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Weak approximation of the stochastic wave equation

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ABSTRACT

We investigate the accuracy of approximation of $\mathbb{E}[\varphi(u(t))]$, where $\{u(t) : t \in [0, \infty)\}$ is the solution of the stochastic wave equation driven by the space–time white noise and φ is an $\mathbb{R}$-valued function defined on the Hilbert space $L^2(\mathbb{R})$. The approximation is done by the leap-frog scheme. We show that, under certain conditions on φ, the approximation by the leap-frog scheme is of order two.

1. Introduction

The numerical analysis of stochastic partial differential equations is a young topic of research. In most of the articles in the literature, the aim has been to analyse pathwise convergence or the strong error for parabolic SPDEs; see e.g. Gyöngy [1,2], Gyöngy and Millet [3], Hausenblas [4,5], Kloeden and Shot [6], Millet and Morien [7] and Shardlow [8]. For the Kortweg–de Vries equation and the stochastic Schrödinger equation, see e.g. De Bouard and Debussche [9,10] and Debussche and Printems [11]. For the numerical approximation of the stochastic wave equation, see e.g. Quer-Sardanyons and Sanz-Solé [12], Walsh [13] and Kovacs et al. [14].

The strong error depends on the regularity of the noise. Nevertheless, in the best possible case the order of the scheme is $\frac{1}{2}$ of the order of the scheme without noise. However, it can be shown, that considering the Monte Carlo error or the so-called weak error, the order of the scheme can be improved; see e.g. Fichter and Manthey [15], Hausenblas [16] and Shardlow [17] and the very recent works of De Bouard and Debussche [9] and Debussche and Printems [18,19].

Let u be the solution of a one-dimensional quasi-linear stochastic wave equation, $\hat{u}$ its approximation and ϕ be a real-valued function defined on $L^2(\mathbb{R} \times [0, \infty); \mathbb{R})$. Our point of interest is the accuracy with which the entity $\mathbb{E}[\varphi(u)]$ can be computed. In the Monte Carlo simulation, a large number M of independent trajectories $\{\hat{u}^i : 1 \le i \le M\}$ are simulated on a computer. Then, the entity $\mathbb{E}[\varphi(u)]$ is approximated by

$$\frac{1}{M} \sum_{i}^{M} \varphi(\hat{u}^i).$$

The resulting error depends on the choice of the approximation $\hat{u}$ and the parameter M. The effect of M can be described by the Central Limit Theorem, while the effect of the choice of the approximation can be measured by the quantity

$$|\mathbb{E}[\varphi(u)] - \mathbb{E}[\varphi(\hat{u})]|,$$

which is called the weak error. Milstein [20] and Talay [21] were the first who investigated the weak error in finite dimension. Moreover, let us mention that a further work that investigated the weak error was Talay and Tubaro [22].

Therefore, it is known that for example, the Euler scheme is in general of strong order 1/2 and of weak order 1. Our objective is to investigate the weak error of the leap-frog scheme applied to the stochastic wave equation. This scheme

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is widely used in the deterministic case and is, in the deterministic context, of order 2. For more details, we refer to e.g. Iserles [23] or Quarteroni et al. [24]. Now, the question we are interested in, is, whether the order of convergence for the weak error of the leap-frog scheme can be also of order two, and, if yes, under which conditions. We will show that, similar to the Euler scheme, the order of convergence can also be in the best possible case equal to two.

The paper is organised as follows. In the next section, i.e. Section two, we present the main result. In Section 3 three we give some preliminaries of the stochastic wave equation. In Section 4, the numerical approximation is described. The actual proof of our main result is the content of Section 5. In Appendix A the stability of the leap-frog scheme is analysed and in Appendix B we recall some basic facts about finite differences.

2. The main result

As mentioned before, we are concerned with the numerical approximation of a quasi-linear stochastic wave equation in $[0, 1]$ driven by a space–time white noise. First, let us recall the definition of a space–time white noise.

Definition 2.1. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete probability space and $\mathcal{O} \subset \mathbb{R}^d$ a measurable subset. Then a space–time (Gaussian) white noise on $\mathcal{O}$ is a measurable mapping

$$W : (\Omega, \mathcal{F}) \to (M([0,\infty) \times \mathcal{O}), \mathcal{M}([0,\infty) \times \mathcal{O})^1)$$

such that

(i) for all $A \in \mathcal{B}([0,\infty) \times \mathcal{O})$, $W(A)$ is a real-valued, Gaussian random variable with mean 0 and variance $\lambda_{d+1}(A)^2$, provided $\lambda_{d+1}(A) < \infty$;
(ii) if the sets $A_1, A_2 \in \mathcal{B}([0,\infty) \times \mathcal{O})$ are disjoint, then the random variables $W(A_1)$ and $W(A_2)$ are independent and $W(A_1 \cup A_2) = W(A_1) + W(A_2)$.
(iii) for any $A \in \mathcal{B}(\mathcal{O})$ the real-valued process $[0,\infty) \ni t \mapsto W([0,t) \times A) \in \mathbb{R}$ is adapted.

For any $(t, \xi) \in [0,\infty) \times \mathcal{O}$, the Radon–Nikodym derivative of the measure W exists $\mathbb{P}$-a.s. and will be denoted by $\dot{W} = \{\dot{W}(t, \xi) : 0 \leq t < \infty, \xi \in \mathcal{O}\}$.

The stochastic wave equation driven by space–time white noise was initially introduced in [25] (for an introduction of the stochastic wave equation and its applications we refer to e.g. [26–28]) and reads as follow:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u(t,\xi) = \dfrac{\partial^2}{\partial \xi^2} u(t,\xi) + f(u(t,\xi)) + \dot{W}(t,\xi), \quad t \in [0,T], \xi \in [0,1], \\ u(0,\xi) = u_0(\xi), \quad \xi \in [0,1], \\ \dfrac{\partial}{\partial t} u(0,\xi) = v_0(\xi), \quad \xi \in [0,1], \end{cases} \tag{1}$$

where $T > 0$. We consider Dirichlet boundary conditions, that is $u(t, 0) = u(t, 1) = 0$, for all $t \in [0, T]$. The random perturbation $\dot{W}$ here is the Radon Nikodym derivative of a space–time (Gaussian) white noise on $[0, T] \times [0, 1]$ over a given complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. The initial conditions $u_0, v_0 : \mathbb{R} \to \mathbb{R}$ are bounded functions satisfying some regularity conditions and the drift term $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous.

As a solution to Eq. (1), we take the so-called mild solution.

Definition 2.2. We call a process $u = \{u(t, \xi) : (t, \xi) \in [0, T] \times [0, 1]\}$ a mild solution of Eq. (1), if u is an adapted process over $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ such that $\mathbb{P}$-a.s. the following integral equation is satisfied:

$$\begin{aligned} u(t,\xi) = &\int_0^1 G(t,\xi,\zeta) v_0(\zeta)\,d\zeta + \frac{\partial}{\partial t}\left(\int_0^1 G(t,\xi,\zeta) u_0(\zeta)\,d\zeta\right) \\ &+ \int_0^t \int_0^1 G(t-s,\xi,\zeta) f(u(s,\zeta))\,d\zeta + \int_0^t \int_0^1 G(t-s,\xi,\zeta)\, W(ds, d\zeta), \end{aligned}$$

for all $t \in (0, T]$ and $\xi \in (0, 1)$. Here G denotes the Green function associated to the wave equation on $[0, 1]$ with Dirichlet boundary conditions.

In this paragraph, we give a description of the leap-frog scheme applied to Eq. (1). Let τ_h be the time step size corresponding to the grid size h and $k_h := \frac{1}{h}$. We assume, without loss of generality, that k_h is a positive integer. Then, the approximation of $u(j\tau_h)$, for an integer $j \geq 0$, is given by

$$\hat{u}_h^j(\xi) := k_h \sum_{i=0}^{k_h} 1_{(N_i^h, N_{i+1}^h]}(\xi) \left[\frac{(\xi - N_i^h)}{h}\ \hat{u}_{i+1,h}^j + \frac{(N_{i+1}^h - \xi)}{h}\ \hat{u}_{i,h}^j \right], \quad \xi \in [0,1], \tag{2}$$

[1] For a measurable space $(S, \mathcal{S})$ we denote by $M(S)$ the set of all measures on S.
[2] For $d \in \mathbb{N}$, λ_d denotes the d-dimensional Lebesgue measure.

where $\{\hat{u}^j_{i,h} : i = 1, \dots, k_h,\ j = 1, 2, 3, \dots\}$ satisfies the following recursion

$$\begin{cases} \hat{u}^0_{i,h} := u_0(N^h_{i-1}) + hu_0'((N^h_{i-1} + N^h_i)/2), \quad i = 1, \dots, k_h - 1, \\ \hat{u}^1_{i,h} := \hat{u}^0_{i,h} + \tau_h v_0(N^h_{i-1}) + h\tau_h v_0'((N^h_{i-1} + N^h_i)/2), \quad i = 1, \dots, k_h - 1, \\ (\tau_h)^{-2}\left(\hat{u}^{j+1}_{i,h} - 2\hat{u}^j_{i,h} + \hat{u}^{j-1}_{i,h}\right) = h^{-2}\left(\hat{u}^j_{i-1,h} - 2\hat{u}^j_{i,h} + \hat{u}^j_{i+1,h}\right) + h^{-\frac{1}{2}}\Delta^j_{i,h}W + f(\hat{u}^j_{i,h}), \\ \quad i = 1, \dots, k_h - 1,\ j \geq 1. \end{cases} \tag{3}$$

Here,

$$\Delta^j_{i,h}W := h^{-\frac{1}{2}} \int_{j\tau_h}^{(j+1)\tau_h} W(\mathrm{d}s, J^h_i), \quad i = 1, \dots, k_h,$$

and $J^h_i = [(N^h_{i-1} + N^h_i)/2, (N^i_h + N^h_{i+1})/2]$. Notice that, since the leap-frog scheme arises by the explicit mid-point rule, the initial conditions have to be chosen following this pattern as well.

Before stating the main result of the paper, let us introduce the following Sobolev-type spaces. Let $\Lambda = -\frac{\partial^2}{\partial x^2}$ and $H = L^2([0, 1])$. Then, for any $\alpha \geq 0$ we denote by $H^\alpha([0, 1])$ the domain of the operator $(1 + \Lambda)^{\frac{\alpha}{2}}$ in $L^2([0, 1])$ with Dirichlet boundary conditions. The space $H^\alpha([0, 1])$ is equipped with the norm

$$\|w\|_{H^\alpha} := \|(1 + \Lambda)^{\frac{\alpha}{2}} w\|_H, \quad w \in H_\alpha([0, 1]).$$

For $\alpha < 0$, we define $H^\alpha([0, 1])$ as the completion of H with respect to the norm $\| \cdot \|_{H^\alpha}$. Since we will investigate the quality of approximation of functionals defined on $L^2([0, \infty) \times [0, 1]; \mathbb{R})$, we introduce also a scale of Sobolev spaces on $L^2([0, \infty) \times [0, 1]; \mathbb{R})$. Let $\Delta = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \xi^2}$ and for $\alpha \geq 0$ let $H^{-\alpha}([0, \infty) \times [0, 1])$ be the completion of $L^2([0, \infty) \times [0, 1]; \mathbb{R})$ with respect to the norm $\|(I - \Delta)^{-\alpha} \cdot \|$. Now, we can formulate our main result.

Theorem 2.1. *Let $u = \{u(t, \xi) : (t, \xi) \in [0, T_0] \times [0, 1]\}$ be the mild solution of the stochastic wave equation, i.e. the solution of Eq. (1), and $\hat{u}_h := \{\hat{u}^j_{i,h} : j = 1, \dots, K,\ i = 1, \dots\}$ be its approximation given by the leap-frog scheme, i.e. given by the recursion (3) with $\tau_h = h$. Assume that $f \in C^1_b(\mathbb{R})$ and $u_0, v_0 \in C^2_b(\mathbb{R})$ have bounded support.*

Fix $\alpha > \frac{3}{2}$ and $T_0 \geq 1$. Suppose that for a function $\Phi : H^{-\alpha}([0, T_0] \times [0, 1]) \to \mathbb{R}$ there exists a function $\phi : \mathbb{R} \times [0, T_0] \times [0, 1] \to \mathbb{R}$ such that ϕ is four times differentiable in the first variable with bounded derivatives and

$$\Phi(u) = \int_0^{T_0} \int_0^1 \phi\left((I - \Delta)^{-\alpha} u(t, \xi), t, \xi\right) \mathrm{d}\xi\, \mathrm{d}t, \quad u \in H^{-\alpha}([0, 1] \times [0, T]).$$

Then, there exists a constant $C > 0$ such that

$$\left|\mathbb{E}\Phi\left(\hat{u}_h\right) - \mathbb{E}\Phi\left(u\right)\right| \leq Ch^2, \quad 0 \leq k \leq K.$$

Remark 2.1. The function Φ has to be Fréchet differentiable on $H^{-\alpha}([0, T_0] \times [0, 1])$ where $\alpha > \frac{3}{2}$. That means, we can only describe properties, which can be characterised in $H^{-\alpha}([0, T_0] \times [0, 1])$.

Corollary 2.1. *Under the conditions of Theorem 2.1, for all $T_0 \geq 1$ the following holds. Let $\phi : [0, T_0] \times [0, 1] \to \mathbb{R}$ be a function such that $\phi \in C^\alpha_b([0, T_0] \times [0, 1])$, $\alpha > \frac{1}{2}$ and $\phi = 0$ on the boundary of $[0, T_0] \times [0, 1]$. Then, there exists a constant C such that for any $h \in (0, 1]$*

$$\left|\mathbb{E} \sum_{k=0}^{n_h} \sum_{j=0}^{k_h} h\, \tau_h\, \hat{u}^k_{j,h}\, \phi(k\tau_h, jh) - \mathbb{E} \int_0^T \int_0^1 u(t, \xi)\phi(t, \xi)\, \mathrm{d}\xi\, \mathrm{d}t\right| \leq Ch^2, \quad 0 \leq k \leq K.$$

where $n_h \tau_h = T_0$.

Proof. The proof is a combination of Theorem 2.1 and duality arguments. □

3. Preliminaries

There exists different approaches to deal with Eq. (1), e.g. the variational approach (see e.g. Walsh [28]) and the semigroup approach (see e.g. Da Prato and Zabczyk in [27]). Here, in the first part of this section we describe the semigroup approach. In the second part of this section, we consider the wave equation with non homogeneous boundary conditions.

3.1. Lifting of the wave equation

In the semigroup approach we formulate the second order system given in (1) as a first order system (see also [29, Chapter 7.4]). This first order system generates a C_0 semigroup, such that a solution to (1) can be defined by using the variation of constant formula. Usually, this kind of solution is also called a mild solution.

Let $\Lambda := -\frac{\partial^2}{\partial \xi^2}$ be the Laplace operator on $H := L^2([0,1])$ with domain

$$D(\Lambda) = \{u \in H : \Lambda u \in H,\ u(0) = u(1) = 0\}.$$

Let V^*, H, V be a Gelfand triple

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*,$$

where $V = D((1+\Lambda)^{\frac{1}{2}})$. Here, V^* denotes the dual of V, and H has been identified with its dual H^*. Notice that all the embeddings in the above diagram are dense and continuous.

Let us define the Hilbert space

$$\mathcal{H} = H \oplus V^* = H \oplus D\left((1+\Lambda)^{-\frac{1}{2}}\right) \tag{4}$$

equipped with the inner product

$$\left\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\rangle_{\mathcal{H}} = \langle u_1, u_2\rangle_H + \left\langle (1+\Lambda)^{-\frac{1}{2}} v_1, (1+\Lambda)^{-\frac{1}{2}} v_2 \right\rangle_H,$$

for all $u_1, u_2 \in H$ and $v_1, v_2 \in V^*$. We define a linear operator $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ by

$$D(\mathcal{A}) = D(\Lambda) \oplus V$$

and

$$\mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\Lambda & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix} \in D(\mathcal{A}). \tag{5}$$

Let us also define an operator $F : \mathcal{H} \to \mathcal{H}$ by

$$F\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) := \begin{pmatrix} 0 \\ N_f(u) \end{pmatrix}$$

where $N_f : H \to H$ is the Nemytskij operator associated to f given by

$$N_f(u)(\xi) = f(u(\xi)), \quad \xi \in \mathbb{R}, u \in H.$$

Putting $v(t,\xi) := \frac{\partial u}{\partial t}(t,\xi)$, for $t \geq 0$ and $\xi \in [0,1]$, Eq. (1) can be rewritten in the following form

$$\begin{cases} \mathrm{d}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \left[\mathcal{A}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + F\left(\begin{pmatrix} u(t) \\ v(t) \end{pmatrix}\right)\right]\mathrm{d}t + \begin{pmatrix} 0 \\ \mathrm{d}W(t) \end{pmatrix}, \\ \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \end{cases} \tag{6}$$

where $\{W(t) : t \in [0,T]\}$ is a cylindrical Wiener process on H. Let us note that a space–time Gaussian white noise on $[0,1]$ can be written as a cylindrical Wiener process in $L^2([0,1])(= H)$. Due to the fact that the embedding $H \hookrightarrow V^*$ is Hilbert–Schmidt, W takes values in V^*. Now, Eq. (6) is equivalent to

$$\begin{cases} \mathrm{d}X(t) = [\mathcal{A}X(t) + F(X(t))]\mathrm{d}t + Q\,\mathrm{d}W(t), \\ X(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \end{cases} \tag{7}$$

where $X(t) = (u(t), v(t))^T$, $t \geq 0$, and the operator Q is defined by $D(Q) = V^*$ and

$$Qz = \begin{pmatrix} 0 \\ z \end{pmatrix}.$$

The operator $\mathcal{A}$ generates a unitary C_0-semigroup of contractions $S = \{S(t) : t \geq 0\}$ on the Hilbert space $\mathcal{H}$. The explicit form of S is given by

$$S(t) = \begin{pmatrix} \cos(\sqrt{\Lambda}t) & \frac{1}{\sqrt{\Lambda}}\sin(\sqrt{\Lambda}t) \\ -\sqrt{\Lambda}\sin(\sqrt{\Lambda}t) & \cos(\sqrt{\Lambda}t) \end{pmatrix}, \quad t \geq 0.$$

For more details we refer to [29, Theorem 4.5, Chapter 7] or [27, Example 5.8].

The mild solution of (7) is given by an adapted and $\mathcal{H}$-valued process X such that X solves $\mathbb{P}$-a.s. the following integral equation

$$X(t) = X(0) + \int_0^t S(t-s)F(X(s))\mathrm{d}s + \int_0^t S(t-s)Q\,\mathrm{d}W(s), \quad t \geq 0. \tag{8}$$

The stochastic convolution term on the right-hand side of (8) is well defined, that is, $S(t-s)Q$ defines a Hilbert–Schmidt operator from V^* with values in $\mathcal{H}$. Existence and uniqueness of a $\mathcal{H}$-valued solution $\{X(t) : t \in [0,T]\}$ to Eq. (8) is a consequence of [27, Theorem 7.4].

Since we will need it later on, let us introduce the following scales of Hilbert spaces. For any $\alpha \in \mathbb{R}$, let

$$\mathcal{H}_\alpha := H_\alpha \oplus H_{\alpha-1}. \tag{9}$$

We equip $\mathcal{H}_\alpha$ with the inner product defined by

$$\left\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\rangle_{\mathcal{H}_\alpha} = \langle u_1, u_2 \rangle_{H_\alpha} + \langle v_1, v_2 \rangle_{H_{\alpha-1}},$$

for $u_1, u_2 \in H_\alpha$ and $v_1, v_2 \in H_{\alpha-1}$.

3.2. The stochastic wave equation with non-homogeneous boundary conditions

To handle the space discretisation of Eq. (1) we will perform in Section 4.1 a change of coordinates by means of reflecting at the t-ξ-axis. This change of coordinates will lead to a stochastic wave equation with non-homogeneous boundary conditions. Note, that there will be also appear initial and terminal conditions. However, since the semigroup generated by the lifted wave equation is unitary, the initial conditions determine the values of the solution at time T and, vice versa, the terminal conditions determine the values of the solution at time 0. Therefore, we will fix in the equation the initial conditions and will not impose any terminal conditions.

Let u_0 and v_0 be fixed and let $L > 0$. For simplicity we assume that the support of

$$\xi \mapsto \int_{\mathbb{R}} G(T_0, \zeta, \xi) u_0(\zeta)\,\mathrm{d}\zeta \quad \text{and} \quad \frac{\partial}{\partial t}\xi \mapsto \int_{\mathbb{R}} G(T_0, \zeta, \xi) v_0(\zeta)\,\mathrm{d}\zeta$$

are included in $[0, 1]$. We consider the following problem:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} w(t,\xi) = \dfrac{\partial^2}{\partial \xi^2} w(t,\xi) - \dot{W}(t,\xi) - f(w(t,\xi)), & t \in [0,L], \xi \in \mathbb{R}_+, \\ w(0,\xi) = w_0(\xi), \qquad w'(0,\xi) = w_1(\xi), \quad \xi \in \mathbb{R}_+, \\ w(t,0) = u_0(t), \quad t \in [0,L], \\ \dfrac{\partial}{\partial \xi} w(t,0) = v_0(t), \quad t \in [0,L], \end{cases} \tag{10}$$

where $w_0(\xi) = u(\xi, 0)$ and $w_1(\xi) = v(\xi, 0)$ for $\xi \in \mathbb{R}_+$, $v(t,\xi) := \frac{\mathrm{d}}{\mathrm{d}t} u(t,\xi)$, $\xi \in \mathbb{R}$, $t \in [0,T]$ and u is a solution to Eq. (1).

In the theory of deterministic PDEs, there exist several approaches to deal with non-homogeneous boundary conditions. Since we are working in the framework of semigroup theory, we have chosen the approach introduced in [30, Chapter 4.11], which has been recently considered in [31] (see also [32]) to study heat and wave equations with a non-homogeneous random input on the boundary. To define a solution to (10), first, we define a *boundary operator* associated to the corresponding boundary problem, and, then, we define the mild solution of (10) as the solution of the homogeneous problem disturbed by a perturbation due to the boundary operator.

We start by considering the Laplace operator Λ on a bounded interval (for simplicity we take $[0, 1]$) with boundary conditions. Namely, $\Lambda := -\frac{\partial^2}{\partial \xi^2}$ with domain

$$D(\Lambda) = \left\{ u \in L^2([0,1]) : \Lambda u \in L^2([0,1]),\ u(0) = 0,\ u'(0) = 0 \right\}.$$

The Dirichlet and Neumann boundary conditions are described by the mappings υ_D and υ_N, respectively, which are defined by

$$\upsilon_D \phi := \phi(0) \in \mathbb{R} \quad \text{and} \quad \upsilon_N \phi := \phi'(0) \in \mathbb{R}, \tag{11}$$

for any $\phi \in L^2([0,1])$ for which the expression above makes sense.

Given $\gamma_D, \gamma_N \in \mathbb{R}$, the inhomogeneous problem is defined by the pair $(\Lambda, D(\Lambda))$, where Λ is the Laplacian and

$$D(\Lambda) = \left\{ u \in L^2([0,1]) : \Lambda u \in L^2([0,1]),\ \upsilon_D u = \gamma_D,\ \upsilon_N u = \gamma_N \right\}.$$

Let $\lambda \in \mathbb{R}$. First, we consider the following auxiliary (deterministic) elliptic problem: for any given $\gamma_D, \gamma_N \in \mathbb{R}$, find u in $\mathcal{S}'([0,1])$[3]satisfying

$$\Lambda u = \lambda u, \qquad \upsilon_D u = \gamma_D \quad \text{and} \quad \upsilon_N u = \gamma_N. \tag{12}$$

Problem (12) defines a *boundary operator* on $\mathbb{R}^2$ (see [31, Definition 1.2]). We write (u, φ) to denote the action of $u \in \mathcal{S}'([0,1])$ on $\varphi \in \mathcal{S}([0,1])$.

Definition 3.1. Let $\gamma_D, \gamma_N \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. We call $u \in \mathcal{S}'([0,1])$ a weak solution to (12), if

$$(u, \Lambda\phi) - (\gamma_D \upsilon_N, \phi) - (\gamma_N \upsilon_D, \phi) = \lambda(u, \phi),$$

for all $\phi \in \mathcal{S}([0,1])$ such that $\phi(0) = \phi'(0) = 0$.

We denote by $B^\lambda(\gamma_D, \gamma_N)$ the weak solution to (12) with boundary conditions γ_D, γ_N. Then, $B^\lambda : \mathbb{R}^2 \to \mathcal{S}'([0,1])$ is called the boundary operator associated to the (weak) problem of Definition 3.1. In our problem, i.e. Problem (10), the boundary operator can be calculated explicitly.

Example 3.1 (*Compare with Example 1.1 in [31, Section 6.1]*)**.** The boundary operator associated to the problem (12) with $\lambda = 1$ is given by

$$B^1(\gamma_D, \gamma_N)(x) = \frac{1}{2}(\gamma_D + \gamma_N)e^x + \frac{1}{2}(\gamma_D - \gamma_N)e^{-x}, \quad x \in [0,1].$$

Let us remind the scales of Hilbert spaces H_α and $\mathcal{H}_\alpha$, $\alpha \in \mathbb{R}$, defined at the end of Section 3.1. We denote by Λ_α the restriction (or self-adjoint extension if $\alpha < 0$) of the Laplacian Λ to H_α, and by $\mathcal{A}_\alpha$ the operator

$$\mathcal{A}_\alpha = \begin{pmatrix} 0 & 1 \\ -\Lambda_\alpha & 0 \end{pmatrix} \quad \text{Dom}\,(\mathcal{A}_\alpha) = \mathcal{H}_\alpha. \tag{13}$$

Recall that $\mathcal{A}_\alpha$ generates a unitary group $\{S_\alpha(t), t \geq 0\}$ on $\mathcal{H}_\alpha$. Let $\{Y(t), \in [0,L]\}$ be an $\mathcal{H}_\alpha$-valued stochastic process given by

$$\begin{aligned} Y(t) &= S_\alpha(t)Y(0) + \int_0^t S_\alpha(t-s)F(Y(s))ds + \int_0^t S_\alpha(t-s)Q\,dW(s) \\ &\quad + \int_0^t (1-\mathcal{A}_\alpha)S_\alpha(t-s)\mathcal{B}^1(u_0(s), v_0(s))ds, \quad t \geq 0, \end{aligned} \tag{14}$$

where $\mathcal{B}^\lambda$, $\lambda \in \mathbb{R}$, denotes the *lifted* boundary operator given by

$$\mathcal{B}^\lambda(\gamma_D, \gamma_N) := \begin{pmatrix} B^\lambda(\gamma_D, \gamma_N) \\ B^\lambda(\gamma_D, \gamma_N) \end{pmatrix}, \quad \gamma_D, \gamma_N \in \mathbb{R}.$$

Putting $(w(t), y(t)) := Y(t)^T$, $t \geq 0$, and $\frac{\partial w}{\partial t} := y$, w solves the deterministic boundary value problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} w(t,\xi) = \dfrac{\partial^2}{\partial \xi^2} w(t,\xi), \quad t \in [0,L], \xi \in \mathbb{R}_+, \\ w(0,\xi) = w_0(\xi), \qquad w'(0,\xi) = w_1(\xi), \quad \xi \in \mathbb{R}_+, \\ w(t,0) = u_0(t), \quad t \in [0,L], \\ \dfrac{\partial}{\partial \xi} w(t,0) = v_0(t), \quad t \in [0,L]. \end{cases} \tag{15}$$

For more details we refer to [31, Section 6.1]. Finally,

$$(\Lambda - 1)B^1(\gamma_D, \gamma_N) = \frac{1}{2}(\gamma_D + \gamma_N)(\delta' + \delta)e^{-M} + \frac{1}{2}(\gamma_D - \gamma_N)(\delta' + \delta)e^{M} - (\gamma_D + \gamma_N)\delta' - (\gamma_D - \gamma_N)\delta + (\gamma_D - 1). \tag{16}$$

Thus, setting $\lambda = 1$ and substituting the calculation before in Example 3.1, the last term disappears and we get

$$\begin{pmatrix} w \\ v \end{pmatrix}(t) = S_\alpha(t)\begin{pmatrix} y_0 \\ z_0 \end{pmatrix} + \int_0^t S_\alpha(t-s)\begin{pmatrix} 0 \\ (\Lambda - 1)B^1(\gamma_D(s), \gamma_N(s)) \end{pmatrix} ds, \quad t \geq 0.$$

[3] $\mathcal{S}([0,1])$ denotes Schwarz space of rapidly decreasing C^∞ functions on $[0,1]$ and $\mathcal{S}'([0,1])$ its dual, i.e. the space of tempered distributions on $[0,1]$.

Setting

$$\gamma_D(t) = \begin{cases} u_0(t) & 0 \le t \le 1 \\ 0, & t \ge 1, \end{cases} \quad \text{and} \quad \gamma_D(t) = \begin{cases} v_0(t), & 0 \le t \le 1 \\ 0, & t \ge 1, \end{cases}$$

Eq. (6) is equivalent to the following equation

$$\begin{cases} \mathrm{d}X(t) = [\mathcal{A}X(t) + F(X(t))]\mathrm{d}t + Q\,\mathrm{d}W(t) + (\mathcal{A} - 1)\mathcal{B}^1(\gamma_D(t), \gamma_N(t)), \\ X(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \end{cases}$$

where $X(t) = (u(t), v(t))^T$, $t \ge 0$, and the operator Q and its domain are given by $D(Q) = H \otimes V^*$ and

$$Qz = \begin{pmatrix} 0 \\ z \end{pmatrix}.$$

4. The numerical scheme

The numerical approximation is done by the leap-frog scheme. This scheme is widely used in the deterministic case; see e.g. [23,24]. In our approach we first introduce the space discretisation by finite differences (or finite elements), then we introduce a time discretisation, and, finally, we end up with the leap-frog scheme.

Let h be the parameter of the subdivision $\{\mathcal{T}_h, 0 < h \le 1\}$, defined in Appendix B, corresponding to the size of the grid and let $\mathcal{V}_h$ be the family of all functions $u \in L^2([0, 1])$ which are linear between the grid points. Let $\tilde{u}_h = \{\tilde{u}(t), 0 \le t < \infty\}$ be the space discretisation of Eq. (1) by finite differences corresponding to the parameter $0 < h \le 1$ (see Appendix B). In particular, $\tilde{u}$ solves the following finite-dimensional equation

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}\tilde{u}_h(t) = \Lambda_h\tilde{u}_h(t) + \mathcal{W}_h(\mathrm{d}t) + \mathcal{N}_h^f(\tilde{u}_h(t)), \quad t > 0, \\ \tilde{u}_h(0) = \mathcal{P}_h u_0, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\tilde{u}_h(0) = \mathcal{P}_h v_0 \end{cases} \tag{17}$$

where $\mathcal{P}_h$, $\mathcal{N}_h$, $\mathcal{W}_h$ and Λ_h are defined in Appendix B. Identifying $\mathcal{V}_h$ with $\mathbb{R}^{k_h}$ and putting $\tilde{\mathbf{u}}_h := \mathbf{P}_h\tilde{u}_h$, obviously, Eq. (17) is equivalent to

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}\tilde{\mathbf{u}}_h(t) = \mathbf{\Lambda}_h\tilde{\mathbf{u}}_h(t) + \mathbf{W}_h(\mathrm{d}t) + \mathbf{N}_h^f(\tilde{\mathbf{u}}_h(t)), \quad t > 0, \\ \tilde{\mathbf{u}}_h(0) = \mathbf{P}_h u_0, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{u}}_h(0) = \mathbf{P}_h v_0. \end{cases} \tag{18}$$

Again, here $\mathbf{P}_h$, $\mathbf{N}_h$, $\mathbf{W}_h$ and $\mathbf{\Lambda}_h$ are defined in Appendix B.

Similar to Section 3.1 we lift Eq. (18) to a first order equation and end up with a (finite dimensional) stochastic differential equation of the same type as in Eq. (7). In particular, we end up with the following $\mathcal{H}$-valued SDE

$$\begin{cases} \mathrm{d}\tilde{u}_h(t) = \tilde{v}_h(t)\mathrm{d}t, \\ \mathrm{d}\tilde{v}_h(t) = \Lambda_h\tilde{u}_h^k\mathrm{d}t + \mathrm{d}W_h(dt) + N_f\tilde{u}_h^k\mathrm{d}t, \\ \tilde{u}_h(0) = \mathcal{P}_h u_0, \qquad \tilde{v}_h(0) = \mathcal{P}_h v_0, \end{cases} \tag{19}$$

or equivalently

$$\begin{cases} \mathrm{d}\tilde{X}_h(t) = \left(\mathcal{A}_h\tilde{X}_h(t) + F_h(\tilde{X}_h(t))\right)\mathrm{d}t + Q\,\mathcal{W}_h(\mathrm{d}t), \\ \tilde{X}_h(0) = \begin{pmatrix} \mathcal{P}_h u_0 \\ \mathcal{P}_h v_0 \end{pmatrix}, \end{cases} \tag{20}$$

where we used the following notations:

$$\tilde{X}_h(t) := \begin{pmatrix} \tilde{u}_h(t) \\ \tilde{v}_h(t) \end{pmatrix}, \qquad \mathcal{A}_h := \begin{pmatrix} 0 & 1 \\ -\Lambda_h & 0 \end{pmatrix}, \qquad D(\mathcal{A}_h) = \mathcal{V}_h.\text{[4]}$$

The drift term is defined by

$$\mathcal{V}_h \otimes \mathcal{V}_h \ni \begin{pmatrix} u \\ v \end{pmatrix} \mapsto F_h\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) := \begin{pmatrix} 0 \\ \mathcal{N}_h^f(v) \end{pmatrix} \in \mathcal{V}_h \otimes \mathcal{V}_h, \quad 0 < h \le 1.$$

4 For a definition, see Appendix B.

Under suitable hypothesis on the drift f and the stability condition

$$\tau_h = h, \tag{21}$$

Eq. (20) has a unique solution $\{\tilde{X}_h(t), t \in [0, T]\}$ with values in $\mathcal{H}$ such that there exists a constant C with

$$\mathbb{E} \left|\tilde{X}_h(t)\right|^2_{\mathcal{H}} \leq C, \quad 0 < h \leq 1.$$

4.1. Reflecting of the approximation

To handle the space approximation we will reflect Eq. (18) at the t–ξ-axis. Then one can write the space approximation of Eq. (1) as an semi implicit Euler scheme in time. Without loss of generality we assume $T = L = 1$. If $T > L$, then we extend the initial condition to the interval $[0, T]$ by setting $u(\xi) = v(\xi) = 0$ for $L < \xi \leq T$. In Appendix B we have seen, that we can associate the finite element space $\mathcal{V}_h$ by $\mathbb{R}^{k_h}$. Substitution of the exact form of the stiffness matrix $\mathbf{\Lambda}_h$ gives for (20) the following system of equations written in the Itô form:

$$\begin{cases} d\tilde{\mathbf{u}}^h_k(t) = \tilde{\mathbf{v}}^h_k(t)dt, \\ d\tilde{\mathbf{v}}^h_k(t) = -h^{-2}\left(\tilde{\mathbf{u}}^h_{k+1}(t) - 2\tilde{\mathbf{u}}^h_k(t) + \tilde{\mathbf{u}}^h_{k-1}(t)\right) dt + f(\mathbf{u}^h_k(t))dt + h^{-1}W(dt, J^h_k), \\ \tilde{\mathbf{u}}^h_k(0) = \hat{\mathbf{u}}^0_{k,h}, \\ \tilde{\mathbf{v}}^h_k(0) = \hat{\mathbf{v}}^0_{k,h}, \quad 1 \leq k \leq k_h. \end{cases} \tag{22}$$

System (22) defines a function $\tilde{\mathbf{u}}^h : [0, 1] \times \{1, \ldots, k_h\} \ni (t, k) \mapsto \tilde{\mathbf{u}}^h(t, N^h_k) := \tilde{\mathbf{u}}^h_k(t) \in \mathbb{R}$, where $[0, 1]$ represents the time and $\{1, \ldots, k_h\}$ the space. That means, for a fixed $t \in [0, 1]$, $\tilde{u}_h(t)$ given by $\left((\tilde{\mathbf{u}}_h(t))_1, \ldots, (\tilde{\mathbf{u}}_h(t))_{k_h}\right)$ belongs to $\mathbb{R}^{k_h}$. Now, the (finite dimensional) space $\{1, \ldots, k_h\}$ is mapped onto the k_h-iterates of the set of functions $\{\bar{u}^h_k : [0, 1] \to \mathbb{R}, k = 1, \ldots, k_h\}$. This is equivalent to apply a change of coordinates in the above Eq. (22). For $k = 1, \ldots, k_h$ and any $(s, z) \in [0, 1]^2$ put $\bar{u}^h_k(s, z) := \tilde{\mathbf{u}}^h(z, -s)$. Then, from the recursion (22) we obtain that $\bar{u}^h_k(z) := \bar{u}_h(N^h_k, z)$ satisfies the following stochastic equation:

$$\begin{cases} \dfrac{d^2}{dz^2}\bar{u}^h_k(z) = -h^{-2}\left(\bar{u}^h_{k+1}(z) - 2\bar{u}^h_k(z) + \bar{u}^h_{k-1}(z)\right) + h^{-1}\dot{W}(z, J^h_k) + f(\bar{u}^h_k(z)), \\ \bar{u}^h_k(0) = (\mathcal{P}_h u_0)(N^h_k), \\ \bar{v}^h_k(0) = (\mathcal{P}_h v_0)(N^h_k), \quad k = 1, \ldots, k_h. \end{cases} \tag{23}$$

The notation $\dot{W}(z, J^h_k)$ is the formal notation for $\frac{d}{dz}W(z, J^h_k)$. Put

$$\bar{\mathbf{v}}^h_k(z) := h^{-1}\left(\bar{u}^h_{k+1}(z) - \bar{u}^h_k(z)\right).$$

Then

$$\begin{aligned} \frac{d^2}{dz^2}\bar{u}^h_k(z) &= -h^{-2}\left(\bar{u}^h_{k+1}(z) - 2\bar{u}^h_k(z) + \bar{u}^h_{k-1}(z)\right) + h^{-1}\dot{W}(z, J^h_k) + f(\bar{u}^h_k(z)) \\ &= -h^{-1}\left[h^{-1}\left(\bar{u}^h_{k+1}(z) - \bar{u}^h_k(z)\right) - h^{-1}\left(\bar{u}^h_k(z) - \bar{u}^h_{k-1}(z)\right)\right] h^{-1}\dot{W}(z, J^h_k) + f(\bar{u}^h_k(z)) \\ &= -h^{-1}\left[\bar{v}^h_k(z) - \bar{v}^h_{k-1}(z)\right] + h^{-1}\dot{W}(z, J^h_k) + f(\bar{u}^h_k(z)). \end{aligned}$$

Rearranging gives

$$\begin{cases} h^{-1}\left[\bar{u}^h_{k+1}(z) - u^h_k(z)\right] = \bar{v}^h_k(z)dt, \quad k = 1, \ldots, l_h. \\ h^{-1}\left[\bar{v}^h_{k+1}(z) - v^h_k(z)\right] = -\dfrac{d^2}{dz^2}\bar{u}^h_{k+1}(z) + h^{-1}\dot{W}(z, J^h_k) + f(\bar{u}^h_k(z)), \quad k = 1, \ldots, l_h, \\ \bar{u}^h_0(z) = 0, \quad 0 \leq z \leq 1, \\ \bar{v}^h_0(z) = 0, \quad 0 \leq z \leq 1, \end{cases} \tag{24}$$

with boundary conditions

$$\begin{cases} \bar{u}^h_k(z) = (\mathcal{P}_h u_0)(N^h_k); \quad k = 1, \ldots, l_h, \\ \dfrac{\partial}{\partial z}\, \bar{u}^h_k(z) = (\mathcal{P}_h v_0)(N^h_k); \quad k = 1, \ldots, l_h. \end{cases} \tag{25}$$

Similarly to before, first, Eq. (24) can be lifted on $\mathcal{H}$ and, secondly, the boundary conditions (25) can be handled in the same way as in Section 3.2. So we arrive at

$$\begin{cases} h^{-1}\left[\begin{pmatrix}\bar u_k^h(z)\\ \bar v_k^h(z)\end{pmatrix}-\begin{pmatrix}\bar u_{k-1}^h(z)\\ \bar v_{k-1}^h(z)\end{pmatrix}\right]=\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}\bar u_{k-1}^h(z)\\ \bar v_{k-1}^h(z)\end{pmatrix}+\begin{pmatrix}0&0\\ \Lambda&0\end{pmatrix}\begin{pmatrix}\bar u_k^h(z)\\ \bar v_k^h(z)\end{pmatrix}\\ \quad +(1-\mathcal{A})\mathcal{B}\left((\mathcal{P}_h u_0)(N_k^h),(\mathcal{P}_h v_0)(N_k^h)\right)\\ \quad +\begin{pmatrix}0\\ h^{-1}\Delta_{k-1}\dot W(z)\end{pmatrix}+\begin{pmatrix}0\\ f\left(\bar u_{k-1}^h(z)\right)\end{pmatrix},\quad k=1,\dots,k_h,\end{cases} \tag{26}$$

where $\Delta_k\dot W(z)=\dot W(z,kh+\frac{h}{2})-\dot W(z,kh-\frac{h}{2})$ and $\mathcal{B}:\mathbb{R}^2\to H$ is defined in (3.1), resp. (16). Comparing Eqs. (26) and (33), one can see that both are of the same type apart from the following differences. First, in (26) an additional term due to the boundary conditions appears, and, secondly, the operator Λ_h in (33) is finite dimensional, therefore bounded, and the operator Λ in (26) is unbounded.

We approximate the solution between the grid points $t_k^h = k\tau_h$ and $t_{k+1}^h = (k+1)\tau_h$ in the same way as we have approximated the solution of (33). If $(\bar u_h(t),\bar v_h(t))^T$ denote the approximation at time t, where $\bar{\mathbf{u}}_h(t)=(\bar u_0(t),\dots,\bar u_m(t))$, then the solution will be defined for $t=k\tau_h$, $k\in\mathbb{N}$, by

$$\begin{pmatrix}\bar u_h(t)\\ \bar v_h(t)\end{pmatrix}:=\begin{pmatrix}\bar u_h^k\\ \bar v_h^k\end{pmatrix} \tag{27}$$

and for $t\in(t_k^h,t_{k+1}^h)$, by

$$\begin{cases}\bar u_h(t)=\bar u_h^k+\int_{t_k^h}^t \bar v_h(s)\mathrm{d}s,\\ \bar v_h(t)=\bar v_h^k+(t-t_k^h)\Lambda\bar u_h^k+\left(t-t_k^h\right)^2\Lambda\bar v_h^k+\Delta_h^k W(t)+\left(t-t_k^h\right)N^f\bar u_h^k\\ \quad +(t-t_k^h)(1-\Lambda)B^1\left((\mathcal{P}_h u_0)(N_k^h),(\mathcal{P}_h u_0)(N_k^h)\right).\end{cases} \tag{28}$$

Equivalently,

$$\begin{cases}\mathrm{d}\bar u_h(t)=\bar v_h(t)\mathrm{d}t,\\ \mathrm{d}\bar v_h(t)=\Lambda\bar u_h^k\mathrm{d}t+2(t-t_k^h)\Lambda\bar v_h^k\mathrm{d}t+\mathrm{d}W(\mathrm{d}t)+N_f\bar u_h^k\mathrm{d}t\\ \quad +(t-t_k^h)(1-\Lambda)B^1\left((\mathcal{P}_h u_0)(N_k^h),(\mathcal{P}_h u_0)(N_k^h)\right).\end{cases} \tag{29}$$

5. Time discretisation

Next we discretise Eq. (18) with respect to the time variable. Let τ_h be the time step size corresponding to the subdivision $\{\mathcal{T}_h, 0<h\le 1\}$. The approximation of $u(k\tau_h, ih)$ at the grid points will be denoted by $\hat u_{i,h}^k$. We make use of the classical centred difference method to discretise the second derivative with respect to time appearing in (18), such that we end up with the so-called leap-frog scheme.

The initial values $\hat u_{i,h}^0$ and $\hat u_{i,h}^1$ are defined by

$$\begin{aligned}&\hat u_{i,h}^0 := (\mathbf{P}_h u_0)_i,\quad i=1,\dots,k_h,\\ &\hat u_{i,h}^1-\hat u_{i,h}^0 := \tau_h(\mathbf{P}_h v_0)_i,\quad i=1,\dots,k_h,\end{aligned} \tag{30}$$

where $\mathbf{P}_h : H\to\mathbb{R}^{k_h}$ is defined by (51). Note, by $(\cdot)_i$ we denote the projection onto the ith column, i.e.

$$\mathbb{R}^{k_h}\ni\mathbf{u}=(u_1,\dots,u_{k_h})\mapsto(\mathbf{u})_i=u_i\in\mathbb{R}.$$

The vectors $\hat u_{i,h}^k$, $k\ge 2$, are given by the following recursion

$$(\tau_h)^{-2}\left(\hat u_{i,h}^{k+1}-2\hat u_{i,h}^k+\hat u_{i,h}^{k-1}\right)=h^{-2}\left(\hat u_{i-1,h}^k-2\hat u_{i,h}^k+\hat u_{i+1,h}^k\right)+h^{-\frac12}\Delta_{i,h}^k W+f(\hat u_{i,h}^k),\quad i=1,\dots,k_h, \tag{31}$$

where

$$\Delta_{i,h}^k W := h^{-\frac12}\int_{k\tau_h}^{(k+1)\tau_h} W(\mathrm{d}s, J_i^h),\quad i=1,\dots,k_h,$$

with $W=\{W(t,\xi)|0\le t\le T, 0\le\xi\le M\}$ is the space–time white noise.

Set $\Delta_h^k\mathbf{W}:=(\Delta_{1,h}^k W,\dots,\Delta_{k_h,h}^k W)^T$ and $\hat{\mathbf{u}}_h^k=\left(\hat u_{1,h}^k,\dots,\hat u_{k_h,h}^k\right)^T\in\mathbb{R}^{k_h}$ for all $k=1,\dots,n$. We rewrite (31) as

$$(\tau_h)^{-2}\left(\hat{\mathbf{u}}_h^{k+1}-2\hat{\mathbf{u}}_h^k+\hat{\mathbf{u}}_h^{k-1}\right)=\mathbf{\Lambda}_h\hat{\mathbf{u}}_h^k+\Delta_h^k\mathbf{W}+\mathbf{N}_h^f(\hat{\mathbf{u}}_h^k).$$

Again, we lift the leap-frog scheme to a two-dimensional system of recursions only involving one finite differences. Let

$$\begin{pmatrix} \tilde{\mathbf{u}}_0^h \\ \tilde{\mathbf{v}}_h^0 \end{pmatrix} = \begin{pmatrix} \mathbf{P}_h u_0 \\ \mathbf{P}_h v_0 \end{pmatrix} \tag{32}$$

and put $\hat{\mathbf{u}}_h^n := (\hat{u}_{1,h}^n, \ldots, \hat{u}_{k_h,h}^n)^T$ and $\hat{\mathbf{v}}_n^h := (\hat{v}_{1,h}^n, \ldots, \hat{v}_{k_h,h}^n)^T$. Now, the vector $\left(\hat{\mathbf{u}}_h^k, \hat{\mathbf{v}}_h^k\right)^T$ is given by the recursion

$$\frac{1}{\tau_h}\left(\begin{pmatrix} \hat{\mathbf{u}}_h^k \\ \hat{\mathbf{v}}_h^k \end{pmatrix} - \begin{pmatrix} \hat{\mathbf{u}}_h^{k-1} \\ \hat{\mathbf{v}}_h^{k-1} \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ \mathbf{\Lambda}_h & 0 \end{pmatrix}\begin{pmatrix} \hat{\mathbf{u}}_h^k \\ \hat{\mathbf{v}}_h^k \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{1}_h \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \hat{\mathbf{u}}_h^{k-1} \\ \hat{\mathbf{v}}_h^{k-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \tau_h^{-\frac{1}{2}} \Delta_h^k \mathbf{W} \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{N}_h^f \hat{\mathbf{u}}_h \end{pmatrix}, \tag{33}$$

where $\mathbf{1}_h \in \mathbb{R}^{k_h} \times \mathbb{R}^{k_h}$ denotes the identity. Since

$$\left(\begin{pmatrix} \mathbf{1}_h & 0 \\ 0 & \mathbf{1}_h \end{pmatrix} - \tau_h \begin{pmatrix} 0 & 0 \\ \mathbf{\Lambda}_h & 0 \end{pmatrix}\right)^{-1} = \begin{pmatrix} \mathbf{1}_h & 0 \\ \tau_h \mathbf{\Lambda}_h & \mathbf{1}_h \end{pmatrix},$$

Eq. (33) can be also written in the following explicit form:

$$\begin{pmatrix} \hat{\mathbf{u}}_h^k \\ \hat{\mathbf{v}}_h^k \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ \tau_h \mathbf{\Lambda}_h & \mathbf{1} \end{pmatrix}\left[\begin{pmatrix} \hat{\mathbf{u}}_h^{k-1} \\ \hat{\mathbf{v}}_h^{k-1} \end{pmatrix} + \tau_h \begin{pmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \hat{\mathbf{u}}_h^{k-1} \\ \hat{\mathbf{v}}_h^{k-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta_h^{k-1} \mathbf{W} \end{pmatrix} + \tau_h \begin{pmatrix} 0 \\ \mathbf{N}_h^f \hat{\mathbf{u}}_h^{k-1} \end{pmatrix}\right], \tag{34}$$

or equivalently

$$\begin{aligned} \hat{\mathbf{u}}_h^k &= \hat{\mathbf{u}}_h^{k-1} + \tau_h \hat{\mathbf{v}}_h^{k-1}, \\ \hat{\mathbf{v}}_h^k &= \hat{\mathbf{v}}_h^{k-1} + \tau_h \mathbf{\Lambda}_h \hat{\mathbf{u}}_h^{k-1} + \tau_h^2 \mathbf{\Lambda}_h \hat{\mathbf{v}}_h^{k-1} + \Delta_h^{k-1}\mathbf{W} + \mathbf{N}_h^f \hat{\mathbf{u}}_h^{k-1}, \quad k \in \mathbb{N}. \end{aligned}$$

The solution between the grid points $t_k^h = k\tau_h$ and $t_{k+1}^h = (k+1)\tau_h$ will be interpolated as follows. Let $(\hat{\mathbf{u}}_h(t), \hat{\mathbf{v}}_h(t))^T$ denote the approximation at time t, where $\hat{\mathbf{u}}_h(t) = \left(\hat{u}_0(t), \ldots, \hat{u}_m(t)\right)$ and $\hat{\mathbf{v}}_h(t) = \left(\hat{v}_0(t), \ldots, \hat{v}_m(t)\right)$. If $t = k\tau_h$, $k \in \mathbb{N}$, then we put

$$\begin{pmatrix} \hat{\mathbf{u}}_h(t) \\ \hat{\mathbf{v}}_h(t) \end{pmatrix} := \begin{pmatrix} \hat{\mathbf{u}}_h^k \\ \hat{\mathbf{v}}_h^k \end{pmatrix} \tag{35}$$

and if $t \in (t_k^h, t_{k+1}^h)$, then we put

$$\begin{cases} \hat{\mathbf{u}}_h(t) = \hat{\mathbf{u}}_h^k + \displaystyle\int_{t_k^h}^t \hat{\mathbf{v}}_h(s)\mathrm{d}s, \\ \hat{\mathbf{v}}_h(t) = \hat{\mathbf{v}}_h^k + (t - t_k^h)\mathbf{\Lambda}_h \hat{\mathbf{u}}_h^k + \left(t - t_k^h\right)^2 \mathbf{\Lambda}_h \hat{v}_h^k + \Delta_h^k \mathbf{W}(t) + \left(t - t_k^h\right)\mathbf{N}_h^f \hat{\mathbf{u}}_h^k. \end{cases} \tag{36}$$

Equivalently,

$$\begin{cases} \mathrm{d}\hat{\mathbf{u}}_h(t) = \hat{\mathbf{v}}_h(t)\mathrm{d}t, \\ \mathrm{d}\hat{\mathbf{v}}_h(t) = \mathbf{\Lambda}_h \hat{\mathbf{u}}_h^k \mathrm{d}t + 2(t - t_k^h)\mathbf{\Lambda}_h \hat{\mathbf{v}}_h^k \mathrm{d}t + d\mathbf{W}_h(\mathrm{d}t) + \mathbf{N}_f \hat{\mathbf{u}}_h^k \mathrm{d}t, \quad t \geq 0, \end{cases} \tag{37}$$

or for $\hat{u}_h(t) := \mathcal{I}_h$[5] $\hat{\mathbf{u}}_h(t)$ and $\hat{v}_h(t) := \mathcal{I}_h \hat{\mathbf{v}}_h(t)$, $t \in [0, T]$,

$$\begin{cases} \mathrm{d}\hat{u}_h(t) = \hat{v}_h(t)\mathrm{d}t, \\ \mathrm{d}\hat{v}_h(t) = \Lambda_h \hat{u}_h^k \mathrm{d}t + 2(t - t_k^h)\Lambda_h \hat{v}_h^k \mathrm{d}t + d\mathcal{W}_h(\mathrm{d}t) + \mathcal{N}_f \hat{u}_h^k \mathrm{d}t, \quad t \geq 0. \end{cases} \tag{38}$$

6. Error analysis

This section is devoted to the actual proof of Theorem 2.1. To be precise, we will investigate the Monte Carlo error, i.e. the quantity given by

$$E^h(\Phi) := \left|\mathbb{E}^{\mathcal{P}_h u_0, \mathcal{P}_h v_0} \Phi(\hat{u}_h) - \mathbb{E}^{u_0, v_0} \Phi(u)\right|.$$

Above, $\Phi : H_2^{-\alpha_0}([0, L] \times [0, T_0]) \to \mathbb{R}$ is a fixed mapping and $\mathcal{P}_h$ is given in (50). Similarly to the way the approximation in Section 4 was introduced, the error can be split in two parts, i.e.

$$E^h(\Phi) \leq \mathrm{error}_1^h(\Phi) + \mathrm{error}_2^h(\Phi),$$

where the first entity is the error between the space and the space–time approximation and the second entity is the error between the exact solution and the space approximation. In particular,

$$\begin{aligned} \mathrm{error}_1^h(\Phi) &= \left|\mathbb{E}^{\mathcal{P}_h u_0, \mathcal{P}_h v_0} \Phi(\tilde{u}_h) - \mathbb{E}^{\mathcal{P}_h u_0, \mathcal{P}_h v_0} \Phi(\hat{u}_h)\right|, \quad \text{and} \\ \mathrm{error}_2^h(\Phi) &= \left|\mathbb{E}^{u_0, v_0} \Phi(u) - \mathbb{E}^{\mathcal{P}_h u_0, \mathcal{P}_h v_0} \Phi(\tilde{u}_h)\right|. \end{aligned}$$

[5] See the definition of the interpolant given in Eq. (48).

6.1. The first error

In this section we give an estimate of the first term, i.e. of $\text{error}_1^h(\Phi)$. Identifying $H^{-\alpha}([0,T_0]\times[0,1];\mathbb{R})$ with $H^{-\alpha}([0,T_0];H^{-\alpha}([0,1];\mathbb{R}))$ there exists a function $\rho : H^{-\alpha}([0,T_0])\times[0,T_0]\to H^{-\alpha}([0,1];\mathbb{R})$ such that

$$\Phi(u)=\int_0^{T_0}\rho(u(T),T)\,\mathrm{d}T.$$

Therefore, the Fubini Theorem and the Minkowski inequality give

$$\begin{aligned}\text{error}_1^h(\Phi)&=\left|\mathbb{E}^{\mathcal{P}_hu_0,\mathcal{P}_hv_0}\int_0^{T_0}\rho(\tilde u_h(T),T)\,\mathrm{d}T-\mathbb{E}^{\mathcal{P}_hu_0,\mathcal{P}_hv_0}\int_0^{T_0}\rho(\hat u_h(T),T)\,\mathrm{d}t\right|\\&\le\int_0^{T_0}\left|\mathbb{E}^{\mathcal{P}_hu_0,\mathcal{P}_hv_0}\rho(\tilde u_h(T),T)-\mathbb{E}^{\mathcal{P}_hu_0,\mathcal{P}_hv_0}\rho(\hat u_h(T),T)\right|\,\mathrm{d}T.\end{aligned}$$

We will show in the following, that there exists a constant $C>0$ such that

$$\sup_{0\le T\le T_0}\left|\mathbb{E}^{\mathcal{P}_hu_0,\mathcal{P}_hv_0}\rho(\tilde u_h(T),T)-\mathbb{E}^{\mathcal{P}_hu_0,\mathcal{P}_hv_0}\rho(\hat u_h(T),T)\right|\le C\,\tau_h^2,\quad h\in(0,1].$$

Let us fix $T\in[0,T_0]$ and let us assume that $T=n_h\tau_h$. Similarly to the method of Talay and Tubaro [22] we introduce a function Ψ as follows:

$$\begin{array}{cccc}\Psi: & \mathcal{H}_0\times[0,T] & \to & \mathbb{R}\\ & (x,y,t) & \mapsto & \Psi(x,y,t),\end{array}$$

where

$$\Psi(x,y,t):=\mathbb{E}^{x,y}\left[\rho\left(\tilde u_h(T-t),T\right)\right],\quad t\in[0,T].$$

Recall that the process $\{\tilde u_h(t):t\in[0,T]\}$ corresponds to the space approximation of $\{u_h(t):t\in[0,T]\}$, given by the stochastic differential equation (19), and, written in a lifted form, solves the stochastic differential equation (20). Now, due to the definition of Ψ, it is straightforward that

$$\begin{aligned}\text{error}_1^h(\phi,T)&:=\left|\mathbb{E}^{\mathcal{P}_hu_0,\mathcal{P}_hv_0}\rho(\tilde u_h(T),T)-\mathbb{E}^{\mathcal{P}_hu_0,\mathcal{P}_hv_0}\rho(\hat u_h(T),T)\right|\\&=\left|\mathbb{E}^{\mathcal{P}_hu_0,\mathcal{P}_hv_0}\Psi(\hat u_h^n,\hat v_h^n,T)-\mathbb{E}^{u_0,v_0}\Psi(\mathcal{P}_hu_0,\mathcal{P}_hv_0,0)\right|.\end{aligned}$$

By the tower property of the conditional expectation we infer that

$$\begin{aligned}\text{error}_1^h(\phi,T)&=\mathbb{E}^{\mathcal{P}_hu_0,\mathcal{P}_hv_0}\sum_{i=0}^{n_h-1}\Psi\left(\hat u_h^{n_h-i},\hat v_h^{n_h-i},(n_h-i)\tau_h\right)-\Psi\left(\hat u_h^{n_h-(i+1)},\hat v_h^{n_h-(i+1)},(n_h-(i+1))\tau_h\right)\\&=\mathbb{E}^{\mathcal{P}_hu_0,\mathcal{P}_hv_0}\sum_{i=0}^{n_h-1}\mathbb{E}^{\hat u_h^{n_h-i-1},\hat v_h^{n_h-i-1}}\left[\Psi\left(\hat u_h^1,\hat v_h^1,(n_h-i)\tau_h\right)-\Psi\left(\mathcal{P}_hu^0,\mathcal{P}_hv_h^0,(n_h-i-1)\tau_h\right)\right].\end{aligned}$$

Consequently, we have to investigate the following differences

$$\widehat{\text{error}}_\Psi^h(k,T):=\left|\mathbb{E}^{x,y}\left[\Psi\left(\hat u_h^1,\hat v_h^1,(n_h-k)\tau_h\right)-\Psi\left(x,y,(n_h-k-1)\tau_h\right)\right]\right|,\quad k\in\{0,\dots,n_h-1\},$$

where x,y are random elements of $\mathcal{V}_h$. Note that in order to get convergence of order two, we have to show that there exists a constant $C>0$ such that

$$\widehat{\text{error}}_\Psi^h(k,T)\le C\,\tau_h^3,\quad 0<h\le 1.$$

Putting $\hat u_0:=\mathcal{P}_hu_0$ and $\hat v_0:=\mathcal{P}_hv_0$, the process $\{\hat u_h(t):t\ge 0\}$ defined in (37), satisfies for $t\in(0,\tau_h)$

$$\begin{cases}\mathrm{d}\hat u_h(t)=\hat v_0(t)\mathrm{d}t,\\ \mathrm{d}\hat v_h(t)=\Lambda_h\hat u_0\mathrm{d}t+2t\Lambda_h\hat v_0\mathrm{d}t+\Delta^0\mathcal{W}(t)+N_h^f\hat u_0\mathrm{d}t.\end{cases}\tag{39}$$

Remind that $\Delta^0\mathcal{W}_h(t)=\frac{1}{\sqrt{h}}\sum_{i=1}^{k_h}\int_0^t W(\mathrm{d}s,J_i^h)\,g_i^h$, for $t\in(0,\tau_h)$, where $g_i^h:[0,1]\ni\xi\mapsto 1_{[N_{i-1}^h,N_i^h)}(\xi)(\xi-N_{i-1}^h)+1_{[N_i^h,N_{i+1}^h)}(\xi)(N_i^h-\xi)$. Hence by the Itô formula

$$\begin{aligned}\widehat{\text{error}}_\Psi^h(k,T)=\mathbb{E}^{x,y}\Bigg[&\int_0^{\tau_h}\left\langle D_x\Psi\left(\hat u_h(s),\hat v_h(s),s+(n_h-k-1)\tau_h\right),\hat v_h(s)\right\rangle_{H_0}\mathrm{d}s\\&+\int_0^{\tau_h}\left\langle D_y\Psi\left(\hat u_h(s),\hat v_h(s),s+(n_h-k-1)\tau_h\right),\Lambda_hx+2s\Lambda_hy+\mathcal{N}_h^fx\right\rangle_{V_0^*}\mathrm{d}s\\&+\frac12\int_0^{\tau_h}\mathrm{Tr}\left(D_{yy}^2\Psi\left(\hat u_h(s),\hat v_h(s),s+(n_h-k-1)\tau_h\right)\right)_{V_0^*}\mathrm{d}s\\&+\int_0^{\tau_h}D_t\Psi\left(\hat u_h(s),\hat v_h(s),s+(n_h-k-1)\tau_h\right)\mathrm{d}s\Bigg].\end{aligned}\tag{40}$$

Recall that the pair $(\tilde{u}_h, \tilde{v}_h)$ appearing in the definition of Ψ satisfies Eq. (19). Taking the derivative of Ψ with respect to time corresponds applying the Kolmogorov equation (19) to Ψ. Therefore

$$\begin{aligned} &D_t\Psi\left(\hat{u}_h(s), \hat{v}_h(s), (t-s-(n_h-k-1)\tau_h)\right) \\ &\quad = -\mathbb{E}\left\langle D_x\Psi\left(\hat{u}_h(s), \hat{v}_h(s), t-s-(n_h-k-1)\tau_h\right), \tilde{v}_h(s)\right\rangle_{H_0} \\ &\quad - \mathbb{E}\left\langle D_y\Psi\left(\hat{u}_h(s), \hat{v}_h(s), t-s-(n_h-k-1)\tau_h\right), \Lambda_h\tilde{u}_h(s) + \mathcal{N}_h^f\tilde{u}_h(s)\right\rangle_{V_0^*} \\ &\quad - \frac{1}{2}\,\mathbb{E}\mathrm{Tr}\left[D_{yy}\Psi\left(\hat{u}_h(s), \hat{v}_h(s), t-s-(n_h-k-1)\tau_h\right)\right]_{V_0^*} \end{aligned}$$

We plug this equality in (40) and obtain

$$\begin{aligned} \widehat{\mathrm{error}}_\Psi^h(k,T) = \mathbb{E}^{x,y}\Bigg[&\left\langle D_x\Psi\left(\hat{u}_h(s), \hat{v}_h(s), s+(n_h-k-1)\tau_h\right), \hat{v}_h(s)-\tilde{v}_h(s)\right\rangle_{H_0} \mathrm{d}s \\ &+ \int_0^{\tau_h}\left\langle D_y\Psi\left(\hat{u}_h(s), \hat{v}_h(s), s+(n_h-k-1)\tau_h\right), \Lambda_h x + N_h^f x - \Lambda_h\tilde{u}_h(s) - \mathcal{N}_h^f\tilde{u}_h(s)\right\rangle_{V_0^*} \mathrm{d}s\Bigg]. \end{aligned}$$

Notice that the terms containing the trace have been cancelled out. Put

$$\mathrm{diff}_1(x,y,k) := \mathbb{E}^{x,y}\left[\int_0^{\tau_h}\left\langle D_x\Psi\left(\hat{u}_h(s), \hat{v}_h(s), s+(n_h-k-1)\tau_h\right), \left(\hat{v}_h(s)-\tilde{v}_h(s)\right)\right\rangle_{H_0} \mathrm{d}s\right],$$

$$\mathrm{diff}_2(x,y,k) := \mathbb{E}^{x,y}\left[\int_0^{\tau_h}\langle D_y\Psi\left(\hat{u}_h(s), \hat{v}_h(s), s+(n_h-k-1)\tau_h\right), N_h^f x - \mathcal{N}_h^f\tilde{u}_h(s)\rangle_{V_0^*}\mathrm{d}s\right],$$

$$\mathrm{diff}_3(x,y,k) := \mathbb{E}^{x,y}\left[\int_0^{\tau_h}\langle D_y\Psi\left(\hat{u}_h(s), \hat{v}_h(s), s+(n_h-k-1)\tau_h\right), \Lambda_h x - \Lambda_h\tilde{u}_h(s)\rangle_{V_0^*}\mathrm{d}s\right].$$

We first deal with the term $\mathrm{diff}_1(x, y, k)$. Taking into account that the Kolmogorov equation at the time grid points coincide for $\tilde{u}_h$ and $\hat{u}_h$, we write

$$\begin{aligned} \mathrm{diff}_1(x,y,k) = \mathbb{E}^{x,y}\Bigg[&\int_0^{\tau_h}\Big[\left\langle D_x\Psi\left(\hat{u}_h(s), \hat{v}_h(s), s+(n_h-k-1)\tau_h\right), \tilde{v}_h(s)\right\rangle_{H_0} \\ &- \left\langle D_x\Psi\left(\hat{u}_h(0), \hat{v}_h(0), (n_h-k-1)\tau_h\right), y\right\rangle_{\tilde{H}_0}\Big]\mathrm{d}s\Bigg] \\ &- \mathbb{E}^{x,y}\Bigg[\int_0^{\tau_h}\left\langle D_x\Psi\left(\hat{u}_h(0), \hat{v}_h(0), (n_h-k-1)\tau_h\right), \hat{v}_h(s)\right\rangle_{H_0} \\ &- \left\langle D_x\Psi\left(\hat{u}_h(s), \hat{v}_h(s), s+(n_h-k-1)\tau_h\right), y\right\rangle_{H_0} \mathrm{d}s\Bigg]. \end{aligned}$$

Next, we apply the Itô formula to the following functional:

$$\Theta(s, \hat{X}_h(s)) := \mathbb{E}\left\langle D_x\Psi(\hat{u}_h(s), \hat{v}_h(s), s+(n_h-k-1)\tau_h), \tilde{v}_h(s)\right\rangle_{H_0}.$$

In particular,

$$\begin{aligned} \frac{\partial}{\partial s}\Theta(s, \hat{u}_h(s), \hat{v}_h(s)) = &\left\langle D_{sx}\Psi(\hat{u}_h(s), \hat{v}_h(s), s+(n_h-k-1)\tau_h), \tilde{v}_h(s)\right\rangle_{H_0} \\ &+ \left\langle D_{sy}\Psi(\hat{u}_h(s), \hat{v}_h(s), s+(n_h-k-1)\tau_h), \Lambda_h\tilde{u}_h(s) + \mathcal{N}_h^f\tilde{u}_h(s)\right\rangle_{V_0}. \end{aligned}$$

Observe that under the standing situation, $\tilde{v}_h(0) = y$. Therefore, we end up with the following equality:

$$\begin{aligned} \mathrm{diff}_1(x,y,k) = \mathbb{E}^{x,y}\Bigg[\int_0^{\tau_h}\int_0^s\Big\{&\left\langle D_{xx}\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r+(n_h-k-1)\tau_h\right), \tilde{v}_h(r)\otimes\hat{v}_h(r)\right\rangle_{H_0\otimes H_0} \\ &+ \left\langle D_{xy}\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r+(n_h-k-1)\tau_h\right), \tilde{v}_h(r)\otimes\left(\Lambda_h x + \mathcal{N}_h^f x + 2r\Lambda_h y\right)\right\rangle_{H_0\otimes V_0^*} \end{aligned}$$

$$
\begin{aligned}
&+ \left\langle D_x\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), \Lambda_h \tilde{u}_h(r) + \mathcal{N}_h^f \tilde{u}_h(r)\right\rangle_{H_0} \\
&+ \left\langle D_{x,r}\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), \tilde{v}_h(r)\right\rangle_{H_0} \\
&+ \frac{1}{2}\left\langle D_x \mathrm{Tr}\left[D^h_{yy}\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right)\right]_{V_0^*}, \tilde{v}_h(r)\right\rangle_{H_0} \\
&+ \frac{1}{2}\sum_{i=1}^{\infty}\left\langle D_{xy}\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), e_i \otimes e_i\right\rangle\Bigg\}\mathrm{d}r\mathrm{d}s\Bigg] \\
&- \mathbb{E}^{x,y}\Bigg[\int_0^{\tau_h}\int_0^s \Bigg\{\left\langle D_{xx}\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), \hat{v}_h(r) \otimes \hat{v}_h(r)\right\rangle_{H_0\otimes H_0} \\
&+ \left\langle D_{xy}\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), \hat{v}_h(r) \otimes \left(\Lambda_h x + \tilde{\mathcal{N}_f} x + 2r\Lambda_h y\right)\right\rangle_{H_0\otimes V_0^*} \\
&+ \left\langle D_x\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), \Lambda_h x + 2r\Lambda_h y + \mathcal{N}_h^f x\right\rangle_{H_0} \\
&+ \left\langle D_{x,r}\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), \hat{v}_h(r)\right\rangle_{H_0} \\
&+ \frac{1}{2}\left\langle D_x \mathrm{Tr}\left[D_{yy}\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right)\right]_{V_0^*}, \hat{v}_h(r)\right\rangle_{H_0} \\
&+ \sum_{i=1}^{\infty}\left\langle D_{xy}\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), e_i \otimes e_i\right\rangle\Bigg\}\mathrm{d}r\mathrm{d}s\Bigg].
\end{aligned}
$$

Rearranging the summands gives

$$
\begin{aligned}
\mathrm{diff}_1(x, y, k) = {} & \mathbb{E}^{x,y}\left[\int_0^{\tau_h}\int_0^s \left\langle D_{xx}\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), \left(\tilde{v}_h(r) - \hat{v}_h(r)\right) \otimes \hat{v}_h(r)\right\rangle_{H_0\otimes H_0} \mathrm{d}r\mathrm{d}s\right] \\
&+ \mathbb{E}^{x,y}\Bigg[\int_0^{\tau_h}\int_0^s \Big\langle D_{xy}\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), \left(\tilde{v}_h(r) - \hat{v}_h(r)\right) \\
&\otimes \left(\Lambda_h x + \mathcal{N}_h^f x + 2r\Lambda_h y\right)\Big\rangle_{H_0\otimes V_0^*} \mathrm{d}r\mathrm{d}s\Bigg] \\
&+ \frac{1}{2}\,\mathbb{E}^{x,y}\left[\int_0^{\tau_h}\int_0^s \left\langle D_x \mathrm{Tr}\left[D_{yy}\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right)\right]_{V_0^*}, \left(\tilde{v}_h(r) - \hat{v}_h(r)\right)\right\rangle_{H_0} \mathrm{d}r\mathrm{d}s\right] \\
&+ \mathbb{E}^{x,y}\left[\int_0^{\tau_h}\int_0^s \left\langle D_{x,r}\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), \left(\tilde{v}_h(r) - \hat{v}_h(r)\right)\right\rangle_{H_0} \mathrm{d}r\mathrm{d}s\right] \\
&+ \mathbb{E}^{x,y}\Bigg[\int_0^{\tau_h}\int_0^s \Big\langle D_x\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), \Lambda_h \tilde{u}_h(r) + \mathcal{N}_h^f \tilde{u}_h(r) \\
&- \left(\Lambda_h x + 2r\Lambda_h y + \mathcal{N}_h^f x\right)\Big\rangle_{H_0} \mathrm{d}r\mathrm{d}s\Bigg] \\
=: {} & \mathrm{diff}_{11}(x, y, k) + \mathrm{diff}_{12}(x, y, k) + \mathrm{diff}_{13}(x, y, k) + \mathrm{diff}_{14}(x, y, k) + \mathrm{diff}_{15}(x, y, k).
\end{aligned}
$$

Again, taking into account that the Kolmogorov equation at the time grid points coincide for $\tilde{u}_h$ and $\hat{u}_h$, we write for the first difference

$$
\begin{aligned}
\mathrm{diff}_{11} = {} & \mathbb{E}^{x,y}\Bigg[\int_0^{\tau_h}\int_0^s \Bigg\{\left\langle D_{xx}\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), \tilde{v}_h(r) \otimes \hat{v}_h(r)\right\rangle_{H_0\otimes H_0} \\
&- \left\langle D_{xx}\Psi\left(x, y, (n_h - k - 1)\tau_h\right), y \otimes y\right\rangle_{H_0\otimes H_0}\Bigg\} \\
&- \Bigg\{\left\langle D_{xx}\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), \hat{v}_h(r) \otimes \hat{v}_h(r)\right\rangle_{H_0\otimes H_0} \\
&- \left\langle D_{xx}\Psi\left(x, y, (n_h - k - 1)\tau_h\right), y \otimes y\right\rangle_{H_0\otimes H_0}\Bigg\}\mathrm{d}r\mathrm{d}s\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E}^{x,y}\Bigg[\int_0^{\tau_h}\int_0^{s_0}\int_0^{s_1}\Big\{\big\langle D_{xxx}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n_h-k-1)\tau_h\right),\tilde{v}_h(s_2)\otimes\hat{v}_h(s_2)\otimes\hat{v}_h(s_2)\big\rangle_{H_0\otimes H_0\otimes H_0}\\
&+\Big\langle D_{xxy}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n_h-k-1)\tau_h\right),\tilde{v}_h(s_2)\otimes\hat{v}_h(s_2)\otimes\Big(\Lambda_hx+2s_2\Lambda_hy+\mathcal{N}_h^fx\Big)\Big\rangle_{H_0\otimes H_0\otimes V_0^*}\\
&+\Big\langle D_{xx}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n_h-k-1)\tau_h\right),\Big(\Lambda_h\tilde{u}_h(s_2)+\mathcal{N}_h^f\tilde{u}_h(s_2)\Big)\otimes\hat{v}_h(s_2)\Big\rangle_{H_0\otimes H_0}\\
&+\Big\langle D_{xx}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n_h-k-1)\tau_h\right),\tilde{v}_h(s_2)\otimes\Big(\Lambda_hx+2s_2\Lambda_hy+\mathcal{N}_h^fx\Big)\Big\rangle_{H_0\otimes H_0}\\
&+\frac{1}{2}\sum_{i=0}^{\infty}\big\langle D_{xxyy}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n_h-k-1)\tau_h\right),\tilde{v}_h(s_2)\otimes\hat{v}_h(s_2)\otimes e_i\otimes e_i\big\rangle_{H_0\otimes H_0\otimes V_0^*\otimes V_0^*}\\
&+\frac{1}{2}\sum_{i=1}^{\infty}\big\langle D_{xxy}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n_h-k-1)\tau_h\right),e_i\otimes\hat{v}_h(s_2)\otimes e_i\big\rangle_{H_0\otimes H_0\otimes H_0}\\
&+\frac{1}{2}\sum_{i=1}^{\infty}\big\langle D_{xxy}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n_h-k-1)\tau_h\right),\tilde{v}_h(s_2)\otimes\hat{v}_h(s_2)\otimes e_i\big\rangle_{H_0\otimes H_0\otimes H_0}\\
&+\frac{1}{2}\sum_{i=1}^{\infty}\big\langle D_{xx}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n_h-k-1)\tau_h\right),e_i\otimes e_i\big\rangle_{H_0\otimes H_0\otimes H_0}\Big\}\,\mathrm{d}s_2 ds_1\mathrm{d}s_0\Bigg].
\end{aligned}
$$

Similar representation can be found for the remaining differences, i.e.

$$
\begin{aligned}
\mathrm{diff}_{12} &= \mathbb{E}^{x,y}\Bigg[\int_0^{\tau_h}\int_0^{s}\Big\langle D_{xy}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),r+(n_h-k-1)\tau_h\right),\tilde{v}_h(r)\otimes\Big(\Lambda_hx+\mathcal{N}_h^fx+2r\Lambda_hy\Big)\Big\rangle_{H_0\otimes V_0^*}\\
&-\Big\langle D_{xy}\Psi\left(x,y,(n_h-k-1)\tau_h\right),y\otimes\Big(\Lambda_hx+\mathcal{N}_h^fx+2r\Lambda_hy\Big)\Big\rangle_{H_0\otimes V_0^*}\\
&-\Big\langle D_{xy}\Psi\left(\hat{u}_h(r),\hat{v}_h(r),r+(n_h-k-1)\tau_h\right),\hat{v}_h(r)\otimes\Big(\Lambda_hx+\mathcal{N}_h^fx+2r\Lambda_hy\Big)\Big\rangle_{H_0\otimes V_0^*}\\
&-\Big\langle D_{xy}\Psi\left(x,y,(n_h-k-1)\tau_h\right),y\otimes\Big(\Lambda_hx+\mathcal{N}_h^fx+2r\Lambda_hy\Big)\Big\rangle_{H_0\otimes V_0^*}\,\mathrm{d}r\mathrm{d}s\Bigg]\\
&= \mathbb{E}^{x,y}\Bigg[\int_0^{\tau_h}\int_0^{s_1}\int_0^{s_0}\big\langle D_{xxy}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n_h-k-1)\tau_h\right),\hat{v}_h(s_2)\otimes\tilde{v}_h(s_2)\\
&\otimes\Big(\Lambda_hx+\mathcal{N}_h^fx+2s_2\Lambda_hy\Big)\Big\rangle_{H_0\otimes H_0\otimes V_0^*}\\
&+\Big\langle D_{yxy}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n_h-k-1)\tau_h\right),\Big(\Lambda_hx+\mathcal{N}_h^fx+2s_2\Lambda_hy\Big)\\
&\otimes\tilde{v}_h(s_2)\otimes\Big(\Lambda_hx+\mathcal{N}_h^fx+2s_2\Lambda_hy\Big)\Big\rangle_{V_0^*\otimes H_0\otimes V_0^*}\\
&+\Big\langle D_{rxy}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n_h-k-1)\tau_h\right),\tilde{v}_h(s_2)\otimes\Big(\Lambda_hx+\mathcal{N}_h^fx+2s_2\Lambda_hy\Big)\Big\rangle_{H_0\otimes V_0^*}\\
&+\frac{1}{2}\sum_{i=0}^{\infty}\big\langle D_{yxyy}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n_h-k-1)\tau_h\right),\hat{v}_h(s_2)\otimes\tilde{v}_h(s_2)\otimes e_i\otimes e_i\big\rangle_{V_0^*\otimes H_0\otimes V_0^*\otimes V_0^*}\\
&+\frac{1}{2}\sum_{i=0}^{\infty}\big\langle D_{yxy}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n_h-k-1)\tau_h\right),e_i\otimes\tilde{v}_h(s_2)\otimes e_i\big\rangle_{V_0^*\otimes H_0\otimes V_0^*}\\
&+\frac{1}{2}\sum_{i=0}^{\infty}\big\langle D_{yxy}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n_h-k-1)\tau_h\right),\hat{v}_h(s_2)\otimes e_i\otimes\tilde{v}_h(s_2)\big\rangle_{V_0^*\otimes H_0\otimes V_0^*}\\
&+\frac{1}{2}\sum_{i=0}^{\infty}\big\langle D_{yx}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n_h-k-1)\tau_h\right),e_i\otimes e_i\big\rangle_{V_0^*\otimes H_0}\\
&+\Big\langle D_{xy}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n_h-k-1)\tau_h\right),\Big(\Lambda_h\tilde{u}_h(s_2)+\mathcal{N}_h^f\tilde{u}_h(s_2)\Big)\otimes\Big(\Lambda_hx+\mathcal{N}_h^fx+2s_2\Lambda_hy\Big)\Big\rangle_{H_0\otimes V_0^*}
\end{aligned}
$$

$$- \left\langle D_{xxy}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \hat{v}(s_2) \otimes \hat{v}_h(s_2) \otimes \left(\Lambda_h x + \mathcal{N}_h^f x + 2s_2 \Lambda_h y\right)\right\rangle_{H_0 \otimes H_0 \otimes V_0^*}$$
$$- \left\langle D_{yxy}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \left(\Lambda_h x + \mathcal{N}_h^f x + 2s_2 \Lambda_h y\right)\right.$$
$$\left. \otimes \hat{v}_h(s_2) \otimes \left(\Lambda_h x + \mathcal{N}_h^f x + 2s_2 \Lambda_h y\right)\right\rangle_{V_0^* \otimes H_0 \otimes V_0^*}$$
$$- \left\langle D_{rxy}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \hat{v}_h(s_2) \otimes \left(\Lambda_h x + \mathcal{N}_h^f x + 2s_2 \Lambda_h y\right)\right\rangle_{H_0 \otimes V_0^*}$$
$$\left. - \left\langle D_{xy}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \hat{v}_h(s_2) \otimes 2\Lambda_h y\right\rangle_{H_0 \otimes V_0^*} \mathrm{d}s_2 \mathrm{d}s_1 \mathrm{d}s_0 \right].$$

Setting

$$\tilde{\Psi}(x, y, r) := \mathrm{Tr}\left[D_{yy}\Psi(x, y, r)\right]_{V_0^*},$$

we obtain for the third term

$$\mathrm{diff}_{13} = \mathbb{E}^{x,y}\left[\int_0^{\tau_h}\int_0^{s_1} \left\langle D_x\tilde{\Psi}\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), \tilde{v}_h(r)\right\rangle - \left\langle D_x\tilde{\Psi}(x, y, (n_h - k - 1)\tau_h), y\right\rangle_{H_0}\right.$$
$$\left. - \left\{\left\langle D_x\tilde{\Psi}\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), \hat{v}_h(r)\right\rangle - \left\langle D_x\tilde{\Psi}(x, y, (n_h - k - 1)\tau_h), y\right\rangle_{H_0}\right\}\mathrm{d}r\mathrm{d}s\right]$$
$$= \mathbb{E}^{x,y}\left[\int_0^{\tau_h}\int_0^{s_0}\int_0^{s_1} \left\langle D_{xx}\tilde{\Psi}\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \left(\tilde{v}_h(s_2) - \hat{v}_h(s_2)\right) \otimes \hat{v}_h(s_2)\right\rangle_{H_0 \otimes H_0}\right.$$
$$+ \left\langle D_{xy}\tilde{\Psi}\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \left(\tilde{v}_h(s_2) - \hat{v}_h(s_2)\right) \otimes \left(\Lambda_h x + \mathcal{N}_h^f x + 2s_2 \Lambda_h y\right)\right\rangle_{H_0 \otimes V_0^*}$$
$$+ \left\langle D_{xr}\tilde{\Psi}\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \left(\tilde{v}_h(s_2) - \hat{v}_h(s_2)\right)\right\rangle_{H_0}$$
$$+ \mathrm{Tr}_{V_0^*}\left\langle D_{xyy}\tilde{\Psi}\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \left(\tilde{v}_h(s_2) - \hat{v}_h(s_2)\right)\right\rangle_{H_0}$$
$$\left. + \left\langle D_x\tilde{\Psi}\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \left(\tilde{\Lambda}_h u_h(s_2) + \mathcal{N}_h u_h(s_2) - \Lambda_h x - \mathcal{N}_h^f x - 2s_2 \Lambda_h y\right)\right\rangle_{H_0} \mathrm{d}s_2 \mathrm{d}s_1 \mathrm{d}s_0\right].$$

Next, we get for the fourth term

$$-\mathrm{diff}_{14} = \mathbb{E}^{x,y}\left[\int_0^{\tau_h}\int_0^{s}\left\{\left\langle D_{xx}\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), \left(\tilde{v}_h(r) - \hat{v}_h(r)\right) \otimes \tilde{v}_h(r)\right\rangle_{H_0 \otimes H_0}\right.\right.$$
$$\left.\left. + \left\langle D_{xy}\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), \left(\tilde{v}_h(r) - \hat{v}_h(r)\right) \otimes \left(\Lambda_h \tilde{u}_h(r) + \mathcal{N}_h^f \tilde{u}_h(r)\right)\right\rangle_{H_0 \otimes V_0^*}\right\}\mathrm{d}r\mathrm{d}s\right].$$

Again, rearranging and taking into account that the Kolmogorov equation at the time grid points coincide for $\tilde{u}_h$ and $\hat{u}_h$, we write

$$-\mathrm{diff}_{14} = \mathbb{E}^{x,y}\left[\int_0^{\tau_h}\int_0^{s_0}\int_0^{s_1}\left\{\left\langle D_{xxx}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \tilde{v}_h(s_2)\right.\right.\right.$$
$$\left. \otimes\ \tilde{v}_h(s_2) \otimes \left(\hat{v}_h(s_2) - \tilde{v}_h(s_2)\right)\right\rangle_{H_0 \otimes H_0 \otimes H_0} + \left\langle D_{xxy}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \tilde{v}_h(s_2)\right.$$
$$\left. \otimes\ \tilde{v}_h(s_2) \otimes \left(\Lambda_h x + \mathcal{N}_h^f x + 2s_2 \Lambda_h y - \Lambda_h \tilde{u}_h(s_2) - \mathcal{N}_h \tilde{u}_h(s_2)\right)\right\rangle_{H_0 \otimes H_0 \otimes V_0^*}$$
$$+ 2\left\langle D_{xx}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \left(\Lambda_h \tilde{u}_h(s_2) + \mathcal{N}_h \tilde{u}_h(s_2)\right)\right.$$
$$\left. \otimes\ \tilde{v}_h(s_2)\right\rangle_{H_0 \otimes H_0} \left\langle D_{xx}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \hat{v}_h(s_2) \otimes \left(\Lambda_h \tilde{u}_h(s_2) + \mathcal{N}_h^f \tilde{u}_h(s_2)\right)\right\rangle_{H_0 \otimes H_0}$$
$$- \left\langle D_{xxx}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \hat{v}_h(s_2) \otimes \tilde{v}_h(s_2) \otimes \left(\hat{v}_h(s_2) - \tilde{v}_h(s_2)\right)\right\rangle_{H_0 \otimes H_0 \otimes H_0}$$
$$+ \left\langle D_{xxy}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \hat{v}_h(s_2)\right.$$
$$\left. \otimes\ \tilde{v}_h(s_2) \otimes \left(\Lambda_h x + \mathcal{N}_h^f x + 2s_2 \Lambda_h y - \Lambda_h \tilde{u}_h(s_2) - \mathcal{N}_h \tilde{u}_h(s_2)\right)\right\rangle_{H_0 \otimes H_0 \otimes V_0^*}$$

$$
\begin{aligned}
&+ 2\left\langle D_{xx}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \left(\Lambda_h x + \Lambda_h s_2 y + \mathcal{N}_h^f x\right) \otimes \tilde{v}_h(s_2)\right\rangle_{H_0\otimes H_0} \\
&\times \langle D_{xx}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \hat{v}_h(s_2) \otimes \left(\Lambda_h \tilde{u}_h(s_2) + \mathcal{N}_h^f \tilde{u}_h(s_2)\right)\rangle_{H_0\otimes H_0} \\
&\times \langle D_{xyx}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \tilde{v}_h(s_2) \otimes \tilde{v}_h(s_2) \otimes \left(\hat{v}_h(s_2) - \tilde{v}_h(s_2)\right)\rangle_{H_0\otimes H_0\otimes H_0} \\
&+ \langle D_{xyy}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \tilde{v}_h(s_2) \\
&\otimes \tilde{v}_h(s_2) \otimes \left(\Lambda_h x + \mathcal{N}_h^f x + 2s_2\Lambda_h y - \Lambda_h \tilde{u}_h(s_2) - \mathcal{N}_h \tilde{u}_h(s_2)\right)\rangle_{H_0\otimes H_0\otimes V_0^*} \\
&+ 2\langle D_{xy}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \left(\Lambda_h \tilde{u}_h(s_2) + \mathcal{N}_h \tilde{u}_h(s_2)\right) \otimes \tilde{v}_h(s_2)\rangle_{H_0\otimes H_0} \\
&\times \left\langle D_{xy}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \hat{v}_h(s_2) \otimes \left(\Lambda_h \tilde{u}_h(s_2) + \mathcal{N}_h^f \tilde{u}_h(s_2)\right)\right\rangle_{H_0\otimes H_0} \\
&- \left\langle D_{xyx}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \hat{v}_h(s_2) \otimes \tilde{v}_h(s_2) \otimes \left(\hat{v}_h(s_2) - \tilde{v}_h(s_2)\right)\right\rangle_{H_0\otimes H_0\otimes H_0} \\
&+ \left\langle D_{xyy}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \hat{v}_h(s_2)\right. \\
&\left.\otimes\ \tilde{v}_h(s_2) \otimes \left(\Lambda_h x + \mathcal{N}_h^f x + 2s_2\Lambda_h y - \Lambda_h \tilde{u}_h(s_2) - \mathcal{N}_h \tilde{u}_h(s_2)\right)\right\rangle_{H_0\otimes H_0\otimes V_0^*} \\
&+ 2\left\langle D_{xy}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \left(\Lambda_h x + \Lambda_h s_2 y + \mathcal{N}_h^f x\right) \otimes \tilde{v}_h(s_2)\right\rangle_{H_0\otimes H_0} \\
&\times \left\langle D_{xy}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \hat{v}_h(s_2) \otimes \left(\Lambda_h \tilde{u}_h(s_2) + \mathcal{N}_h^f \tilde{u}_h(s_2)\right)\right\rangle_{H_0\otimes H_0} ds_2 ds_1 ds_0.
\end{aligned}
$$

The last term is a little more delicate. In particular we have

$$
\begin{aligned}
\mathrm{diff}_{15} &= \mathbb{E}^{x,y}\Bigg[\int_0^{\tau_h}\int_0^s \Big\{\left\langle D_x\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), \Lambda_h \tilde{u}_h(r) + \mathcal{N}_h^f \tilde{u}_h(r)\right\rangle \\
&\quad - \left\langle D_x\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), \Lambda_h x + 2r\Lambda_h y + \mathcal{N}_h^f x\right\rangle_{H_0} dr ds\Bigg] \\
&= \mathbb{E}^{x,y}\Bigg[\int_0^{\tau_h}\int_0^s \Big\{\left\langle D_x\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), \Lambda_h \tilde{u}_h(r) + \mathcal{N}_h^f \tilde{u}_h(r)\right\rangle \\
&\quad - \left\langle D_x\Psi\left(x, y, (n_h - k - 1)\tau_h\right), \Lambda_h x + \mathcal{N}_h^f x\right\rangle_{H_0}\Big\} - \Big\{\langle D_x\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), \Lambda_h x \\
&\quad +\ \mathcal{N}_h^f x\rangle_{H_0} - \left\langle D_x\Psi\left(x, y, (n_h - k - 1)\tau_h\right), \Lambda_h x + \mathcal{N}_h^f x\right\rangle_{H_0}\Big\} \\
&\quad - \left\langle D_x\Psi\left(\hat{u}_h(r), \hat{v}_h(r), r + (n_h - k - 1)\tau_h\right), 2r\Lambda_h y\right\rangle_{H_0} dr ds\Bigg].
\end{aligned}
$$

Again, taking into account that the Kolmogorov equation at the time grid points coincide for $\tilde{u}_h$ and $\hat{u}_h$, we write

$$
\begin{aligned}
\mathrm{diff}_{15} &= \mathbb{E}^{x,y}\Bigg[\int_0^{\tau_h}\int_0^{s_0}\int_0^{s_1} \Big\{\left\langle D_{xx}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \left(\Lambda_h \tilde{u}_h(s_2) + \mathcal{N}_h^f \tilde{u}_h(s_2)\right) \otimes \hat{v}_h(s_2)\right\rangle_{H_0\times H_0} \\
&\quad + \left\langle D_{xy}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), r + (n_h - k - 1)\tau_h\right), \left(\Lambda_h \tilde{u}_h(s_2) + \mathcal{N}_h^f \tilde{u}_h(s_2)\right) \otimes \left(\Lambda_h x + 2s_2\Lambda_h y + \mathcal{N}_h^f x\right)\right\rangle_{H_0\times V_0^*} \\
&\quad + \left\langle D_{xr}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \Lambda_h \tilde{u}_h(s_2) + \mathcal{N}_h^f \tilde{u}_h(s_2)\right\rangle_{H_0} \\
&\quad + \left\langle D_x\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \Lambda_h \tilde{v}_h(s_2) + D\mathcal{N}_h^f \tilde{u}_h(s_2)\right\rangle_{H_0} \\
&\quad \times \left\langle D_{xx}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \left(\Lambda_h x + \mathcal{N}_h^f x\right) \otimes \hat{v}_h(s_2)\right\rangle_{H_0\times H_0} \\
&\quad - \left\langle D_{xy}\Psi\left(\hat{u}_h(s_2), \hat{v}_h(s_2), s_2 + (n_h - k - 1)\tau_h\right), \left(\Lambda_h x + \mathcal{N}_h^f x\right) \otimes \left(\Lambda_h x + 2s_2\Lambda_h y + \mathcal{N}_h^f x\right)\right\rangle_{H_0\times V_0^*}
\end{aligned}
$$

$$\times \left\langle D_{xs_2}\Psi\left(\hat{u}_h(s_2),\,\hat{v}_h(s_2),\,s_2+(n_h-k-1)\tau_h\right),\,\Lambda_h x+\mathcal{N}_h^f x\right\rangle_{H_0}$$

$$\times \left\langle D_x\Psi\left(\hat{u}_h(s_2),\,\hat{v}_h(s_2),\,s_2+(n_h-k-1)\tau_h\right),\,2s_2\Lambda_h y\right\rangle_{H_0}\mathrm{d}s_2\mathrm{d}s_1\mathrm{d}s_0\Bigg].$$

Analysing term by term of $\mathrm{diff}_{11},\ldots,\mathrm{diff}_{15}$ one can verify that the terms consisting of $s_0\Lambda_h y$ and living in H_0 are the worst cases. In particular, these terms are the following

$$\int_0^{\tau_h}\int_0^{s_0}\int_0^{s_1}\mathbb{E}^{x,y}\left\langle D_{xx}\Psi\left(\hat{u}_h(s_2),\,\hat{v}_h(s_2),\,s_2+(n_h-k-1)\tau_h\right),\,s_2\Lambda_h y\otimes\hat{v}_h(s_2)\right\rangle_{H_0\otimes H_0}\mathrm{d}s_2\mathrm{d}s_1\mathrm{d}s_0,$$

$$\int_0^{\tau_h}\int_0^{s_0}\int_0^{s_1}\mathbb{E}^{x,y}\left\langle D_x\tilde{\Psi}\left(\hat{u}_h(s_2),\,\hat{v}_h(s_2),\,s_2+(n_h-k-1)\tau_h\right),\,s_2\Lambda_h y\right\rangle_{H_0}\mathrm{d}s_2\mathrm{d}s_1\mathrm{d}s_0,$$

and finally, if the worst comes to the worst, the term

$$\mathbb{E}^{x,y}\left[\int_0^{\tau_h}\int_0^{s_0}\left\langle D_x\Psi\left(\hat{u}_h(s_1),\,\hat{v}_h(s_1),\,s_1+(n_h-k-1)\tau_h\right),\,2s_1\Lambda_h y\right\rangle_{H_0}\mathrm{d}s_1\mathrm{d}s_0.\right.$$

But, since $\alpha>\frac{3}{2}$ and $\tau_h\sim h$ there exists a constant C such that

$$\sup_{0\le s\le T}\mathbb{E}\,|\Lambda_h y|^2\le C,\quad h\in(0,1],$$

$$\sup_{0\le r\le T}\mathbb{E}\,|\tilde{v}_h(r)|^4,\ \sup_{0\le r\le T}\mathbb{E}\left|\hat{v}_h(r)\right|^4\le C,\quad h\in(0,1],$$

and, finally,

$$\sup_{0\le k\le N_h}\mathbb{E}\left|\Lambda_h u_h^k(r)\right|^4,\ \sup_{0\le k\le N_h}\mathbb{E}\left|2\tau_h\Lambda_h v_h^k(r)\right|^4\le C,\quad h\in(0,1].$$

Since ϕ is four times Fréchet differentiable, there exists a constant $C<\infty$ such that

$$\mathrm{diff}_{11}(x,y,k),\,\mathrm{diff}_{12}(x,y,k),\,\mathrm{diff}_{13}(x,y,k)\text{ and }\mathrm{diff}_{14}(x,y,k)\le C\,\tau_h^3,\quad h\in(0,1].$$

For the second difference we have

$$\begin{aligned}\mathrm{diff}_2(x,y,k)&:=\int_0^{\tau_h}\left[\left\langle D_y\Psi\left(\hat{u}_h(s),\,\hat{v}_h(s),\,s+(k-1)\tau_h\right),\,\left(\mathcal{N}_f(x)\right)-\left(\mathcal{N}_f(\tilde{u}_h(s))\right)\right\rangle_{V_0^*}\right]\mathrm{d}s\\&=\int_0^{\tau_h}\Big[\left\langle D_y\Psi\left(\hat{u}_h(s),\,\hat{v}_h(s),\,s+(k-1)\tau_h\right),\,\left(\mathcal{N}_f(x)\right)\right\rangle_{V_0^*}-\left\langle D_y\Psi\left(\hat{u}_h(0),\,\hat{v}_h(0),\,s+(k-1)\tau_h\right),\,\mathcal{N}_f(x)\right\rangle_{V_0^*}\Big]\mathrm{d}s\\&\quad+\int_0^{\tau_h}\Big[\left\langle D_y\Psi\left(\hat{u}_h(0),\,\hat{v}_h(0),\,s+(k-1)\tau_h\right),\,\left(\mathcal{N}_f(x)\right)\right\rangle_{V_0^*}-\left\langle D_y\Psi\left(\hat{u}_h(s),\,\hat{v}_h(s),\,s+(k-1)\tau_h\right),\,\mathcal{N}_f(\tilde{u}_h(s))\right\rangle_{V_0^*}\Big]\mathrm{d}s.\end{aligned}$$

Note, that $\mathrm{d}\hat{u}_{i,h}(s)=y_{i,h}\mathrm{d}s$ and $\mathrm{d}\mathcal{N}_h^f(\tilde{u}_h(s))_i=f'(\tilde{u}_{i,h}(s))d\tilde{u}_{i,h}(s)=f'(\tilde{u}_{i,h}(s))\tilde{u}_{i,h}(s)\mathrm{d}s=f'(\tilde{u}_{i,h}(s))\tilde{u}_{i,h}(s)_{i,h}\mathrm{d}s$. Setting

$$D^h\mathcal{N}_h^f(\tilde{u}_h(s))=(f'(\tilde{u}_{1,h}(s))\tilde{u}_{1,h}(s),\ldots,f'(\tilde{u}_{k_h,h}(s))\tilde{u}_{k_h,h}(s))$$

a second application of the Itô formula yields

$$\begin{aligned}\mathrm{diff}_2(x,y,k)&=\int_0^{\tau_h}\int_0^{s}\Bigg\{\left\langle D_{yx}\Psi\left(\hat{u}_h(r),\,\hat{v}_h(r),\,r+(k-1)\tau_h\right),\,\mathcal{N}_h^f(x)\otimes y\right\rangle_{V_0^*\otimes H_0}\\&\quad+\left\langle D_{yy}\Psi\left(\hat{u}_h(r),\,\hat{v}_h(r),\,r+(k-1)\tau_h\right),\,\mathcal{N}_h^f(x)\otimes\Lambda_h x+\mathcal{N}_h^f(x)+2r\Lambda_h y\right\rangle_{V_0^*\otimes V_0^*}\\&\quad+\mathrm{Tr}\left[\left\langle D_{yyy}\Psi\left(\hat{u}_h(r),\,\hat{v}_h(r),\,r+(k-1)\tau_h\right),\,\mathcal{N}_h^f(x)\right\rangle_{V_0^*}\right]_{V_0^*}\left\langle D_{yr}\Psi\left(\hat{u}_h(r),\,\hat{v}_h(r),\,r+(k-1)\tau_h\right),\,\mathcal{N}_h^f(x)\right\rangle_{V_0^*}\\&\quad-\left[\int_0^{\tau_h}\int_0^{s}\left\langle D_{yx}\Psi\left(\hat{u}_h(r),\,\hat{v}_h(r),\,r+(k-1)\tau_h\right),\,\mathcal{N}_h^f(\tilde{u}_h(r))\otimes y\right\rangle_{V_0^*\otimes H_0}\right.\\&\quad+\left\langle D_{yy}\Psi\left(\hat{u}_h(r),\,\hat{v}_h(r),\,r+(k-1)\tau_h\right),\,\mathcal{N}_h^f(\tilde{u}_h(r))\otimes\left(\Lambda_h x+\mathcal{N}_f^h(x)+2r\Lambda_h y\right)\right\rangle_{V_0^*\otimes V_0^*}\end{aligned}$$

$$+\operatorname{Tr}\left[\left\langle D_{yyy}\Psi\left(\hat u_h(r),\hat v_h(r),r+(k-1)\tau_h\right),\mathcal{N}_h^f(\tilde u_h(r))\right\rangle_{\tilde V_0^*}\right]_{V_0^*}\Bigg]$$
$$\times\left\langle D_{yr}\Psi\left(\hat u_h(r),\hat v_h(r),r+(k-1)\tau_h\right),\mathcal{N}_h^f(\tilde u_h(r))\right\rangle_{V_0^*}$$
$$+\left\langle D_{y}\Psi\left(\hat u_h(r),\hat v_h(r),r+(k-1)\tau_h\right),D\mathcal{N}_h^f(\tilde u_h(r))\tilde v_h(r)\right\rangle_{\tilde V_0^*}\Bigg\}\mathrm{d}r\mathrm{d}s.$$

Again, as before, we split the inner part of diff_2 into four differences and write each summand as an integral. The first difference we treat similarly to the term diff_{11}. Applying the Itô formula and using (38) and (19) we get for the first difference

$$\left\langle D_{yx}\Psi\left(\hat u_h(r),\hat v_h(r),r+(k-1)\tau_h\right),\mathcal{N}_h^f(x)\otimes y\right\rangle_{V_0^*\otimes H_0}-\left\langle D_{yx}\Psi\left(\hat u_h(r),\hat v_h(r),r+(k-1)\tau_h\right),\mathcal{N}_h^f(\tilde u_h(r))\otimes y\right\rangle_{V_0^*\otimes H_0}$$
$$=\int_0^r\Bigg(\left\langle D_{yxx}\Psi\left(\hat u_h(r_1),\hat v_h(r_1),r+(k-1)\tau_h\right),\mathcal{N}_h^f(x)\otimes y\otimes\hat v_0(r_1)\right\rangle_{V_0^*\otimes H_0\otimes H_0}$$
$$+\left\langle D_{yxy}\Psi\left(\hat u_h(r_1),\hat v_h(r_1),r_1+(k-1)\tau_h\right),\mathcal{N}_h^f(x)\otimes y\otimes(\Lambda_h\hat u_0+2r_1\Gamma_h\hat v_0+\mathcal{N}_h^f(x))\right\rangle_{V_0^*\otimes H_0\otimes V_0^*}$$
$$+\left\langle D_{yxt}\Psi\left(\hat u_h(r_1),\hat v_h(r_1),r_1+(k-1)\tau_h\right),\mathcal{N}_h^f(x)\otimes y\right\rangle_{V_0^*\otimes H_0}\Bigg)\mathrm{d}r_1$$
$$-\int_0^r\Bigg(\left\langle D_{yxx}\Psi\left(\hat u_h(r_1),\hat v_h(r_1),r+(k-1)\tau_h\right),\mathcal{N}_h^f(\tilde u_h(r))\otimes y\otimes\hat v_0(r_1)\right\rangle_{V_0^*\otimes H_0\otimes H_0}$$
$$-\left\langle D_{yxy}\Psi\left(\hat u_h(r_1),\hat v_h(r_1),r_1+(k-1)\tau_h\right),\mathcal{N}_h^f(\tilde u_h(r))\otimes y\otimes(\Lambda_h\hat u_0+2r_1\Gamma_h\hat v_0+\mathcal{N}_h^f(x))\right\rangle_{V_0^*\otimes H_0\otimes V_0^*}$$
$$-\left\langle D_{yxt}\Psi\left(\hat u_h(r_1),\hat v_h(r_1),r_1+(k-1)\tau_h\right),\mathcal{N}_h^f(\tilde u_h(r))\otimes y\right\rangle_{V_0^*\otimes H_0}$$
$$-\left\langle D_{yx}\Psi\left(\hat u_h(r_1),\hat v_h(r_1),r_1+(k-1)\tau_h\right),\left\langle D\mathcal{N}_h^f(\tilde u_h(r))\otimes\tilde v_h(t)\right\rangle\otimes y\right\rangle_{V_0^*\otimes H_0}\Bigg)\mathrm{d}r_1.$$

Similarly to the term diff_{12} we obtain for the next term

$$\left\langle D_{yy}\Psi\left(\hat u_h(r),\hat v_h(r),r+(k-1)\tau_h\right),\mathcal{N}_h^f(x)\otimes\left(\Lambda_h x+\mathcal{N}_h^f(x)+2r\Lambda_h y\right)\right\rangle_{V_0^*\otimes V_0^*}$$
$$-\left\langle D_{yy}\Psi\left(\hat u_h(r),\hat v_h(r),r+(k-1)\tau_h\right),\mathcal{N}_h^f(\tilde u_h(r))\otimes\left(\Lambda_h x+\mathcal{N}_f^h(x)+2r\Lambda_h y\right)\right\rangle_{V_0^*\otimes V_0^*}$$
$$=\int_0^r\Bigg[\left\langle D_{yyx}\Psi\left(\hat u_h(r_1),\hat v_h(r_1),r_1+(n-k-1)\tau_h\right),\mathcal{N}_h^f(x)\otimes\left(\Lambda_h x+\mathcal{N}_h^f(x)+2r\Lambda_h y\right)\otimes\hat v_h(r_1)\right\rangle_{V_0^*\otimes V_0^*\otimes H_0}$$
$$+\Big\langle D_{yyy}\Psi\left(\hat u_h(r_1),\hat v_h(r_1),r_1+(n-k-1)\tau_h\right),\mathcal{N}_h^f(x)$$
$$\otimes\left(\Lambda_h x+\mathcal{N}_h^f(x)+2r\Lambda_h y\right)\otimes(\Lambda_h\hat u_0+2r_1\Gamma_h\hat v_0+\mathcal{N}_h^f(x))\Big\rangle_{V_0^*\otimes V_0^*\otimes V_0^*}$$
$$+\left\langle D_{yyt}\Psi\left(\hat u_h(r_1),\hat v_h(r_1),r_1+(n-k-1)\tau_h\right),\mathcal{N}_h^f(x)\otimes\left(\Lambda_h x+\mathcal{N}_h^f(x)+2r\Lambda_h y\right)\right\rangle_{V_0^*\otimes V_0^*}$$
$$+\left\langle D_{yyx}\Psi\left(\hat u_h(r_1),\hat v_h(r_1),r_1+(n-k-1)\tau_h\right),\mathcal{N}_h^f(\tilde u_h(r_1))\otimes\left(\Lambda_h x+\mathcal{N}_h^f(x)+2r\Lambda_h y\right)\otimes\hat v_h(r_1)\right\rangle_{V_0^*\otimes V_0^*\otimes H_0}$$
$$+\Big\langle D_{yyy}\Psi\left(\hat u_h(r_1),\hat v_h(r_1),r_1+(n-k-1)\tau_h\right),\mathcal{N}_h^f(\tilde u_h(r_1))$$
$$\otimes\left(\Lambda_h x+\mathcal{N}_h^f(x)+2r\Lambda_h y\right)\otimes(\Lambda_h\hat u_0+2r_1\Gamma_h\hat v_0+\mathcal{N}_h^f(x))\Big\rangle_{V_0^*\otimes V_0^*\otimes V_0^*}$$
$$+\left\langle D_{yyt}\Psi\left(\hat u_h(r_1),\hat v_h(r_1),r_1+(n-k-1)\tau_h\right),\mathcal{N}_h^f(\tilde u_h(r_1))\otimes\left(\Lambda_h x+\mathcal{N}_h^f(x)+2r\Lambda_h y\right)\right\rangle_{V_0^*\otimes V_0^*}$$
$$+\left\langle D_{yy}\Psi\left(\hat u_h(r_1),\hat v_h(r_1),r_1+(n-k-1)\tau_h\right),\left\langle D\mathcal{N}_h^f(\tilde u_h(r))\otimes\tilde v_h(t)\right\rangle_{H_0}\otimes y\right\rangle_{V_0^*\otimes H_0}\mathrm{d}r_1.$$

Also,

$$\mathrm{Tr}\left[\left\langle D_{yyy}\Psi\left(\hat{u}_h(r),\hat{v}_h(r),r+(k-1)\tau_h\right),\mathcal{N}_h^f(x)\right\rangle_{V_0^*}\right]_{V_0^*}$$
$$-\mathrm{Tr}\left[\left\langle D_{yyy}\Psi\left(\hat{u}_h(r),\hat{v}_h(r),r+(k-1)\tau_h\right),\mathcal{N}_h^f(\tilde{u}_h(r))\right\rangle_{\tilde{V}_0^*}\right]_{V_0^*}$$

is similar to diff_{31}. The difference

$$\left\langle D_{yr}\Psi\left(\hat{u}_h(r),\hat{v}_h(r),r+(k-1)\tau_h\right),\mathcal{N}_h^f(x)\right\rangle_{V_0^*}-\left\langle D_{yr}\Psi\left(\hat{u}_h(r),\hat{v}_h(r),r+(k-1)\tau_h\right),\mathcal{N}_h^f(\tilde{u}_h(r))\right\rangle_{V_0^*},$$

can be treated in the same way as the difference diff_{14}.

Finally, we deal with the term $\mathrm{diff}_3(x,y,k)$, i.e.

$$\mathrm{diff}_3(x,y,k):=\mathbb{E}^{x,y}\left[\int_0^{\tau_h}\left\langle D_y\Psi\left(\hat{u}_h(s),\hat{v}_h(s),s+(n-k-1)\tau_h\right),\Lambda_hx-\Lambda_h\tilde{u}_h(s)\right\rangle_{V_0^*}\mathrm{d}s\right].$$

Again, by the same arguments as before we write

$$\begin{aligned}
\mathrm{diff}_3(x,y,k) &:= \mathbb{E}^{x,y}\left[\int_0^{\tau_h}\int_0^s\left\{\left\langle D_{yx}\Psi\left(\hat{u}_h(r),\hat{v}_h(r),r+(n-k-1)\tau_h\right),\Lambda_h\tilde{u}_h(r)\otimes(\hat{v}_h(r)-\tilde{v}_h(r))\right\rangle_{V_0^*\otimes H_0}\right.\right.\\
&\quad+\left\langle D_{yy}\Psi\left(\hat{u}_h(r),\hat{v}_h(r),r+(n-k-1)\tau_h\right),\Lambda_h\tilde{u}_h(r)\otimes(\Lambda_hx+r\Lambda_hy+\mathcal{N}_h^fx-\Lambda_h\tilde{u}_h(r)-\mathcal{N}_h^f\tilde{u}_h(r))\right\rangle_{V_0^*\otimes V_0^*}\\
&\quad\left.\left.+\left\langle D_y\Psi\left(\hat{u}_h(r),\hat{v}_h(r),r+(n-k-1)\tau_h\right),\Lambda_h\tilde{u}_h(r)-\mathcal{N}_h^f\tilde{u}_h(r)\right\rangle_{V_0^*}\right\}\mathrm{d}r\mathrm{d}s\right]\\
&= \mathbb{E}^{x,y}\left[\int_0^{\tau_h}\int_0^{s_0}\int_0^{s_1}\left\{\left\langle D_{yxx}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n-k-1)\tau_h\right),\Lambda_h\tilde{u}_h(s_2)\right.\right.\right.\\
&\quad\otimes\left.(\hat{v}_h(s_2)-\tilde{v}_h(s_2))\otimes(\hat{v}_h(s_2)-\tilde{v}_h(s_2))\right\rangle_{V_0^*\otimes H_0\otimes H_0}+\left\langle D_{yxy}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n-k-1)\tau_h\right),\Lambda_h\tilde{u}_h(s_2)\right.\\
&\quad\otimes\left.(\hat{v}_h(s_2)-\tilde{v}_h(s_2))\otimes(\Lambda_hx+s_2\Lambda_hy+\mathcal{N}_h^fx-\Lambda_h\tilde{u}_h(s_2)-\mathcal{N}_h^f\tilde{u}_h(s_2))\right\rangle_{V_0^*\otimes H_0\otimes V_0^*}\\
&\quad+\left\langle D_{yx}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n-k-1)\tau_h\right),\Lambda_h\tilde{v}_h(s_2)\otimes(\hat{v}_h(s_2)-\tilde{v}_h(s_2))\right\rangle_{V_0^*\otimes H_0}\\
&\quad+\left\langle D_{yx}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n-k-1)\tau_h\right),\Lambda_h\tilde{u}_h(s_2)\otimes(\Lambda_hx+s_2\Lambda_hy+\mathcal{N}_h^fx-\Lambda_h\tilde{u}_h(s_2)-\mathcal{N}_h^f\tilde{u}_h(s_2))\right\rangle_{V_0^*\otimes H_0}\\
&\quad-\left\langle D_{yyx}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n-k-1)\tau_h\right),\Lambda_h\tilde{u}_h(s_2)\otimes(\hat{v}_h(s_2)-\tilde{v}_h(s_2))\otimes(\hat{v}_h(s_2)-\tilde{v}_h(s_2))\right\rangle_{V_0^*\otimes V_0^*\otimes H_0}\\
&\quad+\left\langle D_{yyy}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n-k-1)\tau_h\right),\Lambda_h\tilde{u}_h(s_2)\right.\\
&\quad\otimes\left.(s_2\Lambda_hy-\Lambda_h\tilde{u}_h(s_2)-\mathcal{N}_h^f\tilde{u}_h(s_2))\otimes(s_2\Lambda_hy-\Lambda_h\tilde{u}_h(s_2)-\mathcal{N}_h^f\tilde{u}_h(s_2))\right\rangle_{V_0^*\otimes V_0^*\otimes V_0^*}\\
&\quad+\left\langle D_{yy}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n-k-1)\tau_h\right),\Lambda_h\tilde{v}_h(s_2)\otimes(s_2\Lambda_hy-\Lambda_h\tilde{u}_h(s_2)-\mathcal{N}_h^f\tilde{u}_h(s_2))\right\rangle_{V_0^*\otimes V_0^*}\\
&\quad\left.\left.+\left\langle D_{yy}\Psi\left(\hat{u}_h(s_2),\hat{v}_h(s_2),s_2+(n-k-1)\tau_h\right),\Lambda_h\tilde{u}_h(s_2)\otimes(\Lambda_hy-\Lambda_h\tilde{v}_h(s_2)-D\mathcal{N}_h^f\tilde{u}_h(s_2))\right\rangle_{V_0^*\otimes V_0^*}\right\}\mathrm{d}s_2\mathrm{d}s_1\mathrm{d}s_0\right].
\end{aligned}$$

Again, analysing term by term one can verify that the worst terms are bounded uniformly in h. In particular, there exists a C such that

$$|\mathrm{diff}_3(x,y,k)|\le C\tau_h^3,\quad h\in(0,1].$$

Summing up leads to

$$\mathrm{error}_1^h(\Phi,T)\le C\ \tau_h^2.$$

6.2. The second error

In Section 4.1 we have seen, that the solution of Eq. (20) coincides in distribution with the solution of the reflected wave equation approximated by the implicit Euler scheme. In particular, if $\mathcal{A}$ is defined in (5) it holds

$$\begin{cases} \frac{1}{\tau_h}\left[\begin{pmatrix}\tilde{u}_i(t)\\ \tilde{v}_i(t)\end{pmatrix} - \begin{pmatrix}\tilde{u}_{i-1}(t)\\ \tilde{v}_{i-1}(t)\end{pmatrix}\right] = \mathcal{A}\begin{pmatrix}\tilde{u}_i(t)\\ \tilde{v}_i(t)\end{pmatrix} + \begin{pmatrix}0\\ \mathcal{I}_h \Delta W(t)\end{pmatrix} + F_h\begin{pmatrix}\tilde{u}_i(t)\\ \tilde{v}(t)\end{pmatrix}, \\ \quad +\mathcal{B}\begin{pmatrix} h^{-1}\left[u_0(N_i^h) - u_0(N_{i-1}^h)\right] \\ h^{-1}\left[v_0(N_{i-1}^h) - v_0(N_{i-1}^h)\right]\end{pmatrix}, \quad i = 1, \dots, k_h, \\ \begin{pmatrix}\tilde{u}(0)\\ \tilde{v}(0)\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}. \end{cases}$$

The second difference will be handled in the same way as the first difference, where we followed Talay and Tubaro [22]. Nevertheless, some changes have to be taken into account. For simplicity we assume that $T_0 = 1$ (see Section 4.1). First, note, that identifying $H^{-\alpha}([0, 1] \times [0, 1]; \mathbb{R})$ by $H^{-\alpha}([0, 1]; H^{-\alpha}([0, 1]; \mathbb{R}))$ there exists a function ρ such that

$$\Phi(u) = \int_0^1 \rho(u(\xi), \xi)\, d\xi.$$

Let $\{\tilde{u}_k^h : 0 \le k \le k_h\}$ be the solution of System (22) and let $\{\bar{u}_h^k : 0 \le k \le k_h\}$ be the solution to System (29). Similarly, let $u = \{u(t, \xi) : (t, \xi) \in [0, 1] \times [0, 1]\}$ be the solution of system (1) and $w = \{w(t, \xi) : (t, \xi) \in [0, 1] \times [0, 1]\}$ be a solution of System (10). Then $w(t, \xi) = u(\xi, t)$, $(t, \xi) \in [0, 1] \times [0, 1]$, and $\tilde{u}_j^h(\xi) = \bar{u}_h^j(t)$ for $t = \xi = t_j^h, j = 1, \dots, k_h$. The interpolation between the grid points $\{N_i^h : i = 1, \dots, k_h\}$ of $\hat{u}$ is given by formula (2). To be consistent we define the interpolation between the grid points $\{N_i^h : i = 1, \dots, k_h\}$ of $\tilde{u}$ by the same formula, i.e. by

$$\tilde{u}_h(t, \xi) := \sum_{i=0}^{k_h} 1_{(N_i^h, N_{i+1}^h]}(\xi)\left[\frac{(\xi - N_i^h)}{h}\ \tilde{u}_{i+1,h}(t) + \frac{(N_{i+1}^h - \xi)}{h}\ \tilde{u}_{i,h}(t)\right], \quad \xi \in [0, 1]. \tag{41}$$

Reflecting and a short calculation give for $\bar{u}$ the interpolation between the time points $\{t_j^h : j = 1, \dots .k_h$

$$\bar{u}_h(t, \xi) := \sum_{i=0}^{k_h - 1} 1_{(t_i^h, t_{i+1}^h]}(t)\left[\bar{u}_{i,h}(\xi) + (\xi - N_i^h)\ \bar{v}_{i,h}(\xi)\right], \tag{42}$$

where $\bar{v}_{i,h} = \frac{1}{h}[\bar{u}_{i+1} - \bar{u}_i]$. Note, that there does not exist any condition how to interpolation $\bar{v}$ between the time grid. Therefore, we can choose an interpolation method between the time grid points which is appropriate for our purpose. Since we would like that the equation for $\bar{u}$ of System (29) coincides with the equation given in Definition (42), we chose the interpolation method given by System (29). Now having defined $\bar{u}$ as a mapping from $[0, T_0] \times [0, 1] \to \mathbb{R}$ we obtain for the functional following identities

$$\Phi(u) = \int_0^1 \rho(u(\xi), \xi)\, d\xi = \int_0^1 \rho(w(t), t)\, dt,$$

and

$$\Phi(\bar{u}_h) = \int_0^1 \rho(\tilde{u}(\xi), \xi)\, d\xi := \int_0^1 \rho(\bar{u}(t), t)\, dt.$$

Again, the Fubini Theorem and the Minkowski inequality give

$$\begin{aligned} \text{error}_2^h(\Phi) &\le \int_0^1 \left|\mathbb{E}^{\mathcal{P}_h u_0, \mathcal{P}_h v_0}\rho(w(t), t) - \mathbb{E}^{\mathcal{P}_h u_0, \mathcal{P}_h v_0}\rho(\bar{u}(t), t)\right|\, dt \\ &\le \sup_{0 \le t \le 1}\left|\mathbb{E}^{\mathcal{P}_h u_0, \mathcal{P}_h v_0}\rho(w(t), t) - \mathbb{E}^{\mathcal{P}_h u_0, \mathcal{P}_h v_0}\rho(\bar{u}(t), t)\right|. \end{aligned}$$

We will show in the following, that there exists a constant $C > 0$ such that

$$\sup_{0 \le T \le 1}\left|\mathbb{E}^{\mathcal{P}_h u_0, \mathcal{P}_h v_0}\rho(w(T), T) - \mathbb{E}^{\mathcal{P}_h u_0, \mathcal{P}_h v_0}\rho(\bar{u}_h(T), T)\right| \le C\,\tau_h^2, \quad h \in (0, 1].$$

Let us fix $T \in [0, 1]$. Similarly to the method of Talay and Tubaro [22] we introduce a function Ψ as follows:

$$\Psi : \begin{array}{ccc} \mathcal{H}_0 \times [0, T] & \to & \mathbb{R} \\ (x, y, t) & \mapsto & \Psi(x, y, t), \end{array}$$

where

$$\Psi(x,y,t) := \mathbb{E}^{x,y}\left[\rho\left(w(T-t),T\right)\right], \quad t\in[0,T].$$

Now, due to the definition of Ψ, it is straightforward that

$$\begin{aligned}\mathrm{error}_1^h(\phi,T) &:= \left|\mathbb{E}^{\mathcal{P}_h u_0,\mathcal{P}_h v_0}\rho(w(T),T) - \mathbb{E}^{\mathcal{P}_h u_0,\mathcal{P}_h v_0}\rho(\bar u_h(T),T)\right| \\ &= \left|\mathbb{E}^{\mathcal{P}_h u_0,\mathcal{P}_h v_0}\Psi(\bar u_h^{k_h},\bar v_h^{k_h},T) - \mathbb{E}^{u_0,v_0}\Psi(\mathcal{P}_h u_0,\mathcal{P}_h v_0,0)\right|.\end{aligned}$$

By the tower property of the conditional expectation we infer that

$$\begin{aligned}\mathrm{error}_1^h(\phi,T) &= \mathbb{E}^{\mathcal{P}_h u_0,\mathcal{P}_h v_0}\sum_{i=0}^{k_h-1}\Psi\left(\bar u_h^{k_h-i},\bar v_h^{k_h-i},(k_h-i)\tau_h\right) - \Psi\left(\bar u_h^{k_h-(i+1)},\bar v_h^{k_h-(i+1)},(k_h-(i+1))\tau_h\right) \\ &= \mathbb{E}^{\mathcal{P}_h u_0,\mathcal{P}_h v_0}\sum_{i=0}^{k_h-1}\mathbb{E}^{\bar u_h^{k_h-i-1},\bar v_h^{k_h-i-1}}\left[\Psi\left(\bar u_h^1,\bar v_h^1,(k_h-i)\tau_h\right) - \Psi\left(\mathcal{P}_h u^0,\mathcal{P}_h v_h^0,(k_h-i-1)\tau_h\right)\right].\end{aligned}$$

Similar to before, put

$$\widetilde{\mathrm{error}}_\Psi^h(k) := \left|\mathbb{E}^{x,y}\left[\Psi\left(\bar u_h^1,\bar v_h^1,(k_h-k)\tau_h\right) - \Psi\left(x,y,(k_h-k-1)\tau_h\right)\right]\right|,$$

where x, y are some random elements of $\mathcal{V}_h$ and $k\in\{0,\dots,k_h-1\}$. In order to get convergence of order two, we have again to show that there exists a constant C such that

$$\widetilde{\mathrm{error}}_\Psi^h(k) \le C\,\tau_h^3, \quad h\in(0,1].$$

In particular, comparing Eqs. (29) and (37) we see that plugging in the Kolmogorov equation in (40) we get in Eq. (40) the additional term

$$\begin{aligned}\widetilde{\mathrm{error}}_\Psi^h(k) &= \widehat{\mathrm{error}}_\Phi^h(k) + \int_0^h \mathbb{E}^{x,y}\Big[\langle D_y\Psi\left(\bar u_h(s),\bar v_h(s),s+(n-k-1)\tau_h\right), \\ &\qquad (\Lambda-1)B^1\left(u_0(s),v_0(s)\right) - (\Lambda-1)B^1\left(\mathcal{P}_h u_0(s),\mathcal{P}_h v_0(s)\right)\rangle_{V_0^*}\mathrm{d}s\Big].\end{aligned}$$

Proceeding in the very same way as done many times before, we write

$$\begin{aligned}&\int_0^h \mathbb{E}^{x,y}\Big[\langle D_y\Psi\left(\bar u_h(s),\bar v_h(s),s+(k_h-k-1)\tau_h\right), \\ &\quad (\Lambda-1)B^1\left(u_0(s),v_0(s)\right) - (\Lambda-1)B^1\left(\mathcal{P}_h u_0(s),\mathcal{P}_h v_0(s)\right)\rangle_{V_0^*}\mathrm{d}s\Big] \\ &= \int_0^h\Big[\mathbb{E}^{x,y}\langle D_y\Psi\left(\bar u_h(s),\bar v_h(s),s+(k_h-k-1)\tau_h\right),(\Lambda-1)B^1\left(u_0(s),v_0(s)\right)\rangle_{V_0^*} \\ &\quad \times\mathbb{E}^{x,y}\langle D_y\Psi\left(\bar u_h(s),\bar v_h(s),s+(k_h-k-1)\tau_h\right),(\Lambda-1)B^1\left(u_0(s),v_0(s)\right)\rangle_{V_0^*} \\ &\quad +\mathbb{E}^{x,y}\langle D_y\Psi\left(\bar u_h(s),\bar v_h(s),s+(k_h-k-1)\tau_h\right),(\Lambda-1)B^1\left(u_0(s),v_0(s)\right)\rangle_{V_0^*} \\ &\quad -\mathbb{E}^{x,y}\langle D_y\Psi\left(\bar u_h(s),\bar v_h(s),s+(k_h-k-1)\tau_h\right),(\Lambda-1)B^1\left(\mathcal{P}_h u_0(s),\mathcal{P}_h v_0(s)\right)\rangle_{V_0^*}\Big]\mathrm{d}s.\end{aligned}$$

Observe that the boundary operator is bilinear. In addition, the boundary conditions, i.e. the functions u_0 and v_0, are purely deterministic. As a consequence we write

$$\begin{aligned}&= \int_0^h\int_0^{s_1}\Big[+\mathbb{E}^{x,y}\left\langle D_y\Psi\left(\bar u_h(s_1),\bar v_h(s_1),s_1+(k_h-k-1)\tau_h\right),(\Lambda-1)B^1\left(u_0'(s_2),v_0(s_1)\right)\right\rangle_{V_0^*} \\ &\quad +\mathbb{E}^{x,y}\left\langle D_y\Psi\left(\bar u_h(s_1),\bar v_h(s_1),s_1+(k_h-k-1)\tau_h\right),(\Lambda-1)B^1\left(u(s_1),v_0'(s_2)\right)\right\rangle_{V_0^*} \\ &\quad -\mathbb{E}^{x,y}\left\langle D_y\Psi\left(\bar u_h(s_1),\bar v_h(s_1),s_1+(k_h-k-1)\tau_h\right),(\Lambda-1)B^1\left(u_0'(h/2),v_0(0)\right)\right\rangle_{V_0^*} \\ &\quad -\mathbb{E}^{x,y}\left\langle D_y\Psi\left(\bar u_h(s_1),\bar v_h(s_1),s_1+(k_h-k-1)\tau_h\right),(\Lambda-1)B^1\left(u_0(0),v_0'(h/2)\right)\right\rangle_{V_0^*}\Big]\mathrm{d}s_2\mathrm{d}s_1 \\ &\quad +\int_0^h\mathbb{E}^{x,y}\left\langle D_y\Psi\left(\bar u_h(s),\bar v_h(s),s+(k_h-k-1)\tau_h\right),(\Lambda-1)B^1\left(su_0'(h/2),sv_0'(h/2)\right)\right\rangle_{V_0^*}\,\mathrm{d}s\end{aligned}$$

$$= \int_0^h \int_0^{s_1} \Big[\mathbb{E}^{x,y} \left\langle D_y \Psi\left(\bar{u}_h(s_1), \bar{v}_h(s_1), s_1 + (k_h - k - 1)\tau_h\right), (\Lambda - 1) B^1 \left(u_0'(s_2) - u_0'(h/2), v_0(s_1) \right) \right\rangle_{V_0^*}$$

$$+ \left\langle D_y \Psi\left(\bar{u}_h(s_1), \bar{v}_h(s_1), s_1 + (k_h - k - 1)\tau_h\right), (\Lambda - 1) B^1 \left(u_0'(h/2), v_0(s_1) - v_0(0) \right) \right\rangle_{V_0^*}$$

$$+ \left\langle D_y \Psi\left(\bar{u}_h(s_1), \bar{v}_h(s_1), s_1 + (k_h - k - 1)\tau_h\right), (\Lambda - 1) B^1 \left(u_0(s_1), v_0'(s_2) - v_0(h/2) \right) \right\rangle_{V_0^*}$$

$$+ \left\langle D_y \Psi\left(\bar{u}_h(s_1), \bar{v}_h(s_1), s_1 + (k_h - k - 1)\tau_h\right), (\Lambda - 1) B^1 \left(u_0(s_1) - u_0(0), v_0'(h/2) \right) \right\rangle_{V_0^*} \Big] ds_2 ds_1$$

$$+ \int_0^h \mathbb{E}^{x,y} \left\langle D_y \Psi\left(\bar{u}_h(s), \bar{v}_h(s), s + (k_h - k - 1)\tau_h\right), (\Lambda - 1) B^1 \left(s u_0'(h/2), s v_0'(h/2) \right) \right\rangle_{V_0^*} ds$$

$$= \int_0^h \int_0^{s_1} \int_{s_2}^{h/2} \Big[\mathbb{E}^{x,y} \langle D_y \Psi\left(\bar{u}_h(s_1), \bar{v}_h(s_1), s_1 + (k_h - k - 1)\tau_h\right), (\Lambda - 1) B^1 \left(u_0''(s_3), v_0(s_1) \right) \rangle_{V_0^*}$$

$$+ \mathbb{E}^{x,y} \left\langle D_y \Psi\left(\bar{u}_h(s_1), \bar{v}_h(s_1), s_1 + (k_h - k - 1)\tau_h\right), (\Lambda - 1) B^1 \left(u_h(s_1), v_0''(s_3) \right) \right\rangle_{V_0^*} \Big] ds_3 ds_2 ds_1$$

$$+ \int_0^h \int_0^{s_1} \int_0^{s_1} \Big[\mathbb{E}^{x,y} \left\langle D_y \Psi\left(\bar{u}_h(s_1), \bar{v}_h(s_1), s_1 + (k_h - k - 1)\tau_h\right), (\Lambda - 1) B^1 \left(u_0''(s_3), v_0'(h/2) \right) \right\rangle_{V_0^*}$$

$$+ \mathbb{E}^{x,y} \left\langle D_y \Psi\left(\bar{u}_h(s_1), \bar{v}_h(s_1), s_1 + (k_h - k - 1)\tau_h\right), (\Lambda - 1) B^1 \left(u_0'(h/2), v_0''(s_3) \right) \right\rangle_{V_0^*} ds_3 ds_2 ds_1$$

$$+ \int_0^h \mathbb{E}^{x,y} \left\langle D_y \Psi\left(\bar{u}_h(s), \bar{v}_h(s), s + (k_h - k - 1)\tau_h\right), (\Lambda - 1) B^1 \left(s\, u_0'(h/2), s u_0'(h/2) \right) \right\rangle_{V_0^*} \Big] ds.$$

Since the initial conditions are belonging to $C^2(\mathcal{O})$, $(\Lambda - 1)B^1 : H_\alpha \to H_\alpha$ is bounded for any $\alpha < -1$ and $H_\alpha \hookrightarrow V_0^*$, the inner part of the RHS in the inequality above is bounded. Taking into account that we can make the same calculation each time when the derivative up to the boundary operator appears, we can show that the second error is of second order. Because the exact calculations are quite technical, we omit them.

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Appendix A. Stability leap-frog scheme

The leap-frog scheme is a linear multi-step method, in particular, an explicit two step method. In the finite-dimensional case without any perturbation, the stability can be characterised by the characteristic polynomial and their roots. In the infinite-dimensional case, one has to take into account the eigenvalues of the discrete Laplacian.

Analysing the stability of the scheme by analysing the eigenvalues and eigenvectors of the associated stiffness matrix, we can show the following proposition

Proposition A.1. *Let $\{(\hat{\mathbf{u}}_h^k, \hat{\mathbf{v}}_h^k)^T : k \in \mathbb{N}\}$ be a Markov chain given by the following recursion*

$$\begin{pmatrix} \hat{\mathbf{u}}_h^k \\ \hat{\mathbf{v}}_h^k \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \tau_h \\ \tau_h \mathbf{\Lambda}_h & \mathbf{1} + \tau_h \mathbf{\Lambda}_h \end{pmatrix} \begin{pmatrix} \hat{\mathbf{u}}_h^{k-1} \\ \hat{\mathbf{v}}_h^{k-1} \end{pmatrix} + \begin{pmatrix} \mathbf{1} & 0 \\ \tau_h \mathbf{\Lambda}_h & \mathbf{1} \end{pmatrix} \left[\begin{pmatrix} 0 \\ \Delta_h^k \mathbf{W} \end{pmatrix} + \tau_h \begin{pmatrix} 0 \\ \mathbf{N}_h^f \hat{\mathbf{u}}_h^k \end{pmatrix} \right].$$

If for $K < \infty$,

$$\sum_{k=1}^K \mathbb{E} \left\| \Delta_h^k \mathbf{W} \right\|_{V_0^*}^2 < \infty$$

and $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, then there exist some constants C_1 and C_2 such that

$$\left\| \begin{pmatrix} \hat{\mathbf{u}}_h^k \\ \hat{\mathbf{v}}_h^k \end{pmatrix} \right\|_{\mathcal{H}} \le C_1 \exp\left(C_2 K \tau_h\right), \quad 0 \le k \le K\tau_h, h > 0.$$

Proof. The proof is done by first by showing stability of the scheme without perturbation. To be precise, we will show that there exists some $C_1, C_2 < \infty$ such that we have for all $h > 0$

$$\left| \begin{pmatrix} \mathbf{1} & \tau_h \\ \tau_h \mathbf{\Lambda}_h & \mathbf{1} + \tau_h \mathbf{\Lambda}_h \end{pmatrix}^k \right|_{L(\mathcal{H},\mathcal{H})} \le C_1 \exp(C_2 k \tau_h), \quad k \in \mathbb{N},$$

This is done by analysing the eigenvalues of

$$\begin{pmatrix} 0 & \tau_h \\ \tau_h \mathbf{\Lambda}_h & \tau_h \mathbf{\Lambda}_h \end{pmatrix}.$$

The next step is expand $(\hat{\mathbf{u}}_h^k, \hat{\mathbf{v}}_h^k)^T$ as the following sum

$$\begin{pmatrix} \hat{\mathbf{u}}_h^k \\ \hat{\mathbf{v}}_h^k \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \tau_h \\ \tau_h \mathbf{\Lambda}_h & \mathbf{1} + \tau_h \mathbf{\Lambda}_h \end{pmatrix}^{k-1} \begin{pmatrix} \hat{\mathbf{u}}_h^0 \\ \hat{\mathbf{v}}_h^0 \end{pmatrix} + \sum_{l=0}^{k-1} \begin{pmatrix} \mathbf{1} & \tau_h \\ \tau_h \mathbf{\Lambda}_h & \mathbf{1} + \tau_h \mathbf{\Lambda}_h \end{pmatrix}^{k-1-l} \left[\begin{pmatrix} 0 \\ \Delta_h^l \mathbf{W} \end{pmatrix} + \tau_h \begin{pmatrix} 0 \\ \mathbf{N}_h^f \hat{\mathbf{u}}_h^l \end{pmatrix} \right].$$

Now, estimating term by term and applying the Gronwal Lemma leads to the assertion.

Due to Lemma 10.5 [23, p.198], (291), the eigenvalues of $\mathbf{\Lambda}_h$ are given by

$$q_j^h = \frac{2}{h^2} \left(\cos(j\pi h) - 1\right) = -2 \sin^2 \left(\frac{\pi j h}{2} \right), \quad j = 1, 2, \ldots, k_h,$$

each with corresponding orthogonal eigenvector $\mathbf{e}_i = (e_{1i}, \ldots, e_{k_h i})^T$, $i = 1, \ldots, k_h$, where

$$e_{j,l} = \sqrt{2h} \sin\left(h 2\pi j l\right), \quad j, l = 1, 2, \ldots, k_h.$$

Direct computations gives that the matrix

$$\begin{pmatrix} 0 & \tau_h \\ \tau_h \mathbf{\Lambda}_h & \tau_h \mathbf{\Lambda}_h \end{pmatrix} \tag{43}$$

has eigenvalues

$$\lambda_j^h = -\tau_h \frac{q_j^h}{q_j^h - 1},$$

with eigenvectors

$$\frac{\begin{pmatrix} \mathbf{e}_j \\ r_j^h \mathbf{e}_j \end{pmatrix}}{\left| \begin{pmatrix} r_j^h \mathbf{e}_j \\ \mathbf{e}_j \end{pmatrix} \right|} \quad \text{and} \quad \frac{\begin{pmatrix} r_j^{2,h} \mathbf{e}_j \\ \mathbf{e}_j \end{pmatrix}}{\left| \begin{pmatrix} r_j^h \mathbf{e}_j \\ \mathbf{e}_j \end{pmatrix} \right|};$$

where

$$r_j^h = -\frac{q_j^h}{q_j^h - 1}.$$

In order to verify the convergence, we have to analyse the limit of λ_j^i, $j = 1, \ldots, k_h$, $i = 1, 2$, as $h \to 0$. Since there exists C_1 and C_2 such that

$$-\frac{C_1}{h^2} \le q_j^h \le -C_2 h^2, \quad j = 1, \ldots, k_h,$$

the stability condition (21) implies, that there exists some constant $C_0 < \infty$ such that

$$\lambda_j h \le 1 + C_0 \tau_h, \quad j = 1, \ldots, k_h,$$

and therefore,

$$\left(1 + \lambda_j\right)^{\tau_h n} \le C_0 \exp\left(C_1 \tau_h n\right) \quad j = 1, \ldots, k_h, \ n \in \mathbb{N}.$$

It follows, that

$$\left\| \begin{pmatrix} \mathbf{1} & \tau_h \\ \tau_h \mathbf{\Lambda}_h & \mathbf{1} + \tau_h \mathbf{\Lambda}_h \end{pmatrix}^k \begin{pmatrix} \mathbf{e}_j \\ r_j^h \mathbf{e}_j \end{pmatrix} \right\| \le C_0 \left(1 + C_1 \tau_h\right)^k \left\| \begin{pmatrix} \mathbf{e}_j \\ r_j^h \mathbf{e}_j \end{pmatrix} \right\| \le C_0 \exp\left(C_1 \tau_h k\right) \left\| \begin{pmatrix} \mathbf{e}_j \\ r_j^h \mathbf{e}_j \end{pmatrix} \right\|, \quad 0 < h \le 1. \tag{44}$$

Let $\mathbf{E}_h \in \mathbb{R}^{k_h} \times \mathbb{R}^{k_h}$ be the unitary matrix, which maps $:= \left(\mathbf{e}_1, \ldots, \mathbf{e}_{k_h}\right)$. By recursion, the solution can be written as

$$\begin{pmatrix} \hat{\mathbf{u}}_h^k \\ \hat{\mathbf{v}}_h^k \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \tau_h \\ \tau_h \mathbf{\Lambda}_h & \mathbf{1} + \tau_h \mathbf{\Lambda}_h \end{pmatrix}^{m_h} \mathbf{E}_h \begin{pmatrix} \hat{\mathbf{u}}_h^0 \\ \hat{\mathbf{v}}_h^0 \end{pmatrix} + \sum_{l=1}^{m_h} \begin{pmatrix} \mathbf{1} & \tau_h \\ \tau_h \mathbf{\Lambda}_h & \mathbf{1} + \tau_h \mathbf{\Lambda}_h \end{pmatrix}^{m_h - j} \mathbf{E}_h \left[\begin{pmatrix} 0 \\ \Delta_h^j \mathbf{W} \end{pmatrix} + \tau_h \begin{pmatrix} 0 \\ \mathbf{N}_h^f \hat{\mathbf{u}}_h^j \end{pmatrix} \right]. \tag{45}$$

Since $\mathbf{E}_h$ is unitary, (44) and (45) and the Gronwal Lemma gives stability of the scheme. □

Appendix B. Finite differences

In this section we recall some basic facts about finite differences and introduce the notation used in the Sections before. For simplicity, we assume we take as underlying Hilbert space $H = L^2([0,1])$. In addition, we denote by V the space $H_2^1([0,1])$.

Fix the parameter $h > 0$, put $k_h = 1/h$ and assume for simplicity that k_h is an integer. Put $\mathcal{T}_h = \left\{N_1^h, \dots, N_{k_h}^h\right\}$, where $N_i^h := ih$. We call $\mathcal{T}_h$ the family of nodal variables. For a function $v \in V$ we define the interpolant by

$$(\mathcal{I}_h v)(\xi) := k_h \sum_{i=1}^{k_h} 1_{(N_i^h, N_{i+1}^h]}(\xi)\, \frac{1}{h}\, \left(v(N_{i+1}^h)(\xi - N_i^h) + v(N_i^h)(N_{i+1}^h - \xi)\right), \quad \xi \in [0,1].$$

The Nemytskij operator N^f induced by f is approximated by $\mathcal{N}_h^f$, where $\mathcal{N}_h^f(v)$ is defined by

$$\mathcal{N}_h^f(v)(t) := k_h \sum_{i=1}^{k_h} 1_{(N_i^h, N_{i+1}^h]}(\xi)\, \frac{1}{h}\, \left(f(v(N_{i+1}^h))(\xi - N_i^h) + f(v(N_i^h))(N_{i+1}^h - \xi)\right), \quad \xi \in [0,1]. \tag{46}$$

The Laplace operator is approximated by its corresponding variational problem. In particular, for a given subdivision $\mathcal{T}_h$ let $\mathcal{V}_h$ be the space of continuous functions, which are linear between the grid points. Now, the *Ritz –Galerkin approximation* of the Laplace operator is the unique operator Λ_h on $\mathcal{V}_h$ which satisfies

$$\langle \Lambda u, v\rangle = \langle \Lambda_h u, v\rangle, \quad \forall u, v \in \mathcal{V}_h. \tag{47}$$

Since $\mathcal{V}_h$ is finite dimensional, we can identify $\mathcal{V}_h$ with $\mathbb{R}^{k_h}$. Doing so, we introduce for a function $u \in \mathcal{V}_h$ the vector $\mathbf{u} \in \mathbb{R}^{k_h}$, which is defined by

$$\mathcal{V}_h \ni u \mapsto \mathbf{u} = \left(u(N_1^h), \dots, u(N_{k_h}^h)\right).$$

By $(\mathbf{u})_i$ we denote the projection onto the ith column, i.e.

$$\mathbb{R}^{k_h} \ni \mathbf{u} = (u_1, \dots, u_{k_h}) \mapsto (\mathbf{u})_i = u_i \in \mathbb{R},$$

and by $\mathcal{I}_h$ we denote the interpolant of a vector defined by

$$\mathbb{R}^{k_h} \ni \mathbf{u} = (u_1, \dots, u_{k_h}) \mapsto \mathcal{I}_h \mathbf{u} = k_h \sum_{i=1}^{k_h} 1_{(N_i^h, N_{i+1}^h]}(\xi) \frac{1}{h} \left(u(N_{i+1}^h)\frac{1}{h}(\xi - N_i^h) + u(N_i^h)\frac{1}{h}(N_{i+1}^h - \xi)\right). \tag{48}$$

Similarly to $\mathcal{I}_h$, for any $u \in V$ we define the interpolation operator by $\mathbf{I}_h$ by

$$V \ni u \mapsto \mathbf{I}_h u := \left(u(N_1), \dots, u(N_{k_h})\right) \in \mathbb{R}^{k_h}. \tag{49}$$

In the same way, by identifying $\mathcal{V}_h$ with $\mathbb{R}^{k_h}$ one can associate to Λ_h a stiffness matrix $\mathbf{\Lambda}_h$. Straightforward calculations show, that $\mathbf{\Lambda}_h$ is given by $(a_{i,j}^h)_{i,j=1}^{k_h}$, where

$$a_{i,j}^h = h^{-2} \times \begin{cases} 0 & \text{if } i \neq j, j+1, j-1, \\ -1 & \text{if } i = j+1, j-1, \\ 2 & \text{if } i = j. \end{cases}$$

The leap-frog scheme is constructed by a mid point rule. This is reflected in the way the initial conditions are approximated. To be precise, we had to take an approximation constructed by the midpoint rule. As a consequence we introduce the following interpolation operator $\mathcal{P}_h$ by

$$V \ni u \mapsto \mathcal{P}_h u(\xi) := \sum_{i=1}^{k_h} 1_{[N_{i-1}^h, N_i^h)}(\xi) \left[u(N_{i-1}^h) + (\xi - N_{i-1}^h) u'((N_{i-1}^h + N_i^h)/2)\right], \quad \xi \in [0,1]. \tag{50}$$

Again, identifying $\mathcal{V}_h$ with $\mathbb{R}^{k_h}$ we introduce a projection operator $\mathbf{P}_h$ by

$$V \ni u \mapsto \mathbf{P}_h u := \left(u(N_0) + hu'((N_0 + N_1)/2), \dots, u(N_{k_h - 1}) + hu'((N_{k_h-1} + N_{k_h})/2)\right) \in \mathbb{R}^{k_h}. \tag{51}$$

In our problem W denotes the space–time white noise. For any $t > 0$ we define the interpolant by

$$\mathcal{W}_h(t,\xi) = k_h^{\frac{1}{2}} \sum_{i=1}^{k_h} 1_{(N_i^h, N_{i+1}^h]}(\xi) \left(W([0,t], J_{i+1}^h)(\xi - N_i^h) + W([0,t], J_i^h)(N_{i+1}^h - \xi)\right). \tag{52}$$

where $J_i^h = (ih - \frac{h}{2}, ih + \frac{h}{2}]$. Again, identifying $\mathcal{V}_h$ with $\mathbb{R}^{k_h}$ we can associate for $t > 0$ to $\mathcal{W}_h(t)$ a vector

$$\mathbf{W}_h(t) = \left(k_h^{\frac{1}{2}}\ W(t, J_1^h), \ldots, k_h^{\frac{1}{2}}\ W(t, J_{k_h}^h) \right).$$

Since for any $A \in \mathcal{B}(\mathbb{R})$, the process $[0, \infty) \ni t \mapsto W([0, t) \times A)$ is a Brownian motion with variance $\lambda(A)$, we can infer that $\mathbf{W}_h(t)$ equals in distribution to an $\mathbb{R}^{k_h}$-dimensional Brownian motion with covariance matrix $k_h^{\frac{1}{2}}\ I$. In other words,

$$(k_h^{\frac{1}{2}} \beta_k^i(t), \ldots, k_h^{\frac{1}{2}} \beta_k^{k_h}(t)) \in \mathbb{R}^{k_h},$$

where β_k^i, $i = 1, \ldots, k_h$, are independent one-dimensional Brownian motions with mean zero and variance $\tau_h \lambda(J_i^h)$. Thus, the approximation of

$$\Delta_h^k \mathbf{W} := \mathbf{W}_h(k\tau_h) - \mathbf{W}_h((k+1)\tau_h)$$

at time $k\tau_h$ is given by the random vector

$$\chi_k^h = (\chi_{1,k}^h, \ldots, \chi_{k_h,k}^h),$$

where $\chi_{i,k}^h$, $i = 1, \ldots, k_h$, $k \in \mathbb{N}$, are Gaussian with mean zero and variance $\tau_h k_h$.

Proposition B.1. *The $\mathcal{V}_h$-valued process $\mathcal{W}(t)$ defined in (52) fulfills the following properties:*

1. *There exists some constant C such that we have*

$$\mathbb{E} \, \|\mathcal{W}_h(t)\|_{V^*}^2 \le Ct, \quad h \in (0, 1].$$

Proof. Thus, we have by the isometry

$$\begin{aligned}
\mathbb{E} \, \|\mathcal{W}_h(t)\|_{V^*}^2 &= k_h^{\frac{1}{2}} \sum_{i=1}^{k_h} 1_{(N_i^h, N_{i+1}^h]}(\xi) \left(W([0, t], J_{i+1}^h)(\xi - N_i^h) + W([0, t], J_i^h)(N_{i+1}^h - \xi) \right) \\
&= \mathbb{E} \left\| \sum_{i=1}^{k_h} W([0, t], J_i^h) h^{-\frac{1}{2}} \left[(\xi - N_{i-1}^h) + (N_{i+1}^h - \xi) \right] \right\|_{V^*}^2 \\
&= \mathbb{E} \sum_{i=1}^{k_h} \left\| W([0, t], J_i^h) h^{-\frac{1}{2}} \left[(\xi - N_{i-1}^h) + (N_{i+1}^h - \xi) \right] \right\|_{V^*}^2 \\
&\le \mathbb{E} \sum_{i=1}^{k_h} \left| W([0, t], J_i^h) \right|^2 h^{-1} \left\| (\xi - N_{i-1}^h) + (N_{i+1}^h - \xi) \right\|_{V^*}^2 .
\end{aligned}$$

But, putting $g_i^h(\xi) := 1_{[N_{i-1}^h, N_i^h)}(\xi)\ (\xi - N_{i-1}^h) + 1_{[N_i^h, N_{i+1}^h)}(\xi)\ (N_{i+1}^h - \xi)$, $\xi \in [0, 1]$, we get

$$\left\| g_i^h \right\|_{V^*}^2 = \sum_{k,l \in \mathbb{N}} \left\langle g_i^h, \Lambda^{-\frac{1}{2}} e_k \right\rangle \left\langle g_i^h, \Lambda^{-\frac{1}{2}} e_l \right\rangle \langle e_k, e_l \rangle ,$$

where $\{e_k : k \in \mathbb{N}\}$ are the eigenfunction of Λ. Thus, we have

$$\begin{aligned}
\left\| g_i^h \right\|_{V^*}^2 &= \sum_{k \in \mathbb{N}}^{\infty} \left\langle g_i^h, \Lambda^{-\frac{1}{2}} e_k \right\rangle^2 = C \sum_{k \in \mathbb{N}}^{\infty} k^{-2} \left\langle g_i^h, e_k \right\rangle^2 \\
&\le \sum_{k \in \mathbb{N}}^{\infty} k^{-2} \left\| g_i^h \right\|_H^2 \le Ch.
\end{aligned}$$

Substituting the result in the equation before we obtain

$$\mathbb{E} \, \|\mathcal{W}(t)\|_{V^*}^2 \le \mathbb{E} \sum_{i=1}^{k_h} th \le Ct. \quad \square$$

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