom}_R(\mathrm{g}, M) \xrightarrow{r} M$$

is the operator ∇_E. We claim that r is injective with image $k[\Lambda \setminus \{c\}] \subseteq M$. In fact, if $\phi \in \mathrm{Hom}_R(\mathrm{g}, M)$ with $\phi(E) = 0$, then $\phi(t^w E) = t^w \phi(E) = 0$ since M is torsion free. Therefore r is injective. Consider t^λ with $\lambda \in \Lambda \setminus \{c\}$. Since $\lambda \notin S$, we have that $\Gamma^{(1)} + \lambda \subseteq \Lambda$. Therefore, $\phi(E) = t^\lambda$ and $\phi(t^w E) = t^w t^\lambda$ for $w \in \Gamma^{(1)}$ defines a well-defined R-linear map $\phi : \mathrm{g} \to M$. Moreover, we clearly have $t^c \notin \mathrm{im}(r)$. Since r is a graded homomorphism, this proves that $\mathrm{im}(r) = k[\Lambda \setminus \{c\}] \subseteq M$. If $\phi \in \mathrm{Hom}_R(\mathrm{g}, M)$ is homogenous of degree w, then $\phi(E) \in M$ is homogenous of degree w and $w \neq c$. This implies that $d^0(\phi(E)/(w - c)) = \phi$ and therefore d^0 is surjective. We conclude that $\mathrm{H}^1_{\mathrm{Rin}}(\mathrm{g}, M) = 0$. Since $\mathrm{C}^i_{\mathrm{Rin}}(\mathrm{g}, M, \nabla) = \mathrm{Hom}_R(\wedge^i \mathrm{g}, M) = 0$ for $i \geqslant 2$, $\mathrm{H}^i_{\mathrm{Rin}}(\mathrm{Der}_k(R), M, \nabla) = 0$ for $i \geqslant 1$. □

Given a graded torsion free R-module M of rank one on a monomial curve R, it does not necessarily exist a connection on M, see Section 5.2 in Eriksen and Gustavsen [6]. However, if there exists a connection, it is integrable and unique up to analytic isomorphism by Theorem 2.4 and the proposition above.

4. The case of an isolated normal singularity

In this section, we assume that R is a noetherian Cohen–Macaulay domain over k of dimension $d \geqslant 2$ with a unique isolated singularity. For any finitely generated torsion free R-module M of rank one, $\overline{R} = \mathrm{End}_R(M) = R$ from Proposition 2.1 since R is normal by Serre's normality criterion. Moreover, if M admits a connection ∇, then the induced connection $\overline{\nabla}$ on $\overline{R}$ is the standard action of $\mathrm{Der}_k(R)$ on R.

From Proposition 2.2, we know that $\mathrm{H}^2_{\mathrm{Rin}}(\mathrm{Der}_k(R), R)$ contains the obstruction for the existence of an integrable connection on M. If it vanishes, then it follows from Theorem 2.4 that $\mathrm{H}^1_{\mathrm{Rin}}(\mathrm{Der}_k(R), R)$ is a moduli space for the integrable connections on M, up to analytic equivalence.

When $k = \mathbb{C}$ is the complex numbers, there are also other interpretations of the k-vector spaces $\mathrm{H}^1_{\mathrm{Rin}}(\mathrm{Der}_k(R), R)$ and $\mathrm{H}^2_{\mathrm{Rin}}(\mathrm{Der}_k(R), R)$: In this section, we will relate the Lie–Rinehart cohomology $\mathrm{H}^*_{\mathrm{Rin}}(\mathrm{Der}_k(R), R)$ to the topological cohomology $\mathrm{H}^*(U_{\mathrm{an}}, \mathbb{C})$ where $U = X \setminus \{x\}$, $X = \mathrm{Spec}(R)$ and $x \in X$ is the singular point. The Lie–Rinehart cohomology $\mathrm{H}^*_{\mathrm{Rin}}(\mathrm{Der}_k(R), R)$ is also closely related to the punctured de Rham cohomology of Huang, Luk and Yau [10] and therefore to the μ and τ invariants.

Theorem 4.1. *Let R be a finitely generated Cohen–Macaulay domain over $\mathbb{C}$ of dimension $d \geqslant 2$ with a unique isolated singularity $x \in X = \mathrm{Spec}(R)$. Then there is a natural exact sequence*

$$0 \to \mathrm{H}^1_{\mathrm{Rin}}\big(\mathrm{Der}(R), R\big) \to \mathrm{H}^1(U_{\mathrm{an}}, \mathbb{C}) \to E_2^{0,1} \to \mathrm{H}^2_{\mathrm{Rin}}\big(\mathrm{Der}(R), R\big) \to \mathrm{H}^2(U_{\mathrm{an}}, \mathbb{C}),$$

where $E_2^{0,1} = \ker(\mathrm{H}^1(U, \mathcal{O}_X) \to \mathrm{H}^1(U, \Omega^1_X))$ and $U = X \setminus \{x\}$. Moreover, if $d \geqslant 3$, then $E_2^{0,1} = 0$.

Proof. The definition of the Lie–Rinehart complex generalizes to give a complex $\mathcal{C}^*_{\mathrm{Rin}}(\Theta_X, \mathcal{O}_X)$ of sheaves on X, given by

$$\mathcal{C}^n_{\mathrm{Rin}}(\Theta_X, \mathcal{O}_X) = \mathcal{H}om_{\mathcal{O}_X}\big(\wedge^n \Theta_X, \mathcal{O}_X\big)$$

with the natural action of the tangent sheaf Θ_X on $\mathcal{O}_X$. In particular, there is a restricted complex $\mathcal{C}^*_{\mathrm{Rin}}|_U = \mathcal{C}^*_{\mathrm{Rin}}(\Theta_U, \mathcal{O}_U)$ of sheaves on U. Denote by $\mathbb{H}^i = \mathbb{H}^i(U, \mathcal{C}^*_{\mathrm{Rin}}|_U)$ the hypercohomology of the sheafified Lie–Rinehart complex, see for instance 5.7.9 in Weibel [19]. From the five term sequence, we get

$$0 \to E_2^{1,0} \to \mathbb{H}^1 \to E_2^{0,1} \to E_2^{2,0} \to \mathbb{H}^2,$$

where $E_2^{p,q} = {}^I E_2^{p,q} \cong \mathrm{H}^p(\mathrm{H}^q(U, \mathcal{C}^*_{\mathrm{Rin}}|_U))$. Consider in particular the vector spaces

$$E_2^{p,0} = \mathrm{H}^p\big(\mathrm{H}^0\big(U, \mathcal{C}^*_{\mathrm{Rin}}\big|_U\big)\big).$$

Since $\mathcal{C}^p_{\mathrm{Rin}}|_U$ is sheaf of reflexive modules for $p \geqslant 0$ by Corollary 1.2 in Hartshorne [9], we get from Proposition 1.6(iii) in [9] that $\mathrm{H}^0(U, \mathcal{C}^*_{\mathrm{Rin}}|_U) = \mathrm{C}^*_{\mathrm{Rin}}(\mathrm{Der}(R), R)$. Thus $E_2^{p,0} \cong \mathrm{H}^p_{\mathrm{Rin}}(\mathrm{Der}(R), R)$. Note further that since U is smooth, $\mathcal{C}^*_{\mathrm{Rin}}|_U$ coincides with the de Rham complex, so by Grothendieck's algebraic de Rham theorem, $\mathbb{H}^i = \mathbb{H}^i(U, \mathcal{C}^*_{\mathrm{Rin}}|_U) \cong \mathrm{H}^i(U_{\mathrm{an}}, \mathbb{C})$, see [8]. On the other hand, we see that

$$E_2^{0,1} = \mathrm{H}^0\big(\mathrm{H}^1\big(U, \mathcal{C}^*_{\mathrm{Rin}}\big|_U\big)\big) \cong \ker\big(\mathrm{H}^1(U, \mathcal{O}_X) \to \mathrm{H}^1\big(U, \Omega^1_X\big)\big).$$

For the last part, we notice that $\mathrm{H}^1(U, \mathcal{O}_X) = \mathrm{H}^2_{\{x\}}(\mathcal{O}_X)$, where the last group is the local cohomology with respect to the closed subscheme $\{x\}$, see for instance 4.6.2 in Weibel [19]. By Corollary 4.6.9 in [19], this group vanishes if $d \geqslant 3$. In particular, it follows that $E_2^{0,1} = 0$ in this case. □

For surface singularities, it is in general difficult to compute $E_2^{0,1}$ directly. We have the following partial results:

Remark 4.2. For a rational complex surface singularity with link L, it is known that $\mathrm{H}^1(L,\mathbb{C})=0$, see Mumford [15], and by Poincarè duality, $\mathrm{H}^2(L,\mathbb{C})=0$. For simplicity, we assume that R is quasi-homogenous (for instance a quotient singularity). In this case, we have $\mathrm{H}^i(U_{\mathrm{an}},\mathbb{C})=\mathrm{H}^i(L,\mathbb{C})=0$ for $i=1$ and 2. Thus $\mathrm{H}^1_{\mathrm{Rin}}(\mathrm{Der}(R),R)=0$ and $E_2^{0,1}\cong\mathrm{H}^2_{\mathrm{Rin}}(\mathrm{Der}(R),R)$.

Remark 4.3. For simple elliptic complex surface singularities, Kahn has shown that $\mathrm{H}^1_{\mathrm{Rin}}(\mathrm{Der}(R),R)\cong\mathbb{C}$, see [12]. One the other hand, $\mathrm{H}^1(U_{\mathrm{an}},\mathbb{C})\cong\mathbb{C}^2$, so $E_2^{0,1}\neq 0$ in this case.

Remark 4.4. Lie–Rinehart cohomology is related to the punctured local holomorphic de Rham cohomology $H^i_h(V,x)$ introduced in Huang, Luk and Yau [10] for a germ (V,x) of a complex analytic space. In fact, for a hypersurface singularity R of dimension $d\geqslant 2$, $H^i_h(V,x)$ coincides with $\mathrm{H}^i_{\mathrm{Rin}}(\mathrm{Der}_{\mathbb{C}}(R),R)$ for $i\geqslant 1$, see Lemma 2.7 in [10] and Proposition 1.6(iii) in Hartshorne [9]. The main theorem in Huang, Luk and Yau [10] states that when $R=\mathbb{C}[[x_0,\dots,x_d]]/(f)$ is an isolated singularity of dimension $d\geqslant 2$, then

(1) $\dim_{\mathbb{C}}\mathrm{H}^i_{\mathrm{Rin}}(\mathrm{Der}_{\mathbb{C}}(R),R)=0$ for $1\leqslant i\leqslant d-2$,
(2) $\dim_{\mathbb{C}}\mathrm{H}^d_{\mathrm{Rin}}(\mathrm{Der}_{\mathbb{C}}(R),R)-\dim_{\mathbb{C}}\mathrm{H}^{d-1}_{\mathrm{Rin}}(\mathrm{Der}_{\mathbb{C}}(R),R)=\mu-\tau$,

where μ is the Milnor number and τ is the Tjurina number of the singularity.

5. The case of a quasi-homogenous surface

In this section, we compute the Lie–Rinehart cohomology $\mathrm{H}^*_{\mathrm{Rin}}(\mathrm{Der}_k(R),R)$ in the case of a integral quasi-homogenous surface singularity $R=k[x_1,x_2,x_3]/(f)$. We write $d_i=\deg x_i$ for $i=1,2,3$, $d=\deg f\geqslant 2$ and put $\omega_i=d_i/d$, $\delta=\omega_1+\omega_2+\omega_3-1$.

The Lie–Rinehart complex $\mathrm{C}^*=\mathrm{C}^*_{\mathrm{Rin}}(\mathrm{Der}_k(R),R)$ in the present case is given as

$$\mathrm{C}^0=R\xrightarrow{d^0}\mathrm{C}^1=\mathrm{Hom}_R\big(\mathrm{Der}_k(R),R\big)\xrightarrow{d^1}\mathrm{C}^2=\mathrm{Hom}_R\big(\wedge^2\mathrm{Der}_k(R),R\big)\to 0$$

since $\wedge^3\mathrm{Der}_k(R)$ is supported at the singular locus of $\mathrm{Spec}(R)$. The map d^0 is given by $d^0(r)(D)=D(r)$ for $r\in R$, $D\in\mathrm{Der}_k(R)$ and d^1 is given by

$$d^1(\varphi)\big((D_1\wedge D_2)\big)=D_1\big(\varphi(D_2)\big)-D_2\big(\varphi(D_1)\big)-\varphi\big([D_1,D_2]\big)$$

for $\varphi\in\mathrm{Hom}_R(\mathrm{Der}_k(R),R)$ and $D_1,D_2\in\mathrm{Der}_k(R)$. Note that $\mathrm{H}^i_{\mathrm{Rin}}(\mathrm{Der}_k(R),R)=0$ for $i\geqslant 3$.

It is clear that $\mathrm{C}^1=\mathrm{Hom}_R(\mathrm{Der}_k(R),R)$ and $\mathrm{C}^2=\mathrm{Hom}_R(\wedge^2\mathrm{Der}_k(R),R)$ are graded, and that d^0 and d^1 are homogenous of degree zero. It follows from properties of an isolated quasi-homogenous singularity that $\mathrm{Der}_k(R)$ is naturally generated by the Euler derivation E (homogenous of degree 0) and the Kozul derivations D_1, D_2, D_3 (homogenous of degree $d-d_1-d_2$, $d-d_1-d_3$, $d-d_2-d_3$ respectively), given by

$$E=\omega_1x_1\frac{\partial}{\partial x_1}+\omega_2x_2\frac{\partial}{\partial x_2}+\omega_3x_3\frac{\partial}{\partial x_3},\qquad D_1=\frac{\partial f}{\partial x_2}\frac{\partial}{\partial x_1}-\frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_2},$$
$$D_2=\frac{\partial f}{\partial x_3}\frac{\partial}{\partial x_1}-\frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_3},\qquad D_3=\frac{\partial f}{\partial x_3}\frac{\partial}{\partial x_2}-\frac{\partial f}{\partial x_2}\frac{\partial}{\partial x_3}.$$

In fact, we may consider $\mathrm{Der}_k(R)$ as the submodule of $R\otimes\mathrm{Der}_k(k[x_1,x_2,x_3])$ of derivations D such that $D(f)\in(f)$. Replacing D with $D+rE$ for some $r\in k[x_1,x_2,x_3]$, we may assume that $D(f)=0$. Since R has an isolated singularity, $\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\frac{\partial f}{\partial x_3}$ is a regular sequence, and from this it follows that

D is in the submodule of $\operatorname{Der}_k(R)$ generated by D_1, D_2 and D_3. For details on derivations on quasi-homogenous singularities, see Saito [17] and Scheja and Wiebe [18].

To give a description of $C^1 = \operatorname{Hom}_R(\operatorname{Der}_k(R), R)$, we define

$$\varphi = \begin{pmatrix} f_1 & f_2 & f_3 & 0 \\ \omega_2 x_2 & -\omega_1 x_1 & 0 & f_3 \\ \omega_3 x_3 & 0 & -\omega_1 x_1 & -f_2 \\ 0 & \omega_3 x_3 & -\omega_2 x_2 & f_1 \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} \omega_1 x_1 & f_2 & f_3 & 0 \\ \omega_2 x_2 & -f_1 & 0 & f_3 \\ \omega_3 x_3 & 0 & -f_1 & -f_2 \\ 0 & \omega_3 x_3 & -\omega_2 x_2 & \omega_1 x_1 \end{pmatrix},$$

where $f_i = \partial f / \partial x_i$.

Lemma 5.1. *The matrices (φ, ψ) give a matrix factorization of* $\operatorname{Der}_k(R)$ *and the transposed matrices (φ^T, ψ^T) give a matrix factorization of* $\operatorname{Hom}_R(\operatorname{Der}_k(R), R)$.

Proof. This follows from Lemma 1.5 in Yoshino and Kawamoto [20] and Proposition 2.1 in Behnke [1]. For the last part see for instance Lemma 11 in Eriksen and Gustavsen [6]. □

Mapping $\operatorname{Hom}_R(\operatorname{Der}_k(R), R)$ into R^4 by evaluation on (E, D_1, D_2, D_3), we obtain the rows $\psi^{(i)}$ in ψ as generators for $\operatorname{Hom}_R(\operatorname{Der}(R), R)$ in R^4. We see that $\deg \psi^{(i)} = d_i$ for $i = 1, 2, 3$, and $\deg \psi^{(4)} = d_1 + d_2 + d_3 - d = d\delta$.

To give a description of $\mathrm{C}^2 = \operatorname{Hom}_R(\wedge^2 \operatorname{Der}_k(R), R)$, we consider the element $\Delta = \frac{\partial f}{\partial x_3} \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} - \frac{\partial f}{\partial x_2} \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} + \frac{\partial f}{\partial x_1} \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}$ of degree $d - d_1 - d_2 - d_3 = -d\delta$. A calculation gives

$$\begin{aligned} E \wedge D_1 &= \omega_3 x_3 \Delta, & E \wedge D_2 &= -\omega_2 x_2 \Delta, & E \wedge D_3 &= \omega_1 x_1 \Delta, \\ D_1 \wedge D_2 &= \frac{\partial f}{\partial x_1} \Delta, & D_1 \wedge D_3 &= \frac{\partial f}{\partial x_2} \Delta, & D_2 \wedge D_3 &= \frac{\partial f}{\partial x_3} \Delta, \end{aligned}$$

and we conclude that $\wedge^2 \operatorname{Der}_k(R) = (x_1, x_2, x_3)\Delta$. From this, we get the following isomorphisms of graded modules:

$$\operatorname{Hom}_R\bigl(\wedge^2 \operatorname{Der}_k(R), R\bigr) = \operatorname{Hom}_R(\mathfrak{m}\Delta, R) \cong \operatorname{Hom}_R(R\Delta, R) \cong R[-\deg \Delta].$$

We compute the map d^1 and get

$$\begin{aligned} d^1\bigl(r\psi^{(1)}\bigr)(E \wedge D_3) &= E\bigl(r\psi^{(1)}\bigr) - D_3\bigl(r\psi^{(1)}(E)\bigr) - r\psi^{(1)}\bigl([E, D_3]\bigr) = -\omega_1 x_1 D_3(r), \\ d^1\bigl(r\psi^{(2)}\bigr)(E \wedge D_2) &= E\bigl(r\psi^{(2)}\bigr) - D_2\bigl(r\psi^{(2)}\bigr)(E) - r\psi^{(2)}\bigl([E, D_2]\bigr) = -\omega_2 x_2 D_2(r), \\ d^1\bigl(r\psi^{(3)}\bigr)(E \wedge D_1) &= E\bigl(r\psi^{(3)}\bigr) - D_1\bigl(r\psi^{(3)}\bigr)(E) - r\psi^{(3)}\bigl([E, D_1]\bigr) = -\omega_3 x_3 D_1(r), \\ d^1\bigl(r\psi^{(4)}\bigr)(E \wedge D_1) &= E\bigl(r\psi^{(4)}\bigr) - D_1\bigl(r\psi^{(4)}\bigr)(E) - r\psi^{(4)}\bigl([E, D_1]\bigr) = \omega_3 x_3\bigl(E(r) + \delta r\bigr), \end{aligned}$$

using

$$[E, D_1] = (1 - \omega_1 - \omega_2)D_1, \qquad [E, D_2] = (1 - \omega_1 - \omega_3)D_2, \qquad [E, D_3] = (1 - \omega_2 - \omega_3)D_3.$$

From this we conclude that

$$\begin{aligned} d^1\bigl(r\psi^{(1)}\bigr)(\Delta) &= -D_3(r), & d^1\bigl(r\psi^{(2)}\bigr)(\Delta) &= D_2(r), \\ d^1\bigl(r\psi^{(3)}\bigr)(\Delta) &= -D_1(r), & d^1\bigl(r\psi^{(4)}\bigr)(\Delta) &= E(r) + \delta r, \end{aligned}$$

and in conclusion we have reached a very concrete description of the Lie–Rinehart complex $\mathrm{C}^*_{\mathrm{Rin}}(\mathrm{Der}_k(R), R)$. Using this, we are able to prove the following result:

Theorem 5.2. *Let $R = k[x_1, x_2, x_3]/(f)$ be an integral quasi-homogenous surface singularity. Then the grading on R induces a grading on $\mathrm{H}^*_{\mathrm{Rin}}(\mathrm{Der}_k(R), R)$, and*

$$\begin{aligned} &\mathrm{H}^0_{\mathrm{Rin}}\big(\mathrm{Der}_k(R), R\big) = \mathrm{H}^0_{\mathrm{Rin}}\big(\mathrm{Der}_k(R), R\big)_0 \cong R_0 = k, \\ &\mathrm{H}^i_{\mathrm{Rin}}\big(\mathrm{Der}_k(R), R\big) = \mathrm{H}^i_{\mathrm{Rin}}\big(\mathrm{Der}_k(R), R\big)_0 \cong R_{d-d_1-d_2-d_3} \quad \text{for } i = 1, 2, \\ &\mathrm{H}^i_{\mathrm{Rin}}\big(\mathrm{Der}_k(R), R\big) = 0 \quad \text{for } i \geqslant 3, \end{aligned}$$

as graded k-vector space, where $d_i = \deg x_i$ for $i = 1, 2, 3$ and $d = \deg f \geqslant 2$.

Proof. For simplicity, we write $\mathrm{H}^i = \mathrm{H}^i_{\mathrm{Rin}}(\mathrm{Der}_k(R), R)$ for $i \geqslant 0$. We have that $\mathrm{H}^0 = \{r \in R\colon D(r) = 0,\ \forall D \in \mathrm{Der}_k(R)\}$. If $r \in \mathrm{H}^0_\omega$, then $E(r) = \omega r = 0$ implies that $\omega = 0$ or $r = 0$. Thus $\mathrm{H}^0 = R_0 = k$.

We have that $\mathrm{H}^2 = \operatorname{coker} d^1 = \mathrm{C}^2/\operatorname{im} d^1 \cong R[-d\delta]/\operatorname{im} d^1$ and $\operatorname{im} d^1$ is spanned by $D_i(r)$ for $i = 1, 2, 3$ and $E(r) + \delta r = (\frac{\deg r}{d} + \delta) r$ as r runs through all homogenous elements in R. Since $\deg D_i = -d\delta + d_i$, for $i = 1, 2, 3$, it follows that $D_i(r) \in R_{-d\delta + d_i + \deg r} = \mathrm{C}^2_{d_i + \deg r}$ for $i = 1, 2, 3$. Furthermore, $\frac{\deg r}{d} + \delta = 0$ if and only if $\deg r = -\delta d$. We conclude that $\operatorname{im} d^1 = \mathrm{C}^2_{\neq 0}$, and hence $\mathrm{H}^2 = \mathrm{C}^2_0 \cong R_{d-d_1-d_2-d_2}$.

To compute H^1, we note that $\operatorname{im} d^0_0 = 0$ and $\ker d^1_0 \cong R_{-d\delta} \cdot \psi^{(4)}$ by the argument above. It follows that $\mathrm{H}^1_0 \cong R_{-d\delta} \cdot \psi^{(4)} \cong R_{d-d_1-d_2-d_2}$. We claim that $\mathrm{H}^1_\omega = 0$ for $\omega \neq 0$. To prove the claim, we first note that since $\mathrm{H}^2_\omega = 0$ for $\omega \neq 0$, it follows that d^1_ω induces an isomorphism $\mathrm{C}^1_\omega / \ker d^1_\omega \cong \mathrm{C}^2_\omega$ for $\omega \neq 0$. Also, $\operatorname{im} d^0_\omega \cong \mathrm{C}^0_\omega$ for $\omega \neq 0$. Thus

$$\dim_k \mathrm{H}^1_\omega = \dim_k \ker d^1_\omega - \dim_k \operatorname{im} d^0_\omega = \dim_k \mathrm{C}^1_\omega - \dim_k \mathrm{C}^2_\omega - \dim_k \mathrm{C}^0_\omega$$

for $\omega \neq 0$. To compute these dimensions, recall the Auslander sequence

$$0 \to \omega_R \to E \to \mathfrak{m} \to 0,$$

where ω_R is the canonical module, E is the fundamental module and $\mathfrak{m}$ is the maximal graded ideal of R. From (the proof of) Proposition 2.1 in Behnke [1], we have that $\omega_R \cong \mathrm{Hom}_R(\wedge^2 \mathrm{Der}_k(R), R) = \mathrm{C}^2$ and $E \cong \mathrm{Hom}_R(\mathrm{Der}_k(R), R) = \mathrm{C}^1$ as graded modules, since R is quasi-homogenous, see also Lemma 1.2 in Yoshino and Kawamoto [20]. Since there are homogenous isomorphisms $\mathrm{Ext}^1_R(\mathfrak{m}, \omega_R) \cong \mathrm{Ext}^2_R(R/\mathfrak{m}, \omega_R) \cong R/\mathfrak{m}$ of degree zero, see Definition 3.6.8, Example 3.6.10 and Proposition 3.6.12 in Bruns and Herzog [2], it follows that the Auslander sequence is homogenous of degree zero. Thus $\dim \mathrm{C}^1_\omega = \dim \mathrm{C}^2_\omega + \dim \mathrm{C}^0_\omega$ for $\omega \neq 0$. This proves the claim that $\mathrm{H}^1_\omega = 0$ for $\omega \neq 0$. □

Remark 5.3. It follows from Theorem 4.4 that $\mathrm{H}^1_{\mathrm{Rin}}(\mathrm{Der}_k(R), R) \cong \mathrm{H}^2_{\mathrm{Rin}}(\mathrm{Der}_k(R), R)$ when $R \cong k[x_1, x_2 x_3]/(f)$ is a quasi-homogenous surface singularity, since it is known that $\mu = \tau$ in the quasi-homogenous case. From Theorem 5.2 it follows that $\mathrm{H}^1_{\mathrm{Rin}}(\mathrm{Der}_k(R), R) \cong \mathrm{H}^2_{\mathrm{Rin}}(\mathrm{Der}_k(R), R)$ as graded k-vector spaces.

Remark 5.4. We see that all cohomology is concentrated in degree 0 in the case covered by the theorem. It follows from the proof of Proposition 2.2 that integrability class $\mathrm{ic}(M)$ lies in $\mathrm{H}^2_{\mathrm{Rin}}(\mathrm{Der}_k(R), R)_0$ for any graded torsion free rank one module M. Moreover, if $\mathrm{ic}(M) = 0$, it follows from the proof of Theorem 2.4 and Theorem 5.2 that $\mathrm{H}^1_{\mathrm{Rin}}(\mathrm{Der}_k(R), R)_0$ is a moduli space for integrable connections. Hence up to analytic equivalence, all integrable connections are homogenous.

Example 5.5. We consider the singularity $R=\mathbb{C}[x_1,x_2,x_3]/(f)$, where f is homogenous of degree d, with $d_1=d_2=d_3=1$. Then $\mathrm{H}^i_{\mathrm{Rin}}(\mathrm{Der}_k(R),R)\cong R_{d-1-1-1}=R_{d-3}$ so that

$$\dim_{\mathbb{C}}\mathrm{H}^i_{\mathrm{Rin}}\big(\mathrm{Der}_k(R),R\big)=\binom{d-3+2}{d-3}=\binom{d-1}{d-3}=\frac{(d-1)(d-2)}{2}$$

for $i=1,2$. This number is the genus of the curve $V(f)$ in $\mathbb{P}^2$, and therefore also the genus the exceptional curve in the minimal resolution of R.

Example 5.6. The minimally elliptic singularity $\mathbb{C}[x_1,x_2,x_3]/(x_1^3+x_2^4+x_3^4)$ has $d=12, d_1=4, d_2=d_3=3$, so $\mathrm{H}^i_{\mathrm{Rin}}(\mathrm{Der}_k(R),R)\cong R_{12-4-3-3}=R_2=0$ for $i=1,2$.

Corollary 5.7. *Let $R=k[x_1,x_2,x_3]/(f)$ be an integral quasi-homogeneous surface singularity, and let M be any finitely generated torsion free graded R-module of rank one. Then any homogenous connection on M is integrable.*

Proof. Let ∇ be an arbitrary homogenous connection on M, and let

$$0\leftarrow M\leftarrow L_0\xleftarrow{d_0}L_1$$

be a graded presentation of M, where $\{e_i\}$ and $\{f_i\}$ are homogeneous bases of L_0 and L_1, and $d_0=(a_{ij})$ is the matrix of d_0 with respect to these bases. Then we have $\deg(a_{ij})=\deg(f_j)-\deg(e_i)$ for all i,j. We consider the diagonal matrix P with entries $\epsilon_j=(\deg(e_j)-\deg(e_1))/d$ on the diagonal. Since we have

$$E(d_0)=\frac{1}{d}\deg(a_{ij})(a_{ij})=\frac{1}{d}\big(\deg(f_j)-\deg(e_i)\big)(a_{ij}),$$

we see that $E(d_0)+Pd_0=d_0Q$ for some $Q\in\mathrm{End}_R(L_1)$. Therefore, $\nabla'_E=E+P\in\mathrm{End}_k(L_0)$ induces an operator $\nabla'_E\in\mathrm{End}_k(M)$ such that $\nabla'_E(rm)=E(r)m+r\nabla'_E(m)$ for all $r\in R$ and $m\in M$. Since $\nabla_E-\nabla'_E\in\mathrm{End}_R(M)_0=R_0=k$, it follows that $\nabla_E=E+P+\lambda I$ for some $\lambda\in k$.

We claim that the curvature $K_\nabla=0$. Since $K_\nabla\in\mathrm{Hom}_R(\wedge^2\mathrm{Der}_k(R),R)$, it follows from the calculations preceding Theorem 5.2 that it is enough to show that $K_\nabla(E\wedge D_1)=0$. We also have $[E,D_1]=\frac{1}{d}(d-d_1-d_2)D_1$. Write $\nabla_{D_1}=D_1+Q$, where $Q=(q_{ij})\in\mathrm{End}_R(L_0)$ and $\deg(q_{ij})=\deg(e_j)-\deg(e_i)+(d-d_1-d_2)$. Then:

$$\begin{aligned}K_\nabla(E\wedge D_1)&=\nabla_E\nabla_{D_1}-\nabla_{D_1}\nabla_E-\nabla_{[E,D_1]}\\&=(E+P+\lambda I)(D_1+Q)-(D_1+Q)(E+P+\lambda I)-\frac{1}{d}(d-d_1-d_2)(D_1+Q)\\&=E(Q)-D_1(P+\lambda I)+[P+\lambda I,Q]-\frac{1}{d}(d-d_1-d_2)Q\\&=E(Q)+[P,Q]-\frac{1}{d}(d-d_1-d_2)Q.\end{aligned}$$

A direct computation gives $[P,Q]=(-\frac{1}{d})(\deg(e_j)-\deg(e_i))(q_{ij})$, and we clearly have $E(Q)=\frac{1}{d}(d-d_1-d_2)Q+\frac{1}{d}(\deg(e_j)-\deg(e_i))(q_{ij})$. $\square$

Example 5.8. Let $R=k[x,y,z]/(x^3+y^3+z^3)$. The module M with presentation matrix

$$\begin{pmatrix} x & -y^2+yz-z^2 \\ y+z & x^2 \end{pmatrix}$$

is a maximal Cohen–Macaulay of rank one, see Laza et al. [13]. The derivation module $\mathrm{Der}_k(R)$ is generated by the four derivations

$$E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \qquad D_1 = 3y^2\frac{\partial}{\partial x} - 3x^2\frac{\partial}{\partial y},$$
$$D_2 = 3z^2\frac{\partial}{\partial x} - 3x^2\frac{\partial}{\partial z}, \qquad D_3 = 3z^2\frac{\partial}{\partial y} - 3y^2\frac{\partial}{\partial z},$$

where E is the Euler derivation and D_1, D_2 and D_3 are the Kozul derivations. Using our Singular [7] library CONNECTIONS.LIB [5], we find that a connection is represented by

$$\nabla_E = E + \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}, \qquad \nabla_{D_1} = D_1 + \begin{pmatrix} 0 & 2x \\ -2y+z & 0 \end{pmatrix},$$
$$\nabla_{D_2} = D_2 + \begin{pmatrix} 0 & 2x \\ y-2z & 0 \end{pmatrix}, \qquad \nabla_{D_3} = D_3 + \begin{pmatrix} -2y+2z & 0 \\ 0 & y-z \end{pmatrix}.$$

Again using [5], we check that this is an integrable connection. Further one finds that the connection represented by

$$\nabla'_E = E + \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}, \qquad \nabla'_{D_1} = D_1 + \begin{pmatrix} xz & 2x \\ -2y+z & xz \end{pmatrix},$$
$$\nabla'_{D_2} = D_2 + \begin{pmatrix} -xy & 2x \\ x^2+y-2z & xz \end{pmatrix}, \qquad \nabla'_{D_3} = D_3 + \begin{pmatrix} x^2-2y+2z & 0 \\ 0 & x^2+y-z \end{pmatrix}$$

is not integrable.

The integrability class $\mathrm{ic}(M) = 0$ in $\mathrm{H}^2_{\mathrm{Rin}}(\mathrm{Der}_k(R), R)$ which means that ∇' becomes integrable after removing terms of degree different from zero. In fact, we see that this gives ∇.

We also find that $\mathrm{H}^1_{\mathrm{Rin}}(\mathrm{Der}_k(R), R) \cong k$, which means that there is a one parameter family of integrable connections on M.

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