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Exploring the behavior of LWR continuum models of traffic flow in presence of shock waves

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Abstract

LWR (Lighthill-Whitham-Richards) models represent the behavior of traffic streams through the continuity equation and an assumed equilibrium speed-density relationship. Such models assume traffic streams to be always in equilibrium. That is it assumes that the speed and density values at any point in the stream at any time are according to the equilibrium relation. This paper shows that in the presence of shocks this assumption is not valid for every equilibrium speed-density relation.

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keywords: Traffic flow; LWR model; shock wave; shock speed

1. Introduction

LWR (Lighthill-Whitham-Richards) models represent the behavior of traffic streams through the continuity equation and an assumed equilibrium speed-density relationship. Such models assume traffic streams to be always in equilibrium. That is it assumes that the speed and density values at any point in the stream at any time are according to the equilibrium relation. This paper shows that in the presence of shocks this assumption is not valid for every equilibrium speed-density relation.

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Nomenclature

| | | |
|--------------------------------------|------------------------------------|---------------------|
| q : traffic flow rate | k : traffic density | u : traffic speed |
| k_0 : traffic density at max. flow | u_0 : traffic speed at max. flow | k_j : jam density |
| u_j : free-flow speed | | |

The paper is divided into five sections of which this is the first. Section 2 briefly describes LWR model. The third section presents numerical experiments to illustrate that the traffic speed and traffic density travel at different speeds at shocks for certain equilibrium speed-density relationships. It is also shown that these speeds are the same for other equilibrium relationships. Section 4 analytically demonstrates the fact that shock speeds for traffic speed and traffic density need not be equal for every speed-density relationships. Section 5 concludes the paper by highlighting that in light of the analysis presented in this paper the assumption of equilibrium in LWR models, especially in the presence of shocks, can be made only for certain speed-density relationships.

2. Background

LWR models were proposed independently by Lighthill and Whitham (1955) and Richards (1956). An LWR model consists of continuity equation, borrowed from fluid mechanics, fundamental equation of traffic flow ($q = uk$) and an equilibrium speed-density ($u-k$) relationship. Equation 1 presents the continuity equation. In traffic engineering, continuity equation as presented in Equation 1 states the conservation of number of vehicles on a section of a road with no entry or exit in the section.

$$\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0 \quad (1)$$

In LWR models, it is assumed that traffic always remains in equilibrium (i.e. $u = u(k)$). Hence Equation 1 takes the form given in Equation 2. This form is henceforth referred to as the LWR^k form.

$$\frac{\partial k}{\partial t} + \frac{dq}{dk} \frac{\partial k}{\partial x} = 0 \quad (2)$$

Equation 2 casts Equation 1 in k by using the fact that $u = u(k)$. Similarly Equation 1 can be cast in u since $u = u(k)$ is a one to one mapping between u and k . Hence one can write:

$$\frac{dk}{du} \frac{\partial u}{\partial t} + \frac{dq}{dk} \frac{dk}{du} \frac{\partial u}{\partial x} = 0 \quad (3)$$

Since, $u(k)$ is not a constant with respect to k (i.e., $\frac{dk}{du} \neq 0$). Equation 3 implies the LWR^u form as :

$$\frac{\partial u}{\partial t} + \frac{dq}{dk} \frac{\partial u}{\partial x} = 0 \quad (4)$$

Now, according to LWR models, traffic always remains in equilibrium. This means that the information of traffic density (k) and traffic speed (u) always travel together. This also means that if one variable (i.e., u or k) is computed through LWR model then the other one can be computed using the assumed equilibrium u - k relation. Therefore, if the information of u is computed using LWR^u and the corresponding information of k is obtained using the equilibrium u - k relation then this information on k should match with the information on k obtained from the corresponding LWR^k model. It is expected that this property should be present irrespective of the assumed u - k relation. However, it is seen that this property is absent for many u - k relations especially at shocks. It is also shown that only for certain u - k relations this property is present in LWR streams under all conditions.

In order to illustrate the inconsistency in u and k speeds that appear in certain LWR streams numerical experiments on LWR streams with Greenberg's u - k relation are presented. In order to also show that for certain assumed u - k relationships this inconsistency does not arise. LWR streams with Greenshields' u - k relation are presented. These experiments are presented in the next section.

3. Numerical demonstration of inconsistency in shock wave speed

In this section, numerical experiments with LWR^k and LWR^u models using Greenberg and Greenshields' u - k relation are presented. These models are in the form of PDE's and are solved using FEM formulation developed by Vikram et al. (2011). The PDE's are solved with suitable initial and boundary conditions. This section is divided into two subsections where the first subsection is dedicated to the numerical study of LWR^k and LWR^u models using Greenberg's u - k relation and the second subsection is dedicated to the numerical study of LWR^k and LWR^u models using Greenshields' u - k relation.

3.1 Study of LWR^k and LWR^u models using Greenberg's u - k relation

A test case is selected such that it leads to formation of a shock wave. Therefore, the initial condition of LWR^k corresponds to a sudden increase in traffic density, midway on the road. At $t = 0$, the density increases from 15 veh/km at $x = 495$ m to 65 veh/km at $x = 500$ m. Consequently the initial condition of LWR^u corresponds to a sudden drop in traffic speed. At $t = 0$, the speed decreases from 74.9 km/h at $x = 495$ m to 22.1 km/hr at $x = 500$ m. The boundary condition for solving the LWR^k model is applied at the inlet of the road where the density remains constant at 15 veh/km; similarly, the boundary condition for solving the LWR^u model is applied at the inlet of the road where the speed remains constant at 74.9 km/h over the time.

Figure 1 presents the results of the test case solved using both LWR^k and LWR^u models. Note in this case dq/dk in Equation 2 and 4 are obtained from the Greenberg's u - k relation; in this experiment the u - k relation used is $u = 10 \text{ m/s} \ln\left(\frac{120 \text{ veh/km}}{k}\right)$. Figure 1(a) presents the variation of traffic density, over the road at three instants of time, obtained by solving LWR^k model. The corresponding variation in traffic speed using Greenberg's u - k relation is presented in Figure 1(b). The variation in traffic speed obtained by solving LWR^u model is presented in Fig 1(c). The corresponding variation in traffic density obtained by using Greenberg's u - k is presented in Figure 1(d).

In this experiment two paths are followed in order to generate the u and k profiles of the stream. In Path 1, the density variation (k -profile) is generated using the LWR^k model and the speed variation (u -profile) is obtained by determining the speeds using Greenberg's u - k relation for the densities in the k -profile. The profiles from Path 1 are presented in Figures 1 (a) and (b). In Path 2, the u -profile is generated using LWR^u model and the k -profile is obtained by determining the densities using Greenberg's u - k relation for the speeds in the u -profile. The profiles from Path 2 are presented in Figures 1 (c) and (d).

Given that in LWR models the stream is always assumed to be in equilibrium, the profiles generated by Path 1 should match with those generated by Path 2. A quick comparison of the figures shows that this is not the case. In fact it can be easily seen that the shock fronts in Figure 1 (a) (or (b)) and Figure 1 (d) (or (c)) are moving at different speeds.

3.2 Study of LWR^k and LWR^u models based on Greenshields' $u-k$ relation

In this experiment also, a test case is selected such that it leads to formation of a shock wave. Therefore, the initial condition of LWR^k corresponds to a sudden increase in traffic density, midway on the road. At $t = 0$, the density increases from 15 veh/km at $x = 495$ m to 65 veh/km at $x = 500$ m. Consequently the initial condition of LWR^u corresponds to a sudden drop in traffic speed. At $t = 0$, the speed decreases from 87.5 km/h at $x = 495$ m to 45.8 km/hr at $x = 500$ m. The boundary condition for solving the LWR^k model is applied at the inlet of the road where the density remains constant at 15 veh/km; similarly, the boundary condition for solving the LWR^u model is applied at the inlet of the road where the speed remains constant at 87.5 km/h over the time.

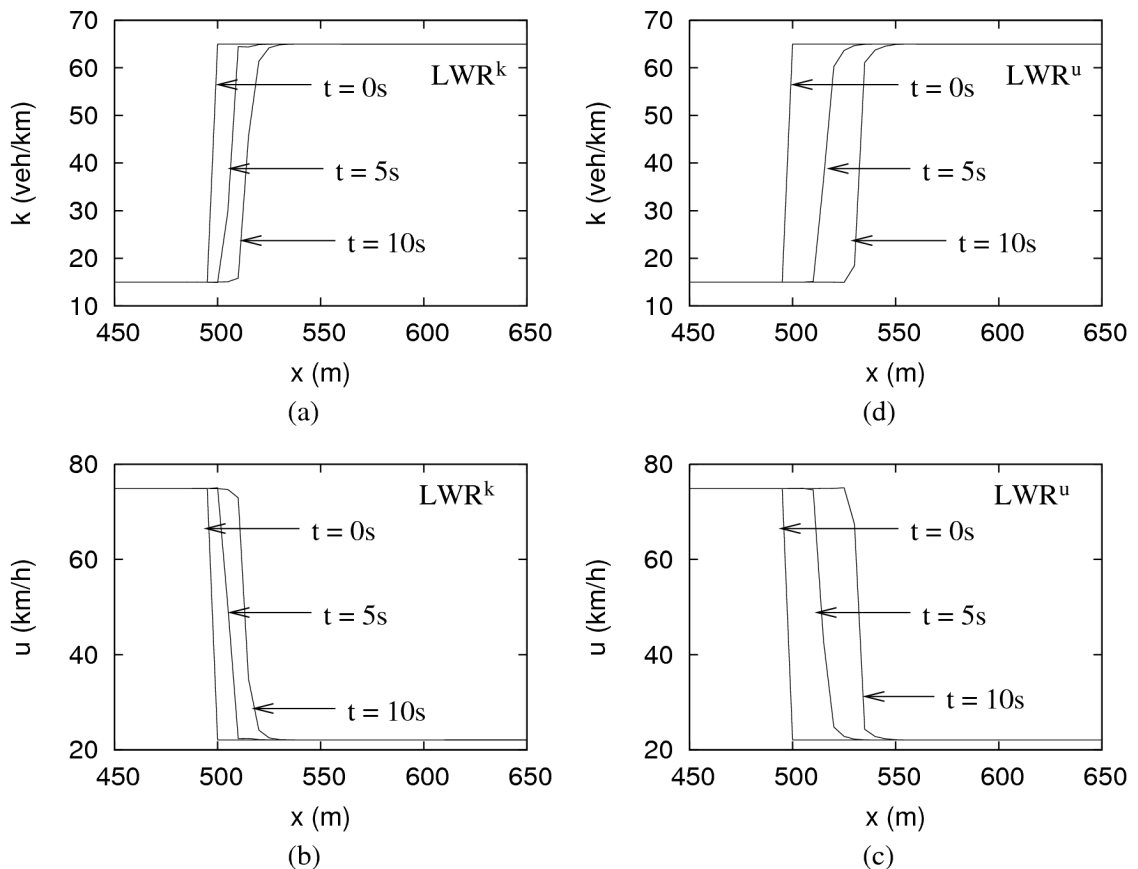


Fig 1: Variation of traffic (a) density, k and (c) speed, u obtained by solving LWR^k with Greenberg model, variation of traffic (b) density, k and (d) speed, u obtained by solving LWR^u with Greenberg model.

As before two paths are followed to generate the u and k profiles shown in Figure 2. Note in this case dq/dk in Equations 2 and 4 are obtained using the Greenshields' $u-k$ relation. In this experiment the $u-k$ relation used is

$$u = 100 \text{ km/h} \left(1 - \frac{120 \text{ veh/km}}{k} \right)$$

. Figures 2 (a) and 2 (b) respectively give the k -profile and u -profile using Path 1

(with Greenshields' model as the u - k relation). Figures 2 (c) and 2 (d) respectively give the u -profile and k -profile using Path 2 (with Greenshields' model as u - k relation).

Unlike in the experiment in Section 3.1, here both the paths generate identical profiles. Thus in this case there is no inconsistency in the LWR^k and LWR^u models. That is, although LWR model showed inconsistencies when the underlying u - k relation is assumed to be Greenberg's, the LWR model did not demonstrate any inconsistency when the underlying u - k relation is assumed to be Greenshields' relation.

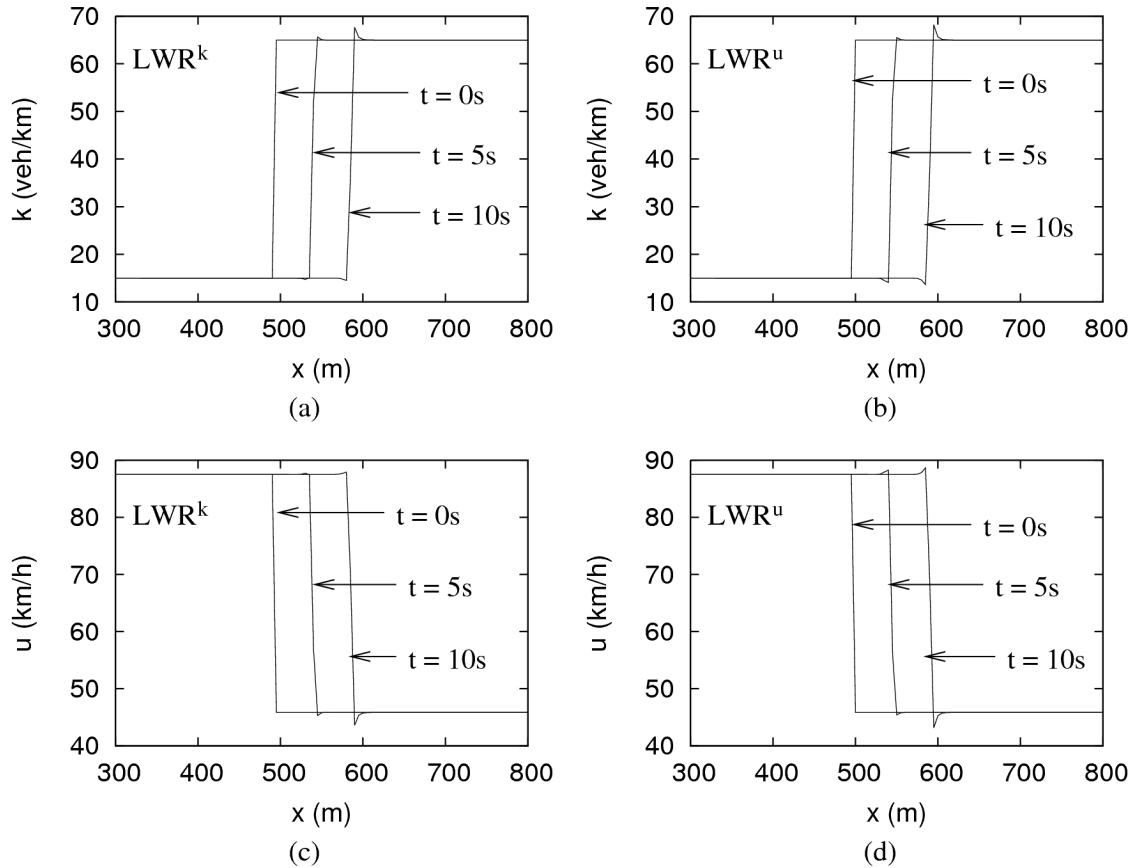


Fig 2: Variation of traffic (a) density, k and (c) speed, u obtained by solving LWR^k with Greenshields' model, variation of traffic (b) density, k and (d) speed, u obtained by solving LWR^u with Greenshields' model.

Fig 2 presents the results of a test case solved using both LWR^k and LWR^u models. Fig 2(a) presents the variation of traffic density, over the road at three instants of time, obtained by solving LWR^k model. The corresponding variation in traffic speed one should be able to obtain by making use of Greenshields' u - k relation or by solving the corresponding LWR^u model. The corresponding variation in traffic speed using Greenshields' u - k relation is presented in Fig 2(c). The corresponding variation in traffic speed obtained by solving LWR^u model is presented in Fig 2(d). If the variation in traffic speed obtained through solving LWR^u model is

considered then the corresponding variation in traffic density can be obtained by making use of Greenshields' $u-k$ relation. It is this variation in traffic density which is presented in Fig 2(b).

These results were intriguing and encouraged the authors to explore more on the issue of shock wave speeds of LWR^k and LWR^u models based on different $u-k$ relations. Analytical studies on the shock wave speeds of LWR^k and LWR^u models based on different $u-k$ relations are carried out in the next section to throw more light on this issue.

4. Analytical study on shock wave speeds of LWR^k and LWR^u models for various $u-k$ relations

As mentioned earlier, for any assumed equilibrium $u-k$ relation the LWR model can be viewed in two different forms, namely the LWR^k and LWR^u forms. In this section shock speeds as implied by these two forms for four different equilibrium $u-k$ relations are derived. The equilibrium $u-k$ relations used are (i) Greenberg's relation: $u = u_0 \ln(k_j / k)$, (ii) Underwood's relation: $u = u_f e^{-k/k_0}$, (iii) Northwestern's relation:

$$u = u_f e^{-k^2/2k_0^2} \text{ and (iv) Greenshields' relation: } u = u_f - (u_f / k_j)k$$

4.1. Shock wave speeds from LWR^k and LWR^u forms using Greenberg's $u-k$ relation

The LWR^k form based on Greenberg's $u-k$ relation is obtained by replacing dq/dk appropriately in Equation 2. This is shown in Equation 5, which, however, is not in conservation form. Equation 5 is brought to conservation form by using integration by parts; the conservation form of LWR^k for Greenberg's $u-k$ relation is presented in Equation 6.

$$\frac{\partial k}{\partial t} + u_0 \left(\ln \left(\frac{k_j}{k} \right) - 1 \right) \frac{\partial k}{\partial x} = 0 \tag{5}$$

$$\frac{\partial k}{\partial t} + \left(u_0 k \ln \left(\frac{k_j}{k} \right) \right) \frac{\partial k}{\partial x} = 0 \tag{6}$$

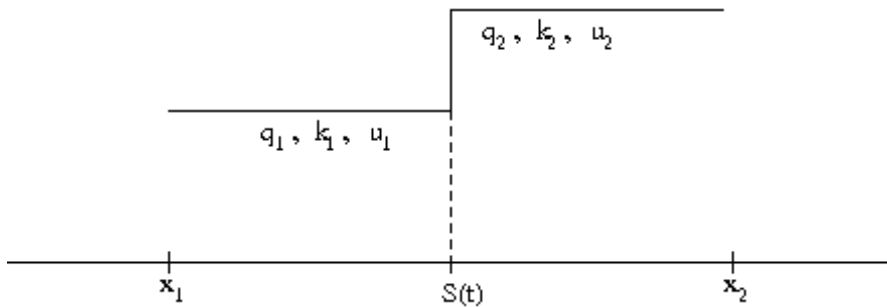


Fig. 3 A schematic of the section of a road with a shock wave at $S(t)$.

A schematic of a section of a road with a shock wave at $S(t)$ is presented in Figure 3. The flow of traffic is considered to be from x_1 to x_2 . A shock wave in traffic flow is discontinuity in traffic variables and this is indicated by a vertical line in Figure 3. From the conservation form of the LWR^k presented in Equation 6, the integral form of LWR^k can be written as follows,

$$\frac{\partial}{\partial t} \int_{x_1}^{x_2} k dx = -u_0 \left(k_2 \ln \frac{k_j}{k_2} - k_1 \ln \frac{k_j}{k_1} \right) \tag{7}$$

From Figure 3, Equation 7 can be expanded as follows:

$$\frac{\partial}{\partial t} \left(\int_{x_1}^{S(t)} k dx + \int_{S(t)}^{x_2} k dx \right) = -u_0 \left(k_2 \ln \frac{k_j}{k_2} - k_1 \ln \frac{k_j}{k_1} \right) \tag{8}$$

Since, x_1 and x_2 are fixed locations, Equation 8 can be written as:

$$k_1 \frac{\partial}{\partial t} (S(t) - x_1) + k_2 \frac{\partial}{\partial t} (x_2 - S(t)) = u_0 \left(k_1 \ln \frac{k_j}{k_1} - k_2 \ln \frac{k_j}{k_2} \right) \tag{9}$$

or,

$$k_1 \dot{S}(t) - k_2 \dot{S}(t) = u_0 \left(k_1 \ln \frac{k_j}{k_1} - k_2 \ln \frac{k_j}{k_2} \right) \tag{10}$$

The shock wave speed derived from LWR^k can be obtained by rearranging the terms in Equation 10:

$$\dot{S}(t) = \frac{u_0}{k_1 - k_2} \left(k_1 \ln \frac{k_j}{k_1} - k_2 \ln \frac{k_j}{k_2} \right) \tag{11}$$

Having obtained the shock wave speed from LWR^k in Equation 11, the shock wave speed implied by the LWR^u form is now derived. Since the u-k relation is typically represented as $u = u(k)$, for the purpose of determining $\dot{S}(t)$ from LWR^u, it is advantageous to rewrite LWR^u in Equation 4 in the following form:

$$\frac{\partial u}{\partial t} + \frac{dq}{dk} \frac{du}{dk} \frac{\partial k}{\partial x} = 0 \tag{12}$$

Using Greenberg’s u-k relation, Equation 12 takes the following form:

$$\frac{\partial u}{\partial t} + \left(-\frac{u_0^2}{k} \ln \frac{k_j}{k} + \frac{u_0^2}{k} \right) \frac{\partial k}{\partial x} = 0 \tag{13}$$

Equation 13 can be written in conservation form as:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(\frac{u_0^2}{2} \left(\ln \frac{k_j}{k} \right)^2 - u_0^2 \ln \frac{k_j}{k} \right) = 0 \tag{14}$$

By following steps similar to those mentioned previously, the shock wave speed from LWR^u is:

$$\dot{S}(t) = \frac{u_0^2 \left[\left(\ln \frac{k_j}{k_1} \right)^2 - \left(\ln \frac{k_j}{k_2} \right)^2 \right]}{2u_0 \left(\ln \frac{k_j}{k_1} - \ln \frac{k_j}{k_2} \right)} - \frac{u_0^2 \left(\ln \frac{k_j}{k_1} - \ln \frac{k_j}{k_2} \right)}{u_0 \left(\ln \frac{k_j}{k_1} - \ln \frac{k_j}{k_2} \right)} \tag{15}$$

Equation 15 can be simplified as:

$$\dot{S}(t) = \frac{u_0}{2} \ln \frac{k_j^2}{k_1 k_2} - u_0 \tag{16}$$

By comparing the shock wave speed obtained from LWR^k form (as given in Equation 11) with that obtained from LWR^u form (as given in Equation 16) it can be easily concluded that they are different. An outcome of this difference could also be seen in Figure 1. As mentioned earlier, ideally, this inconsistency in the shock speeds should not be present.

4.2. Shock wave speeds from LWR^k and LWR^u forms using Underwood’s and Northwestern’s *u-k* relation

Analyzing along the lines shown in the previous section, the shock wave speeds, $\dot{S}(t)$ with Underwood’s *u-k* relation are obtained as follows. The $\dot{S}(t)$ from the LWR^k form is:

$$\dot{S}(t) = u_f \frac{k_1 e^{-k_1/k_0} - k_2 e^{-k_2/k_0}}{k_1 - k_2} \tag{17}$$

The $\dot{S}(t)$ from the LWR^u form is:

$$\dot{S}(t) = \frac{u_f}{4} \left(e^{-k_1/k_0} + e^{-k_2/k_0} \right) - \frac{u_f}{2k_0} \left(\frac{k_1 e^{-2k_1/k_0} - k_2 e^{-2k_2/k_0}}{e^{-k_1/k_0} - e^{-k_2/k_0}} \right) \tag{18}$$

As with Greenberg’s *u-k* relation in this case also the $\dot{S}(t)$ expression from LWR^k and LWR^u forms, respectively given in Equations 17 and 18 are different.

The shock wave speeds $\dot{S}(t)$ with Northwestern’s *u-k* relation are obtained as follows. The $\dot{S}(t)$ from the LWR^k form is:

$$\dot{S}(t) = \frac{u_f \left(k_1 e^{-k_1^2/2k_0^2} - k_2 e^{-k_2^2/2k_0^2} \right)}{k_1 - k_2} \tag{19}$$

The $\dot{S}(t)$ from the LWR^u form is:

$$\dot{S}(t) = \frac{u_f \left(k_1^2 e^{-k_1^2/2k_0^2} - k_2^2 e^{-k_2^2/2k_0^2} \right)}{k_0^2 \left(e^{-k_1^2/2k_0^2} - e^{-k_2^2/2k_0^2} \right)} \tag{20}$$

As with Greenberg’s and Underwood’s u - k relations, in this case also the $\dot{S}(t)$ expression from the LWR^k and LWR^u forms, respectively given in Equations 19 and 20 are different.

4.3. Shock wave speeds from LWR^k and LWR^u forms using Greenshields’ u - k relation

The LWR^k form based on Greenshields’ u - k relation is obtained by replacing dq/dk appropriately in Equation 2. This is shown in Equation 21, which, however, is not in conservation form. Equation 21 is brought to conservation form by using integration by parts; the conservation form of LWR^k for Greenshields’ u - k relation is presented in Equation 22.

$$\frac{\partial k}{\partial t} + \left(1 - \frac{2k}{k_j} \right) \frac{\partial k}{\partial x} = 0 \tag{21}$$

$$\frac{\partial k}{\partial t} + \frac{\partial}{\partial x} \left(u_f k \left(1 - \frac{k}{k_j} \right) \right) = 0 \tag{22}$$

By following steps similar to those mentioned in Section 4.1, the shock wave speed from LWR^k is:

$$\dot{S}(t) = u_f \frac{k_1 \left(1 - \frac{k_1}{k_j} \right) - k_2 \left(1 - \frac{k_2}{k_j} \right)}{k_1 - k_2} \tag{23}$$

which on simplification becomes,

$$\dot{S}(t) = u_f \left(1 - \frac{k_1 + k_2}{k_j} \right) \tag{24}$$

Having obtained the shock wave speed from LWR^k in Equation 24, the shock wave speed implied by the LWR^u form is now derived. Proceeding along the same lines as in Section 4.1 the LWR^u with Greenshields’ u - k relation can be written as:

$$\frac{\partial u}{\partial t} + \frac{u_f^2}{k_j} \left(\frac{2k}{k_j} - 1 \right) \frac{\partial k}{\partial x} = 0 \tag{25}$$

As before, Equation 25 is written in conservation form as follows:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(u_f^2 \frac{k}{k_j} \left(\frac{k}{k_j} - 1 \right) \right) = 0 \quad (26)$$

By following steps similar to those mentioned previously, the shock wave speed from LWR^u is:

$$\dot{S}(t) = u_f^2 \frac{\frac{k_1}{k_j} \left(\frac{k_1}{k_j} - 1 \right) - \frac{k_2}{k_j} \left(\frac{k_2}{k_j} - 1 \right)}{u_f \left(1 - \frac{k_1}{k_j} - 1 + \frac{k_2}{k_j} \right)} \quad (27)$$

Equation 27 can be simplified as:

$$\dot{S}(t) = u_f \left(1 - \frac{k_1 + k_2}{k_j} \right) \quad (28)$$

Equation 28 presents the shock wave speed derived from LWR^u using Greenshields' u - k relation. It can be observed that the shock wave speeds derived from LWR^k (see Equation 24) and LWR^u (see Equation 28) are equal. This was also observed in the numerical experiment results presented in Figure 2.

5. Conclusions

The analysis and numerical experiments in this paper shows that in the presence of shock the properties of LWR model become dependent on the choice of the equilibrium u - k relation. For certain u - k relations the shock speeds show a difference depending on whether the stream is simulated in terms of density or speed. In other words, for these cases, the speed information and the density information travel at different speeds in presence of a shock. In effect, for these cases the stream ceases to be in equilibrium in presence of shocks. Such a property is, for obvious reasons, not desirable in a theory (like LWR) that assumes the traffic stream to be in equilibrium at all times.

Surprisingly this fact about LWR models remained hidden. This is possibly because LWR models (being a 1-equation model) were always used to simulate the traffic stream in one variable (typically k) and the other variable (typically u) was calculated using the u - k relation.

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