JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 16, 1-14 (1966)

The Laplace Transform and Initial-Boundary Value Problems

R. C. MACCAMY*

Carnegie Institute of Technology, Pittsburgh, Pennsylvania

Submitted by R. J. Duffin

1. INTRODUCTION

This paper concerns the use of the Laplace transform in some mixed initial-boundary value problems for equations of the form $LU = U_t$ or $LU = U_{tt}$, where L is an elliptic differential operator. The procedure is familiar in the case of one space variable where one obtains ordinary differential equations for the transform and, in many cases, explicit solutions.

The ease of solution in the one-dimensional case often obscures the fact that useful information can be obtained without having the explicit solution. In this paper we discuss some problems in more than one space variable. The central idea is that the behavior of U for large t can be obtained by studying boundary-value problems for the (elliptic) transform equation. For the situations we consider these transform problems would be quite amenable to numerical solution.

The results we give are intended to be illustrative only. They have been or could be obtained by other methods. This is characteristic of Laplace transform techniques. They are useful mainly as heuristic devices. One may hope that the techniques given here would serve this purpose in more complicated situations including, for example, initial value problems in elasticity and electromagnetic theory. Related techniques have been used in acoustic theory [1] and in ship motion problems in [2] and [3].

Although the methods are heuristic, we do want to indicate the kinds of facts one needs to make them rigorous. We shall observe a sharp distinction between parabolic ($LU = U_t$) and hyperbolic ($LU = U_{tt}$) problems which is probably representative of the general situation. We carry through the procedure for the simple parabolic equation $LU = \Delta U - q(x) U = U_t$, $q \geqslant 0$, and find that complete proofs can be given using only elementary facts concerning integral equations.

There exists what amount to an abstract formulation of the procedure for

* This work was supported by the Air Force Office of Scientific Research.

409/16/1-1

parabolic problems in the work of Lax and Milgram [4]. Our work is simpler and more detailed than that of [4], but this is achieved at a drastic reduction in the generality. We remark that general results of the type we give for parabolic problems are also obtained by the methods of Friedman [5].

The physically interesting hyperbolic problems appear more difficult mainly because they require solutions in unbounded regions. We take as a representative problem the solution of the wave equation $\Delta U = U_{tt}$ in the region exterior to an obstacle. Some interesting results for this problem have been obtained by Lax, Morawetz, and Phillips [6]. We obtain similar results,[1] in a partially heuristic manner, by studying exterior problems for the equation,

$$\Delta u - S^2 u = 0. \tag{1.1}$$

Difficulties arise in our procedure because there are unanswered questions concerning (1.1). In particular we need facts about the behavior of solutions of (1.1) for large S. An asymptotic theory for large S has been given by Keller [7] but its validity has not yet been established. This prevents us from completing our proofs.

It is known that the behavior of solutions of (1.1) is different in two and three dimensions. In Section 4 we use this fact to show how the results of [6] should be modified in two-dimensional problems.

The principal tool in both [4] and [6] was semi-group theory. The present work illustrates the close connection between that theory and the Laplace transform. Our conditions for the inversion of the Laplace transform (Section 2) correspond to the hypothesis of the Hille-Yoshida theorem.

2. Singularities of the Laplace Transform

In this section we collect some facts concerning the Laplace transform. The proofs are quite simple and are essentially contained in Doetsch [8]. We write

$$L_S(F) = \int_0^\infty e^{-St} F(t)\, dt, \qquad \lambda_\beta(f) = (2\pi)^{-1} e^{\beta t} \int_{-\infty}^{+\infty} e^{i\eta t} f(\beta + i\eta)\, d\eta.$$

It is known that under some conditions L_S and λ_β are inverse operators. A simple example is given by the functions

$$F_k(t, S_0) = e^{S_0 t}\, t^{-k}; \qquad f_k(S, S_0) = \Gamma(1-k)\,(S - S_0)^{k-1},$$
$$k = 0, -1, -2, \cdots .$$

[1] Our results are for a convex body while those of [6] are for star-shaped bodies.

If

$$S_0 = a + ib \qquad \text{and} \qquad S = \xi + i\eta$$

then

$$L_S(F_k) = f_k \quad \text{in} \quad \xi > a, \quad \lambda_\beta(f_k) = F_k \quad \text{for} \quad \beta > a. \tag{2.1}$$

A second example is provided by the functions

$$F_k(t, S_0) = 0 \quad \text{in} \quad 0 \leqslant t < 1, \qquad e^{S_0 t}\, t^{-k} \quad \text{in} \quad t \geqslant 1, \qquad k = 1, 2, \cdots .$$

and their transforms $f_k(S, S_0) = L_S(F_k)$. Relation (2.1) holds here also. The functions f_k cannot be determined explicitly but one has the following result.

LEMMA 2.1. *$f_k(S, S_0)$ is analytic in $\xi \geqslant a$ except at S_0 and,*

$$f_k - \frac{(-1)^k (S - S_0)^{k-1}}{(k-1)!} \log (S - S_0)$$

is analytic in a neighborhood of S_0.

We want to discuss the transforms of a certain class of functions $F(t)$ with respect to their singularities and their behavior for large $|S|$. The class is chosen because it is easy to work with and yet contains those functions usually encountered in physical problems. The techniques would be valid for a wider class. A function $F(t)$ will be called *S_0-regular* if $F(t) \in C^\infty[0, \infty]$ and

$$F(t) = e^{S_0 t}\, \Phi(t), \qquad \Phi(t) \sim \sum_{k=-n}^{\infty} a_k t^{-k} \quad \text{as} \quad t \to \infty, \tag{2.2}$$

where it is assumed that the asymptotic series can be differentiated termwise arbitrarily often.

LEMMA 2.2. *Let $F(t)$ be S_0-regular with $S_0 = a + ib$. Then if $S = \xi + i\eta$ the function $f(S) = L_S(F)$ satisfies the following conditions:*

(C_1) *$f(S)$ is analytic in $\xi > a$*

(C_2) $f(S) = \sum_{k=-n}^{m} a_k f_k(S, S_0) + \psi_m(S), \qquad m = 1, 2, \cdots,$

where

$$\lim_{\xi \downarrow a} \psi_m^{(k)}(\xi + i\eta) \quad \textit{exists for all } \eta \textit{ if} \quad k \leqslant m - 1.$$

(C_3) $f(S) = F(0)\, S^{-1} + F'(0)\, S^{-2} + \chi(S), \quad \chi^{(k)}(S) = 0(S^{-3})$

as $S \to \infty$ in $\xi > a$, $k = 0, 1, 2, \cdots$.

Note that $f(S)$ is an entire function if $F(t)$ has compact support.

The next result proceeds in the other direction, from the transform to the function. We say a function $f(S)$ is (r, a) admissible, for a non-negative integer r and a real number a, if it satisfies (C_1) and the following analogs of (C_2) and (C_3):

$$(C_2') \quad f(S) = \sum_{k=-n}^{m} a_k f_k(S, S_0) + \psi(S),$$

for some integers m and n and where,

$$\lim_{\xi \downarrow a} \psi^{(k)}(\xi + i\eta) \qquad \text{exists for all } \eta \text{ if} \qquad k \leqslant r.$$

$$(C_3') \quad f^{(r)}(S) = 0(S^{-3}), \quad f^{(k)}(S) \to 0 \quad \text{for} \quad k < r;$$

as $S \to \infty$ in $\xi \geqslant a$. If $r = 0$ the statement about $f^{(k)}$ is omitted. Observe that if $F(t)$ is S_0-regular then $f(S) = L_S(F)$ is (r, a) admissible for all $r \geqslant 2$.

LEMMA 2.3. *Let $f(S)$ be (r, a) admissible for some r and a. Then $\lambda_\beta(f)$ exists and is independent of β in $\beta > a$. It defines a function $F(t)$ which is continuously differentiable in $t > 0$ and $f(S) = L_S(F)$ in $\xi > a$. Moreover,*

$$F(t) - \sum_{k=-n}^{m} a_k F_k(t, S_0) = o(e^{S_0 t}\, t^{-r}) \qquad \text{as} \qquad t \to \infty.$$

3. PARABOLIC PROBLEMS

Let D be a bounded domain in E^n with boundary B. We study the equation,

$$L[U] \equiv \Delta U - q(x)\, U = U_t \qquad \text{in} \qquad D \times (0, \infty), \qquad \text{(E)},$$

where $q(x)$ is a continuous, non-negative function in $D + B$. U is to be subject to the conditions,

$$U(x, 0) = g(x) \qquad \text{in} \qquad D \qquad \text{(A)}$$

$$U_\nu + \beta(x)\, U = F(x, t) \qquad \text{on} \qquad B, \qquad \text{(B)}$$

where ν is the exterior normal and $\beta(x)$ is non-negative.[2] The special case q and β both zero is discussed at the end of this section and we assume here they are not both identically zero.

The function $g(x)$ is to be of class $C^{(1)}$ in $D \cup B$ and F is to be continuous in x and t in $B \times [0, \infty)$. Moreover F is to be S_0-regular, $S_0 = a + ib$, for

[2] See the remark in Section 4.

each x. F then has the form (2.2) with coefficients $a_k(x)$ and it is assumed that the asymptotic series is uniform in x. Finally we assume the validity of the continuity condition,

$$g_\nu + \beta g = F(x, 0) \qquad \text{on} \qquad B. \tag{C}$$

We denote by (P) the above problem.

Suppose (P) has a solution and let $u(x, S) = L_S(U)$. Then we have,

$$Lu - Su = -g(x) \qquad \text{in} \qquad D \tag{E$'$}$$

$$u_\nu + \beta u = f(x, S) = L_S(F) \qquad \text{on} \qquad B. \tag{B$'$}$$

We shall show that this problem has a solution for all complex S in a certain half plane. Singularities in the solution will be produced by those present in $f(x, S)$ but poles will also be produced by the solution process itself, independent of f. The existence of u in a half plane yields the existence of a solution U of (P) and the singularities serve to determine the asymptotic behavior of U through Lemma 2.3.

We say that a solution $U(x, t)$ of (P) *follows* F if

$$U(x, t) - \sum_{k=-n}^{m} A_k(x) F_k(t, S_0) = o(e^{S_0 t}\, t^{-m}) \qquad \text{as} \qquad t \to \infty \tag{3.1}$$

for any $m \geqslant 0$ and any $x \in D$. We can now state the result.

THEOREM. *There exists a solution* $U(x, t)$ of (P) *and a number* $\alpha < 0$ *such that*

(i) U follows F if $a > \alpha$,

(ii) $U = 0(e^{\alpha t})$ *as* $t \to \infty$ if $a < \alpha$.

We comment on the case $a = \alpha$ later. The number α will be S_1 where S_1 is the largest eigenvalue of the problem (E′), (B′) and the $A_k(x)$ in (3.1) will be solutions of elliptic boundary value problems.

The problem (E′), (B′) can be handled as follows. Standard procedures show the existence of a function $u^0(x, S)$ satisfying

$$Lu^0 = 0 \quad \text{in} \quad D \qquad u_\nu{}^0 + \beta u^0 = f(x, S) \quad \text{on} \quad B,$$

and also the existence of a symmetric Green's function $G(x, y)$ for the operator L in D with $G_\nu - \beta G = 0$ on B. The function u^0 can be written in the form

$$u^0(x, S) = \int_B K(x, y) f(x, S)\, dy, \tag{3.2}$$

where K is independent of S. Hence if F is S_0-regular then u^0 is (r, a) admissible for any $r \geqslant 2$.[3]

We seek a solution of (E′), (B′) in the form $u = u^0 + v$ where

$$Lv - Sv = -g(x) + Su^0 \quad \text{in} \quad D, \qquad v_\nu + \beta v = 0 \quad \text{on} \quad B.$$

Then v can be expressed in the form

$$v(x, S) = \int_D \sigma(y)\, G(x, y)\, dy,$$

where $\sigma = Lv$ and σ is a solution of the integral equation.

$$\sigma(x) - S \int_D \sigma(y)\, G(x, y)\, dy = -g(x) + Su^0(x) \equiv H(x) \quad \text{in} \quad D \tag{3.3}$$

The integral equation (3.3) has a unique solution for all values of S at which u^0 is regular except for the eigenvalues of problem (E′), (B′). These form a denumerable set $\{S_i\}$ and are all negative, $0 > S_1 > S_2 \cdots$. The solution of (3.3) can be written in the resolvent form

$$\sigma(x) = H(x) + S \int_D H(y)\, R(x, y, S)\, dy,$$

where R is a meromorphic function of S with poles only at the S_i. Note that

$$v(x) = (Lv - H(x))\, S^{-1} = (\sigma(x) - H(x))\, S^{-1} = \int_D H(y)\, R(x, y, S)\, dy. \tag{3.4}$$

We deduce the following facts from (3.4). Suppose first that $a > \alpha = S_1$; then R is analytic in $\operatorname{Re} S \geqslant a$ so that v and hence also u satisfy (C_1). The behavior of f on $\operatorname{Re} S = S_0$ is determined by (C_2) and (3.4) shows that v has the same behavior as does u^0 and hence u. It follows that u satisfies (C_1) and (C_2') with $m = 0$ and a replaced by α.

We want to indicate how to calculate the coefficients $A_k(x)$ in condition (C_2') for u. For a given m we write (C_2) in the form

$$f(S) = B(S) + C(S) \log (S - S_0) + \psi_m(S),$$

where C is a polynomial in $(S - S_0)$ and $B(S)$ a polynomial in $(S - S_0)^{-1}$. This gives a corresponding decomposition of u in the form

$$u = b + c \log (S - S_0) + u_m .$$

[3] This fact also follows from the maximum principle without the use of the formula (3.2).

The above development shows that b has a pole of order $(n+1)$ at S_0 and c is regular near S_0 and $\partial^k u_m/\partial S^k$, $k \leqslant m$ exists in Re $S \geqslant a$. We can write then

$$b = \sum_{k=-n-1}^{\infty} b^k(x)\,(S - S_0)^k, \qquad c = \sum_{k=0}^{\infty} c^k(x)\,(S - S_0)^k.$$

The b_k will be determined recursively by the formulas,

$$Lb^{-n-1} - S_0 b^{-n-1} = g \quad \text{in } D, \quad b_\nu^{-n-1} + \beta b^{-n-1} = \Gamma(n+1)\, a_{-n}(x) \quad \text{on } B$$
$$Lb^k - S_0 b^k = b^{k-1} \quad \text{in } D, \qquad b_\nu^{\,k} + \beta b^k = \Gamma(1-k)\, a_k(x) \quad \text{on } B,$$

$k = -n, -n+2, \cdots, -1$. The c_k's will be determined in the same manner.

It remains to establish the condition (C_3'). We do this by using a technique suggested in [9]. Let $\varphi_i(x)$ be a complete orthonormal set of eigenfunctions for $G(x, y)$. Then we can write (3.4) in the form

$$v(x, S) = \sum_{k=1}^{\infty} \frac{(H, \varphi_k)\,\varphi_k(x)}{S_k - S}, \qquad (H, \varphi_k) = \int_D H\varphi_k\, dy. \tag{3.5}$$

Consider the function $H(x)$. We write u^0 in the form

$$u^0(x, S) = S^{-1}w^0(x) + S^{-2}w^1(x) + w^2(x, S) \tag{3.6}$$

where $Lw^i = 0$ in D and, in the notation of (C_3),

$$w_\nu^{\,0} + \beta w^0 = F(x, 0), \qquad w_\nu^{\,1} + \beta w^1 = F_t(x, 0), \qquad w_\nu^{\,2} + \beta w^2 = \chi \quad \text{on } B.$$

It follows from (C_3) and (3.2) that $w^2 = 0(S^{-3})$ as $S \to \infty$ in $\xi \geqslant a$. We obtain a corresponding decomposition of H as $H^0 + H^1 + H^2$, where

$$H^0 = -g(x) + w^0(x), \qquad H^1 = -S^{-1}w^1, \qquad H = Sw^2.$$

We observe that (C) implies $H_\nu^{\,0} + \beta H^0 = 0$ on B hence H^0 is sourcewise representable in D, and the series

$$\sum_{k=1}^{\infty} (H^0, \varphi_k)\,\varphi_k(x)$$

converges absolutely and uniformly to H^0. Note also that the series

$$\sum_{k=1}^{\infty} S_k^{-1}(H^1, \varphi_k)\,\varphi_k$$

converges absolutely by Bessel's inequality since H^1 and G are square integrable and $S_k^{-1}\varphi_k(x)$ is the kth Fourier coefficient of G. Finally we have $(H^2, \varphi_k) = 0(S^{-2})$. Hence we deduce that

$$v = 0(S^{-1}) \quad \text{as} \quad S \to \infty, \quad \text{in} \quad \xi \geqslant a.$$

Differentiation of the formulas we have used will yield in the same way

$$v_S = 0(S^{-2}), \qquad \frac{\partial^k v}{\partial S^k} = 0(S^{-3}), \qquad k \geqslant 2, \qquad \text{as} \qquad S \to \infty \qquad \text{in} \qquad \xi \geqslant a.$$

The same estimates hold for u^0 and (C_3') is established.

The proof of the theorem now follows from Lemma 2.3 with two further remarks. The first is that differentiations with respect to x under the integral sign in $\lambda_\beta(u)$ can be justified by noting that all the estimates we have obtained can be differentiated with respect to x. The second is that the validity of condition (A) can be proved by integrating (3.5) termwise and setting $t = 0$.

REMARKS. (1). Formula (3.5) could be used to obtain a "separation of variables" solution of the original problem (P). We want to emphasize though that this procedure is really not necessary, since it is only the first eigenfunction which ever figures in asymptotic behavior and possibly not even this one.

(2) The modifications for the case $a = \alpha$ are fairly clear. If $S_0 = \alpha + i\eta$ $\eta \neq 0$ then the solution will follow F except that an extra term of the form $F_{-1}(x, \alpha)$ will be introduced by the pole at α. If $S_0 = \alpha$ the form (C_2) of u changes. In addition to the singular terms in u which were present before there will be two new terms of the form f_{-n-1} and $(S - S_0)^{-1} \log (S - S_0)$. The first produces $F_{-n-1}(t)$ in U, while the second can be shown to yield a term in U of the form

$$e^{\alpha t}\, t^{-1} \log t \qquad \text{as} \qquad t \to \infty.$$

(3) The special case $\Delta U = U_t$ in D, $U_\nu = F(x, t)$ on B can be treated similarly. There is a slight difference in that the transform problem has zero as an eigenvalue. It is not hard to see that this produces the following result. Suppose $F(x, t) = F(x)$. The steady state problem $\Delta U = 0$ in D, $U = F(x)$ on B has no solution unless the constant

$$\gamma = \int_B F\, dS$$

is zero. However the steady state problem for $F_0 = F - \gamma A^{-1}$, A the area of B, will have a steady-state solution $U^0(x)$. One can show by methods like those of this section that the solution of the initial-value problem satisfies the estimate,

$$U(x, t) - V^{-1}\gamma t - U^0(x) = 0(e^{-\alpha t}),$$

where V is the volume of B and $-\alpha$ the first nonzero eigenvalue of the transform problem (see [10]).

4. Problems for the Wave Equation

In this section we consider exterior problems for the wave equation. In particular we study the problem

$$\Delta U = U_{tt} \qquad \text{in} \quad D \tag{4.1}$$

$$U = F(x, t) \qquad \text{on} \quad B_u \tag{4.2}$$

$$U(x, 0) = U_t(x, 0) = 0 \qquad \text{in} \quad D, \tag{4.3}$$

where D is the region exterior to a convex curve or surface B. The problem will be studied in two and three dimensions and the corresponding solutions will be denoted by U^2 and U^3.

Remark. We note that the general initial value problem can always be reduced to (4.3) by subtracting known solutions of (4.1) in all of space. This would only change (4.2). In the parabolic problems treated in the preceding section we could in the same way have assumed, without loss of generality, that $g(x) = 0$. On the other hand it would also have been sufficient to have solved (P) with $F \equiv 0$. This is fact is what we did in the S plane. In the wave equation problem however it is not clear that one can obtain the solution of the general initial value problem by solving (4.1) with $U \equiv 0$ on B and the nonhomogeneous conditions

$$U(x, 0) = f(x), \qquad U_t(x, 0) = g(x) \qquad \text{in} \quad D. \tag{4.4}$$

It is the latter problem which was studied in [6] under the assumption that f and g have compact support. We choose to work with the (more general) conditions (4.2) and (4.3).

We shall also assume here that $F(x, t)$ has compact support in t although it will be clear that the results could be modified easily for the functions of Section 2. We assume that F satisfies the conditions

$$F(x, 0) = F_t(x, 0) = 0 \qquad \text{on} \quad B \tag{4.5}$$

that (4.2), (4.3) and continuity would require.

It has already been indicated that the results here cannot quite be proved. What we show is that the following are at least plausible.

(1) There exist solutions U^2 and U^3.

(2) $U^2(x, t) = o(1)$ as $t \to \infty$

(3) $U^3(x, t) = o(e^{-at})$ as $t \to \infty$ for some $a > 0$.

Consider the transformed problem, that is,

$$\Delta u^{2,3} - S^2 u^{2,3} = 0 \qquad \text{in} \qquad D \tag{4.6}$$

$$u^{2,3} = f = L_S(F) \qquad \text{on} \qquad B. \tag{4.7}$$

We have seen that there are two aspects of the functions u to be considered; their analyticity as the functions of S and their behavior for large S. The analyticity question is answered by the following result that is proved in the Appendix.

LEMMA 4.1. *$u^3(x, S)$ is a meromorphic function of S in the entire S-plane with poles only in* $\operatorname{Re} S < 0$. *$u^2(x, S)$ is an analytic function of S in* $\operatorname{Re} S > 0$ *and*

$$\lim_{\xi \downarrow 0} u^2(x, \xi + i\eta) \text{ exists for all } \eta. \tag{4.8}$$

We study the second question, that of large S behavior, and then we shall draw some conclusions concerning properties (1), (2), and (3). In studying large S behavior it is convenient to use the following device. We determine a function $V(x, S, t)$ as a solution of the problem,

$$\Delta V - S^2 V = 0 \quad \text{in} \quad D, \qquad V = F(x, t) \quad \text{on} \quad B$$

with t as a parameter. Then we can obtain a formal solution of (4.6) and (4.7) as

$$u(x, S) = \int_0^\infty e^{-St}\, V(x, S, t)\, dt. \tag{4.9}$$

The point of the above procedure is that it makes the boundary data for V independent of S. Then the asymptotic theory of Keller [7] becomes particularly simple. B is convex hence we can introduce a co-ordinate system in D by drawing normals to B at each point μ on B and using μ and distance τ along the normals as coordinates. Then one can find a formal asymptotic expansion of the form

$$V(\tau, \mu, S, t) \sim e^{-S\tau} \sum_{n=0}^{\infty} S^{-n} v_n(\tau, \mu, t). \tag{4.10}$$

The functions v_n are determined recursively by the formulas

$$v_0(\tau, \mu, t) = F(\mu, t); \qquad v_n(\tau, \mu, t) = \int_0^\tau K(\tau, \mu, \alpha)\, \Delta v_{n-1}(\alpha, \mu, t)\, d\alpha,$$

with a function K which depends only on the surface B.

We recal that F was assumed to have compact support in t. It follows that V and the ν_n's will also. If we assume that the series (4.10) is uniform in t we can substitute this series into (4.9) and integrate. The result will be

$$u \sim e^{-S\tau} \sum_{n=0}^{\infty} L_S(\nu_n). \tag{4.11}$$

Observe that Lemma 2.2, Eq. 4.5 and the construction of the ν_n's show that,

$$L_S(\nu_k) = 0(S^{-3}) \qquad \text{as} \qquad S \to \infty \quad \text{in} \quad \operatorname{Re} S \geqslant a \tag{4.12}$$

for any a.

We assume the validity of the above formal process and then we can verify properties (1)-(3). Lemma 4.1 and Eqs. (4.10) and (4.11) show that $U = \lambda_\beta(u)$ is a solution, in either two or three dimensions, provided only that $\beta > 0$. The result (2) follows immediately if we observe that we have shown that u^2 is (0, 0) admissible and hence we can use Lemma 2.3.

In order to establish (3) we need to make an additional observation. Lemma 4.1 states that the poles of u^3 must lie in $\operatorname{Re} S < 0$. We claim that they must in fact lie in $\operatorname{Re} S \leqslant -a$ for some $a > 0$. For any $\alpha > 0$ there cannot exist infinitely many poles in $\operatorname{Re} S \geqslant -\alpha$ unless their imaginary parts tend to infinity. But a sequence of poles S_n with $\operatorname{Re} S_n \geqslant -\alpha$ and $\operatorname{Im} S_n \to \infty$ would be a violation of the fact that u is bounded as $S \to \infty$ in $\operatorname{Re} S \geqslant -\alpha$, a fact which (4.10) and (4.11) show. Hence there is a pole S_0 of maximum real part a. Then u^3 is (r, a^1) admissible for any r and any $a^1 > a$.

The proofs of (1), (2), and (3) are thus reduced to establishing the validity of the Keller theory and this remains an open question. We remark that the validity of (4.10) has been established for $\operatorname{Re} S \geqslant S_0$, where S_0 is a certain positive constant, by Miranker [11]. This fact would enable one to complete the proof of (1) but does not validate the estimates (2) and (3).

Appendix

Proof of Lemma 4.1

The existence of a unique solution of the problem (4.6) and (4.7) for $\operatorname{Re} S \geqslant 0$ has been established by Weyl [12]. The solutions are required to satisfy the radiation conditions

$$u^2 \sim A\,|\,x\,|^{-1/2}\,e^{-Sx}, \qquad u^3 \sim A\,|\,x\,|^{-1}\,e^{-Sx} \qquad \text{as} \qquad |\,x\,| \to \infty. \tag{A.1}$$

The procedure is to formulate the problem as an integral equation. It has the difficulty that the homogeneous integral equation has solutions when $\operatorname{Re} S = 0$. The following modification of Weyl's procedure, which avoids this difficulty, was communicated to the author by Professor Peter Werner.

We seek the solution u^3 in the form

$$u^3(x) = (4\pi)^{-1} \int_B \sigma(y) \left(\frac{\partial}{\partial \nu_y} - i \right) \{ | x - y |^{-1} e^{-S|x-y|} \} \, dy, \qquad \text{(A.2)}$$

and we obtain the integral equation

$$\sigma(x) - (2\pi)^{-1} \int_B \sigma(y) \left(\frac{\partial}{\partial \nu_y} - i \right) \{ | x - y |^{-1} e^{-S|x-y|} \} \, dy = -2f(x, S) \quad \text{on } B, \qquad \text{(A.3)}$$

for σ. In these formulas, ν denotes the exterior normal. The adjoint homogeneous equation for (A.3) is

$$\sigma(x) - (2\pi)^{-1} \int_B \sigma(y) \left(\frac{\partial}{\partial \nu_x} - i \right) \{ | x - y |^{-1} e^{-S|x-y|} \} \, dy = 0 \quad \text{on} \quad B \qquad \text{(A.4)}$$

We are going to show that there exists no nonzero solution of (A.4). Let σ be a solution of (A.4) and set

$$u(x) = (4\pi)^{-1} \int_B \sigma(y) \{ | x - y |^{-1} e^{-S|x-y|} \} \, dy.$$

$u(x)$ is then a solution of (4.6) both in D and the region D' interior to B. Equation (A.4) states that

$$\left(\frac{\partial u}{\partial \nu} \right)^- + iu^- = 0 \qquad \text{on} \qquad B, \qquad \text{(A.5)}$$

where the $-$ sign denotes limits from D'. By Green's theorem, Eqs. (4.6) and (A.5), we have

$$\int_{D'} [| \operatorname{grad} u |^2 + S^2 | u |^2] \, dV = -i \int_B | u |^2 \, dS. \qquad \text{(A.6)}$$

If $S = \xi + i\eta$ with $\xi \geqslant 0$, $\eta \geqslant 0$ (A.6) shows that $u \equiv 0$ in D and consequently,

$$\left(\frac{\partial u}{\partial \nu} \right)^- = 0 \qquad \text{on} \qquad B. \qquad \text{(A.7)}$$

If $S = i\eta$ then (A.6) yields $u^- \equiv 0$ on B hence by (A.5) Eq. (A.7) still holds.

Now consider u for x in D. u is continuous across B so that $u^+ \equiv 0$ on B and hence by the uniqueness theorem for D $u \equiv 0$ in D. But then

$$\left(\frac{\partial u}{\partial \nu}\right)^+ = 0 \qquad \text{on} \qquad B$$

and it follows that

$$\sigma = \left(\frac{\partial u}{\partial \nu}\right)^+ - \left(\frac{\partial u}{\partial \nu}\right)^- = 0.$$

We can now appeal to Fredholm theory to deduce the existence of a unique solution of (A.3), and hence a solution of our problem, for $S = \xi + i\eta$ in $\xi, \eta \geqslant 0$. We can obtain the solution in $\xi \geqslant 0$, $\eta \leqslant 0$ by observing that the solution we have obtained is real for $\eta = 0$, hence we can obtain u in $\eta \leqslant 0$ by reflection.

The formulation of the problem as an integral equation actually gives us more information. Observe that both the kernel and the known right hand side of (A.3) are entire functions of S. It was shown by Tamarkin [13][4] that under these circumstances the solution will be a meromorphic function of S with poles only at values of S for which the adjoint homogeneous equation has nontrivial solutions. We have seen that these must lie in $\operatorname{Re} S < 0$ and this completes the proof of Lemma 4.1 for u^3.

The situation for u^2 is more complicated. Here the kernel in (A.2) and (A.3) must be replaced by the singular Bessel function $H_0^{(1)}(iS \mid x - y \mid)$. This function has a branch point at $S = 0$. The preceding integral equation can be applied again so as to establish the analyticity of u^2 in $\operatorname{Re} S \geqslant 0$, $S = 0$. However it is shown in [10] that u^2 will in general have a branch point of a very complicated type at $S = 0$. It is shown in [10] that Eq. (4.8) is valid hence u^2 is (0.0) admissible, but this is all that can be said, hence the conclusion (2).

References

1. J. Radlow and Y. M. Chen. Time decay in plane pulse diffraction by a smooth convex cylinder. Unpublished note.
2. J. Kotik and J. Lurye. Some topics in the theory of coupled ship motions. Fifth Symposium on Naval Hydrodynamics, Bergen, Norway, 1964.
3. F. Ursell. "The decay of the free motion of a floating body," *J. Fluid Mech.* **19** (1964), 305-319.
4. P. D. Lax and A. N. Milgram. Parabolic equations. *In* "Contributions to the Theory of Partial Differential Equations." Ann. of Math. Studies, Princeton Univ. Press (1954), 167-190.

[4] Tamarkin's result was for a kernel which is L^2 in x but it is not difficult to modify it to our case.

5. A. FRIEDMAN. "Partial Differential Equations of Parabolic Type." Prentice-Hall, Englewood Cliffs, New York, 1964.
6. P. D. LAX, C. S. MORAWETZ, AND R. S. PHILLIPS. Exponential decay of solutions of the wave equation in the exterior of a starshaped obstacle. *Comm. Pure Appl. Math.* **16** (1963), 477-486.
7. J. B. KELLER, R. M. LEWIS, AND B. B. SECKLER. Asymptotic solutions of some diffraction problems. *Comm. Pure Appl. Math.* **9** (1956), 207-265.
8. G. DOETSCH. "Theorie und Andwendung der Laplace-Transformation," Dover, New York, 1943.
9. E. ROTHE. Über asymptotische Entwicklungen bei Randwertaufgaben elliptischer partieller Differentialgleichungen. *Math. Ann.* **108** (1933), 578-594.
10. R. C. MACCAMY. Low Frequency acoustic oscillations. *Quart. J. Appl. Math.* To appear.
11. W. L. MIRANKER. Parametric theory of $\Delta u + k^2 u = 0$. *Arch. Rat. Mech.* **1** (1957), 139-153.
12. H. WEYL. "Kapazität von Strahlungfeldern" *Math. Zeit.* **55** (1952), 187-198.
13. J. D. TAMARKIN. "On Fredholm Integral Equations whose Kernels are Analytic in a Parameter." *Ann. of Math.* **28** (1927), 127-152.