Augmenting the connectivity of geometric graphs

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Abstract

Let $G$ be a connected plane geometric graph with $n$ vertices. In this paper, we study bounds on the number of edges required to be added to $G$ to obtain 2-vertex or 2-edge connected plane geometric graphs. In particular, we show that for $G$ to become 2-edge connected, $\frac{2n}{3}$ additional edges are required in some cases and that $\frac{6n}{7}$ additional edges are always sufficient. For the special case of plane geometric trees, these bounds decrease to $\frac{n}{2}$ and $\frac{2n}{3}$, respectively.

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1. Introduction

A classical problem in graph theory is that of augmenting the connectivity of a graph $G$ by adding to it as few edges as possible. The problem of increasing the connectivity of a connected graph to make it 2-vertex or 2-edge connected using the smallest possible number of edges can be solved in linear time [6]. For $k = 3, 4$ polynomial time algorithms for augmenting a $(k - 1)$-vertex connected graph to a $k$-vertex connected graph have been known for some time (see [12,22]); only recently a polynomial time algorithm for this problem has been found for any fixed $k$ [14]. A survey in which these problems are described within a more generic framework is given in [17]. Other related results appear in [3,5,7,8,18,21].

The problem of increasing the connectivity of planar graphs was studied by Kant [15,16]. He proved that it is NP-hard to determine the minimum number of edges required to be added to augment a given planar graph into

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a 2-vertex connected planar graph. The corresponding problem for 2-edge connectivity, i.e., determining the minimum number of edges we have to add to augment a given planar graph into a 2-edge connected planar graph, is open.

We consider here similar problems for graphs with the additional constraint that a given geometric embedding has to be preserved. A geometric graph is a graph $G$ such that its vertex set is a set of points on the plane in general position (no three points being collinear), and its edge set is a set of line segments joining pairs of vertices of $G$. A geometric graph $G$ is plane if no two edges of $G$ intersect except at a common vertex. This terminology is standard in combinatorial geometry [19], yet it is worth mentioning that the term planar straight line graph is also often used, particularly in works on meshing and graph drawing.

In this paper we study the following problem: let $G$ be a connected plane geometric graph. How many edges must be added to $G$ in such a way that the plane geometric graph we obtain is 2-edge or 2-vertex connected? We show that for $G$ to become 2-edge connected, $\frac{2n}{3}$ additional edges are required in some cases and that $\frac{2m}{7}$ additional edges are always sufficient. If $G$ is a plane geometric tree (a plane connected geometric graph with $n$ vertices and $n - 1$ edges), the addition of $\frac{2n}{3}$ edges is always sufficient to make it 2-edge connected and $\frac{2m}{7}$ edges are sometimes required. Moreover, if $G$ has $b$ blocks, then it can be augmented to a 2-vertex connected plane geometric graph by adding at most $b - 1$ edges.

A closely related problem for geometric graphs was studied by Rappaport [20]. He proved that the problem of deciding whether a plane geometric graph $G$ which is a set of polygonal chains admits a simple circuit (a geometric graph which is a cycle) is NP-complete. This paper was perhaps the first of a series of papers by several authors in which the objective is to find planar geometric graphs that have some specific structure [1,9–11,13].

The paper is organized as follows. In Section 2 we study the problem of obtaining 2-vertex connected graphs, in Section 3 we study the problem of adding edges to a plane geometric tree to obtain a 2-edge connected plane geometric graph, and in Section 4 we study the problem of obtaining 2-edge connected plane geometric graphs from generic plane geometric graphs. We make some concluding remarks in Section 5.

Let us recall some standard notations and definitions. Let $G = (V, E)$ be a graph. A graph is $k$-vertex connected (resp. $k$-edge connected) if the deletion of any set of at most $k - 1$ vertices (resp. $k - 1$ edges) of $G$ results in a connected graph. If $G$ is a graph and $e$ one of its edges, $G - e$ will denote the graph obtained by removing $e$ from $G$. Similarly $G + e$ will denote the graph obtained by adding to $G$ an edge $e$ not in $G$. An edge $e$ with vertices $u$ and $v$ will be denoted by $uv$.

A vertex $v$ of a graph $G$ is called a cut vertex if $G - v$ is not connected. Given a graph $G$, a maximal subgraph that has no cut vertices is called a block of $G$. Observe that if $G$ is a tree, the blocks of $G$ are its edges. A block of a graph with at least 3 vertices is 2-vertex connected. An edge $e$ of a graph $G$ is called a bridge if $G - e$ is not connected.

All geometric graphs considered here will be plane. We will also assume that all our graphs have $n$ vertices, $n \geq 3$, and that they are connected. A plane geometric graph $G$ is called a triangulation if all of its faces, except perhaps for the unbounded face of $G$, are triangles.

2. Two-vertex connected plane geometric graphs

A set of points in the plane is in convex position if the elements of the set are the vertices of a convex polygon. Observe that if $G$ is a plane geometric graph with at least two vertices whose vertices are in convex position, then $G$ is an outerplanar graph, and thus it has at least two vertices of degree two or less. Therefore the edge and vertex connectivity of $G$ are at most two. Hence we only study problems regarding the completion of plane geometric graphs to 2-vertex and 2-edge connected graphs.

In this section, we solve the problem of finding the maximum number of edges that must be added to a plane geometric graph to obtain a 2-vertex connected plane geometric graph.

Theorem 1. Let $G$ be a connected plane geometric graph with $b$ blocks. Then $G$ can be augmented to a 2-vertex connected plane geometric graph by adding at most $b - 1$ edges to $G$. This bound is tight.

Proof. The proof proceeds by induction on the number of blocks of $G$. Recall that $G$ contains at least three vertices. If $G$ has exactly one block, then it is already 2-connected and no edges need to be added.

Suppose, then, that $G$ has at least two blocks, and let $uv$ be a cut vertex of $G$. We now prove that we can add one edge to $G$ so that we obtain a plane geometric graph with fewer blocks than $G$. This will prove our result. Let us divide
the vertices of $G - v$ into two disjoint sets: a set $V_1$ formed by the vertices of one of the components of $G - v$, and a set $V_2$ containing the remaining vertices of $G - v$. By construction, the vertices of $V_1$ are in a block different from those containing the elements of $V_2$. A folklore result for plane geometric graphs asserts that any plane geometric graph $G$ can be completed to a triangulation $T$. Observe next that no triangulation contains a cut vertex. It follows now that there is an edge $e$ in $T$ that joins two vertices, one in $V_1$ and the other in $V_2$, for otherwise $v$ would be a cut vertex in $T$. By adding $e = (v_1, v_2)$ to $G$ we obtain a plane geometric graph in which the edges contained in any simple path from $v_1$ to $v_2$ are in a new common block.

To prove that the bound is tight, we observe that if $G$ is a zig-zag path whose vertices are in convex position, it has exactly $n - 1$ blocks (its edges), and to make it 2-vertex connected, we need to add to $G$ exactly $n - 2$ edges. □

3. Two-edge connected plane geometric graphs from trees

We observe next that a similar idea to that used in the proof of Theorem 1 can be used to increase the edge connectivity of a plane geometric graph.

Let $e$ be a bridge of a plane geometric graph $G$, and let $H_1$ and $H_2$ be the components of $G - e$. As in the proof of Theorem 1, if we add edges to $G$ until we get a triangulation, there must be an edge $f \neq e$ joining a vertex in $H_1$ to a vertex in $H_2$. Clearly $e$ is no longer a bridge in $G + f$. Thus we have:

**Lemma 1.** Let $G$ be a plane geometric graph with $k$ bridges. Then $G$ can be augmented to a 2-edge connected plane geometric graph by adding at most $k$ edges to $G$.

It is straightforward to see that in some cases, $k$ edges are necessary.

3.1. Method 1 for trees

The next lemma will be useful for improving the previous bound for the case when $G$ is a tree.

**Lemma 2.** Let $G$ be a plane geometric graph and $e = uv$ a bridge of $G$. Let $H_1$ and $H_2$ be the components of $G - e$ such that $u \in H_1$. Then if $H_1$ has more than one vertex, there is an edge $f = v'w$ such that $G + f$ is planar, $v' \in H_2$, $w \in H_1$, $w \neq u$, and thus $e$ is no longer a bridge of $G + f$.

**Proof.** Let $T$ be a triangulation that contains $G$ as a subgraph. In $T$ we must necessarily have at least one edge $f = v'w$ connecting a vertex $w \neq u$ of $H_1$ to $v$, for otherwise $u$ would be a cut vertex of $T$. □

We can now prove:

**Lemma 3.** Let $G$ be a plane geometric tree with $h$ leaves. Then $G$ can be augmented to a 2-edge connected plane geometric graph by adding to it at most $\left\lceil \frac{n+h-2}{2} \right\rceil$ edges.

**Proof.** Let $S$ be the set of edges of $G$ such that none of its vertices is a leaf; let $|S| = m$, $m + h = n - 1$. Observe that for any edge $e \in S$, the components of $G - e$ have at least two vertices.

The proof is constructive, and in each step we add a new edge creating a cycle that contains at least two bridges of $G$. We start with a leaf, $v$, of $G$. Let $e = uv$ be the edge of $G$ incident to $v$. Since the component of $G - e$ containing $u$ has at least two vertices, then by Lemma 2 there is an edge $f = vwG$ such that $G_1 = G + f$ is a plane geometric graph, $u \neq w$. Moreover $G_1$ contains a cycle $C_1$ with at least two edges of $G$. Let $H_1 = C_1$, the only 2-edge connected component of $G_1$.

If $G_1$ is not 2-edge connected, let $e = uv$ (if it exists) be a bridge of $G_1$ such that $v$ is a vertex of $H_1$ and the second component $F$ of $G_1 - e$ has at least two vertices. Note that $F$ is a tree. By Lemma 2, there is an edge $f = v'w$, $u \neq w$, such that $w$ is a vertex of $F$ and $G_1 + f$ is plane. Observe that when we add $f$ to $G_1$ we create a cycle $C_2$ containing at least two bridges of $G_1$, $e$ and a bridge in the path from $u$ to $w$ in $G$. Let $G_2 = G_1 + f$, and $H_2$ be the subgraph of $G_2$ obtained by adding to $H_1$ the edges of $C_2$ not in $H_1$. Thus, the only 2-edge connected component of $G_2$ is $H_2$. 
We iterate this process by adding in each step a new cycle to the 2-edge connected component. In a generic step $i$, we search for a bridge $e = uv$ of $G_i$ such that $v$ is a vertex of $H_i$ and the second component $F$ of $G_i - e$ has at least two vertices, and then we define the graphs $G_{i+1}$ and $H_{i+1}$ as before. The process stops when the graph obtained, $G_i$, is 2-edge connected or all the edges of $G_i$ not in $H_i$ are leaves, that is, until the bridge $e$ we were seeking above does not exist. In this last case, we use an extra edge to eliminate the bridges on the remaining leaves. Clearly during the process we added at most $\lceil \frac{m+1}{2} \rceil + h - 1 = \lfloor \frac{n+h-2}{2} \rfloor$ edges. \hfill \Box

As a corollary we have:

**Corollary 1.** Let $G$ be a plane geometric graph that is a path. Then it can always be augmented to a 2-edge connected plane geometric graph by adding at most $\lfloor n/2 \rfloor$ edges; the bound is tight.

The bound is again achieved when $G$ is a zig-zag path and its vertices are the vertices of a convex polygon.

The bound given in Lemma 3 is, however, poor if $G$ has many leaves. We present below a different method which gives better results for this case.

### 3.2. Two lemmas

Our objective now is to show that any plane geometric tree can be augmented to a 2-edge connected plane geometric graph with the addition of at most $\frac{3n}{2}$ edges.

Before we prove this result, some remarks are in order. In general, we want to increase the edge connectivity of a plane geometric graph $G$. To achieve this we will take a second plane geometric graph $H$ (not necessarily connected) and consider the union of $G$ and $H$. The main requirement is that $G \cup H$ also be a plane geometric graph. We will also use the following technical trick that will allow us to simplify our proofs. In what follows it could happen that $H$ has some edges in common with $G$. If an edge $uv$ is an edge in $G$ and $H$, we will consider $u$ and $v$ to be joined by two edges, i.e., we will admit multiple edges. We will color the edges of $G$ black, and the edges of $H$ red. Thus if $u$ and $v$ are joined by two edges, one will be black, and the other red. The black edges will always remain, while the red edges may be deleted and inserted throughout the procedures.

In all of the figures here, red edges will be represented by dashed curves. We stress that the edge represented by the dashed curve is in fact a straight line segment, and that for our purposes we will consider double edges as non-intersecting.

For a trivial example, to obtain a 2-edge connected plane geometric graph from a plane geometric tree $G$, we can proceed as follows. Let $H$ be isomorphic to $G$, and let $G' = G \cup H$. Then $G'$ is 2-edge connected. Indeed any two vertices in $G'$ are joined by two edge disjoint paths, a black path in $G$, and a red path in $H$.

The next lemma will prove useful and will allow duplicated edges to be eliminated.

**Lemma 4.** Let $G' = G \cup H$ be a planar geometric graph such that $G'$ is 2-edge connected. Edges of $G$ are colored black and edges of $H$ are colored red. Let $u$ and $v$ be two vertices of $G'$ that are joined by a black and a red edge, $e$ and $e'$ respectively. Then we can either eliminate $e'$ or substitute it by another red edge $f$ such that $G' - e' + f$ is 2-edge connected. In the second case, $f$ can be chosen such that it does not create a new double edge.

**Proof.** If there is a cycle which uses $e$ and bypasses $e'$, we can eliminate $e'$, and $G' - e'$ remains 2-edge connected. Suppose there is no such cycle. Then $e$ is a bridge of $G' - e'$. By Lemma 2, there is an edge $f \neq e$ such that $G' - e' + f$ is a plane geometric graph. It is easy to see that $G' - e' + f$ is 2-edge connected. Clearly $f$ is not part of a double edge. \hfill \Box

A plane geometric perfect matching of a point set $P$ with $2m$ elements is a set of $m$ disjoint segments joining pairs of elements of $P$. We will use the two following results which are interesting in themselves.

**Lemma 5.** Let $P$ be a simple polygon and let $R = \{r_1, \ldots, r_l\}$ be the set of reflex vertices of $P$. Let $A$ be a subset of the vertex set of $P$ with an even number of elements such that $R \subset A$. Then there is a plane geometric perfect matching $M$ of $A$ such that the line segments determined by $M$ are contained in the interior or lie on the boundary of $P$.  


Proof. The proof is by induction on the number of reflex vertices of $P$. The result is clearly true if $P$ is convex.

Suppose then that $P$ has at least one reflex vertex $r$. Let $\ell$ be an open line segment contained in the interior of $P$ such that $\ell$ splits $P$ into two polygons $P_1$ and $P_2$ such that one endpoint of $\ell$ is $r$, and $r$ is a convex vertex in both of $P_1$ and $P_2$; see Fig. 1.

Let $A_1$ and $A_2$ be the subsets of $A - \{r\}$ that are vertices of $P_1$ and $P_2$ respectively. One of them, say $A_1$, has an even number of elements, while $A_2$ has an odd number of elements. Since $r$ is no longer a reflex vertex, both $A_1$ and $A_2$ have fewer reflex elements than $A$. By adding $r$ to $A_2$, and applying induction on $P_1$ and $A_1$, and $P_2$ and $A_2$, respectively, the result follows.

Lemma 6. Let $G$ be a plane geometric tree with an even number of vertices. Then there is a perfect matching $\mathcal{M}$ on the set of vertices of $G$ such that the graph $G'$ obtained by adding to $G$ the edges of $\mathcal{M}$ is a plane geometric graph, possibly with multiple edges. If two vertices of $G'$ are joined by two edges, one of them belongs to $G$ and the other to $\mathcal{M}$.

Proof. Let $G$ be the given plane geometric tree and $\Delta$ a triangle that encloses all the vertices of $G$, such that one vertex of $G$, say $u = v_0$, is also a vertex of $\Delta$. By duplicating the edges of $G$ and traversing externally the edges, starting at $u$, we obtain a weakly simple polygon $P'$ with $2(n - 1) + 3$ edges (a weakly simple polygon is a closed polygonal chain without self-crossings; a precise and longer formal definition can be found in [4]). Figure 2b shows how this weakly simple polygon $P'$ is built.

It is well known that any weakly simple polygon $P'$ can be combinatorially considered and algorithmically processed as a simple polygon $P$ with vertices infinitesimally close to the vertices of $P'$ (see for example [2,4,11]). In our case, we proceed as follows. We traverse the boundary of $P'$. Each time that a vertex $v_j$ of $G$ appears, say in the sequence $v_i, v_j, v_k$, we substitute vertex $v_j$ by a new vertex $v'_j$ on the bisector of the clockwise angle $v_i v_j v_k$ and place $v'_j$ infinitesimally close to $v_j$ (at a distance $\epsilon$ from $v_j$). Linking these new vertices in the order in which they were created we obtain the simple polygon $P$ being sought (see Fig. 2c). Note that $G$ lies on the complement of $P$.

Let us call the newly added vertices of type $v'_j$ copies of $v_j$. If $v$ has degree $k$, the set $S_u$ of copies of $v$ has $k$ elements, except for $S_u$ that contains as many copies of $u$ as the degree of $u$ plus one. Notice that, by construction, at most one copy of each vertex $v$ can be reflex in $P$.

Then, we can define a set $A$ of $n$ copies by choosing in each set $S_u$ the reflex copy of $v$, if it exists, or an arbitrary copy of $v$ otherwise, and we can form a perfect matching of these $n$ copies in the interior of $P$ by applying Lemma 5. Besides, if $\epsilon$ is small enough we can substitute each segment $v'u'$ between copies by the segment $vu$ between vertices of $G$, without producing crosses (but perhaps producing duplicated edges), thus obtaining the matching sought (see Fig. 2d).

3.3. Method 2 for trees

We now outline how to complete a plane geometric tree $G$ to a 2-edge connected plane geometric graph using few edges. Remember that the edges of $G$ are colored black and the other edges are colored red. In a nutshell, the algorithm used to accomplish this task is as follows:
Algorithm 1

Phase 1. Matching
Given \( G \), construct \( \mathcal{M} \) and \( G' \) as in Lemma 6. Let \( C_1, \ldots, C_s \) be the 2-edge connected components of \( G' \).

Phase 2. Merging components with 2 and 4 vertices
Add and delete some suitable red edges of \( G' \) to obtain a geometric graph \( G'' \) such that \( G \) is still a subgraph of \( G'' \), \( G' \) and \( G'' \) have the same number of edges, and all the 2-edge connected components of \( G'' \) contain at least 6 nodes.

Phase 3. Merging components with at least 6 edges
Add some extra edges to \( G'' \) by using the techniques described in Lemma 2 to eliminate all bridges.

End Algorithm 1

Observe that at the end of Phase 1, the components \( C_1, \ldots, C_s \) have the following properties:

1. They are vertex disjoint.
2. They are joined by exactly \( s - 1 \) bridges. Moreover, the graph \( T' \) whose vertices are \( C_1, \ldots, C_s \), two of which are adjacent if there is a bridge that joins them, is a tree.
3. If we add an edge joining a vertex of \( C_i \) to a vertex in \( C_j \), the resulting graph has a new 2-edge connected component containing all the vertices of the components in the path joining \( C_i \) to \( C_j \) in \( T' \).
4. If a component \( C_i \) of \( G' \) has \( k \) edges of \( \mathcal{M} \), then \( C_i \) has exactly \( 2k \) nodes, since \( \mathcal{M} \) is a perfect matching. Moreover, these nodes are connected by exactly \( 2k - 1 \) edges of \( G \).

For the tree shown in Fig. 3 (left), the graph \( G' \) obtained by adding \( \mathcal{M} \) to it has two 2-edge connected components, one of which has two vertices, and the other six. They are joined by the bridge \( uv \).
Now we show the details of Phase 2. Complete $G'$ to a triangulation $T$. Given any 2-edge component $C_i$ of $G'$, as $T$ has to be 2-edge connected, then there must exist in $T$ at least one empty triangle $uvw$ such that $u, v \in C_i$ and $w \notin C_i$. These empty triangles $uvw$ will be used in the process of merging components, where some of the red edges of $G'$ will be changed for other edges of $T$, using the method given in the following lemma.

**Lemma 7.** Let $G'$ be a plane geometric graph obtained by adding a set $E$ of red edges to a plane geometric tree $G$ (the black edges). Let $C_i$ be a 2-edge connected component of $G'$. Suppose that there is an empty triangle $uvw$ such that $u, v \in C_i$, $w \notin C_i$, and that in the path $\Pi$ from $u$ to $v$ in $G$ (a path formed by black edges contained in $C_i$) there is a double edge, i.e., a black edge $e = uv'$ and a red copy $e'$. Then we can join $C_i$ with the component containing the vertex $w$, using the same number of red edges as $G'$.

**Proof.** Without loss of generality, suppose that, when edge $u'v'$ is deleted from the tree $G$, $u, u'$ and $w$ are in the same component of $G$, and $v$ and $v'$ are in the other component. Let $\Pi_1$ be the path joining $u'$ and $w$ in $G$ and let $\Pi_2$ be the path joining $v'$ and $v$ in $G$. As $\Pi_1$, edge $u'v'$, $\Pi_2$ and edge $v'v$ form a cycle, necessarily $v'v$ is neither an edge of $G$ nor an edge of $E$. Then, by adding the new red edge $v'v$ to $G'$ and deleting the red copy $e'$ we keep the 2-edge connectivity of $C_i$ and we join it to the 2-edge component containing $w$. □

As a component $C_i$ of $G'$ with two vertices consists of a double edge, a black edge $uv$ and a red copy $e'$, and there is an empty triangle $uvw$ in $T$, then we can apply the previous lemma to join $C_i$ to the 2-edge component containing $w$, by adding $f = uvw$ (or $f = vuv$) and deleting $e'$. See Fig. 3 for an example. Note that $G' + f - e'$ is a plane geometric graph. This process can be applied to the new graph obtained, to successively eliminate all the 2-edge connected components with two vertices. Thus, a new graph $G''$ is obtained with the same number of edges as $G'$ and no 2-edge connected components with two vertices.

We now show how to join the 2-edge connected components of $G''$ with four vertices to other components of $G''$ without increasing the number of edges of $G''$. Let $C_i$ be a 2-edge connected component of $G''$ with four vertices. There are five different ways (up to isomorphisms) in which the black and red edges are distributed in $C_i$. These are depicted in Fig. 4. Cases a), b) and c) correspond to the different ways in which $C_i$ can appear in $G'$ or $G''$, knowing that the red edges are in $M$. Cases d) and e) can occur when we join a 2-edge connected component with two vertices to another one using the previous method.

In cases b) and d) we can eliminate the red edges $u'v$ and $u'v'$, respectively, retaining the 2-edge connectivity of $C_i$. We can then join $C_i$ to another component of $G''$ by using an extra edge of $T$. In case a) we can eliminate edge $uu'$ and add a red edge connecting $u$ to $v$, reducing the configuration to case e).

We now show how to deal with cases c) and e). In both cases, $C_i$ consists of a triangle together with a double edge. Let us assume that the vertices of $C_i$ are labeled as in c) and e) in Fig. 4, where the triangle is, counterclockwise, $uu'v'$, and the double edge is $uv$.

Case c) We know that there is at least one empty triangle $\Delta$ of $T$ with two vertices in $C_i$ and the other vertex $w \notin C_i$. Then, if the two vertices of $\Delta$ in $C_i$ are $uu'$ or $vv'$ or $vv'$, we can apply the method given in the previous lemma. Otherwise, since segment $uv$ must belong to at least one empty triangle of $T$ (or two of them if $uv$ is not an edge on the external face of $T$), then $uvu'$, or $uvu'$, or both must be empty triangles of $T$. Hence, without loss of
generality, suppose that $uvu'$ is an empty triangle of $T$. Then, we can modify the red edges of $C_i$ by taking as new red edges $vu'$ and $uv'$ (this last edge becoming a double edge), instead of $u'v'$ and $uv$. This is the case shown in Fig. 4c.

Again, if the two vertices of $\Delta$ in $C_i$ are $v'u$, or $v'u'$, we can apply the method of the previous lemma. Otherwise, necessarily $u'v'$, or $uv$ or $uv'$, or both, are empty triangles of $T$ and the only possibility left for $\Delta$ is to be a triangle of type $uu'w$. However, we have the two empty triangles $uvu'$ and $uu'w$, and since it is impossible to have three empty triangles with the same side $uu'$ in $T$, then $uvu'$ cannot be an empty triangle and necessarily $uv'v$ must be the empty triangle. Then taking as new red edges of $C_i$ the edges $uv$ and $uv'$, we can apply the method of Lemma 7.

Case e) This is solved in a similar way. Without loss of generality, suppose that the red edge in the triangle $uu'v'$ is the edge $uv'$. If the two vertices of $\Delta$ in $C_i$ are $vu$, or $vu'$ or $vv'$, we apply the previous lemma. Otherwise, $uvv'$, or $u'uv'$, or both, are empty triangles of $T$. If the triangle $uvv'$ is empty, then we can delete the two red edges of $C_i$, add the red edge $vv'$ and use an extra red edge to join $C_i$ with another 2-edge component. Suppose then, that $uvu'$ is an empty triangle of $T$. We can modify the red edges of $C_i$ by taking as new red edges $vu'$ and $u'v'$, this last edge becoming a double edge (see Fig. 4e).

Again, if the two vertices of $\Delta$ in $C_i$ are $v'v$, or $v'u'$, we can apply the previous lemma. Otherwise, necessarily $v'v'u$, or $v'u'v$, or both, are empty triangles, and $\Delta$ must be a triangle of type $uu'w$. As before, triangles $uu'w$, $uu'v$ and $uu'v'$ cannot be empty at the same time, so the empty triangle is $v'v'u$. But then, we can delete the red edges $vu'$ and $u'v'$, add the red edge $vu'$ and use an extra red edge to join $C_i$ with another 2-edge component.

Summarizing, we have just proved:

**Lemma 8.** Let $G$ be a plane geometric tree with $n$ vertices. If $n$ is even, then by adding at most $\frac{n}{2}$ edges to $G$, we can obtain a plane geometric graph all of whose 2-edge connected components have at least six vertices.

We can now prove:

**Theorem 2.** Any plane geometric tree $G$ with $n \geq 6$ vertices can be augmented to a 2-edge connected plane geometric graph by adding at most $\lfloor \frac{2n}{3} \rfloor - 1$ edges if $n$ is even, and at most $\lfloor \frac{2(n+1)}{3} \rfloor - 1$ edges if $n$ is odd.
Proof. For \( n \) even, and using Lemma 8, by adding at most \( \frac{n}{2} \) edges to \( G \) we can obtain a plane geometric graph such that all its 2-edge connected components have at least 6 vertices. We then have at most \( s = \lceil n/6 \rceil \) components joined by exactly \( s - 1 \) bridges. Each of the bridges can be eliminated by adding an extra edge to \( G'' \) as in Lemma 2.

For \( n \) odd, we choose an arbitrary vertex \( v \) and we add a new vertex \( v' \) (as if \( v \) had been duplicated) connected to \( v \) by an edge of length \( \epsilon \), arbitrarily small. We apply the lemma for this new tree, delete \( v' \), and connect \( v \) to the edges adjacent to \( v' \).

Observe that at this point we may have some double edges, which can now be eliminated as in Lemma 4 without increasing the total number of edges.

With respect to the computational complexity of the proposed method, notice that phase 1 of Algorithm 1, building a matching using vertices of a simple polygon \( P \), as described in Lemma 5, can be theoretically done in linear time. First, triangulate \( P \), and delete some diagonals to obtain a convex partition of \( P \). This process can be done in linear time. The other two steps, the assignment of an even number of vertices to each convex region and the matching among vertices in each region, can be done again with the same complexity.

Similarly, phases 2 and 3 of the algorithm are linear. This is clear for phase 3 because we can obtain a compatible triangulation in linear time, and also the 2-edge-connected components and the bridges can be calculated with the same complexity. In phase 2, merging components with 2 or 4 vertices, some of the edges of the triangulation are chosen as new red edges, but the number of changes and candidates for changes are again \( O(n) \). In order to choose one of these new edges, we have to decide, given an empty triangle \( uvw \), whether the duplicated edge \( u'v' \) is in the black path from \( w \) to \( v \) or not. However, by doing a linear time preprocessing step in the tree, this decision can be made in constant time.

4. Connectivity for arbitrary plane geometric graphs

In the previous section, we provided a bound on the maximum number of edges that need to be added to a plane geometric tree to obtain a 2-edge connected plane geometric graph. In this section we study the same problem for arbitrary plane connected geometric graphs.

We now construct plane geometric graphs that need at least \( \frac{2n-2}{3} \) added edges to make them 2-edge connected.

Let \( G_1 \) be a triangulation with \( n_1 \) vertices, \( k_1 \) of which belong to the external face of \( G \). Then \( G_1 \) has \( f_1 = 2n_1 - k_1 - 1 \) faces and \( e_1 = 3n_1 - k_1 - 3 \) edges. In each internal face of \( G_1 \), place an extra vertex adjacent to a vertex of the face. We also add \( k_1 \) vertices in the external face, close enough to the edges of the external face of \( G_1 \). Each added vertex is adjacent to one vertex of \( G_1 \) as shown in Fig. 5. Let \( G_2 \) be the graph thus obtained. \( G_2 \) has \( n = 3n_1 - 2 \) vertices, \( 2n_1 - 2 \) of which have degree one. Clearly, to make \( G_2 \) 2-edge connected, we need to add an edge for each of the \( 2n_1 - 2 \) vertices of \( G_2 \) of degree one.

By deleting one or two vertices of degree one from this construction, if necessary, we have proved:

Fig. 5. A graph that requires \( \frac{2n-2}{3} \) extra edges.
Lemma 9. For each positive integer \( n \geq 3 \) there is a plane connected geometric graph \( G \) with \( n \) vertices such that at least \( \left\lfloor \frac{2n-2}{3} \right\rfloor \) edges must be added to \( G \) in order to make \( G \) 2-edge connected.

We can also prove:

Lemma 10. Any plane connected geometric graph \( G \) can be augmented to a 2-edge connected plane geometric graph by adding at most \( \frac{6n}{7} \) edges.

Proof. Observe first that any plane geometric graph with \( n \) vertices, \( c \geq 2 \) faces, \( a \) edges, and \( b \) bridges satisfies: \( 3c + 2b \leq 2a \). Using Euler’s polyhedral formula, \( n + c = a + 2 \), we obtain \( 2b \leq 3n - a - 6 \). Then any geometric graph with \( a \geq \frac{9n}{7} \) edges has at most \( \frac{6n}{7} \) bridges, and thus can be augmented to a 2-edge connected graph by adding at most one edge per bridge.

Suppose then that \( a \) is less than \( \frac{9n}{7} \). Choose any spanning tree \( T \) of \( G \). For any edge \( e = uv \) of \( G \) not in \( T \), choose a new vertex \( v_e \) close enough to \( v \) in the edge \( uv \), remove \( e \) from \( G \) and add the new edge \( uv_e \) (essentially edge \( uv_e \) is the same as edge \( uv \)). The graph thus obtained is a tree with exactly \( a + 1 \) vertices. Then by Theorem 2 the graph can be augmented by adding to it at most

\[
\frac{2\left(\frac{9n}{7} + 1\right)}{3} - 1 \leq \frac{6n}{7}
\]

edges. Lastly, by deleting each added vertex \( v_e \) and connecting the edges that were adjacent to \( v_e \) back to \( v \), where \( v \) is the endpoint of the edge \( e = uv \) closest to \( v_e \), we obtain the desired result. \( \square \)

Let \( k(n) \) be the smallest integer such that any connected plane geometric graph with \( n \) vertices can be augmented to a 2-edge connected plane geometric graph by adding at most \( k(n) \) edges. Combining Lemmas 9 and 10 we have:

Theorem 3. For \( n \geq 6 \),

\[
\frac{2n - 2}{3} \leq k(n) \leq \frac{6n}{7}.
\]

We conclude by presenting a result that can in some instances help to reduce the number of edges that need to be added to some plane geometric graphs to make them 2-edge connected. We say that a vertex of a graph \( G \) is odd if it has odd degree. A simple polygon such that some of its edges belong to \( G \) and the others are line segments joining pairs of visible vertices of \( G \) will be called a compatible cycle.

Lemma 11. Let \( G \) be a plane geometric graph. If there is a compatible cycle that contains all the odd vertices of \( G \), then \( G \) can be augmented to a 2-edge connected plane geometric graph by adding at most \( \lfloor n/2 \rfloor \) edges.

Proof. Recall first that any graph has an even number of odd vertices. Let \( e = uv \) be a bridge of \( G \), and let \( G_1 \) and \( G_2 \) be the components of \( G - e \), \( u \in G_1 \). Since \( G_1 \) and \( G_2 \) must have an even number of odd vertices, and \( u \) and \( v \) are the only vertices that change their degrees in \( G_1 \) and \( G_2 \) with respect to the degrees in \( G \) of the vertices, then necessarily \( G_1 \) and \( G_2 \) contain an odd number of odd vertices of \( G \). In particular, each of them contains at least one odd vertex of \( G \).

Suppose that \( G \) contains \( 2k \) odd vertices, \( k \geq 1 \), and let \( P \) be a compatible cycle of \( G \) containing all the odd vertices of \( G \). Suppose that the odd vertices of \( G \) appear in the order \( i_0, \ldots, i_{2k-1} \) along \( P \). For \( j = 0, \ldots, 2k - 1 \), let \( P_j \) be the path contained in \( P \) joining \( i_j \) to \( i_{j+1} \), addition taken mod \( 2k \).

Let \( S_0 \) be the set containing all \( P_j \) for \( j \) even, and \( S_1 \) the set of paths \( P_j \), \( j \) odd. We claim that if we add to \( G \) a red edge for each edge of a path in \( S_0 \), the resulting graph \( G' \) contains no bridges, and is therefore 2-edge connected. Observe that double edges are allowed as before.

Suppose otherwise that \( G' \) contains a bridge \( e \). As we proved above, each of the components of \( G - e \), \( G_1 \) and \( G_2 \), has an odd number of odd vertices of \( G \). It follows that at least one of the paths in \( S_0 \) must join an odd vertex of \( G_1 \) to one in \( G_2 \). This path generates a red path in \( G' \) from \( G_1 \) to \( G_2 \) that bypasses \( e \), contradicting the assumption that \( e \) is a bridge.
By symmetry, the graph obtained from $G$ by adding a red edge for each edge in a path of $S_1$ contains no bridges, and thus is 2-edge connected. Observe that by Lemma 4, double edges can be eliminated without increasing the total number of edges, keeping the graphs 2-edge connected.

Since either $S_0$ or $S_1$ contains at most $\left\lfloor \frac{n}{2} \right\rfloor$ edges, the result follows. \qed

In particular, this result implies that any connected plane geometric graph whose vertices are in convex position can be augmented to a 2-edge connected plane geometric graph with at most $\left\lfloor \frac{n}{2} \right\rfloor$ edges.

5. Conclusions

In this paper we have considered the problem of calculating the minimum number of edges that can make every connected plane geometric graph with $n$ vertices 2-edge or 2-vertex connected. Upper and lower bounds were obtained for arbitrary connected plane geometric graphs, and for plane geometric graphs which are trees.

Finally, we close with two conjectures:

**Conjecture 1.** Any plane geometric tree with $n$ vertices can be augmented to a 2-edge connected plane geometric graph by adding at most $\frac{n^2}{2} \pm c$ edges, $c$ being a constant.

**Conjecture 2.** Any connected plane geometric graph with $n$ vertices can be augmented to a 2-edge connected plane geometric graph by adding at most $2n^3 \pm c$ edges, $c$ being a constant.

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