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Euler–Chern–Simons gravity from Lovelock–Born–Infeld gravity

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Abstract

In the context of a gauge theoretical formulation, higher-dimensional gravity invariant under the AdS group is dimensionally reduced to Euler–Chern–Simons gravity. The dimensional reduction procedure of Grignani–Nardelli [Phys. Lett. B 300 (1993) 38] is generalized so as to permit reducing D-dimensional Lanczos–Lovelock gravity to $d = D - 1$ dimensions.

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1. Introduction

Odd-dimensional gravity may be cast as a gauge theory for the (A)dS groups [1]. The Lagrangian is the Euler–Chern–Simons form in $D = 2n - 1$ dimensions [2,3]

$$S_{\mathrm{CS}}^{(D)} = \int \sum_{p=0}^{[D/2]} \alpha_p^{(D)} \mathcal{L}_p^{(D)},$$

where

$$\mathcal{L}_p^{(D)} = \varepsilon_{A_1 \cdots A_D} R^{A_1 A_2} \cdots R^{A_{2p-1} A_{2p}} e^{A_{2p+1}} \cdots e^{A_D},$$

$$\alpha_p^{(D)} = \kappa \frac{l^{-(D-2p-1)}}{(D-2p)} \binom{\frac{D-1}{2}}{p},$$

whose exterior derivative is Euler's topological invariant in $2n$ dimensions. The constants κ and l are related to Newton's constant G and to the cosmological constant Λ through $1/\kappa = 2(D-2)!\Omega_{D-2}G$ (where Ω_{D-2} is the area of the $(D-2)$ unit sphere) and $\Lambda = \pm(D-1)(D-2)/2l^2$.

The Chern–Simons Lagrangian remains invariant under local Lorentz rotations in tangent space $\delta e^A = \lambda^A{}_B e^B$, $\delta\omega^{AB} = -D\lambda^{AB}$, and changes by a total derivative under an infinitesimal (A)dS boost $\delta e^A = -D\lambda^A$, $\delta\omega^{AB} = \frac{1}{\ell^2}(\lambda^A e^B - \lambda^B e^A)$. With appropriate boundary conditions, this means that the action is left invariant by the (A)dS gauge transformations. Furthermore, the vielbein and the spin connection correspond to the gauge fields associated with (A)dS boosts and Lorentz rotations, respectively. Thus, odd-dimensional Chern–Simons gravity is a good gauge theory for the (A)dS group, but its usefulness is limited to odd dimensions. This is related to the fact that no topological invariants have been found in odd dimensions, and therefore the derivative of an even-dimensional Lagrangian cannot be made equal to any of them [3].

The requirement that the equations of motion fully determine the dynamics for as many components of

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the independent fields as possible may also be used in the even-dimensional case, leading to the so-called Lovelock–Born–Infeld action [2,3]

$$S_{\mathrm{BI}}^{(2n)} = \frac{\kappa l^2}{2n} \int \varepsilon_{A_1 \cdots A_{2n}} \bar{R}^{A_1 A_2} \cdots \bar{R}^{A_{2n-1} A_{2n}}, \tag{1}$$

where $\bar{R}^{AB} \equiv R^{AB} + \frac{1}{l^2} e^A e^B$ and l is a length. For $D = 2n = 4$, (1) reduces to the EH action with a cosmological constant $\Lambda = \pm 3/l^2$ plus Euler's topological invariant with a fixed weight factor.

When one considers the spin connection ω^{AB} and the vielbein e^A as components of a connection for the (A)dS group, one finds that the action (1) is invariant under local Lorentz rotations while, under infinitesimal (A)dS boosts, it changes by

$$\delta S_{\mathrm{BI}}^{(2n)} = -\kappa \int \varepsilon_{A_1 \cdots A_{2n}} \bar{R}^{A_1 A_2} \cdots \bar{R}^{A_{2n-3} A_{2n-2}} \times T^{A_{2n-1}} \lambda^{A_{2n}},$$

where λ^A is the infinitesimal parameter of the transformation. It simply makes no sense to use the equations of motion associated with the action (1) to enforce the invariance; *any* action is on-shell invariant under *any* infinitesimal transformation, just by the definition of the equations of motion. On the other hand, one could try to set the torsion equal to zero by fiat, and impose $T^A = 0$ as an off-shell identity. This is unsatisfactory in the sense that, in the (A)dS gauge picture, torsion and curvature stand on a similar footing as fields strengths of the connection whose components are e^A and ω^{AB}. It would seem rather odd to have some components of the field strength set arbitrarily to zero while the others remain untouched. The gauge interpretation of even-dimensional Lovelock–Born–Infeld gravity is thus spoiled by the lack of invariance of the action (1) under infinitesimal (A)dS boosts.

A truly (A)dS-invariant action for even as well as for odd dimensions was constructed in Ref. [4] using the Stelle–West formalism [5] for non-linear gauge theories. The action is

$$S_{\mathrm{SW}}^{(D)} = \int \sum_{p=0}^{[D/2]} \alpha_p \varepsilon_{A_1 \cdots A_d} \mathcal{R}^{A_1 A_2} \cdots \mathcal{R}^{A_{2p-1} A_{2p}} \times V^{A_{2p+1}} \cdots V^{A_D},$$

where

$$\mathcal{R}^{AB} = dW^{AB} + W^A{}_C W^{CB},$$

$$V^A = \Omega^A{}_B (\cosh z) e^B + \Omega^A{}_B \left(\frac{\sinh z}{z} \right) D_\omega \zeta^B,$$

$$W^{AB} = \omega^{AB} + \frac{\sigma}{l^2} \left[\left(\frac{\sinh z}{z} \right) e^C + \left(\frac{\cosh z - 1}{z^2} \right) D_\omega \zeta^C \right] \times \left(\zeta^A \delta_C^B - \zeta^B \delta_C^A \right),$$

with

$$\Omega^A{}_B (u) \equiv u \delta_B^A + (1-u) \frac{\zeta^A \zeta_B}{\zeta^2}. \tag{2}$$

Here ζ^A corresponds to the so-called AdS coordinate, which parametrizes the coset space $\mathrm{SO}(D+1)/\mathrm{SO}(D)$, and $z = \zeta / l$.

The method devised in Ref. [4] allows for the even-dimensional action to become (A)dS gauge invariant. The same construction can be applied in odd dimensions, where its only outcome is the addition of a boundary term to the Chern–Simons action.

$(2n-1)$-dimensional gravity has attracted a growing attention in recent years, both as a good theoretical laboratory for a possible quantum theory of gravity and as a limit of the so-called M-theory. In this context it is then interesting to establish a clear link between $D = 2n$ and $D = 2n - 1$ gravities by a dimensional reduction. This is the aim of the present Letter and it is achieved in the framework of a gauge theoretical formulation of both theories. In fact, as is shown in [3,4], gravity in $2n-1$ and $2n$ dimensions can be formulated as a gauge theory of the AdS group. In $2n-1$ dimensions this formulation is especially attractive as the Lanczos–Lovelock action becomes the Chern–Simons term of the AdS group. Such a Chern–Simons action with the correct AdS gauge transformations can then be derived by dimensionally reducing the $2n$ dimensional Lanczos–Lovelock action in its gauge theoretical formulation.

In [6] was proved, in the context of a Poincaré gauge theoretical formulation, that pure gravity in $3+1$ dimensions can be dimensionally reduced to gravity in $2+1$ dimensions. However, the mechanism of Grignani–Nardelli is not applicable in the context of an AdS gauge theoretical formulation. One of the goals of this Letter is to find a generalization of the procedure of Grignani–Nardelli that permits, in the context of an AdS gauge theoretical formulation, to reduce D-dimensional LL gravity to $d \equiv D-1$ dimensions.

2. Grignani–Nardelli procedure and AdS invariance

Latin letters from the beginning of the alphabet will be used for tangent space indices; $A, B, C = 1, 2, \ldots, D$, and $a, b, c = 1, 2, \ldots, d$. Greek letters and Latin letters from the middle of the alphabet will denote space–time indices; $\lambda, \mu, \nu = 1, 2, \ldots, D$, and $i, j, k = 1, 2, \ldots, d$. Fields belonging to $d = D-1$ dimensions will be distinguished by underlining, as in $\underline{\omega}^{ab}$. Exterior derivatives will be denoted as $d = dx^{\mu}\,\partial_{\mu}$ and $\underline{d} = dx^{i}\,\partial_{i}$. First, we consider the dimensional reduction from $3+1$ to $2+1$ dimensions. The Lovelock–Born–Infeld Lagrangian in $D=4$ is given by

$$L_{\mathrm{BI}}^{(4)} = \frac{\kappa}{4}\varepsilon_{ABCD}\left(\mathcal{R}^{AB} + \frac{1}{l^2}\mathbf{V}^A\mathbf{V}^B\right) \times \left(\mathcal{R}^{CD} + \frac{1}{l^2}\mathbf{V}^C\mathbf{V}^D\right),$$

where

$$\mathcal{R}^{AB} \equiv dW^{AB} + W^{A}{}_{C}W^{CB}$$

is the curvature tensor. This Lagrangian can be written in the form

$$L_{\mathrm{BI}}^{(4)} = \kappa\,\varepsilon_{abc}\left(\mathcal{R}^{ab} + \frac{1}{l^2}\mathbf{V}^a\mathbf{V}^b\right)\left(\mathcal{R}^{c4} + \frac{1}{l^2}\mathbf{V}^c\mathbf{V}^4\right),$$

where we have used $\varepsilon_{abc4} = \varepsilon_{abc}$.

Method 1. Table A of [6] can be written as

$$e^A = \left(\mathbf{e}^a, \mathbf{e}^{2n}\right) = \left(e^a, dx^{2n}\right),$$

$$\omega^{AB} = \begin{bmatrix} \omega^{ab} & \omega^{a,2n} \\ \omega^{2n,b} & \omega^{2n,2n} \end{bmatrix} = \begin{bmatrix} \omega^{ab} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\zeta^A = \left(\xi^a, \zeta^{2n}\right) = \left(\zeta^a, 0\right).$$

This means

$$\mathbf{V}^a = V^a,$$

$$\mathbf{V}^4 = (\cosh z)\,dx^4,$$

$$\mathcal{R}^{ab} = R^{ab},$$

$$\mathcal{R}^{a4} = -m^2 D_W\left[\left(\frac{\sinh z}{z}\right)\zeta^a\right]dx^4.$$

By substituting these results in $L_{\mathrm{BI}}^{(4)}$ one gets

$$L_{\mathrm{red}}^{(3)} = \frac{\kappa}{\ell}\varepsilon_{abc}\left\{(\cosh z)\left(R^{ab} + \frac{1}{\ell^2}V^aV^b\right)V^c - \left(R^{ab} + \frac{1}{\ell^2}V^aV^b\right) \times D_W\left[\left(\frac{\sinh z}{z}\right)\zeta^c\right]\right\}.$$

Since

$$\varepsilon_{abc}V^aV^bD_WA^c = d\left(\varepsilon_{abc}V^aV^bA^c\right) - 2\varepsilon_{abc}\mathcal{T}^aV^bA^c,$$

we have

$$L_{\mathrm{red}}^{(3)} = \frac{\kappa}{\ell}(\cosh z)\varepsilon_{abc}\left(R^{ab} + \frac{1}{\ell^2}V^aV^b\right)V^c + \frac{2\kappa}{\ell^3}\left(\frac{\sinh z}{z}\right)\varepsilon_{abc}\mathcal{T}^aV^b\zeta^c + \text{surface term}$$

which it is very different of the Chern–Simons Lagrangian in $2+1$ dimensions.

Method 2. Table B of [6] can be written as

$$e^A = \left(\mathbf{e}^a, \mathbf{e}^{2n}\right) = \left(e^a, dx^{2n}\right),$$

$$\omega^{AB} = \begin{bmatrix} \omega^{ab} & \omega^{a,2n} \\ \omega^{2n,b} & \omega^{2n,2n} \end{bmatrix} = \begin{bmatrix} \omega^{ab} & \frac{1}{\gamma}V^a \\ -\frac{1}{\gamma}V^b & 0 \end{bmatrix},$$

$$\zeta^A = \left(\zeta^a, \zeta^{2n}\right) = (0, \gamma).$$

This means

$$\mathbf{V}^a = \left(\frac{\sinh z}{z}\right)V^a,$$

$$\mathbf{V}^4 = dx^4,$$

$$\mathcal{R}^{ab} = R^{ab} - \frac{1}{\gamma^2}\left(\cosh^2\hat{z}\right)V^aV^b,$$

$$\mathcal{R}^{a4} = \frac{1}{\gamma}(\cosh z)D_{\omega}V^a,$$

with $z = m\gamma$.

By substituting these results in $L_{\mathrm{BI}}^{(4)}$ one gets

$$L_{\mathrm{red}}^{(3)} = \frac{\kappa}{l}\left(\frac{\sinh z}{z}\right)\varepsilon_{abc}\left(R^{ab} - \frac{1}{\gamma^2}V^aV^b\right)V^c,$$

which it is again different from the Chern–Simons Lagrangian in $2+1$ dimensions. Similar results are obtained for higher dimensions.

3. Generalization

We now consider a generalization of the mechanism of Grignani–Nardelli. The basic idea is the following. Let S_D be an action functional defined as the integral of a Lagrangian D-form $\mathcal{L}_D$ over a D-dimensional manifold M_D. Let us explicitly perform one integration out of the D. We are then left with the integral of a d-form over a d-dimensional manifold M_d. Relabel the fields in this integral in a convenient way and call the d-form $\mathcal{L}_d$. Its integral over M_d then becomes what we call S_d. Therefore, in general,

$$S_D = \int_{M_D} \mathcal{L}_D = \int_{M_d} \mathcal{L}_d = S_d, \tag{3}$$

where

$$\mathcal{L}_d = \int_{1\,\mathrm{dim}} \mathcal{L}_D. \tag{4}$$

Consider a surface $\sigma(x) = \mathrm{const}$ and a vector field n not lying on the surface; i.e., $I_n(d\sigma) \neq 0$, where I_ξ is the contraction operator. It will prove convenient to normalize the vector n to make it fulfill the condition $I_n(d\sigma) = 1$.

A p-form field ψ living in D dimensions can always be decomposed according to

$$\psi = \hat{\psi} + \check{\psi}, \tag{5}$$

where we have defined $\hat{\psi} \equiv d\sigma I_n \psi$, $\check{\psi} \equiv I_n(d\sigma\psi)$. The two fields $\hat{\psi}$ and $\check{\psi}$ together carry the same information as the original field ψ, as can be seen from (5). The $\check{\psi}$-component of ψ lies entirely on the surface $\sigma(x) = \mathrm{const}$, while $\hat{\psi}$ retains that part of ψ which goes in the direction of n.

When this decomposition is applied to the D-form Lagrangian $\mathcal{L}_D$, one finds $\hat{\mathcal{L}}_D = d\sigma I_n \mathcal{L}_D$, $\check{\mathcal{L}}_D = 0$. This means that it is possible to integrate $\mathcal{L}_D = \hat{\mathcal{L}}_D$ over σ to obtain the d-dimensional Lagrangian $\mathcal{L}_d$ as

$$\mathcal{L}_d = \int_{\sigma} d\sigma\, I_n \mathcal{L}_D. \tag{6}$$

The action is written as

$$S_D = \int_{M_d} \mathcal{L}_d, \tag{7}$$

where M_d is a d-dimensional manifold that belongs to the equivalence class of manifolds induced by σ (an example is shown below). It is perhaps interesting to note that so far there is no need to assume that the integrated direction has any especial feature such as being compact or extremely curved; the reduction procedure remains well-defined whether we make these assumptions or not.

Now we consider the simple case $\sigma(x) = x^D$; that is, we deal with slices of constant x^D across D-dimensional space–time. A natural choice for n is thus $n = \partial_D \equiv \partial/\partial x^D$, which satisfies $I_n\, d\sigma = I_{\partial_D}\, dx^D = 1$. With these choices, the d-form Lagrangian $\mathcal{L}_d$ may be written as

$$(\mathcal{L}_d)_{i_1\cdots i_d} = \int_{x^D} (\mathcal{L}_D)_{i_1\cdots i_d D}\, dx^D, \tag{8}$$

and M_d is simply any $x^D = \mathrm{const}$ manifold. For *all* of them, the integral

$$S_d = \int_{M_d} \mathcal{L}_d \tag{9}$$

has the same value. However, the relabeling of the fields that remains to be done in (9) may be more natural on one specific surface.

There is much freedom in the way this field-relabeling process is performed, as the only strong constraints come from symmetries. In general, the original fields in $\mathcal{L}_D$ transform locally under a group G over M_D, leaving $\mathcal{L}_D$ invariant. On the other hand, the relabeled fields that enter $\mathcal{L}_d$ must transform locally under a group G' over M_d, leaving $\mathcal{L}_d$ invariant. However, the Lagrangian $\mathcal{L}_d$ is still invariant under the G group, which gets realized now in a different way. Thus, the relabeling process must be carried out in such a way that this requirement is satisfied. A good example is provided by the spin connection ω^{AB}. Under a local, infinitesimal Lorentz transformation $\Lambda = 1 + \frac{1}{2}\lambda^{AB}\mathbf{J}_{AB}$ defined over M_D, it changes by $\delta\omega^{AB} = -D\lambda^{AB}$. Our question now is, what components of this $\mathrm{SO}(D)$ connection may be relabeled as the $\mathrm{SO}(d)$ connection $\underline{\omega}^{ab}$? To find the answer, perform an M_d-local $\mathrm{SO}(d)$ transformation on ω^{AB}, i.e., demand that the $\mathrm{SO}(D)$ infinitesimal parameters λ^{AB} satisfy the conditions $\partial_D\lambda^{AB} = 0$, $\lambda^{A,D} = 0$. These conditions turn the remaining λ^{ab} into the right parameters for a $\mathrm{SO}(d)$ infinitesimal transformation. It is straightforward to show that, when this is the case, we have $\delta\check{\omega}^{ab} = -\check{D}\lambda^{ab}$, $\delta\hat{\omega}^{ab} = \lambda^a{}_c\hat{\omega}^{cb} + \lambda^b{}_c\hat{\omega}^{ac}$, where $\check{D}$ is the exterior covariant derivative in the connection $\check{\omega}^{ab}$. These last equations express that $\check{\omega}^{ab}$ transforms as a SO(d)-connection, while $\hat{\omega}^{ab}$ behaves as a SO(d)-tensor. Therefore, an identification such as

$$\check{\omega}^{ab} \to \underline{\omega}^{ab} + \left[\mathrm{SO}(d)\text{-tensor}\right]^{ab} \tag{10}$$

seems quite reasonable. In general, one simple way to respect the relevant symmetries is to identify the components of a D-dimensional field with those of the corresponding d-dimensional one.

Here we shall perform this kind of identifications in their simplest possible form. Many components of the fields will be frozen to zero; this corresponds to our early assertion that we are only interested (by now) in showing the possibility of getting d-dimensional gravity from its higher-dimensional version. This is no longer true, of course, when we face the full dimensional reduction procedure, for in this case freezing some components of the fields yields a reduced gauge group as well.

4. The Lanczos–Lovelock action

We shall start our dimensional reduction process with the D-dimensional LL action

$$S_D = \int \sum_{p=0}^{[D/2]} \alpha_p \varepsilon_{A_1\cdots A_d} \mathcal{R}^{A_1A_2} \cdots \mathcal{R}^{A_{2p-1}A_{2p}} \times V^{A_{2p+1}} \cdots V^{A_d}. \tag{11}$$

A more suitable version of action (11) is

$$\begin{aligned} S_D = {}& \int_{M_D} \sum_{p=0}^{[(D-1)/2]} (D-2p)\alpha_p \varepsilon_{a_1\cdots a_d} \\ & \times \mathcal{R}^{a_1a_2} \cdots \mathcal{R}^{a_{2p-1}a_{2p}} V^{a_{2p+1}} \cdots V^{a_d} V^D \\ & + 2(-1)^D \int_{M_D} \sum_{p=1}^{[D/2]} p\alpha_p \varepsilon_{a_1\cdots a_d} \\ & \times \mathcal{R}^{a_1a_2} \cdots \mathcal{R}^{a_{2p-3}a_{2p-2}} \mathcal{R}^{a_{2p-1},D} \\ & \times V^{a_{2p}} \cdots V^{a_d}. \end{aligned} \tag{12}$$

The first term in (12) shows that it is *always* possible to obtain a LL action in d dimensions starting from a LL action defined in $D = d+1$.

The coefficients α_p in (11) are selected according to the criterion that the equations of motion fully determine the dynamics for as many components of the independent fields as possible. This analysis leads to [4]

$$\alpha_p = \begin{cases} \frac{\kappa_D}{D-2p} l^{-(D-2p-1)} \binom{\frac{D-1}{2}}{p} & \text{when } D \text{ is odd}, \\ \kappa_D \frac{\kappa_D}{D} l^{-(D-2p-1)} \binom{\frac{D}{2}}{p} & \text{when } D \text{ is even}. \end{cases} \tag{13}$$

A well-defined dynamics in D dimensions leads to a well-defined dynamics in d dimensions; however, we shall additionally demand that the purely gravitational terms in the reduced action produce well-defined dynamics as well. This means that the coefficients α_p must be reduced accordingly; that is, the coefficients in the reduced gravitational Lagrangian must correspond to $\alpha_p^{(d)}$ as given in (13), with $D \to d$.

We consider the dimensional reduction from $D =$ even to $d = D - 1 =$ odd. First we note that the action (11) includes Euler's topological invariant for $p = D/2$, and we may write it as

$$\begin{aligned} S_D = {}& \int_{M_D} \sum_{p=0}^{[d/2]} \alpha_p^{(D)} \varepsilon_{A_1\cdots A_D} \mathcal{R}^{A_1A_2} \cdots \mathcal{R}^{A_{2p-1}A_{2p}} \\ & \times V^{A_{2p+1}} \cdots V^{A_D} \\ & + \alpha_{D/2}^{(D)} \int_{M_D} \varepsilon_{A_1\cdots A_D} \mathcal{R}^{A_1A_2} \cdots \mathcal{R}^{A_{D-1}A_D}. \end{aligned}$$

This action is decomposed as

$$\begin{aligned} S_D = {}& \int_{M_D} \sum_{p=0}^{[d/2]} (D-2p)\alpha_p^{(D)} \varepsilon_{a_1\cdots a_d} \\ & \times \mathcal{R}^{a_1a_2} \cdots \mathcal{R}^{a_{2p-1}a_{2p}} V^{a_{2p+1}} \cdots V^{a_d} V^D \\ & + 2 \int_{M_D} \sum_{p=1}^{[d/2]} p\alpha_p^{(D)} \varepsilon_{a_1\cdots a_d} \\ & \times \mathcal{R}^{a_1a_2} \cdots \mathcal{R}^{a_{2p-3}a_{2p-2}} \mathcal{R}^{a_{2p-1}D} V^{a_{2p}} \cdots V^{a_d} \\ & + \alpha_{D/2}^{(D)} \int_{M_D} \varepsilon_{A_1\cdots A_D} \mathcal{R}^{A_1A_2} \cdots \mathcal{R}^{A_{D-1}A_D}. \end{aligned} \tag{14}$$

In this case the relation between the α_p coefficients in D and d dimensions is given by [cf. Eq. (13)]

$$\frac{1}{l}\frac{\kappa_D}{\kappa_d}(d-2p)\alpha_p^{(d)} = (D-2p)\alpha_p^{(D)}. \tag{15}$$

Plugging (15) into (14), we are led to

$$S_D = S_D^{(\mathrm{G1})} + S_D^{(\mathrm{I})} + S_D^{(\mathrm{G2})}, \tag{16}$$

where

$$S_D^{(\mathrm{G1})} = \frac{1}{l}\frac{\kappa_D}{\kappa_d}\int\limits_{M_D}\sum_{p=0}^{[d/2]}\alpha_p^{(d)}(d-2p)\varepsilon_{a_1\cdots a_d} \times \mathcal{R}^{a_1a_2}\cdots\mathcal{R}^{a_{2p-1}a_{2p}} \times V^{a_{2p+1}}\cdots V^{a_d}V^D, \tag{17}$$

$$S_D^{(\mathrm{I})} = 2\int\limits_{M_D}\sum_{p=1}^{[d/2]}p\alpha_p^{(D)}\varepsilon_{a_1\cdots a_d} \times \mathcal{R}^{a_1a_2}\cdots\mathcal{R}^{a_{2p-3}a_{2p-2}}\mathcal{R}^{a_{2p-1}D} \times V^{a_{2p}}\cdots V^{a_d}, \tag{18}$$

$$S_D^{(\mathrm{G2})} = \frac{\kappa_D l}{D}\int\limits_{M_D}\varepsilon_{A_1\cdots A_D}\mathcal{R}^{A_1A_2}\cdots\mathcal{R}^{A_{D-1}A_D}. \tag{19}$$

Now we show that it is possible to obtain an action for $d = D-1$ dimensional gravity from the corresponding action in D dimensions. Any identification in the spirit of (10) will do the job; for example,

$$\check{V}^a \to \underline{V}^a, \qquad \hat{V}^D \to dx^D, \qquad \check{W}^{ab} \to \underline{W}^{ab}. \tag{20}$$

Strictly speaking, one must perform a series expansion of the fields on the x^D coordinate. The x^D-independent term in this expansion leads, with the given identifications, to gravity in $d = D-1$ dimensions. The last term, which does not contribute to the D-dimensional equations of motion, is the Chern–Simons action for ∂M_D, i.e.,

$$S_D^{(\mathrm{G2})} = \frac{\kappa_D}{\kappa_d}\int\limits_{\partial M_D}\mathcal{L}_d^{(\mathrm{CS})}.$$

The Chern–Simons term, $S_D^{(\mathrm{G2})}$, forces us to take as M_d the boundary of M_D, that is, $M_d = \partial M_D$. In this way the freedom we initially had to pick any manifold out of the equivalence class induced by σ is lost, and we are left with a precise choice for M_d.

It is clear from the form of $S_D^{(\mathrm{G1})}$ that, in order to obtain well-defined dynamics for the gravitational sector alone, one needs to integrate the fields over the x^D coordinate *before* performing the identification process. This is due to the presence of the extra $(d-2p)$ factor, which precludes $S_D^{(\mathrm{G1})}$ from leading to well-defined dynamics. Also, the zero mode hypothesis used in odd dimensions seems useless here, because we need some kind of dependence on the x^D-coordinate to have a well-defined action for gravity in d dimensions. With this in mind, we shall take $M_d = \partial M_D$ and parametrize the x^D-coordinate in such a way that it ranges through $-\infty < x^D \leqslant 0$, with $x^D = 0$ corresponding to the boundary of M_D. We shall additionally assume that the vielbein may be written as

$$V^a = \exp\left(kx^D\right)V_0^a,$$
$$V^D = dx^D,$$

where k is a real, positive constant with dimensions of $[\text{length}]^{-1}$ and V_0^a and W^{ab} are taken to be x^D-independent. Clearly, V_0^a corresponds to $V_0^a = V^a(x^D = 0)$.

With these assumptions, the action (17) takes the form

$$S_D^{(\mathrm{G1})} = \frac{1}{l}\frac{\kappa_D}{\kappa_d}\int\limits_{M_D}\sum_{p=0}^{[d/2]}\alpha_p^{(d)}(d-2p) \times \exp\bigl(k(d-2p)x^D\bigr)\varepsilon_{a_1\cdots a_d} \times \mathcal{R}^{a_1a_2}\cdots\mathcal{R}^{a_{2p-1}a_{2p}}V_0^{a_{2p+1}}\cdots V_0^{a_d}\,dx^D.$$

Integration over x^D from $x^D = -\infty$ to $x^D = 0$ leads to

$$S_D^{(\mathrm{G1})} = \frac{1}{kl}\frac{\kappa_D}{\kappa_d}\int\limits_{M_d}\sum_{p=0}^{[d/2]}\alpha_p^{(d)}\varepsilon_{a_1\cdots a_d} \times \mathcal{R}^{a_1a_2}\cdots\mathcal{R}^{a_{2p-1}a_{2p}}V_0^{a_{2p+1}}\cdots V_0^{a_d}. \tag{21}$$

The further addition of $S_D^{(\mathrm{G1})}$ and $S_D^{(\mathrm{G2})}$ finally yields

$$S_d^{(\mathrm{red})} = \kappa_d^{(\mathrm{red})}\int\limits_{M_d}\sum_{p=0}^{[d/2]}\alpha_p^{(d)}\varepsilon_{a_1\cdots a_d} \times \mathcal{R}^{a_1a_2}\cdots\mathcal{R}^{a_{2p-1}a_{2p}}V_0^{a_{2p+1}}\cdots V_0^{a_d}, \tag{22}$$

where

$$\kappa_d^{(\mathrm{red})} = \frac{\kappa_D}{\kappa_d}\left[\frac{1}{kl}+1\right]. \tag{23}$$

It is now clear that the identifications (20) lead to a well-defined action for gravity in d dimensions, since the coefficients in the action (22) correspond to those given in (13).

From Eqs. (56)–(59) of [4] we can see that (22) can be written as

$$S_d^{(\mathrm{red})} = \kappa_d^{(\mathrm{red})} \int_{M_d} \mathcal{L}_{\mathrm{CS}}^{(2n-1)},$$

where

$$\mathcal{L}_{\mathrm{CS}}^{(2n-1)} = \sum_{p=0}^{n-1} \frac{l^{-(2n-1-2p)}}{(2n-1-2p)} \varepsilon_{a_1 \cdots a_d} \times R^{a_1 a_2} \cdots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \cdots e^{a_d}.$$

It is perhaps necessary to note that the procedure here developed is also valid in the dimensional reduction from $D = \text{odd}$ to $d = D - 1$. In this case

$$(D - 2p)\alpha_p^{(D)} = \frac{d}{l} \frac{\kappa_D}{\kappa_d} \alpha_p^{(d)}, \tag{24}$$

so that the action (11) can be written as

$$\begin{aligned} S_D = \frac{d}{l} \frac{\kappa_D}{\kappa_d} \int_{M_D} \sum_{p=0}^{d/2} \alpha_p^{(d)} \varepsilon_{a_1 \cdots a_d} & \\ \times \mathcal{R}^{a_1 a_2} \cdots \mathcal{R}^{a_{2p-1} a_{2p}} V^{a_{2p+1}} \cdots V^{a_d} V^D & \\ - 2 \int_{M_D} \sum_{p=1}^{d/2} p \alpha_p^{(D)} \varepsilon_{a_1 \cdots a_d} & \\ \times \mathcal{R}^{a_1 a_2} \cdots \mathcal{R}^{a_{2p-3} a_{2p-2}} \mathcal{R}^{a_{2p-1}, D} V^{a_{2p}} \cdots V^{a_d}. & \end{aligned}$$

From this expression it is apparent that it is possible to obtain an action for d-dimensional gravity from the corresponding action in $D = d + 1$ dimensions. Any identification in the spirit of (10) will do the job; for example (20). Strictly speaking, one must perform a series expansion of the fields on the x^D coordinate. The x^D-independent term in this expansion leads, with the given identifications, to gravity in $d = D - 1$ dimensions. This means that even-d gravity obtained from its odd-D partner always posses an empty universe solution, no matter what choice is made for the M_d-manifold inside the equivalence class induced by σ.

5. Conclusions

We have generalized the dimensional reduction mechanism of Grignani–Nardelli [6] in a way that permits obtaining Euler–Chern–Simons gravity from Lovelock–Born–Infeld gravity. The failure of the procedure of Grignani–Nardelli in obtaining of the appropriate coefficients that lead to the Chern–Simons has its origin in the fact that both in even dimensions and in odd dimensions the coefficients can be written uniquely as

$$\alpha_p^{(d)} = \kappa \frac{l^{-(d-2p)}}{d-2p} \binom{n-1}{p}$$

with $n = [(d+1)/2]$ which can split as

$$\alpha_p^{(2n-1)} = \kappa \frac{l^{-(2n-1-2p)}}{2n-1-2p} \binom{n-1}{p},$$

$$\alpha_p^{(2n)} = \frac{\kappa}{2n} l^{-(2n-2p)} \binom{n}{p}.$$

The binomial coefficients that appear in both cases depend on n and not on the dimensionality of space–time. When we go from an even dimension D to an odd dimension $d = D - 1$, n remains constant. Thus the coefficients are not reduced in a way that leads to a Chern–Simons theory.

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