Stochastic user equilibrium with reference-dependent route choice and endogenous reference points

Paolo Delle Site\textsuperscript{a,b,*}, Francesco Filippi\textsuperscript{a,b}

\textsuperscript{a} DICEA Department of Civil Architectural and Environmental Engineering, University of Rome “La Sapienza”, Rome 00184, Italy
\textsuperscript{b} CTL Centre for Transport and Logistics, University of Rome “La Sapienza”, Rome 00184, Italy

Abstract

We consider the application of reference-dependent consumer choice theory to traffic assignment on transportation networks. Route choice is modeled based on random utility maximisation with systematic utility embodying loss aversion for the travel time and money expenditure attributes. Stochastic user equilibrium models found in the literature have considered exogenously given reference points. The paper proposes a model where reference points are determined consistently with the equilibrium flows and travel times. The reference-dependent stochastic user equilibrium (RDSUE) is defined as the condition where (i) no user can improve her utility by unilaterally changing path, (ii) the reference points are the current states, and (iii) if each user updates the reference point to her current path the observed path flows do not change. These conditions are formally equivalent to a multi-class stochastic equilibrium where each class is associated with a path and has as reference point the current state on the path, and the number of users in each class equals the current flow on the path. The RDSUE is formulated as a fixed point problem in the path flows. Existence of RDSUE is guaranteed under usual assumptions. The model is illustrated by an application to a two link network which uses a reference-dependent route choice model calibrated on stated preference data. The impact on the equilibrium of different assumptions on the degree of loss aversion with respect to the travel time attribute are investigated.

Keywords: reference-dependent theory; riskless choice; loss aversion; stochastic user equilibrium; endogenous reference point

1. Introduction

A large body of field and experimental evidence suggests that choices are best explained by assuming that carriers of utility are not states but gains and losses relative to a reference point. There is an asymmetry in choices in the sense...
that gains are valued differently from losses; this asymmetry has the sign of loss aversion, i.e. losses are valued more heavily than gains. Loss aversion effects are found in data across choices in several domains (Kahneman et al., 1991; Ho et al., 2006).

Tversky and Kahneman have proposed new theories of choice where the above suppositions are incorporated. Reference-dependent theory considers riskless choices (Tversky and Kahneman, 1991). In this theory, the utilities of the alternatives are characterised by n attributes whose outcomes are certain. Prospect theory (Kahneman and Tversky, 1979), which has evolved into cumulative prospect theory (Tversky and Kahneman, 1992), considers risky choices. The alternatives are characterised by a prospect value which depends on attributes whose outcomes are uncertain. Reference-dependent theory and (cumulative) prospect theory belong to the approach of behavioural economics which integrates psychological insights into formal economic models.


Both De Borger and Fosgerau (2008) and Hess et al. (2008) estimate a random utility version of their models based on multinomial logit assumptions for the stochastic terms. Both find clear evidence of asymmetrical response in gains and losses.

A further step consists in the use of reference-dependent route choice models in network analysis. We consider stochastic user equilibrium (SUE). SUE is the name for the fixed point solution to the problem of finding flows and travel times over a network where users’ route choices are made according to a random utility model and link travel times are dependent on the link flows (Daganzo and Sheffi, 1977). The paper addresses the version of SUE where a reference-dependent, i.e. riskless, route choice model is adopted.

A few authors have considered the application of cumulative prospect theory to equilibrium problems (Avineri, 2006; Connors and Sumalee, 2009; Sumalee et al., 2009; Xu, Lou, Yin, Zhou, 2011). Travel times are uncertain and the attitude of users towards risk is modelled. The users are assumed to know the distribution of travel time on each route. The equilibrium is deterministic: an extension of Wardrop’s principle is considered with equilibrium being defined as the condition where all used routes have equal maximum prospect value. In the paper by Connors and Sumalee (2009), a stochastic version of the equilibrium is also considered: the perception of the uncertain travel times varies from user to user with equilibrium being defined as the condition where each user chooses the route of maximum random prospect value. This latter model by Connors and Sumalee includes as a particular instance the riskless case.

Delle Site and Filippi (2011) have incorporated a reference-dependent route choice model within a SUE setting. They use an asymmetric logit model with two attributes: travel time and monetary cost. The feature that distinguishes their model from that in Connors and Sumalee (2009) is the assumption on the reference point.

How the reference point is fixed is an open research question in the application of prospect theory where the problem is to set a reference point in terms of travel time which the theory treats as uncertain. This suggests that the reference point should be related to users’ expectations (the issue is dealt with in: Avineri, 2006; Gao et al., 2010; Xu, Lou, Yin, Zhou, 2011; Xu, Zhou, Xu, 2011). Connors and Sumalee (2009) assume in their model that there is an exogenous single reference point for all users of each origin-destination (OD) pair. They argue that, with a stochastic equilibrium, future research should consider a random distribution of reference points across users of the same OD.

Delle Site and Filippi (2011) develop their model starting from the recognition that in reference-dependent theory the attributes of routes are assumed to be known with certainty and it is most natural to interpret reference points as the status quo. They assume that there is a multiplicity of reference points for users of the same OD pair because in the status quo users choose different routes, each having a distinct (certain, hence deterministic) travel time. This gives rise to an equilibrium problem with multiple user classes, with each class having a distinct reference point determined by the travel time in the status quo.

They address the so-called reflexivity issue which has been considered in the context of trade equilibrium (Munro
and Sugden, 2003; Munro, 2009). They show that the equilibrium is maintained when the reference points are updated to the new status quo if the additional assumption is made that the random terms of the route choice model do not change with the updating.

However, other assumptions on the dynamics of the stochastic terms can be considered to take into account intra-personal preference variation or changes in unobserved attributes of the routes. If the stochastic terms are redrawn, the relative convenience of the alternatives may change when the reference point is updated. Therefore, a new equilibrium would need to be computed on the basis of the updated systematic utilities. The reference points would be further adjusted and a new equilibrium computed.

The present paper formulates a new version of the reference-dependent SUE where the reference points are not exogenously given but determined at equilibrium consistently with the equilibrium flows and travel times. The conjecture is made that travellers choose as reference points the status quo corresponding to the current state of the network. The approach is similar to the model in Xu, Lou, Yin and Zhou (2011) who have considered a prospect-based user equilibrium with endogenous reference points but in a deterministic choice setting.

The paper is organised as follows. Section 2 provides the definition of the equilibrium condition and presents the mathematical model. Section 3 presents some numerical results relating to an illustrative network. Directions for further research conclude the paper.

2. Network equilibrium

2.1. Network representation and assumptions

Let $G = (N, A)$ be a strongly connected road transportation network, with $N$ and $A$ being the sets of nodes and links, respectively. Let $a$ be the link index. Origins (O) and destinations (D) constitute a subset of $N$. Let $R$ be the set of OD pairs and $r$ the OD pair index. Let $K'$ be the set of simple paths of OD pair $r$, and $k$ the path index.

For each path $k \in K'$, $F^{k,r}$ denotes the corresponding path flow. We denote by $z_a$ the flow on link $a \in A$. The link flows are obtained from the path flows by:

$$z_a = \sum_{r \in R} \sum_{k \in K'} \delta_{a}^{k,r} \cdot F^{k,r} \quad a \in A \quad (1)$$

where $\delta_{a}^{k,r}$ is the element of the link-path incidence matrix whose value is 1 if path $k$ includes link $a$, is 0 otherwise.

The demand flow of the OD pair $r$ is denoted by $q^r$. We have the demand constraints:

$$q^r = \sum_{k \in K'} F^{k,r} \quad r \in R \quad (2)$$

The feasible path flows are all the non-negative $F^{k,r}$ satisfying the demand constraints (2). Therefore, the set of feasible path flows is non empty, compact and convex.

Let $T^{k,r}$ denote the travel time on path $k$ of OD pair $r$. Let $t_a$ denote the travel time on link $a$. The link travel times are continuous functions of the link flows: $t_a = t_a(z_a, a \in A)$. The path travel times are obtained from the link travel times by the standard link-additive model:

$$T^{k,r} = \sum_{a \in A} \delta_a^{k,r} \cdot t_a(z_a, a \in A) \quad k \in K', r \in R \quad (3)$$
2.2. Reference-dependent route choice

The users of an OD pair perceive a utility on each path. This path utility is a random variable given by the sum of a systematic, i.e. deterministic, component and a stochastic term. The stochastic terms summarise factors that are unobserved by the modeller.

There are two interpretations of the stochastic terms. One accounts for inter-individual variability of tastes as it is assumed that the stochastic terms are individual specific, i.e. utility is deterministic for the decision maker, stochastic only for the modeller. The other accounts for intra-individual variability of tastes as it is assumed that the individual draws from a distribution each time a choice is made, i.e. utility is stochastic also for the decision maker.

A reference-dependent model is adopted for the path systematic utility according to the following hypotheses. The path systematic utility

(i) depends on two attributes: expenditure in travel time \( T \) and expenditure in money \( M \);
(ii) depends on gains \( G \) and losses \( L \) in the two attributes defined relative to a reference point, and increases with gains and decreases with losses;
(iii) is linear in gains and losses and steeper for losses than for gains.

The users of each OD pair \( r \) are grouped into classes, with each class denoted by \( j \) and identified by a reference point in terms of path travel time and money spent. Let \( J^r \) be the set of classes of OD pair \( r \).

The path utilities have the additive form:

\[
U_{j}^{k,r} = V_{j}^{k,r} + \varepsilon_{j}^{k,r}
\]

\[
V_{j}^{k,r} = \beta_{GT} \cdot GT_{j}^{k,r} + \beta_{LT} \cdot LT_{j}^{k,r} + \beta_{GM} \cdot GM_{j}^{k,r} + \beta_{LM} \cdot LM_{j}^{k,r}
\]

\[
GT_{j}^{k,r} = \max\left(T_{j}^{r} - T_{j}^{k,r}, 0\right)
\]

\[
LT_{j}^{k,r} = \max\left(T_{j}^{k,r} - T_{j}^{r}, 0\right)
\]

\[
GM_{j}^{k,r} = \max\left(M_{j}^{r} - M_{j}^{k,r}, 0\right)
\]

\[
LM_{j}^{k,r} = \max\left(M_{j}^{k,r} - M_{j}^{r}, 0\right)
\]

where:

\( U_{j}^{k,r} \) is the path perceived utility,
\( V_{j}^{k,r} \) is the path systematic utility,
\( \varepsilon_{j}^{k,r} \) is the stochastic term,
\( \beta_{GT}, \beta_{GM} \) are the gain coefficients,
\( \beta_{LT}, \beta_{LM} \) are the loss coefficients,
\( GT_{j}^{k,r}, GM_{j}^{k,r} \) are the gain, respectively, in travel time and in money,
\( LT_{j}^{k,r}, LM_{j}^{k,r} \) are the loss, respectively, in travel time and in money,
\( M_{j}^{k,r} \) is the money spent on the path;
\( T_{j}^{r}, M_{j}^{r} \) are the reference point for, respectively, the travel time and the money spent.

Hypothesis (ii) implies that the systematic utility is decreasing in each attribute, i.e. the gain coefficients are positive and the loss coefficients are negative. Hypothesis (iii) implies loss aversion, i.e. \( |\beta_{LT}| > |\beta_{GT}| \) and \( |\beta_{LM}| > |\beta_{GM}| \): in absolute values, the loss coefficient is larger than the gain coefficient for each attribute.
The systematic utility in eqns (4) has two terms for each attribute: a gain term and a loss term. If there is a gain in the attribute the gain term is positive and the loss term is zero. Conversely, if there is a loss the loss term is positive and the gain term is zero.

The single-attribute part of the systematic utility is piecewise linear in the attribute with a kink in the reference point. Thus the function is everywhere continuous in the attribute but non-differentiable in the reference point. If the absolute values of the gain and loss coefficients were equal, the function would be symmetric about the reference point. For different coefficients the function is asymmetric with slope steeper in losses than in gains if the coefficients satisfy loss aversion. This is shown in Figure 1 where the attribute, i.e. the expenditure in travel time or in money, is denoted by X.

A constant additive term may be included in the systematic utility in eqns (4), e.g. to represent other time-independent path attributes; however, without loss of generality it is left out because it does not affect the developments below.

Users of class j of OD pair r who choose path k are those who perceive this path to maximise their utility. The choice probabilities are defined as:

\[
P_{j}^{k,r} = Pr\left(U_{j}^{k,r} \geq U_{j}^{m,r} \quad \forall m \neq k \in K'\right) \quad k \in K', \ j \in J', \ r \in R
\] (5)

We assume that the stochastic terms \( e_{j}^{k,r} \) have a non-degenerate joint probability density function that is continuous, strictly positive, and independent of the path systematic utility. We assume that the choice probabilities are single-valued and continuous in the path systematic utilities:

\[
P_{j}^{k,r} = P_{j}^{k,r}(s_{j}^{k,r}, \ k \in K') \quad k \in K', \ j \in J', \ r \in R
\] (6)

The hypotheses are sufficiently general to admit a range of behavioural assumptions through the form of the joint distribution for the stochastic terms, thus encompassing various additive models, including, but not restricting to, multinomial logit. In the case of multinomial logit the probability function takes an asymmetric “S” shape with a kink in the reference point due to the loss aversion assumption (this is illustrated graphically in Suzuki et al., 2001).

Let \( f_{j}^{k,r} \) denote the flow on path k of class j of OD pair r. The choice model is expressed in terms of these class-specific path flows as:
\[ f_{j}^{k,r} = q_{j}^{r} \cdot P_{j}^{k,r} \quad k \in K', \quad j \in J', \quad r \in R \]  

(7)

where \( q_{j}^{r} \) denotes the number of users of class \( j \) of OD pair \( r \), with

\[ \sum_{j \in J'} q_{j}^{r} = q^{r} \quad r \in R \]  

(8)

2.3. Reference-dependent stochastic user equilibrium with endogenous reference points

The reference-dependent stochastic user equilibrium (RDSUE) model with endogenous reference points seeks the condition where:

• no user can improve her reference-dependent utility by unilaterally changing path,
• the reference points are the current states,
• if each user updates the reference point to her current path the observed path flows do not change.

This condition is obtained when:

• each user chooses the path with the maximum utility,
• each user class is associated with a path and its reference point is the current state on that path,
• the number of users in each class equals the current flow on the corresponding path.

In fact, the flow of a path of an OD pair is given by the union of the following two sets of users: the users who have as reference that path and choose it, and the users who choose the path while having as reference other paths. The number of users in the second set equals the number of users who have as reference the path while choosing other paths.

Mathematically, the equilibrium conditions are defined by the following:

\[ J' \equiv K' \quad r \in R \]  

(9)

\[ q_{j}^{r} = F_{j}^{r} \quad j \in K', \quad r \in R \]  

(10)

where:

\[ F_{j}^{r} = \sum_{k \in K'} f_{j}^{k,r} \quad j \in K', \quad r \in R \]  

(11)

A RDSUE is a solution to the fixed point problem in the path flows \( F_{j}^{r} \):

\[ F_{j}^{r} = \sum_{k \in K'} f_{j}^{k,r} \quad j = 1, \ldots, |K'| - 1 \quad r \in R \]  

(12)

\[ F_{j}^{r} = q^{r} - \sum_{j=1}^{|K'|-1} F_{j}^{r} \quad r \in R \]  

(13)

with

\[ f_{j}^{k,r} = P_{j}^{k,r} \cdot F_{j}^{r} \quad k \in K', \quad j \in K', \quad r \in R \]  

(14)

where \( |K'| \) denotes the cardinality of the set \( K' \).

The dependence of the probabilities on the path flows which appears in eqns (14) is obtained by chaining the expressions (4) of the systematic utilities in the path travel times, the expressions (3) of the path travel times in the link travel times, the link travel times in the link flows, and the expressions (1) of the link flows in the path flows.
A solution to the fixed point problem (12), (13) and (14) uniquely determines the link flows $z_a$, the link travel times $t_a$, the path travel times $T^{k,r}$, as well as the class-specific path flows $f_{j}^{k,r}$.

In the light of the Brower’s fixed point theorem, a solution to RDSUE exists as the feasible set is non empty, compact and convex and all the functions composed to form the fixed point formulation are continuous.

The RDSUE collapses to a conventional SUE when the absolute values of the loss and gain coefficients are equal, i.e. $|\beta_{LT}| = |\beta_{GT}|$ and $|\beta_{LM}| = |\beta_{GM}|$. In fact, due to the model additivity, when these conditions occur choice probabilities are not affected by reference points:

$$P_{j}^{k,r} = P^{k,r} \quad k \in K', \; j \in J', \; r \in R \quad (15)$$

The RDSUE fixed point problem (12), (13) and (14) reduces then to the conventional SUE fixed point problem:

$$F^{j,r} = q^{r} \cdot P^{j,r} \quad j \in K', \; r \in R \quad (16)$$

3. Illustrative example

We use a logit route choice model estimated on the basis of data from a stated preference survey which took place in Rome in 2007 (Delle Site and Filippi, 2011). The results of the estimation are in Table 1.

All the coefficients have the right sign. Individuals value positively gains and negatively losses. All the coefficients are statistically significant (at 5% significance level, two-tailed). The findings support the hypothesis of loss aversion. In absolute value, the loss coefficient is higher than the gain coefficient for both travel time and money. The degree of loss aversion, defined as the ratio between the loss coefficient and the gain coefficient, is higher for money ($\beta_{LM} / \beta_{GM} = 1.34$) than for time ($\beta_{LT} / \beta_{GT} = 1.16$), i.e. individuals are more loss averse in the money dimension than in the time dimension.

We tested the statistical significance of the assumption that responses are asymmetric. We considered the null hypothesis that the difference in absolute value between the gain coefficient and the loss coefficient is zero, i.e. a symmetrical response. Based on the t-statistic in Table 1, for the time attribute we can reject the null hypothesis at 10% significance level (one-tailed, with sign consistent with loss aversion), for the money attribute we can reject the null hypothesis at 5% significance (two tailed). Therefore data support asymmetry to a different extent according to the attribute.

Table 1. Estimation results for the route choice model

<table>
<thead>
<tr>
<th></th>
<th>coefficient</th>
<th>t statistic</th>
<th>t statistic for difference in absolute values</th>
</tr>
</thead>
<tbody>
<tr>
<td>time gain (minutes)</td>
<td>0.10545</td>
<td>9.521</td>
<td>-1.5318</td>
</tr>
<tr>
<td>time loss</td>
<td>-0.12270</td>
<td>-9.827</td>
<td></td>
</tr>
<tr>
<td>money gain (EUR)</td>
<td>1.25287</td>
<td>9.481</td>
<td>-3.302</td>
</tr>
<tr>
<td>money loss</td>
<td>-1.67346</td>
<td>-14.075</td>
<td></td>
</tr>
<tr>
<td>number of observations:</td>
<td>1068</td>
<td></td>
<td></td>
</tr>
<tr>
<td>final log likelihood</td>
<td>-413.4574</td>
<td></td>
<td></td>
</tr>
<tr>
<td>rho-squared</td>
<td>0.4414</td>
<td></td>
<td></td>
</tr>
<tr>
<td>parameters estimated</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>rho-squared adjusted</td>
<td>0.4393</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We consider a two-link network (Figure 2) representing a town centre route and a bypass route.
We assume a total demand of 1200 veh/h. For supply, BPR time-flow functions derived empirically for similar routes are used. The functions (in hours) are \( T = 0.057 [1 + (f/800)^{0.2}] \) for the town centre route, and \( T = 0.045 [1 + 0.68 (f/1230)^{0.6}] \) for the bypass route.

The RDSUE fixed point problem reduces to the non-linear equation in the town centre route flow \( F^1 \):

\[
F^1 = F^1 \cdot P^1 \left( F^1, 1200 - F^1 \right) + \left( 1200 - F^1 \right) \cdot P^1 \left( F^1, 1200 - F^1 \right)
\]

(17)

Table 2 provides the results in the case where a toll of 1 EUR is charged on the bypass. Table 3 provides the class-specific path flows. Of the 858 veh/h which are found on the town centre route, 641 veh/h have as reference point the current state on the route, while the remaining 217 veh/h are those having as reference point the current state on the bypass route. At the same time, there are 217 veh/h which have as reference point the current state on the town centre route and choose the bypass route. Therefore, if the 217 veh/h having as reference point one route and choosing the other update their reference point to their current route the total flow on each route does not change.

We investigated in the case of absence of tolls the sensitivity of the equilibrium to the assumption on the degree of loss aversion with respect to the travel time attribute. In the estimated model the degree of loss aversion is 1.16. This value is low when compared with the results obtained by Hess et al. (2008). They found for the two demand segments considered a degree of loss aversion in the free flow time attribute of 1.49 and 2.44. This comparison suggests that it is meaningful to explore the sensitivity to the degree of loss aversion. We explored the range from 1 to 3. The case of degree equal to 1 is that where demand exhibits no loss aversion. The RDSUE collapses then to a conventional SUE and the solution is obtained by solving the equation:

\[
F^1 = 1200 \cdot P^1 \left( F^1, 1200 - F^1 \right)
\]

(18)

Table 4 shows the results of the sensitivity analysis. As the degree of loss aversion increases, the flow on the town centre route, which is the route where travel time is higher, decreases. The variation in the flow values is not large (maximum of 30 veh/h out of a total flow of 1200 veh/h).

Table 2. RDSUE results with toll on the bypass

<table>
<thead>
<tr>
<th>town centre route</th>
<th>bypass route</th>
<th>time expenditure</th>
</tr>
</thead>
<tbody>
<tr>
<td>flow (veh/h)</td>
<td>time (minutes)</td>
<td>flow (veh/h)</td>
</tr>
<tr>
<td>858</td>
<td>8.3</td>
<td>342</td>
</tr>
</tbody>
</table>
Table 3. RDSUE results with toll on the bypass – class-specific path flows

<table>
<thead>
<tr>
<th>Reference point</th>
<th>Flow on town centre route (veh/h)</th>
<th>Flow on bypass route (veh/h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Town centre route</td>
<td>641</td>
<td>217</td>
</tr>
<tr>
<td>Bypass route</td>
<td>217</td>
<td>125</td>
</tr>
</tbody>
</table>

Table 4. RDSUE results with no toll: sensitivity to loss aversion

<table>
<thead>
<tr>
<th>Degree of loss aversion</th>
<th>Town centre route</th>
<th>Bypass route</th>
<th>Time expenditure</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Flow (veh/h)</td>
<td>Time (minutes)</td>
<td>Flow (veh/h)</td>
</tr>
<tr>
<td>1</td>
<td>563</td>
<td>3.97</td>
<td>637</td>
</tr>
<tr>
<td>1.16</td>
<td>560</td>
<td>3.95</td>
<td>640</td>
</tr>
<tr>
<td>1.5</td>
<td>555</td>
<td>3.93</td>
<td>645</td>
</tr>
<tr>
<td>2</td>
<td>547</td>
<td>3.89</td>
<td>653</td>
</tr>
<tr>
<td>2.5</td>
<td>539</td>
<td>3.86</td>
<td>661</td>
</tr>
<tr>
<td>3</td>
<td>532</td>
<td>3.83</td>
<td>668</td>
</tr>
</tbody>
</table>

4. Conclusion

We have formulated a reference-dependent stochastic user equilibrium model with endogenously determined reference points. The conjecture is made that travellers adopt as reference points the status quo corresponding to the current state of the network. Equilibrium conditions are defined where reference points are determined consistently with current flows and travel times.

The equilibrium is formulated as a fixed point in the path flows. It has been proved that the solution exists under conditions usually satisfied in practice. Uniqueness of the solution is an open problem. The natural development of the research presented is that of computation of the equilibrium in networks of realistic size. This calls for path-based algorithms as the reference-dependent path utility is not additive in the constituent links.

References


